TOPOLOGICAL SINGULARITIES IN THE CODIMENSION-TWO STRATUM OF NONCOLLAPSED RICCI LIMITS

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ABSTRACT. We construct classical noncollapsed Ricci limits with a topologically singular point whose unique tangent cone has a topologically Euclidean vertex. Fix an integer $n \geq 4$ and a closed connected oriented smooth manifold Q^n such that, for some smoothly embedded closed n-disc $D^n \subset Q$, $Q \setminus \operatorname{int} D^n$ carries a positive-Ricci metric with unit-round strictly convex boundary. For all sufficiently small $q > 0$, there is a complete pointed $(n+1)$-dimensional Gromov–Hausdorff limit (Y, d, p), realized by complete smooth boundaryless $(n+1)$-manifolds with nonnegative Ricci curvature and uniformly positive based unit-ball volumes, with unique tangent cone $\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})$ at p, where $\mathrm{C}(S^1_{2\pi q})$ is the cone over the circle of length $2\pi q$. A neighbourhood of p is homeomorphic to the open topological cone over $Q\#(-Q)$, where $-Q$ has opposite orientation, and $H_j(Y, Y \setminus \{p\}; A) \cong \widetilde{H}_{j-1}(Q\#(-Q); A)$ for every abelian group A and $j \geq 0$. If $Q\#(-Q)$ is not an integral homology n-sphere, then p is topologically singular. At p, the density is q, and Y has empty Kapovitch–Mondino boundary. For each integer $N \geq 5$, there is an N-dimensional example where p is the unique nonmanifold point and every tangent cone at every point splits $\mathbb{R}^{N-2}$. Thus the Cheeger–Colding codimension-four singular stratum is empty although Y is not a topological manifold, disproving the Cheeger–Colding conjecture that the interior of its complement is a topological manifold.

CONTENTS

1. Introduction

Let $N \geq 2$ be an integer, and let (M_i^N, g_i, p_i) be complete connected pointed smooth Riemannian N-manifolds without boundary. Assume that constants $K \in \mathbb{R}$ and $v > 0$ satisfy, for every i,

$$\mathrm{Ric}_{g_i} \geq K g_i, \qquad \mathrm{Vol}_{g_i}(B_1(p_i)) \geq v,$$

and suppose that $(M_i, d_{g_i}, p_i) \xrightarrow{\text{pGH}} (Z, d, p)$. A pointed limit arising in this way is called a classical noncollapsed Ricci limit of dimension N. Every tangent cone of Z is a metric cone [5]. At a fixed point, tangent cones need not be unique; Colding–Naber constructed examples in which the topology of the cross-section varies and examples in which the dimension of the maximal Euclidean factor varies [8].

We use the following notation in the statements below. The set $\mathrm{Tan}(X, x)$ consists of the pointed-isometry classes of pointed limits of $(X, r_i^{-1} d, x)$ along sequences $r_i \downarrow 0$. A pointed space (T, o) splits $\mathbb{R}^j$ if it is pointed-isometric to $(\mathbb{R}^j \times W, (0, w))$, with the product metric, for some pointed metric space (W, w). For a nonempty compact metric space F, $\mathrm{C}(F)$ denotes its Euclidean metric cone, with vertex o, and S^1_ℓ denotes the intrinsic circle of total length ℓ. For a nonempty compact topological space M and $r > 0$, let $\mathrm{C}_{\mathrm{top},r}(M) := ([0, r) \times M)/(\{0\} \times M)$ be the open truncated topological cone, whose vertex is also denoted by o. If X is an m-dimensional classical noncollapsed Ricci limit, then

$$\mathcal{R}_{\mathrm{met}}(X) := \{x : \mathrm{Tan}(X, x) = \{\mathbb{R}^m\}\}, \qquad \Theta^m(X, x) := \lim_{r \downarrow 0} \frac{\mathcal{H}^m(B_r(x))}{\omega_m r^m},$$

where $\omega_m = \mathcal{H}^m(B_1^{\mathbb{R}^m}(0))$. The symbol ∂X denotes the Kapovitch–Mondino boundary, not the topological boundary; $\mathrm{Stop}(X)$ is the set of points having no open pointed neighbourhood homeomorphic to $(B_1^{\mathbb{R}^m}(0), 0)$; and $\mathrm{Int}_X(A)$ denotes the interior of A in X. Full conventions appear in Section 2.

For the counterexample discussion, assume $N \geq 5$. For $k \in \{0, \dots, N-1\}$, set

$$\mathcal{S}^k(Z) = \{x \in Z : \text{no tangent cone at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

Cheeger and Colding call x k-*weakly Euclidean* if some tangent cone at x splits $\mathbb{R}^k$. Their set $D_k(Z)$ of points which are not $(k+1)$-weakly Euclidean is therefore $\mathcal{S}^k(Z)$. In the noncollapsed N-dimensional category their regular set is $\mathcal{R}_{\mathrm{met}}(Z)$ by Section 2; hence their singular stratum is

$$\mathcal{S}_k^{\mathrm{CC}}(Z) := D_k(Z) \setminus \mathcal{R}_{\mathrm{met}}(Z).$$

Conjecture 0.7 of [5] states that

$$\mathrm{Int}_Z\big(Z \setminus \mathcal{S}_{N-4}^{\mathrm{CC}}(Z)\big) \quad \text{is a topological manifold.} \tag{1.1}$$

We prove that (1.1) fails in every dimension $N \geq 5$.

The known structure theorems delimit the possible locus of such a failure. Cheeger–Jiang–Naber proved that $\mathcal{S}^k$ is k-rectifiable and that, at $\mathcal{H}^k$-almost every point of $\mathcal{S}^k$, every tangent cone splits $\mathbb{R}^k$ [6]. For noncollapsed limits under a two-sided Ricci bound, Cheeger–Naber proved that the singular set is closed and has Hausdorff codimension at least four; its complement is a manifold with a $C^{1,\alpha}$ Riemannian metric for every $0 < \alpha < 1$. If the approximating metrics are Einstein, the limiting metric is smooth on this complement [7, Theorem 1.4 and Section 2.2]. Bruè–Pigati–Semola proved that, if $Z = \mathbb{R}^{N-3} \times Z^3$ as a metric-measure space, then Z^3 is a topological 3-manifold. They also proved that, if every tangent cone at a point is $(N-4)$-symmetric, then their cross-sections are mutually homeomorphic [4]. These results concern, respectively, almost-everywhere stratum structure, two-sided curvature bounds, the product hypothesis $Z = \mathbb{R}^{N-3} \times Z^3$, or symmetry of every tangent at the point. None excludes the exceptional point constructed here.

Menguy constructed four-dimensional noncollapsed Ricci limits that are not topological manifolds even though all their tangent cones are homeomorphic to $\mathbb{R}^4$ [11, Theorem 0.6]; see also [4, Section 1.2]. Thus the property that all tangent cones are topologically Euclidean does not by itself characterize local manifold structure. Colding–Naber constructed a five-dimensional noncollapsed Ricci limit with nonhomeomorphic tangent cones at one point and, in other examples, tangent cones with different Euclidean splitting ranks [8]. Reiser proved that any finite collection of closed smooth manifolds of one fixed dimension, each admitting a core metric, can occur as cross-sections of tangent cones at one point [13]. The examples used for Corollary B exhibit a different combination: the tangent cone at p is unique and isometric to $\mathbb{R}^{N-2} \times \mathrm{C}(S^1_{2\pi q})$; its maximal Euclidean factor has dimension $N-2$; its distinguished point is topologically regular; $\mathrm{Stop}(Y) = \{p\}$; and $\mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = \emptyset$. Moreover, a sufficiently small pointed neighbourhood of p is homeomorphic to the open topological cone over the oriented connected sum $Q\#(-Q)$, where $-Q$ carries the opposite orientation. For the choices used in Corollary B, the resulting local homology is incompatible with a manifold chart at p.

Theorem A. *Let $n \geq 4$ and let Q^n be core-admissible in the sense of Definition 3.1. There exists $q_0 = q_0(n, Q) > 0$ such that, for every $q \in (0, q_0)$, there exist a complete classical noncollapsed Ricci limit (Y^{n+1}, d, p) and a number $r_0 > 0$ with*

$$\mathrm{Tan}(Y, p) = \left\{\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})\right\}, \qquad (B_{r_0}(p), p) \cong \left(\mathrm{C}_{\mathrm{top}, r_0}(Q\#(-Q)), o\right).$$

Moreover,

$$\Theta^{n+1}(Y, p) = q, \qquad \partial Y = \emptyset,$$

and, for every abelian group A,

$$H_j(Y, Y \setminus \{p\}; A) \cong \widetilde{H}_{j-1}(Q\#(-Q); A) \qquad (j \geq 0).$$

The complement $Y \setminus \{p\}$ is diffeomorphic to $(0, \infty) \times (Q\#(-Q))$. The realizing manifolds have nonnegative Ricci curvature and a common positive lower bound for their based unit-ball volumes. If $Q\#(-Q)$ is not an integral homology sphere, then

$$p \in \mathrm{Stop}(Y) \cap \left(\mathcal{S}^{n-1}(Y) \setminus \mathcal{S}^{n-2}(Y)\right).$$

The distinguished point $(0, o)$ of $\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})$ is nevertheless a topological $(n+1)$-manifold point.

Corollary B. *For every integer $N \geq 5$, there exist $q \in (0, 1)$ and a complete classical noncollapsed N-dimensional Ricci limit (Y, d, p) such that*

$$\partial Y = \emptyset, \qquad \mathrm{Tan}(Y, p) = \left\{\mathbb{R}^{N-2} \times \mathrm{C}(S^1_{2\pi q})\right\}.$$

Moreover,

$$\mathrm{Stop}(Y) = \{p\}, \qquad \mathcal{S}^{N-3}(Y) = \emptyset.$$

Consequently,

$$\mathcal{S}^{N-4}(Y) = \mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = \emptyset, \qquad \mathrm{Int}_Y\left(Y \setminus \mathcal{S}^{\mathrm{CC}}_{N-4}(Y)\right) = Y,$$

and Y is not a topological manifold.

Theorem A is Theorem 3.2; Corollary B is Corollary 3.4. For the choices made in Corollary 3.3, one has $H_2(Q\#(-Q); \mathbb{Z}) \cong \mathbb{Z}^2$. Every point of $Y \setminus \{p\}$ is smooth, while the unique tangent at p splits $\mathbb{R}^{N-2}$. Hence $\mathcal{S}^{N-3}(Y) = \emptyset$, and the nesting of the degeneracy sets gives $\mathcal{S}^{N-4}(Y) = \emptyset$. The local homology formula gives $H_3(Y, Y \setminus \{p\}; \mathbb{Z}) \cong \mathbb{Z}^2$, so p admits no manifold chart. Thus $\mathrm{Stop}(Y) = \{p\}$ and $\mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = \emptyset$. Therefore the interior in (1.1) is all of Y, while Y is not a topological manifold.

Here *codimension two* refers to the infinitesimal metric condition $p \in \mathcal{S}^{N-2}(Y) \setminus \mathcal{S}^{N-3}(Y)$, not to the Hausdorff codimension of the topological singular set. In Corollary B, $\mathrm{Stop}(Y) = \{p\}$. Thus $E := \{p\}$ is closed, $\dim_{\mathcal{H}} E = 0$, and $Y \setminus E$ is smooth. The failure concerns the Cheeger–Colding stratum defined above: $\mathcal{S}^{\mathrm{CC}}_{N-4}(Y)$ does not contain the topological singular set, and in the present examples even the larger degeneracy set $\mathcal{S}^{N-4}(Y)$ is empty.

Section 4 separates the two parts of the example. It first constructs on M_Q a Ricci-positive family converging to the singular link $L_{n,q}$, and then feeds that family into the radial construction of Colding–Naber.

2. Preliminaries and conventions

All metric balls are open, and a bar denotes metric closure. We normalize Hausdorff measure so that, on every m-dimensional smooth Riemannian manifold, $\mathcal{H}^m_d$ equals the Riemannian volume measure. Here d is the ambient metric; the subscript is omitted when the metric is fixed. If $A \subset Z$, then $\mathrm{Int}_Z(A)$ is the interior of A in Z. Put $S^0 := \{-1, 1\}$ with $d_{S^0}(-1, 1) = \pi$. For every integer $m \geq 1$, S^m denotes the unit round m-sphere and g_{S^m} its Riemannian metric. Set $\omega_m := \mathcal{H}^m(B_1^{\mathbb{R}^m}(0))$ for $m \geq 1$ and $\sigma_m := \mathcal{H}^m(S^m)$ for $m \geq 0$. For $\ell > 0$, S^1_ℓ is the circle of total length ℓ with its intrinsic length metric. The symbols $\xrightarrow{\text{pGH}}$ and $\xrightarrow{\text{pmGH}}$ denote pointed Gromov–Hausdorff and pointed measured Gromov–Hausdorff convergence, respectively.

Definition 2.1 (Classical noncollapsed Ricci limit). Let $N \geq 2$ be an integer. A pointed proper metric space (Y, d, p) is a *classical noncollapsed Ricci limit of dimension N* if there exist a sequence $(M_i, g_i, p_i)_{i\in\mathbb{N}}$ of complete connected pointed smooth Riemannian N-manifolds without boundary and constants $K \in \mathbb{R}$ and $v > 0$ such that, for every $i \in \mathbb{N}$,

$$\mathrm{Ric}_{g_i} \geq K g_i, \qquad \mathrm{Vol}_{g_i}\left(B_1^{g_i}(p_i)\right) \geq v,$$

and the sequence satisfies $(M_i, d_{g_i}, p_i) \xrightarrow{\text{pGH}} (Y, d, p)$.

With the normalization above, $\mathrm{Vol}_{g_i} = \mathcal{H}^N_{d_{g_i}}$. The smooth lower Ricci bound makes M_i, equipped with d_{g_i} and $\mathcal{H}^N_{d_{g_i}}$, a noncollapsed $\mathrm{RCD}(K, N)$ space [10, p. 446]. Moreover, $\liminf_{i\to\infty} \mathcal{H}^N_{d_{g_i}}(B_1^{g_i}(p_i)) \geq v > 0$. This lower bound excludes alternative (ii) of [9, Theorem 1.2]; hence alternative (i) applies and gives

$$(M_i, d_{g_i}, \mathcal{H}^N_{d_{g_i}}, p_i) \xrightarrow{\text{pmGH}} (Y, d, \mathcal{H}^N_d, p).$$

In particular, $(Y, d, \mathcal{H}^N_d)$ is a noncollapsed $\mathrm{RCD}(K, N)$ space, and $\mathcal{H}^N_d$ has full support.

Let M be a smooth Riemannian manifold with boundary. Along ∂M, let ν be the inward unit normal. For $v, w \in T(\partial M)$, set

$$\mathrm{II}(v, w) := -\langle \nabla_v \nu, w\rangle.$$

If $m \geq 2$, B_ρ is a geodesic ball in the unit round m-sphere, and $0 < \rho < \pi/2$, then

$$\mathrm{II}_{\partial B_\rho} = \cot(\rho)\, g_{\partial B_\rho}.$$

Indeed, the round metric is $dr^2 + \sin^2(r) g_{S^{m-1}}$, the inward normal along ∂B_ρ is $-\partial_r$, and $\nabla_v \partial_r = \cot(\rho) v$ for $v \in T(\partial B_\rho)$.

Let (F, d_F) be a nonempty compact metric space. Its metric cone is

$$\mathrm{C}(F) := ([0, \infty) \times F)/(\{0\} \times F),$$

with vertex $o = [0, F]$ and Euclidean cone metric

$$d_{\mathrm{C}(F)}([r, x], [s, y])^2 = r^2 + s^2 - 2rs \cos\bigl(\min\{d_F(x, y), \pi\}\bigr).$$

In particular, $d_{\mathrm{C}(F)}(o, [r, x]) = r$. If X is a closed Riemannian manifold with metric h, write $\mathrm{C}(X, h)$ for the cone over the metric space (X, d_h). If M is a nonempty compact topological space and $r > 0$, put

$$\mathrm{C}_{\mathrm{top},r}(M) := ([0, r) \times M)/(\{0\} \times M), \qquad \overline{\mathrm{C}}_{\mathrm{top},r}(M) := ([0, r] \times M)/(\{0\} \times M),$$

with the quotient topology. The common collapsed class is denoted by o. Thus $\mathrm{C}_{\mathrm{top},r}(M)$ is the open truncated topological cone, whereas $\overline{\mathrm{C}}_{\mathrm{top},r}(M)$ is the closed truncated topological cone.

Definition 2.2 (Ricci closability). For a pointed metric space (N, d, q) and $0 \leq a < b \leq \infty$, let $A^N_{a,b}(q) := \{z \in N : a < d(q, z) < b\}$, with the distance restricted from N. A closed Riemannian n-manifold (X, g) is *Ricci closable* if, for every $\epsilon \in (0, 1)$, there exist a complete connected pointed smooth noncompact $(n+1)$-manifold $(N_\epsilon, H_\epsilon, q_\epsilon)$ without boundary, a compact connected smooth domain $K_\epsilon \subset N_\epsilon$ with $q_\epsilon \in \mathrm{Int}_{N_\epsilon} K_\epsilon$, and a diffeomorphism of manifolds with boundary

$$\Phi_\epsilon : [1,\infty) \times X \longrightarrow N_\epsilon \setminus \mathrm{Int}_{N_\epsilon} K_\epsilon$$

such that

$$\mathrm{Ric}_{H_\epsilon} \geq 0, \qquad \Phi_\epsilon^* H_\epsilon = dR^2 + R^2(1-\epsilon)g.$$

The pullback identity is an identity of smooth Riemannian tensors. The domain K_ϵ need not be a metric ball, and R is not required to equal $d_{H_\epsilon}(q_\epsilon, \cdot)$. We use this conical-end formulation throughout. An end beginning at a positive coordinate $R = A$ can be normalized to begin at $R = 1$ by replacing H_ϵ with $A^{-2}H_\epsilon$ and R with R/A; the cross-sectional metric is unchanged.

Let $N \geq 2$ be an integer. For a pointed proper metric space (Z, d, z), $\mathrm{Tan}(Z, z)$ is the set of pointed-isometry classes of pointed proper metric spaces (T, d_T, o) for which there is a sequence $r_i \downarrow 0$ such that

$$(Z, r_i^{-1} d, z) \xrightarrow{\text{pGH}} (T, d_T, o).$$

Put $\mathbb{R}^0 := \{0\}$. For an integer $m \geq 0$, a pointed metric space (T, o) *splits* $\mathbb{R}^m$ if it is pointed-isometric to $(\mathbb{R}^m \times W, (0, w))$, with the product metric, for some pointed proper metric space (W, d_W, w).

Suppose that Z is a classical noncollapsed Ricci limit of dimension N. For $k \in \{0, \dots, N-1\}$, put

$$\mathcal{S}^k(Z) := \{x \in Z : \text{no } T \in \mathrm{Tan}(Z, x) \text{ splits } \mathbb{R}^{k+1}\}.$$

If a tangent splits $\mathbb{R}^{k+2}$, then it also splits $\mathbb{R}^{k+1}$. Hence

$$\mathcal{S}^0(Z) \subset \mathcal{S}^1(Z) \subset \cdots \subset \mathcal{S}^{N-1}(Z).$$

The metric regular set is

$$\mathcal{R}_{\mathrm{met}}(Z) := \{x \in Z : \mathrm{Tan}(Z, x) = \{(\mathbb{R}^N, d_{\mathrm{eucl}}, 0)\}\}.$$

A point $x \in Z$ is topologically regular if there are an open neighbourhood $U \subset Z$ of x and a homeomorphism

$$(U, x) \cong (B_1^{\mathbb{R}^N}(0), 0).$$

Put $\mathrm{Stop}(Z) := Z \setminus \{x \in Z : x \text{ is topologically regular}\}$.

Fix a realization of Z as in Definition 2.1, and let K be its common lower Ricci bound. Define

$$s_{K,N}(t) := \begin{cases} \sqrt{(N-1)/K}\ \sin\big(t\sqrt{K/(N-1)}\big), & K > 0, \\ t, & K = 0, \\ \sqrt{(N-1)/|K|}\ \sinh\big(t\sqrt{|K|/(N-1)}\big), & K < 0, \end{cases}$$

and set $v_{K,N}(r) := N\omega_N \int_0^r s_{K,N}(t)^{N-1}\, dt$. The $\mathrm{RCD}(K,N)$ condition implies the $\mathrm{CD}(K,N)$ condition. Hence, for $0 < r < R$ such that $s_{K,N}$ is positive on $(0, R]$, the Bishop–Gromov inequality gives

$$\frac{\mathcal{H}_d^N(B_r(x))}{v_{K,N}(r)} \geq \frac{\mathcal{H}_d^N(B_R(x))}{v_{K,N}(R)} \qquad (x \in Z)$$

[17, Theorem 2.3]. Since $\mathcal{H}_d^N$ has full support, the right-hand side is positive. The ratio on the left has a limit in $(0, \infty]$ as $r \downarrow 0$. Moreover, $s_{K,N}(t) = t + O(t^3)$ and $v_{K,N}(r) = \omega_N r^N + O(r^{N+2})$. Consequently the volume density

$$\Theta^N(Z, x) := \lim_{r \downarrow 0} \frac{\mathcal{H}_d^N(B_r(x))}{\omega_N r^N}$$

has a positive extended limit. Corollary 2.14 of [9] makes this limit finite and gives $0 < \Theta^N(Z, x) \leq 1$ for every $x \in Z$.

Cheeger–Colding call x regular if, for some integer k, every tangent cone at x is pointed-isometric to $(\mathbb{R}^k, 0)$ [5, §0]. In the present noncollapsed N-dimensional setting, their regular set equals $\mathcal{R}_{\mathrm{met}}(Z)$. To prove this, suppose that every tangent at x is $(\mathbb{R}^k, 0)$, and choose $\rho_i \downarrow 0$. Put $\widehat{d_i} := \rho_i^{-1} d$ and $\mu_i := \rho_i^{-N} \mathcal{H}^N_d$. Choose i_0 so that $K\rho_i^2 \geq -1$ for $i \geq i_0$. For such i, the rescaled space is noncollapsed $\mathrm{RCD}(K\rho_i^2, N)$ and hence noncollapsed $\mathrm{RCD}(-1, N)$. For $R > 0$, define

$$D_R := \sup_{0<t\leq 4R} \frac{v_{-1,N}(2t)}{v_{-1,N}(t)}.$$

The quotient in this supremum extends continuously to $t = 0$ with value 2^N; hence $D_R < \infty$. The Bishop–Gromov inequality then gives

$$\mu_i(B^{\widehat{d_i}}_{2s}(y)) \leq D_R \mu_i(B^{\widehat{d_i}}_s(y))$$

for $i \geq i_0$, $y \in Z$, and $0 < s \leq 4R$. Fix $0 < s \leq R$. Properness makes $\overline{B}^{\widehat{d_i}}_s(y)$ compact; choose a finite maximal $s/2$-separated subset $\{y_j\}_{j\in I}$ of $B^{\widehat{d_i}}_s(y)$. The balls $B^{\widehat{d_i}}_{s/4}(y_j)$ are pairwise disjoint and lie in $B^{\widehat{d_i}}_{5s/4}(y)$. Since $B^{\widehat{d_i}}_{5s/4}(y) \subset B^{\widehat{d_i}}_{4s}(y_j)$, four applications of the displayed measure-doubling inequality give

$$\mu_i(B^{\widehat{d_i}}_{s/4}(y_j)) \geq D_R^{-4} \mu_i(B^{\widehat{d_i}}_{5s/4}(y)).$$

Full support makes the last measure positive. Summing over the disjoint balls yields $|I| \leq D_R^4$; maximality gives $B^{\widehat{d_i}}_s(y) \subset \bigcup_{j\in I} B^{\widehat{d_i}}_{s/2}(y_j)$. Fix $R > 0$ and $\varepsilon > 0$, and choose an integer $q \geq 0$ such that $2^{-q}R < \varepsilon$. Iterating this covering q times gives an ε-net of $B^{\widehat{d_i}}_R(x)$ with at most D_R^{4q} points for every $i \geq i_0$. Each of the finitely many compact balls $\overline{B}^{\widehat{d_i}}_R(x)$ with $i < i_0$ has a finite ε-net. Hence one integer $N(R, \varepsilon)$ bounds the cardinality of the required nets for every i. The pointed compactness theorem [2, Theorem 8.1.10] gives a subsequence $r_i := \rho_{j_i} \downarrow 0$ and a pointed proper metric space (T, o) such that

$$(Z, r_i^{-1} d, x) \xrightarrow{\mathrm{pGH}} (T, o).$$

The limit (T, o) belongs to $\mathrm{Tan}(Z, x)$; the hypothesis therefore gives $(T, o) \cong (\mathbb{R}^k, 0)$. Set $d_i := r_i^{-1} d$. Since $\mathcal{H}^N_{d_i} = r_i^{-N} \mathcal{H}^N_d$, the density identity gives

$$\mathcal{H}^N_{d_i}(B^{d_i}_1(x)) \longrightarrow \Theta^N(Z, x)\omega_N > 0.$$

The rescaled space is noncollapsed $\mathrm{RCD}(Kr_i^2, N)$. For all sufficiently large i, $Kr_i^2 \geq -1$, hence it is also noncollapsed $\mathrm{RCD}(-1, N)$. Alternative (i) of [9, Theorem 1.2] now yields

$$(Z, d_i, \mathcal{H}^N_{d_i}, x) \xrightarrow{\mathrm{pmGH}} (\mathbb{R}^k, d_{\mathrm{eucl}}, \mathcal{H}^N, 0),$$

and the limit is a noncollapsed $\mathrm{RCD}(-1, N)$ space. If $k < N$, then $\mathcal{H}^N(B^{\mathbb{R}^k}_1(0)) = 0$, so the displayed limit measure does not have full support. If $k > N$, then $\mathcal{H}^N(B^{\mathbb{R}^k}_1(0)) = \infty$, so it is not locally finite. Both cases contradict the conclusion of the theorem. Hence $k = N$, and $x \in \mathcal{R}_{\mathrm{met}}(Z)$. Conversely, every $x \in \mathcal{R}_{\mathrm{met}}(Z)$ has all tangent cones pointed-isometric to $(\mathbb{R}^N, 0)$ and is regular in the sense of Cheeger–Colding. Their k-weakly Euclidean definition and the definition of D_k in [5, §0] therefore give

$$D_k(Z) = \mathcal{S}^k(Z), \qquad \mathcal{S}^{\mathrm{CC}}_k(Z) = D_k(Z) \setminus \mathcal{R}_{\mathrm{met}}(Z).$$

The symbol ∂Z denotes the RCD boundary of [10, Definition 4.2], not a topological boundary. Let (T, d_T, o) represent an element of $\mathrm{Tan}(Z, x)$, and choose $r_i \downarrow 0$ such that $(Z, r_i^{-1} d, x) \xrightarrow{\mathrm{pGH}} (T, d_T, o)$. The density identity gives $\mathcal{H}^N_{r_i^{-1}d}(B^{r_i^{-1}d}_1(x)) \to \Theta^N(Z, x)\omega_N > 0$. For all sufficiently large i, the rescaled spaces are noncollapsed $\mathrm{RCD}(-1, N)$ spaces. Alternative (i) of [9, Theorem 1.2] upgrades the convergence to pointed measured Gromov–Hausdorff convergence with reference measure $\mathcal{H}^N_{d_T}$ on T. After rewriting [10, equation (2.16)] in the rescaled spaces, pointed measured convergence shows that its normalizing denominators converge to $c_T := \int_{B_1^T(o)}(1 - d_T(o,z))\, d\mathcal{H}^N_{d_T}(z) > 0$. Consequently $(T, d_T, c_T^{-1}\mathcal{H}^N_{d_T}, o)$ is a metric-measure tangent of $(Z, d, \mathcal{H}^N_d, x)$. Thus [10, Lemma 4.1] applies: the underlying pointed metric space is a cone $\mathrm{C}(F)$ over a noncollapsed $\mathrm{RCD}(N-2, N-1)$ link F. Accordingly,

$$\partial Z := \{x \in Z : \text{ some } T \in \mathrm{Tan}(Z,x) \text{ is } \mathrm{C}(F) \text{ with } \partial F \neq \emptyset\},$$

where the boundary of F is defined inductively by this formula in dimension $N-1$. In dimension one, a singleton, a circle, and a line have empty boundary; the boundary of a half-line is its endpoint, and the boundary of a nondegenerate compact interval is its two-point endpoint set.

Whenever homology appears, A denotes an abelian coefficient group. All homology groups are singular homology groups with coefficients in A. For every nonempty space M, reduced homology is normalized by $\widetilde{H}_{-1}(M;A) := 0$.

3. Main results

Throughout Sections 3–10, let $n \geq 4$ be fixed. If Q is oriented, then the underlying smooth manifold of $-Q$ is Q, and its orientation is reversed. Connected sums are oriented and are taken up to orientation-preserving diffeomorphism. Inequalities between symmetric $(0,2)$-tensors are pointwise quadratic-form inequalities.

Definition 3.1 (Core-admissibility). A closed connected oriented smooth n-manifold Q is *core-admissible* if there are a smoothly embedded closed n-disc $D_Q \subset Q$, the compact manifold with boundary $P_Q := Q \setminus \mathrm{int}_Q D_Q$, a diffeomorphism $\beta_Q : S^{n-1} \to \partial P_Q$, a smooth Riemannian metric k_Q on P_Q, and constants $\kappa_Q, \mu_Q > 0$ such that

$$\begin{gathered} \beta_Q^*(k_Q|_{\partial P_Q}) = g_{S^{n-1}}, \\ \beta_Q^* \mathrm{II}_{\partial P_Q, k_Q} \geq \kappa_Q g_{S^{n-1}}, \qquad \mathrm{Ric}_{k_Q} \geq \mu_Q k_Q. \end{gathered} \tag{3.1}$$

The manifold P_Q is connected. Indeed, its boundary is diffeomorphic to the connected sphere S^{n-1} and is therefore contained in one component of P_Q. Any other component would be disjoint from the boundary, hence open and closed in Q, contrary to the connectedness of Q.

A *core metric* in the sense of [15, Definition 1.2] is a Ricci-positive metric containing an isometrically embedded closed hemisphere of the unit round n-sphere. Every closed connected oriented smooth n-manifold that admits a core metric is core-admissible.

Let Q admit a core metric. By [15, Lemma 2.18], there are a smoothly embedded closed n-disc $D_Q \subset Q$ and a Ricci-positive metric $\bar{k}$ on Q such that, for $P_Q := Q \setminus \mathrm{int}_Q D_Q$, the restriction $k := \bar{k}|_{P_Q}$ has round boundary metric and positive-definite boundary second fundamental form in Reiser's outward-normal convention. If ν is our inward unit normal, then Reiser's outward unit normal is $-\nu$, and his boundary tensor is $\langle \nabla_v(-\nu), w\rangle = -\langle \nabla_v \nu, w\rangle = \mathrm{II}(v,w)$. There are $R > 0$ and a diffeomorphism $\beta_Q : S^{n-1} \to \partial P_Q$ such that $\beta_Q^*(k|_{\partial P_Q}) = R^2 g_{S^{n-1}}$. Set $k_Q := R^{-2}k$. Then $\beta_Q^*(k_Q|_{\partial P_Q}) = g_{S^{n-1}}$, $\mathrm{Ric}_{k_Q} = \mathrm{Ric}_k$, and $\beta_Q^*\mathrm{II}_{\partial P_Q, k_Q} = R^{-1}\beta_Q^*\mathrm{II}_{\partial P_Q, k}$. Let $\lambda_{\min}$ denote the least eigenvalue of a self-adjoint endomorphism. The continuous functions

$$x \longmapsto \lambda_{\min}\big(k_Q^{-1}\,\mathrm{Ric}_{k_Q}\big)(x), \qquad y \longmapsto \lambda_{\min}\big(g_{S^{n-1}}^{-1}\beta_Q^*\mathrm{II}_{\partial P_Q, k_Q}\big)(y)$$

are positive on the compact spaces P_Q and S^{n-1}, respectively. Let μ_Q and κ_Q be their respective minima. Then $\mu_Q, \kappa_Q > 0$, and (3.1) holds.

Theorem 3.2. *Let $n \geq 4$ and let Q^n be core-admissible. Put $M_Q := Q\#(-Q)$. There exists $q_0 = q_0(n,Q) > 0$ such that, for every $q \in (0, q_0)$, there exist a complete pointed classical noncollapsed Ricci limit $(Y^{n+1}_{Q,q}, d, p)$ and a number $r_0 > 0$ satisfying:*

(i) $\mathrm{Tan}(Y_{Q,q}, p) = \{\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})\}$;

(ii) $(B_{r_0}(p), p) \cong (\mathrm{C}_{\mathrm{top},r_0}(M_Q), o)$;
(iii) *for every abelian group A and every integer $j \geq 0$,*

$$H_j(Y_{Q,q}, Y_{Q,q} \setminus \{p\}; A) \cong \widetilde{H}_{j-1}(M_Q; A);$$

(iv) $\Theta^{n+1}(Y_{Q,q}, p) = q$;
(v) $\partial Y_{Q,q} = \emptyset$.

There is a diffeomorphism $Y_{Q,q} \setminus \{p\} \cong (0, \infty) \times M_Q$. The realizing sequence may be chosen to consist of complete connected smooth manifolds without boundary, with nonnegative Ricci curvature and a common positive lower bound for the volumes of their based unit balls. If M_Q is not an integral homology n-sphere, then

$$p \in \mathrm{Stop}(Y_{Q,q}) \cap \big(\mathcal{S}^{n-1}(Y_{Q,q}) \setminus \mathcal{S}^{n-2}(Y_{Q,q})\big).$$

The distinguished point $(0, o)$ of $\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})$ is a topological $(n+1)$-manifold point.

Corollary 3.3. *For every integer $N \geq 5$ there exist $q \in (0, 1)$ and a complete pointed classical noncollapsed N-dimensional Ricci limit (Y, d, p) such that*

$$\partial Y = \emptyset, \qquad p \in \mathrm{Stop}(Y) \cap \big(\mathcal{S}^{N-2}(Y) \setminus \mathcal{S}^{N-3}(Y)\big),$$
$$\mathrm{Tan}(Y, p) = \big\{\mathbb{R}^{N-2} \times \mathrm{C}(S^1_{2\pi q})\big\}.$$

For a classical noncollapsed N-dimensional Ricci limit Z with $N \geq 5$, the conventions identify Cheeger–Colding's regular set with $\mathcal{R}_{\mathrm{met}}(Z)$ and give

$$D_{N-4}(Z) = \mathcal{S}^{N-4}(Z), \qquad \mathcal{S}^{\mathrm{CC}}_{N-4}(Z) = D_{N-4}(Z) \setminus \mathcal{R}_{\mathrm{met}}(Z).$$

Cheeger–Colding Conjecture 0.7 asserts that

$$\mathrm{Int}_Z\big(Z \setminus \mathcal{S}^{\mathrm{CC}}_{N-4}(Z)\big) \quad \text{is a topological manifold}$$

[5, Definitions 0.3–0.4 and Conjecture 0.7].

Corollary 3.4. *For every integer $N \geq 5$ there exist $q \in (0, 1)$ and a complete classical noncollapsed N-dimensional Ricci limit (Y, d, p) such that*

$$\partial Y = \emptyset, \qquad \mathrm{Stop}(Y) = \{p\}, \qquad \mathcal{S}^{N-3}(Y) = \emptyset,$$
$$\mathrm{Tan}(Y, p) = \big\{\mathbb{R}^{N-2} \times \mathrm{C}(S^1_{2\pi q})\big\}.$$

The realizing sequence may be chosen to consist of complete connected smooth N-manifolds without boundary, with nonnegative Ricci curvature and a common positive lower bound for the volumes of their based unit balls. Moreover,

$$\mathcal{S}^{N-4}(Y) = \mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = \emptyset, \qquad \mathrm{Int}_Y\big(Y \setminus \mathcal{S}^{\mathrm{CC}}_{N-4}(Y)\big) = \mathrm{Int}_Y\big(Y \setminus \mathcal{S}^{N-4}(Y)\big) = Y,$$

and Y is not a topological manifold.

Remark 3.5 (Scope of the counterexample). For the spaces in Corollary 3.4, set $E_{\mathrm{top}} := \mathrm{Stop}(Y) = \{p\}$. Since Y is a metric space, E_{top} is closed. For every $s > 0$, $\mathcal{H}^s(E_{\mathrm{top}}) = 0$, whereas $\mathcal{H}^0(E_{\mathrm{top}}) = 1$; hence $\dim_{\mathcal{H}} E_{\mathrm{top}} = 0 \leq N - 4$. The proof of Proposition 9.2 identifies $Y \setminus E_{\mathrm{top}}$ with a nonempty smooth Riemannian N-manifold whose Riemannian distance is the restriction of d. Therefore $\dim_{\mathcal{H}}(Y \setminus E_{\mathrm{top}}) = N$, and finite-union stability of Hausdorff dimension gives $\dim_{\mathcal{H}} Y = N$. Thus $\dim_{\mathcal{H}} Y - \dim_{\mathcal{H}} E_{\mathrm{top}} = N \geq 4$. In contrast, the Cheeger–Colding codimension-four singular stratum $E_{\mathrm{CC}} := \mathcal{S}^{\mathrm{CC}}_{N-4}(Y)$ is empty, while $Y \setminus E_{\mathrm{CC}} = Y$ is not a topological manifold. Moreover,

$$D_{N-4}(Y) = \mathcal{S}^{N-4}(Y) = E_{\mathrm{CC}} = \emptyset, \qquad \mathrm{Stop}(Y) \not\subset E_{\mathrm{CC}}.$$

Proof of Corollary 3.3. Fix an integer $N \geq 5$ and put $n := N - 1$.

Suppose first that n is even. Put $m := n/2$ and let $Q := \mathbb{CP}^m$ with its complex orientation. Then $m \geq 2$, and Q is a closed connected oriented smooth n-manifold. By [3, Theorem C], there are a smoothly embedded closed n-disc $D_Q \subset Q$ and a smooth metric k on $P_Q := Q \setminus \mathrm{int}_Q D_Q$ with positive Ricci curvature, round boundary metric, and positive-definite outward second fundamental form. If ν is the inward unit normal, the latter tensor is $\langle \nabla_v(-\nu), w\rangle = -\langle \nabla_v \nu, w\rangle$; it is therefore II in our convention. Choose $R > 0$ and a diffeomorphism $\beta_Q : S^{n-1} \to \partial P_Q$ such that $\beta_Q^*(k|_{\partial P_Q}) = R^2 g_{S^{n-1}}$, and set $k_Q := R^{-2}k$. Constant scaling gives

$$\beta_Q^*(k_Q|_{\partial P_Q}) = g_{S^{n-1}}, \qquad \mathrm{Ric}_{k_Q} = \mathrm{Ric}_k > 0, \qquad \beta_Q^*\mathrm{II}_{\partial P_Q,k_Q} = R^{-1}\beta_Q^*\mathrm{II}_{\partial P_Q,k} > 0.$$

Define

$$\mu_Q := \min_{x \in P_Q} \lambda_{\min}\big(k_Q^{-1}\,\mathrm{Ric}_{k_Q}\big)(x),$$
$$\kappa_Q := \min_{y \in S^{n-1}} \lambda_{\min}\big(g_{S^{n-1}}^{-1}\beta_Q^*\mathrm{II}_{\partial P_Q,k_Q}\big)(y).$$

The two least-eigenvalue functions are continuous and positive on compact spaces. Thus $\mu_Q, \kappa_Q > 0$, and the displayed data satisfy (3.1). Hence Q is core-admissible.

Suppose next that n is odd. Then $n \geq 5$. Let $Q := S^{n-2} \times S^2$, viewed as the unit sphere bundle $S(\underline{\mathbb{R}}^3) \to S^{n-2}$ of the trivial rank-three Euclidean bundle, and give Q the product orientation. The unit round metric on S^{n-2} is a core metric. Reiser's [15, Theorem 5.1], applied to this linear sphere bundle with fibre S^2, implies that Q admits a core metric.

In either case, Q is core-admissible. Fix an embedded closed disc $D_Q \subset Q$ occurring in a choice of core-admissibility data as in Definition 3.1.

We next compute $H_2(Q;\mathbb{Z})$. If n is even, the standard CW decomposition of $\mathbb{CP}^m$ has one cell in each dimension $0, 2, \ldots, 2m$; hence its cellular differential entering or leaving degree 2 vanishes and $H_2(Q;\mathbb{Z}) \cong \mathbb{Z}$. If n is odd, then $n - 2 \geq 3$, and the integral Künneth theorem gives $H_2(S^{n-2} \times S^2;\mathbb{Z}) \cong H_0(S^{n-2};\mathbb{Z}) \otimes H_2(S^2;\mathbb{Z}) \cong \mathbb{Z}$. Thus $H_2(Q;\mathbb{Z}) \cong \mathbb{Z}$.

Define Q_+, Q_-, and M_Q by

$$Q_+ := Q \setminus \mathrm{int}_Q D_Q, \qquad Q_- := (-Q) \setminus \mathrm{int}_{-Q} D_Q,$$
$$M_Q := Q\#(-Q).$$

For either sign, a collar of the embedded disc boundary makes the pair $(\pm Q, Q_\pm)$ a good pair, and the quotient obtained by collapsing $Q_\pm$ is homeomorphic to $D_Q/\partial D_Q \cong S^n$. Since $n \geq 4$,

$$H_j(\pm Q, Q_\pm;\mathbb{Z}) = 0 \qquad (j = 2, 3).$$

The long exact sequences of these pairs therefore give $H_2(Q_\pm;\mathbb{Z}) \cong H_2(Q;\mathbb{Z})$.

Choose a bicollar $c : S^{n-1} \times (-2, 2) \to M_Q$ of the connected-sum sphere so that the negative and positive collar halves lie in Q_+ and Q_-, respectively. Define

$$U_+ := \mathrm{int}_{M_Q}(Q_+) \cup c(S^{n-1} \times (-2, 1)), \qquad U_- := \mathrm{int}_{M_Q}(Q_-) \cup c(S^{n-1} \times (-1, 2)).$$

Then $U_+ \cup U_- = M_Q$, the spaces $U_\pm$ deformation retract onto $Q_\pm$, and $U_+ \cap U_-$ deformation retracts onto S^{n-1}. The relevant Mayer–Vietoris segment is

$$H_2(S^{n-1};\mathbb{Z}) \longrightarrow H_2(Q_+;\mathbb{Z}) \oplus H_2(Q_-;\mathbb{Z}) \longrightarrow H_2(M_Q;\mathbb{Z}) \longrightarrow H_1(S^{n-1};\mathbb{Z}).$$

The two end groups vanish because $n \geq 4$. Hence

$$H_2(M_Q;\mathbb{Z}) \cong H_2(Q;\mathbb{Z}) \oplus H_2(Q;\mathbb{Z}) \cong \mathbb{Z}^2.$$

In particular, M_Q is not an integral homology n-sphere.

Let $q_0 = q_0(n, Q) > 0$ be supplied by Theorem 3.2 and set $q := \min\{q_0/2, 1/2\}$. Then $q \in (0, 1)$ and $q < q_0$. Applying Theorem 3.2 to (n, Q, q) gives a complete pointed classical noncollapsed $(n + 1)$-dimensional Ricci limit (Y, d, p) satisfying

$$\partial Y = \emptyset, \qquad p \in \mathrm{Stop}(Y) \cap \left(\mathcal{S}^{n-1}(Y) \setminus \mathcal{S}^{n-2}(Y)\right),$$
$$\mathrm{Tan}(Y, p) = \left\{\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})\right\}.$$

Since $n+1 = N$, $n-1 = N-2$, and $n-2 = N-3$, these are precisely the conclusions of Corollary 3.3. □

4. Links, caps, and radial realization

Fix $n \geq 4$ and a core-admissible closed connected oriented smooth n-manifold Q. Choose $D_Q, P_Q, \beta_Q, k_Q, \kappa_Q, \mu_Q$ as in Definition 3.1. In the proof of Theorem 8.1, the constant $\gamma_* \in (0, 1)$ is fixed first and then $q_0(n, Q) \in (0, \gamma_*/2)$ is chosen. Decreasing $q_0(n, Q)$ if necessary, assume $q_0(n, Q) < 1/2$. Throughout this section,

$$0 < q < q_0(n, Q), \qquad M_Q := Q\#(-Q).$$

The metric tangent and the topology of a sufficiently small neighbourhood of the distinguished point are determined by different spaces. The link $L_{n,q}$ determines the tangent cone, whereas M_Q determines the topology of the completed radial space. Theorem 8.1 relates them through a smooth family $(g_s)_{s\geq 0}$ on M_Q satisfying

$$\mathrm{Ric}_{g_s} \geq (n-1)g_s, \qquad dv_{g_s} = dv_{g_0},$$
$$(M_Q, g_s) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}.$$

There is $s_* > 0$ such that $g_s = g_0$ for $0 \leq s \leq s_*$, and g_0 is Ricci closable. Define $\ell_t := g_{-t}$ for $t \leq 0$ and $\ell_t := g_0$ for $t \geq 0$. The two formulas agree with g_0 on a neighbourhood of $t = 0$; hence $(\ell_t)_{t\in\mathbb{R}}$ is smooth. It has common volume form, $\mathrm{Ric}_{\ell_t} \geq (n-1)\ell_t$, equals g_0 for $t \geq 0$, and satisfies $(M_Q, \ell_t) \to L_{n,q}$ in Gromov–Hausdorff topology as $t \to -\infty$. The common-volume, Ricci, stationarity, limit, and closability clauses above are the hypotheses of Proposition 9.2 with $X = M_Q$, $\Sigma = L_{n,q}$, and $g_t = \ell_t$. The proposition yields a complete classical noncollapsed Ricci limit whose unique tangent at the added point is $\mathrm{C}(L_{n,q})$, whose punctured space is the smooth manifold $(0,\infty) \times M_Q$, and whose sufficiently small pointed balls are open topological cones over M_Q.

4.1. The tangent-cone link.

Define

$$B_{n,q} := S^{n-3} * S^1_{2\pi q}, \qquad L_{n,q} := S^0 * B_{n,q}.$$

Since $q < 1/2$, all join factors have diameter at most π. The cone formula for spherical joins [1, Proposition I.5.15], together with Lemma 5.1, gives

$$L_{n,q} \cong_{\mathrm{iso}} S^{n-2} * S^1_{2\pi q}, \qquad \mathrm{C}(L_{n,q}) \cong_{\mathrm{iso}} \mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q}),$$
$$L_{n,q} \cong_{\mathrm{top}} S^n, \qquad \mathcal{H}^n(L_{n,q}) = q\sigma_n.$$

Since $\mathrm{diam}\, S^1_{2\pi q} = \pi q < \pi$, the line argument in Lemma 5.1, together with the displayed splitting, gives maximal Euclidean splitting rank $n-1$. The number q fixes the density and the codimension-two tangent geometry, while M_Q fixes the topology of the completed radial space.

The quotient-length suspension family in [8, Example I] has $B_{n,q}$ in its Gromov–Hausdorff closure. Lemma 5.2 replaces the singular suspension metrics by smooth metrics satisfying (5.5)– (5.8). From these estimates, Theorem 5.3 constructs a smooth path h_s and a smooth nonincreasing function γ satisfying

$$\mathrm{Ric}_{h_s} \geq (n-2)h_s, \qquad \mathrm{Vol}(S^{n-1}, h_s) = \gamma(s)\sigma_{n-1},$$
$$(S^{n-1}, h_s) \xrightarrow[s\to\infty]{\mathrm{GH}} B_{n,q}.$$

The function γ is constant near 0 and converges to q. Its initial value is the number γ_* fixed before q_0; hence

$$q < q_0 < \frac{\gamma_*}{2} < \gamma_* < 1, \qquad \gamma_*^{1/(n-1)} < \frac{1}{4}\min\{1, \kappa_Q\}.$$

The first chain of inequalities supplies the cap-volume margin. The second places the round boundary radius below the convexity scale of the core metric.

For each s, Theorem 5.3 also gives metrics $g_{s,u}$ and isotopies $\phi_{s,u}$, $u \in \mathbb{R}$, such that

$$g_{s,u} = \gamma(s)^{2/(n-1)} g_{S^{n-1}} \quad (u \le -2), \qquad g_{s,u} = \phi_{s,2}^* h_s \quad (u \ge 2),$$
$$\mathrm{Ric}_{g_{s,u}} \ge (n-2) g_{s,u}, \qquad dv_{g_{s,u}} = dv_{g_{s,-2}}.$$

For $-2 \le u \le 2$, set $\widehat{g}^{\mathrm{cap}}_{s,u} := (\phi_{s,u}^{-1})^* g_{s,u}$. Pullback gives

$$\widehat{g}^{\mathrm{cap}}_{s,-2} = \gamma(s)^{2/(n-1)} g_{S^{n-1}}, \qquad \widehat{g}^{\mathrm{cap}}_{s,2} = h_s,$$
$$\mathrm{Ric}_{\widehat{g}^{\mathrm{cap}}_{s,u}} \ge (n-2)\widehat{g}^{\mathrm{cap}}_{s,u}, \qquad \mathrm{Vol}(S^{n-1}, \widehat{g}^{\mathrm{cap}}_{s,u}) = \gamma(s)\sigma_{n-1}.$$

This path is constant on neighbourhoods of $u = -2$ and $u = 2$.

4.2. **Caps and connected-sum topology.** Let $I \subset [0, \infty)$ be nonempty and compact, orient S^{n-1}, and put $\omega_{s,u} := dv_{\widehat{g}^{\mathrm{cap}}_{s,u}}$. Fix a background metric on S^{n-1}, with Green operator $\mathcal{G}_S$ and codifferential d_S^*. Since $\int_{S^{n-1}} \partial_u \omega_{s,u} = 0$, the harmonic projection of $\partial_u \omega_{s,u}$ vanishes. Define

$$\eta_{s,u} := d_S^* \mathcal{G}_S(\partial_u \omega_{s,u}), \qquad \iota_{X_{s,u}} \omega_{s,u} := -\eta_{s,u}.$$

The Hodge identity gives $d\eta_{s,u} = \partial_u \omega_{s,u}$. Let $\varphi_{s,u}$ be the flow of $X_{s,u}$ with $\varphi_{s,-2} = \mathrm{Id}$. Smooth dependence for ordinary differential equations on the compact sphere gives a smooth family in (s, u), and Cartan's formula gives

$$\frac{d}{du}(\varphi_{s,u}^* \omega_{s,u}) = \varphi_{s,u}^*\big(\partial_u \omega_{s,u} + d\iota_{X_{s,u}} \omega_{s,u}\big) = 0.$$

This is Moser's construction [12]. Because $\partial_u \omega_{s,u} = 0$ near $u = \pm 2$, the flow is constant there. Hence $\widetilde{g}_{s,u} := \varphi_{s,u}^* \widehat{g}^{\mathrm{cap}}_{s,u}$ has volume form independent of u, satisfies $\mathrm{Ric}_{\widetilde{g}_{s,u}} \ge (n-2)\widetilde{g}_{s,u}$, and is constant near both endpoints. After endpoint extension, the pointwise norms underlying (6.2) are continuous on $I \times [-2, 2] \times S^{n-1}$, vanish for $|u| > 2$, and are uniformly bounded. Thus Theorem 6.1 applies.

The terminal metric is $\varphi_{s,2}^* h_s$. Before the terminal collar pullback in Theorem 7.5, the outer boundary metric is $\sin^2 \varepsilon\, \varphi_{s,2}^* h_s$, and the inward second fundamental form is greater than $\cot \varepsilon$ times this metric. The terminal collar diffeomorphism $\mathcal{F}_s$ satisfies

$$\mathcal{F}_s \circ \beta_Q = \beta_Q \circ \varphi_{s,2}^{-1} \quad \text{on } S^{n-1}.$$

Therefore the final boundary metric, pulled back by β_Q, is $\sin^2 \varepsilon h_s$. The constant scaling in Theorem 6.1 gives

$$\mathrm{Ric}_{c_\varepsilon^2 G_{s,\varepsilon}} \ge (n - 1 - C_n \varepsilon^2) c_\varepsilon^2 G_{s,\varepsilon}.$$

Use P_Q and k_Q from the fixed core-admissibility data. The cap consists of the spherical-deformation region, a round annulus, and a scaled copy of (P_Q, k_Q). At each of the two seams, the restricted metrics agree and the sum of the inward-convention second fundamental forms is positive definite. The two input Ricci tensors are strictly greater than the lower bound retained after smoothing. Thus Reiser–Wraith's Theorem A and Corollary B apply at each seam [16, Theorem A and Corollary B]; the two smoothing supports are disjoint. The compact-parameter construction and its exhaustion on the half-line are carried out in Theorem 7.5 and Proposition 7.6.

Consequently, for some $\varepsilon_0, C, D, V > 0$, there are metrics $K_{s,\varepsilon}$ on P_Q, $s \ge 0$, $0 < \varepsilon < \varepsilon_0$, such that

$$\beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}) = \sin^2 \varepsilon\, h_s, \qquad \beta_Q^* \mathrm{II}_{\partial P_Q, K_{s,\varepsilon}} > \cot \varepsilon\, \beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}),$$
$$\mathrm{Ric}_{K_{s,\varepsilon}} \ge (n - 1 - C\varepsilon^2) K_{s,\varepsilon}, \qquad \mathrm{diam}(P_Q, K_{s,\varepsilon}) \le D\varepsilon,$$
$$\mathrm{Vol}(P_Q, K_{s,\varepsilon}) \le V\varepsilon^n.$$

At each outer attachment, the truncated suspension and cap have boundary metric $\sin^2 \varepsilon\, h_s$. Their inward second fundamental forms are $-\cot \varepsilon$ times that metric and strictly greater than $\cot \varepsilon$ times it; hence their sum is positive definite. For the chosen gluing reserve, the Ricci tensors on both sides are strictly greater than the retained lower bound. Reflection across the suspension equator interchanges the two cap copies and preserves the boundary metrics, normal fields, and second fundamental forms. These are the hypotheses of Proposition 7.4, which gives the reflected glued family.

Attach a copy of P_Q at one pole and an oppositely oriented copy at the other. The resulting closed manifold is

$$P_Q \cup_{S^{n-1}} ([0,1] \times S^{n-1}) \cup_{S^{n-1}} (-P_Q) \cong Q\#(-Q).$$

For the round initial link, let $\mathfrak{r}$ be reflection across the equator and let $\widehat{G}_*$ be the initial glued metric. The synchronized cap and seam constructions give $\mathfrak{r}^* \widehat{G}_* = \widehat{G}_*$ and $\mathrm{Ric}_{\widehat{G}_*} > 0$. Its fixed-point set is the nonempty equatorial hypersurface $\{\pi/2\} \times S^{n-1}$; the suspension direction is a nowhere-vanishing normal field. Hence the normal bundle is trivial. The compact-filling construction in [13, proof of Proposition 3.2], followed by the conical-end attachment in the proof of Theorem 8.1, gives $c_Q > 0$ such that $c_Q^2 \widehat{G}_*$ is Ricci closable in the sense of Definition 2.2.

After $\widehat{G}_*$ and the cap constants have been fixed, the proof of Theorem 8.1 chooses $q_0 \in (0, \gamma_*/2)$ so that (8.6)–(8.8) hold. These inequalities preserve the normalized Ricci bound, keep the initial scaling within the range supplied by Ricci closability, and place the cap-volume error below the available volume margin.

The proof of Theorem 8.1 chooses a smooth function $\varepsilon : [0, \infty) \to (0, \varepsilon_0)$, constant near 0 and tending to 0, and constructs metrics $\widehat{G}_s$ on M_Q. Equation (8.17) gives $(M_Q, \widehat{G}_s) \to L_{n,q}$ in Gromov–Hausdorff topology. The family also satisfies (8.13) and (8.14).

4.3. **Volume normalization and radial realization.** Define the positive scale

$$b_s := \left(\frac{q\sigma_n}{\mathrm{Vol}(M_Q, \widehat{G}_s)} \right)^{1/n} > 0, \qquad g_s^0 := b_s^2 \widehat{G}_s.$$

The functions b_s and g_s^0 are smooth in s. Write $\beta(s) := \gamma(s)/q - 1$. The lower estimate in (8.14) gives $\mathrm{Vol}(M_Q, \widehat{G}_s) \geq q\sigma_n(1 + \beta(s)/2)$, hence $b_s^2 \leq (1 + \beta(s)/2)^{-2/n}$. In the proof of Theorem 8.1, (8.13) has coefficient $(n-1)(1 + \beta(s)/2)^{-2/n}$. Since the covariant Ricci tensor is unchanged under constant scaling, $\mathrm{Ric}_{g_s^0} = \mathrm{Ric}_{\widehat{G}_s} \geq (n-1) g_s^0$. The two-sided volume estimate implies $b_s \to 1$. The convergence of $(M_Q, \widehat{G}_s)$ gives a uniform diameter bound for all sufficiently large s. Since distances for $g_s^0 = b_s^2 \widehat{G}_s$ equal b_s times the corresponding distances for $\widehat{G}_s$, the identity correspondence has distortion at most $|b_s - 1| \operatorname{diam}(M_Q, \widehat{G}_s) \to 0$. Consequently

$$\mathrm{Vol}(M_Q, g_s^0) = q\sigma_n, \qquad \mathrm{Ric}_{g_s^0} \geq (n-1) g_s^0,$$
$$(M_Q, g_s^0) \xrightarrow[s \to \infty]{\mathrm{GH}} L_{n,q}.$$

The path is constant near 0. Equation (8.7), together with stability of Ricci closability under constant scaling down, shows that g_0^0 is Ricci closable.

Put $\omega_s := dv_{g_s^0}$. Since $\int_{M_Q} \omega_s = q\sigma_n$ for all s, orient M_Q, fix a background metric and its top-degree Green operator $\mathcal{G}_M$ and codifferential d_M^*, and define

$$\eta_s := d_M^* \mathcal{G}_M \dot{\omega}_s, \qquad \iota_{Y_s} \omega_s := -\eta_s.$$

The harmonic projection of $\dot{\omega}_s$ vanishes, so $d\eta_s = \dot{\omega}_s$. Let ψ_s be the flow of Y_s with $\psi_0 = \mathrm{Id}$. A smooth time-dependent vector field on the compact manifold M_Q is complete on every finite s-interval; uniqueness of integral curves makes the resulting flows compatible on overlapping intervals and defines ψ_s for all $s \geq 0$. Then

$$\frac{d}{ds}(\psi_s^* \omega_s) = \psi_s^*(\dot{\omega}_s + d\iota_{Y_s} \omega_s) = 0.$$

The vector field vanishes on the initial constant interval. Hence $g_s := \psi_s^* g_s^0$ is constant there, has volume form dv_{g_0}, satisfies $\mathrm{Ric}_{g_s} \geq (n-1)g_s$, is isometric to g_s^0, and has the Ricci-closable initial metric $g_0 = g_0^0$.

Define $(\ell_t)_{t\in\mathbb{R}}$ as at the beginning of the section. Apply the slow reparametrization in the proof of Proposition 9.2 to this family and denote the resulting family by $(\widetilde{\ell}_t)_{t\in\mathbb{R}}$. It has the same common volume form and Ricci bound, equals g_0 for $t \geq 0$, has the same limit as $t \to -\infty$, and satisfies

$$\sup_{t\in\mathbb{R}} \sup_{x\in M_Q} \Big(|\partial_t \widetilde{\ell}_t|_{\widetilde{\ell}_t}(x) + |\partial_t^2 \widetilde{\ell}_t|_{\widetilde{\ell}_t}(x) + |\nabla^{\widetilde{\ell}_t} \partial_t \widetilde{\ell}_t|_{\widetilde{\ell}_t}(x) \Big) \leq 1.$$

The common volume form, the Ricci bound, the constant positive half-line, and the displayed pointwise derivative estimate instantiate Lemma 9.1 with $X = M_Q$ and the family $(\widetilde{\ell}_t)$. For the dimension-only functions h, f supplied by that lemma,

$$\bar{g} := dr^2 + r^2 h(r)^2 \widetilde{\ell}_{f(r)}, \qquad \mathrm{Ric}_{\bar{g}} \geq 0,$$
$$h(r) \to 1, \qquad f(r) \to -\infty, \qquad rf'(r) \to 0 \quad (r \downarrow 0).$$

Using the conical-end fillings of g_0, the smooth-approximation construction in Proposition 9.2 gives complete connected smooth boundaryless Ricci-nonnegative approximants to the metric completion of $((0,\infty) \times M_Q, \bar{g})$, with a common positive lower bound for the volumes of their based unit balls. Denote the pointed limit by $(Y_{Q,q}, d, p)$.

The estimate $rf'(r) \to 0$ gives tangent uniqueness. For $0 < a < b < \infty$,

$$\delta_\rho(a,b) := \max\{|\log a|, |\log b|\} \sup_{\min\{a,1\}\rho \leq r \leq \max\{b,1\}\rho} |rf'(r)| \longrightarrow 0.$$

Integration of f' gives $|f(\rho t) - f(\rho)| \leq \delta_\rho(a,b)$ for $a \leq t \leq b$. The derivative bound for $(\widetilde{\ell}_t)$ then gives the pointwise quadratic-form inequality

$$e^{-\delta_\rho(a,b)} \widetilde{\ell}_{f(\rho)} \leq \widetilde{\ell}_{f(\rho t)} \leq e^{\delta_\rho(a,b)} \widetilde{\ell}_{f(\rho)} \qquad (a \leq t \leq b).$$

Moreover $h(\rho t) \to 1$ uniformly on $[a,b]$, while $f(\rho) \to -\infty$ and $(M_Q, \widetilde{\ell}_t) \to L_{n,q}$. By Proposition 9.2, the rescaled metrics converge on compact annuli to the corresponding annuli in $\mathrm{C}(L_{n,q})$. The rescaled ball of radius a about the vertex has diameter at most $2a$; letting $a \downarrow 0$ upgrades annular convergence to pointed convergence of balls. Therefore every blow-up at p is $\mathrm{C}(L_{n,q}) \cong_{\mathrm{iso}} \mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})$. The same proposition identifies a sufficiently small pointed neighbourhood of p with the open topological cone over M_Q.

Section 10 computes the density and local homology from this radial model. Let (M_i, d_{G_i}, p_i) be the realizing sequence in Proposition 9.2. It satisfies

$$(M_i, d_{G_i}, p_i) \xrightarrow{\text{pGH}} (Y_{Q,q}, d, p),$$
$$\mathrm{Ric}_{G_i} \geq 0, \qquad \mathrm{Vol}_{G_i}(B_1(p_i)) \geq v > 0.$$

Thus $(M_i, d_{G_i}, \mathcal{H}^{n+1}, p_i)$ is a sequence of noncollapsed $\mathrm{RCD}(0, n+1)$ spaces with a common based noncollapse constant, and every M_i is a topological manifold without boundary. Kapovitch–Mondino's theorem [10, Theorem 1.10], applied with $K = 0$ and $N = n+1$, gives $\partial Y_{Q,q} = \emptyset$. Finally, Section 11 identifies the Cheeger–Colding strata and proves Corollary 3.4.

5. Model links and Colding–Naber paths

5.1. **The target link.** Let (A, d_A) and (B, d_B) be nonempty compact metric spaces with diameters at most π. Their spherical join is

$$A * B := ([0, \pi/2] \times A \times B)/\sim,$$

where $\sim$ is generated by

$$(0, a, b) \sim (0, a, b'), \qquad (\pi/2, a, b) \sim (\pi/2, a', b).$$

Its metric is

$$d_{A*B}([t,a,b],[s,a',b']) = \arccos\big(\cos t \cos s \cos d_A(a,a') + \sin t \sin s \cos d_B(b,b')\big)\,.$$

The diameter assumptions make the truncations in [1, Definition I.5.13] redundant. Definition I.5.13 and Proposition I.5.15 of [1] therefore show that this formula defines a metric and that

$$\Phi_{A,B} : \mathrm{C}(A*B) \longrightarrow \mathrm{C}(A)\times\mathrm{C}(B), \qquad \Phi_{A,B}([r,[t,a,b]]) = ([r\cos t, a],[r\sin t, b]),$$

is a pointed isometry. The quotient map from $[0,\pi/2]\times A\times B$ to the metric join is continuous by the distance formula. It is a continuous surjection from a compact space to a Hausdorff space, hence a quotient map. Thus the metric topology on $A*B$ is the quotient topology.

Put $S(A) := S^0 * A$. For $q\in(0,1]$, define

$$B_{n,q} := S^{n-3} * S^1_{2\pi q}, \qquad L_{n,q} := S(B_{n,q}).$$

Lemma 5.1. *For nonempty compact metric spaces A_1, A_2, A_3 of diameter at most π,*

$$(A_1 * A_2) * A_3 \cong_{\mathrm{iso}} A_1 * (A_2 * A_3).$$

For $q\in(0,1)$,

$$\begin{aligned} L_{n,q} &\cong_{\mathrm{iso}} S^{n-2} * S^1_{2\pi q}, \\ L_{n,q} &\cong_{\mathrm{top}} S^n, && (5.1)\\ \mathcal{H}^n(L_{n,q}) &= q\sigma_n, && (5.2)\\ \mathrm{C}(L_{n,q}) &\cong_{\mathrm{iso}} \mathbb{R}^{n-1}\times \mathrm{C}(S^1_{2\pi q}). && (5.3)\end{aligned}$$

The cone in (5.3) *splits $\mathbb{R}^{n-1}$ and does not split $\mathbb{R}^n$.*

Proof. The cone–join isometries identify each of $\mathrm{C}((A_1*A_2)*A_3)$ and $\mathrm{C}(A_1*(A_2*A_3))$ pointed-isometrically with $\mathrm{C}(A_1)\times\mathrm{C}(A_2)\times\mathrm{C}(A_3)$. Their composite preserves the radial coordinate. If F has diameter at most π, then $d_{\mathrm{C}(F)}([1,x],[1,y]) = 2\sin(d_F(x,y)/2)$, which is strictly increasing in $d_F(x,y)\in[0,\pi]$. Restricting the composite to the radius-one spheres proves the associativity assertion.

The map

$$[t,\varepsilon,x] \longmapsto (\varepsilon\cos t, \sin t\, x) \quad \text{from } S^0 * S^{n-3} \text{ to } S^{n-2}\subset\mathbb{R}\times\mathbb{R}^{n-2}$$

is distance-preserving by the join-distance formula. It is surjective: for $(y_0,y')\in S^{n-2}$, take $t = \arccos|y_0|$, choose $\varepsilon\in S^0$ with $\varepsilon|y_0| = y_0$ when $y_0\neq 0$, and take $x = y'/|y'|$ when $y'\neq 0$; the endpoint relations remove the unused choices. Thus it is an isometry. Associativity gives $L_{n,q}\cong_{\mathrm{iso}} S^{n-2} * S^1_{2\pi q}$.

Identify $S^1_{2\pi q}$ with $\mathbb{R}/(2\pi\mathbb{Z})$ endowed with the metric q times the unit-circle metric, and write its coordinate as θ. The formula

$$\Psi_q([t,x,\theta]) := (\cos t\, x, \sin t\cos\theta, \sin t\sin\theta) \in S^n \subset \mathbb{R}^{n-1}\times\mathbb{R}^2$$

is constant on the join equivalence classes and induces a continuous map. If $(y,z)\in S^n$ and $y,z\neq 0$, its unique preimage is determined by $t = \arctan(|z|/|y|)$, $x = y/|y|$, and $(\cos\theta,\sin\theta) = z/|z|$. If $z = 0$ or $y = 0$, the corresponding join relation removes the undetermined circle or sphere coordinate. Thus Ψ_q is bijective. Its domain is compact and S^n is Hausdorff, so it is a homeomorphism. This proves (5.1).

Set $C_q := \mathrm{C}(S^1_{2\pi q})$ and $\mathcal{U} := (0,\pi/2)\times S^{n-2}\times\mathbb{R}/(2\pi\mathbb{Z})$. The quotient map identifies $\mathcal{U}$ with the complement of the two endpoint strata in $S^{n-2} * S^1_{2\pi q}$. On a sufficiently short angular interval, choose a lift $\widetilde{\theta}$ and define

$$\iota_q(t,x,\theta) := (\cos t\, x, \sin t\cos(q\widetilde{\theta}), \sin t\sin(q\widetilde{\theta})) \in S^n.$$

For two points in this coordinate neighbourhood, the round spherical distance formula for their images equals the join-distance formula, because the circle distance is $q|\widetilde{\theta}-\widetilde{\theta}'|$. Hence ι_q is a local isometry. Since it maps each coordinate neighbourhood isometrically onto an open subset of the unit sphere, $\mathcal{H}^n$ on the regular stratum is the Riemannian volume of

$$g_q = dt^2 + \cos^2 t\, g_{S^{n-2}} + q^2 \sin^2 t\, d\theta^2,$$

whose volume form is $q \cos^{n-2} t \sin t\, dt\, dv_{S^{n-2}}\, d\theta$. The endpoint strata are isometric to S^{n-2} and $S^1_{2\pi q}$ and have zero n-dimensional Hausdorff measure. For $q = 1$, the distance formula shows that Ψ_1 is an isometry onto the unit round S^n. Since $dv_{g_q} = q\, dv_{g_1}$ on $\mathcal{U}$,

$$\mathcal{H}^n(L_{n,q}) = \int_{\mathcal{U}} dv_{g_q} = q \int_{\mathcal{U}} dv_{g_1} = q\mathcal{H}^n(L_{n,1}) = q\sigma_n.$$

This proves (5.2). The radial map $[r, x] \mapsto rx$ is a pointed isometry from $\mathrm{C}(S^{n-2})$ to $\mathbb{R}^{n-1}$: the cone-distance formula agrees with the Euclidean distance because $\langle x, y\rangle = \cos d_{S^{n-2}}(x, y)$. Proposition I.5.15 of [1], applied to S^{n-2} and $S^1_{2\pi q}$, now gives (5.3).

Write $T := \mathbb{R}^{n-1} \times C_q$. The product exhibits an $\mathbb{R}^{n-1}$-splitting. Let $\gamma : \mathbb{R} \to T$ be a unit-speed line with $\gamma(0) = (0, o)$. For $t > 0$, write

$$\gamma(-t) = (v_-, [r_-, z_-]), \qquad \gamma(t) = (v_+, [r_+, z_+]).$$

Since both points have distance t from $(0, o)$, $|v_\pm|^2 + r_\pm^2 = t^2$. The triangle inequalities in the two factors and then in $\mathbb{R}^2$ give

$$\begin{aligned} 2t &= d_T(\gamma(-t), \gamma(t)) \\ &\le \left((|v_-| + |v_+|)^2 + (r_- + r_+)^2\right)^{1/2} \\ &\le (|v_-|^2 + r_-^2)^{1/2} + (|v_+|^2 + r_+^2)^{1/2} = 2t. \end{aligned}$$

Equality in the second inequality is equality in Minkowski's inequality; since both summands have norm t, it implies $(|v_-|, r_-) = (|v_+|, r_+)$. Equality in the first implies $d_{C_q}([r_-, z_-], [r_+, z_+]) = r_- + r_+$. If the common radial coordinate were positive, the cone-distance formula would give $d_{S^1_{2\pi q}}(z_-, z_+) = \pi$, contrary to $\operatorname{diam} S^1_{2\pi q} = \pi q < \pi$. Thus $r_- = r_+ = 0$. Since t was arbitrary, every line in T through $(0, o)$ is contained in $\mathbb{R}^{n-1} \times \{o\}$.

Suppose that T split $\mathbb{R}^n$, and let $F : (\mathbb{R}^n \times W, (0, w)) \to (T, (0, o))$ be a pointed splitting isometry. For $v \ne 0$, the image of the unit-speed line $s \mapsto (sv/|v|, w)$ is contained in $\mathbb{R}^{n-1} \times \{o\}$ and contains $F(v, w)$; pointedness gives the same conclusion for $v = 0$. Hence $F(v, w) = (f(v), o)$ for a distance-preserving map $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with $f(0) = 0$. For the standard basis,

$$\langle f(e_i), f(e_j)\rangle = \frac{|f(e_i)|^2 + |f(e_j)|^2 - |f(e_i) - f(e_j)|^2}{2} = \delta_{ij}.$$

This would give n orthonormal vectors in $\mathbb{R}^{n-1}$, a contradiction. Therefore T does not split $\mathbb{R}^n$. □

5.2. **Regularization of the codimension-two edge.** For $0 < t \le 1$ and a nonempty compact length space (Z, d_Z), let $S_t(Z)$ be the quotient of $[0, t\pi] \times Z$ obtained by collapsing $\{0\} \times Z$ and $\{t\pi\} \times Z$ separately. If $c = (r, z) : [a, b] \to (0, t\pi) \times Z$ is absolutely continuous, set

$$L_t(c) := \int_a^b \sqrt{|r'(s)|^2 + \sin^2(r(s)/t)|\dot z(s)|^2}\, ds,$$

where $|\dot z|$ is the metric derivative in Z. The quotient is endowed with the length metric generated by such curves and radial segments ending at either pole. This is a metric. Indeed, the radial coordinate is 1-Lipschitz. If two distinct interior points have the same radial coordinate r_0 and a joining curve has length less than $\frac{1}{2} \min\{r_0, t\pi - r_0\}$, then the curve remains in a strip on which $\sin(r/t)$ has a positive lower bound; its projection to Z has length bounded by a fixed multiple of the original length. Thus two distinct points cannot have zero distance. The quotient projection is continuous, since a radial segment followed by a horizontal curve gives

$$d_{S_t(Z)}([r, z], [r', z']) \le |r - r'| + d_Z(z, z').$$

It is therefore a quotient map from a compact space to a metric space, so the metric topology is the quotient topology. For a closed Riemannian manifold Z, this is the t-suspension of Colding–Naber [8, Example I, p. 8].

For $t > 0$, let $S^1(t)$ denote the round circle of radius t. Given $\mathbf{t} = (t_1, \dots, t_{n-1})$ with $0 < t_{n-1} \le \dots \le t_1 < 1$, define

$$X_{\mathbf{t}} := S_{t_1}\big(S_{t_2}(\cdots S_{t_{n-2}}(S^1(t_{n-1}))\cdots)\big).$$

Iterating the standard suspension homeomorphism identifies $X_{\mathbf{t}}$ with the fixed smooth sphere S^{n-1}. For every $m \ge 2$, we use the single-join identification

$$[t, p, e^{i\theta}] \longmapsto (\cos t\, p, \sin t \cos\theta, \sin t \sin\theta) \in S^m \subset \mathbb{R}^{m-1} \times \mathbb{R}^2,$$

where $0 \le t \le \pi/2$, $p \in S^{m-2}$, and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. We write $\mathrm{g}_{\mathbf{t}}$ for the transported quotient length metric.

Lemma 5.2. *Let $m \ge 3$ and $q \in (0,1)$. There are $d_* = d_*(m,q) > 0$, a constant $C_m > 0$, and a function $\omega_{m,q} : (0, d_*) \to (0,\infty)$ such that $\omega_{m,q}(d) \to 0$ as $d \downarrow 0$ and the following holds. Let $q \le \beta \le 1$, $0 < A < 1$, and $0 < d < d_*$. On $S^{m-2} * S^1_{2\pi\beta}$, identified with the fixed standard smooth sphere S^m by the preceding homeomorphism, let $\mathrm{g}^{\mathrm{raw}}_{A,\beta}$ be the quotient length metric whose restriction to the regular cylinder is*

$$A^2\big(dt^2 + \cos^2 t\, g_{S^{m-2}} + \beta^2 \sin^2 t\, d\theta^2\big).$$

There is a smooth metric $\mathcal{G}(A,\beta,d)$ on S^m whose restriction to $(0,\pi/2) \times S^{m-2} \times S^1$ is

$$A^2\big(dt^2 + \cos^2 t\, g_{S^{m-2}} + b_{\beta,d}(t)^2 d\theta^2\big). \tag{5.4}$$

The map $(A,\beta,d) \mapsto \mathcal{G}(A,\beta,d)$ extends smoothly to an open subset of $(0,\infty)^3$ containing $(0,1) \times [q,1] \times (0,d_*)$. Moreover,

$$\sec_{\mathcal{G}(A,\beta,d)} \ge A^{-2}(1-d), \tag{5.5}$$

$$\mathcal{G}(A,1,d) = A^2 g_{S^m}, \tag{5.6}$$

$$0 \le \mathrm{Vol}(S^m, \mathcal{G}(A,\beta,d)) - \mathcal{H}^m_{\mathrm{g}^{\mathrm{raw}}_{A,\beta}}(S^m) \le C_m A^m d, \tag{5.7}$$

$$d_{\mathrm{GH}}\big((S^m, \mathcal{G}(A,\beta,d)), (S^m, \mathrm{g}^{\mathrm{raw}}_{A,\beta})\big) \le A\omega_{m,q}(d). \tag{5.8}$$

Proof. Choose $\chi \in C^\infty([0,\infty),[0,1])$ such that $\chi = 0$ on $[0,1/2]$, $\chi = 1$ on $[1,\infty)$, and $\chi' \ge 0$. For $0 < \rho \le \pi/4$, set

$$J(\rho) := \int_0^{\pi/2} \chi(t/\rho) \cot t\, dt.$$

The integrand vanishes on $[0,\rho/2]$, and differentiation under the integral sign gives

$$J'(\rho) = -\rho^{-2} \int_0^{\pi/2} t\chi'(t/\rho) \cot t\, dt < 0.$$

The inequality is strict because χ' is positive somewhere in $(1/2,1)$ and the support of $\chi'(t/\rho)$ lies in $(0,\pi/2)$. Moreover, $J(\rho) \ge \int_\rho^{\pi/4} \cot t\, dt$, so $J(\rho) \to \infty$ as $\rho \downarrow 0$.

Choose $d_* > 0$ so that $d_* < 1/16$, $d_* J(\pi/4) < 1/2$, and, for $0 < d < d_*$,

$$\pi^2 e^{-2/d} \le d, \qquad d\,|\log \sin \sqrt{d}| \le 1. \tag{5.9}$$

This is possible because both left-hand sides divided by their asserted bounds tend to zero. Set

$$\ell := -\log\beta, \qquad \delta := d\frac{\ell}{1+\ell}, \qquad H := \frac{1+\ell}{d}.$$

Then $0 \le \delta < d < 1/2$ and $H > J(\pi/4)$. Since J is strictly decreasing from $+\infty$ to $J(\pi/4)$, there is a unique $\rho = \rho(\beta,d) \in (0,\pi/4)$ satisfying $J(\rho) = H$. The implicit function theorem applies to

$$F(\rho,\beta,d) := J(\rho) - \frac{1-\log\beta}{d},$$

because $\partial_\rho F = J'(\rho) \neq 0$. Thus ρ is smooth on

$$\mathcal{U} := \left\{ (\beta, d) : \beta > 0,\ 0 < d < d_*,\ \frac{1-\log\beta}{d} > J(\pi/4) \right\},$$

which contains $[q,1] \times (0, d_*)$.

Define

$$z(t) := -\delta\chi(t/\rho), \qquad y(t) := \exp\left(\int_0^t z(v)\cot v\, dv\right), \qquad b_{\beta,d}(t) := \sin t\, y(t).$$

For $0 \le t \le \rho/2$, one has $b_{\beta,d}(t) = \sin t$. If $\rho \le t \le \pi/2$, then

$$\int_0^t \chi(v/\rho)\cot v\, dv = J(\rho) + \log\sin t.$$

Since $\delta J(\rho) = \ell$,

$$y(t) = \beta(\sin t)^{-\delta}, \qquad b_{\beta,d}(t) = \beta(\sin t)^{1-\delta}. \tag{5.10}$$

Near $t = 0$, the metric in (5.4), divided by A^2, is $dt^2 + \cos^2 t\, g_{S^{m-2}} + \sin^2 t\, d\theta^2$; hence the normal 2-plane has the standard polar metric and the tangential coefficient is a smooth even function of the normal radius. Near $t = \pi/2$, with $s = \pi/2 - t$, it is

$$ds^2 + \sin^2 s\, g_{S^{m-2}} + \beta^2(\cos s)^{2(1-\delta)} d\theta^2.$$

The first two terms are the standard polar metric in the normal $(m-1)$-plane and the final coefficient is smooth and even in s. These expressions give a smooth metric across both strata and depend smoothly on $(A, \beta, d) \in (0,\infty) \times \mathcal{U}$. If $\beta = 1$, then $\ell = \delta = 0$ and $b_{1,d} = \sin t$, proving (5.6).

Divide the metric by A^2, abbreviate $a(t) := \cos t$ and $b(t) := b_{\beta,d}(t)$, and write

$$g := dt^2 + a(t)^2 g_{S^{m-2}} + b(t)^2 d\theta^2.$$

At a point of S^{m-2} choose a geodesic orthonormal frame $e_1, \dots, e_{m-2}$ and set $E_i = a^{-1}e_i$ and $E_\theta = b^{-1}\partial_\theta$. At that point, Koszul's formula gives

$$\begin{gathered}
\nabla_{\partial_t} E_i = 0, \qquad \nabla_{E_i}\partial_t = \frac{a'}{a}E_i, \qquad \nabla_{E_i}E_j = -\delta_{ij}\frac{a'}{a}\partial_t, \\
\nabla_{\partial_t}E_\theta = 0, \qquad \nabla_{E_\theta}\partial_t = \frac{b'}{b}E_\theta, \\
\nabla_{E_\theta}E_\theta = -\frac{b'}{b}\partial_t, \qquad \nabla_{E_i}E_\theta = \nabla_{E_\theta}E_i = 0.
\end{gathered}$$

Substitution in $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, together with $R^{S^{m-2}}(e_i, e_j)e_j = e_i$ for $i \neq j$, shows that the curvature operator is diagonal on

$$\begin{gathered}
\partial_t \wedge TS^{m-2}, \qquad \Lambda^2 TS^{m-2}, \\
TS^{m-2} \wedge TS^1, \qquad \partial_t \wedge TS^1,
\end{gathered}$$

with respective eigenvalues

$$\begin{gathered}
-\frac{a''}{a} = 1, \qquad \frac{1-(a')^2}{a^2} = 1, \\
-\frac{a'b'}{ab} = 1 + z, \qquad -\frac{b''}{b}.
\end{gathered} \tag{5.11}$$

The second summand is absent when $m = 3$. Since $b'/b = (1+z)\cot t$,

$$-\frac{b''}{b} = 1 + z - z'\cot t - z(1+z)\cot^2 t.$$

Now $-d < z \le 0$, $z' \le 0$, and $1 + z > 0$. Every eigenvalue in (5.11) is at least $1 - d$. If $\Omega = \sum_\alpha c_\alpha \Omega_\alpha$ is the unit simple 2-vector of a two-plane in an orthonormal eigenbasis of Λ^2, then $\sec_g(\Omega) = \sum_\alpha c_\alpha^2 \lambda_\alpha \ge 1 - d$. Smoothness extends the estimate to both strata. Scaling by A^2 proves (5.5).

The function y is nonincreasing, $y(0) = 1$, and $y(\pi/2) = \beta$; hence

$$\beta \le y(t) \le 1. \tag{5.12}$$

Furthermore, $J(\rho) \le \int_{\rho/2}^{\pi/2} \cot t \, dt = -\log \sin(\rho/2)$. Since $J(\rho) = H \ge d^{-1}$ and $\sin(\rho/2) \ge \rho/\pi$,

$$\rho \le \pi e^{-1/d}, \qquad \rho^2 \le d \tag{5.13}$$

by (5.9). For $0 < x \le 1$ and $0 \le \delta < 1/2$, the mean-value theorem applied to $r \mapsto x^{-r}$ gives

$$0 \le x^{-\delta} - 1 \le \delta |\log x| x^{-1/2}. \tag{5.14}$$

Let $\Sigma_0 \cong S^{m-2}$ and $\Sigma_1 \cong S^1$ be the strata at $t = 0$ and $t = \pi/2$. Their inclusions are Lipschitz for both metrics, with induced dimensions $m - 2$ and 1, so they have zero m-dimensional Hausdorff measure. On the regular cylinder the Riemannian volume forms give, with $c_m := 2\pi\sigma_{m-2}$,

$$\frac{\mathrm{Vol}(S^m, \mathcal{G}(A, \beta, d)) - \mathcal{H}^m_{g^{\mathrm{raw}}_{A,\beta}}(S^m)}{A^m} = c_m \int_0^{\pi/2} \cos^{m-2} t \, \sin t \, (y(t) - \beta) \, dt.$$

The integral is nonnegative by (5.12). Its part on $[0, \rho]$ is at most $C_m \rho^2$. On $[\rho, \pi/2]$, (5.10) and (5.14) bound it by

$$C_m \delta \int_0^{\pi/2} \cos^{m-2} t \, \sin^{1/2} t \, |\log \sin t| \, dt.$$

The integral is finite: the integrand is $O(t^{1/2} |\log t|)$ at 0 and $O((\pi/2 - t)^m)$ at $\pi/2$. Since $\rho^2 \le d$ and $\delta \le d$, this proves (5.7).

Let d_0 and d_1 be the distances of $g^{\mathrm{raw}}_{A,\beta}$ and $\mathcal{G}(A, \beta, d)$ on the common quotient. We first justify the curve class used below. For $i = 0, 1$, every pair of points and every $\varepsilon > 0$ can be joined by a finite concatenation c whose open nonradial pieces lie in the regular cylinder and whose d_i-length is at most $d_i(x, x') + \varepsilon$. For d_0, this follows from the quotient length definition after replacing an arc in Σ_0 by its parallel copy at $t = \varepsilon$ and an arc in Σ_1 by its parallel copy at $s = \pi/2 - t = \varepsilon$. Write L for the $g_{S^{m-2}}$-length of the arc in Σ_0 and L_θ for the angular length of the arc in Σ_1. The respective length errors are bounded by

$$2A\varepsilon + A|1 - \cos \varepsilon| L, \qquad 2A\varepsilon + A \sup_{0 \le r \le \varepsilon} |b_{\beta,d}(\pi/2 - r) - \beta| L_\theta,$$

and tend to zero; the raw metric has the same property with $b_{\beta,d}$ replaced by $\beta \sin t$. For d_1, begin with a piecewise geodesic curve of length at most $d_1(x, x') + \varepsilon/2$ and cover it by finitely many strongly convex coordinate balls. In one such ball, fix a vertex v outside the strata. If the geodesic segment $[v, w]$ meets Σ_j at $z = \exp_v(s \exp_v^{-1} w)$ with $0 < s < 1$, then $w = \exp_v(s^{-1} \exp_v^{-1} z)$. The bad endpoints are therefore contained in the image of the smooth map $(s, z) \mapsto \exp_v(s^{-1} \exp_v^{-1} z)$ on its domain in $(0, 1) \times \Sigma_j$. This domain has dimension at most $m - 1$, so its image has zero m-dimensional measure. Fubini's theorem on the finite product of small coordinate balls permits the interior vertices to be moved arbitrarily little so that every open geodesic edge avoids both strata. If an endpoint lies in a stratum, a short normal segment moves it to the regular cylinder. Continuity of length under these finite vertex perturbations and the two parallel-copy estimates above prove the assertion.

Since $b_{\beta,d}(t) \ge \beta \sin t$, comparison of regular-cylinder lengths now gives

$$d_0 \le d_1. \tag{5.15}$$

Fix η with $\rho \le \eta < 1/4$. For $t \ge \eta$, (5.10) gives

$$1 \le \frac{y(t)}{\beta} = (\sin t)^{-\delta} \le \exp\big(d |\log \sin \eta|\big) =: L_{d,\eta}. \tag{5.16}$$

Let c be a finite regular concatenation from x to x' with $L_{d_0}(c) \le d_0(x,x') + \varepsilon$. If c meets $\{t \le \eta\}$, let $I = [s_-, s_+]$ be the interval between its first and last visits to this set. Choose compatible angular representatives on the regular pieces of $c|_I$; compatibility across a visit to $t = 0$ is possible because all angular values are identified there. Replace the angular coordinate on I by a constant and insert, at s_- and s_+, shortest angular arcs restoring the original endpoint values. Each inserted arc has d_1-length at most $\pi A b_{\beta,d}(t) \le \pi A \sin t \le \pi A\eta$. The central replacement has no angular velocity, so its d_0- and d_1-lengths are equal and do not exceed the original d_0-length on I. Outside I, (5.16) bounds d_1-length by $L_{d,\eta}$ times d_0-length. The same outer estimate applies to all of c when the inner set is not met. Letting $\varepsilon \downarrow 0$ yields

$$d_1(x,x') \le L_{d,\eta} d_0(x,x') + 2\pi A\eta. \tag{5.17}$$

Two radial segments to Σ_0 and a shortest arc in Σ_0 give $\operatorname{diam}(S^m, d_0) \le 2\pi A$. For the identity correspondence $\mathcal{R}_{\mathrm{id}} := \{(z,z) : z \in S^m\}$, (5.15), (5.17), and the diameter bound imply, whenever $d|\log\sin\eta| \le 1$,

$$\operatorname{dis}(\mathcal{R}_{\mathrm{id}}) \le C_m A\big(\eta + d|\log\sin\eta|\big), \tag{5.18}$$

because $e^r - 1 \le 2r$ for $0 \le r \le 1$. By (5.13), $\rho \le \sqrt{d}$. Take $\eta = \sqrt{d}$ and define

$$\omega_{m,q}(d) := C_m\big(\sqrt{d} + d|\log\sin\sqrt{d}|\big).$$

Equation (5.9) makes (5.18) applicable, and $\sin\sqrt{d} \ge 2\sqrt{d}/\pi$ gives $\omega_{m,q}(d) \to 0$. Since the identity relation is a correspondence and $d_{\mathrm{GH}} \le \frac{1}{2}\operatorname{dis}(\mathcal{R}_{\mathrm{id}})$, (5.8) follows after increasing C_m. □

5.3. The variable-volume path.

Theorem 5.3. *Fix $0 < q < \gamma_* < 1$. There exist a smooth nonincreasing function $\gamma : [0,\infty) \to (q, \gamma_*]$, smooth metrics h_s on S^{n-1}, smooth paths $g_{s,u}$, and smooth isotopies $\phi_{s,u}$, with $(s,u) \in [0,\infty) \times \mathbb{R}$, such that*

$$\operatorname{Ric}_{h_s} \ge (n-2)h_s, \tag{5.19}$$

$$\operatorname{Vol}(S^{n-1}, h_s) = \gamma(s)\sigma_{n-1}, \tag{5.20}$$

$$(S^{n-1}, h_s) \xrightarrow[s\to\infty]{\mathrm{GH}} B_{n,q}. \tag{5.21}$$

The function γ equals γ_* on $[0, s_0]$ for some $s_0 > 0$ and satisfies $\gamma(s) \to q$ as $s \to \infty$. For every $s \ge 0$,

$$g_{s,u} = \gamma(s)^{2/(n-1)} g_{S^{n-1}} \quad (u \le -2), \qquad g_{s,u} = \phi_{s,2}^* h_s \quad (u \ge 2),$$
$$\phi_{s,u} = \mathrm{id} \quad (u \le -2), \qquad \phi_{s,u} = \phi_{s,2} \quad (u \ge 2),$$
$$\operatorname{Ric}_{g_{s,u}} \ge (n-2)g_{s,u}, \qquad dv_{g_{s,u}} = dv_{g_{s,-2}}.$$

The maps $(s,u) \mapsto g_{s,u}$ and $(s,u) \mapsto \phi_{s,u}$ are smooth. There is $\delta_{\mathrm{pl}} > 0$, independent of s, such that the left endpoint identities hold for $u \le -2 + \delta_{\mathrm{pl}}$ and the right endpoint identities hold for $u \ge 2 - \delta_{\mathrm{pl}}$. For $0 \le s \le s_0$ and $u \in \mathbb{R}$,

$$h_s = g_{s,u} = \gamma_*^{2/(n-1)} g_{S^{n-1}}, \qquad \phi_{s,u} = \mathrm{id}.$$

Proof. Put $m = n-1$ and $s_0 = 1$. Define

$$\vartheta(s) = \begin{cases} 0, & 0 \le s \le s_0, \\ \exp\big(-(s-s_0)^{-1}\big), & s > s_0. \end{cases}$$

Then ϑ is smooth and nondecreasing, vanishes on $[0, s_0]$, takes values in $[0,1)$, and tends to 1. Set

$$\gamma(s) = q + (\gamma_* - q)(1 - \vartheta(s)), \qquad a(s) = \big(\gamma_* + (1-\gamma_*)\vartheta(s)\big)^{1/m}.$$

Thus γ has the asserted monotonicity and endpoint behaviour, while

$$a(s)^m - \gamma(s) = (1-q)\vartheta(s) \ge 0, \qquad 0 < a(s) < 1$$

for finite s, and $a(s) \to 1$.

Define

$$\mathbf{t}^+(s) = \left(a(s), \dots, a(s), \frac{\gamma(s)}{a(s)^{m-1}} \right), \qquad \mathbf{t}^-(s) = \left(\gamma(s)^{1/m}, \dots, \gamma(s)^{1/m}\right).$$

Since $\gamma(s) \le a(s)^m$, the last coordinate of $\mathbf{t}^+(s)$ is at most $a(s)$. For $\varepsilon \in \{-, +\}$ and $1 \le j \le m$, put $x_j^\varepsilon(s) = -\log t_j^\varepsilon(s)$. Both endpoint vectors lie in the convex set

$$\mathcal{D}_{\gamma(s)} = \left\{ x \in (0,\infty)^m \,:\, x_1 \le \dots \le x_m, \quad \sum_{j=1}^m x_j = -\log \gamma(s) \right\}.$$

Choose $\chi_0 \in C_c^\infty((-2,2))$ with $\chi_0 \ge 0$ and $\int_{\mathbb{R}} \chi_0 = 1$, and define $\lambda(u) = \int_{-\infty}^u \chi_0(v)\,dv$. Since $\operatorname{supp} \chi_0 \Subset (-2,2)$, there is $\delta_0 > 0$ such that $\lambda(u) = 0$ for $u \le -2 + \delta_0$ and $\lambda(u) = 1$ for $u \ge 2 - \delta_0$. Set

$$x_j(s,u) = (1 - \lambda(u))x_j^-(s) + \lambda(u)x_j^+(s), \qquad t_j(s,u) = e^{-x_j(s,u)}.$$

Convexity of $\mathcal{D}_{\gamma(s)}$ gives $0 < t_m(s,u) \le \dots \le t_1(s,u) < 1$ and $\prod_{j=1}^m t_j(s,u) = \gamma(s)$. The first $m-1$ entries are equal. Define

$$A(s,u) = t_1(s,u), \qquad \beta(s,u) = \frac{t_m(s,u)}{t_1(s,u)}, \qquad \beta_+(s) = \frac{\gamma(s)}{a(s)^m}.$$

Then $\beta(s,u) = \beta_+(s)^{\lambda(u)}$. Since $\gamma(s) \le \beta_+(s) \le 1$ and $0 \le \lambda \le 1$, one has $\beta_+(s)^{\lambda(u)} \ge \beta_+(s)$. Consequently,

$$q < \gamma(s) \le \beta(s,u) \le 1, \qquad A(s,u)^m \beta(s,u) = \gamma(s). \tag{5.22}$$

For every Riemannian metric h, the change of variable $r = At$ gives $dr^2 + \sin^2(r/A)A^2 h = A^2(dt^2 + \sin^2 t\, h)$. The first $m-1$ suspension parameters are $A(s,u)$. Repeated use of this identity factors out $A(s,u)^2$. The remaining unit suspensions are spherical joins with S^0; associativity from Lemma 5.1 identifies the resulting quotient with $S^{m-2} * S^1_{2\pi\beta(s,u)}$. Under the fixed smooth-sphere identification of Lemma 5.2, its quotient length metric is

$$g^{\mathrm{raw}}_{s,u} = A(s,u)^2 \left(dt^2 + \cos^2 t\, g_{S^{m-2}} + \beta(s,u)^2 \sin^2 t\, d\theta^2 \right).$$

On every lifted angular interval, the map

$$(t, x, \theta) \longmapsto A(s,u)\big(\cos t\, x, \sin t \cos(\beta(s,u)\widetilde{\theta}), \sin t \sin(\beta(s,u)\widetilde{\theta})\big)$$

is a local isometry into the round sphere of radius $A(s,u)$. Thus the m-dimensional Hausdorff measure on the regular cylinder is the Riemannian measure of the displayed tensor. The collapsed strata have dimensions $m-2$ and 1, hence zero m-dimensional Hausdorff measure. Using $\sigma_m = 2\pi\sigma_{m-2}/(m-1)$, one obtains

$$\mathcal{H}^m_{g^{\mathrm{raw}}_{s,u}}(S^m) = A(s,u)^m \beta(s,u) \frac{2\pi\sigma_{m-2}}{m-1} = \gamma(s)\sigma_m.$$

Replace the constant d_* in Lemma 5.2 by $\min\{d_*, 1/16\}$, and set

$$d(s,u) = \frac{d_*}{8} \frac{1 - A(s,u)^2}{2 - A(s,u)^2} (1 - \vartheta(s))^4.$$

For finite s, $0 < A(s,u) < 1$, and therefore

$$0 < d(s,u) < d_*, \qquad d(s,u) < 1 - A(s,u)^2.$$

The functions A, β, and d are constant in u near $(-\infty,-2]$ and $[2,\infty)$ and are constant in s on $[0, s_0]$. Define $\widetilde{g}_{s,u} = \mathcal{G}(A(s,u), \beta(s,u), d(s,u))$. The hypotheses of Lemma 5.2 follow from (5.22); hence

$$\sec_{\widetilde{g}_{s,u}} \ge A(s,u)^{-2}(1 - d(s,u)) > 1, \qquad \mathrm{Ric}_{\widetilde{g}_{s,u}} \ge (m-1)\widetilde{g}_{s,u}.$$

Moreover,

$$0 \le \mathrm{Vol}(S^m, \widetilde{g}_{s,u}) - \gamma(s)\sigma_m \le C_m A(s,u)^m d(s,u). \tag{5.23}$$

Let

$$V(s,u) = \mathrm{Vol}(S^m, \widetilde{g}_{s,u}), \qquad \xi(s,u)^m = \frac{\gamma(s)\sigma_m}{V(s,u)}, \qquad \widehat{g}_{s,u} = \xi(s,u)^2 \widetilde{g}_{s,u}.$$

The smooth parameter dependence in Lemma 5.2 implies that V, ξ, and $\widehat{g}$ are smooth. Equation (5.23) gives $0 < \xi \le 1$. Since constant scaling leaves the Ricci tensor unchanged and $\xi^{-2} \ge 1$,

$$\mathrm{Vol}(S^m, \widehat{g}_{s,u}) = \gamma(s)\sigma_m, \qquad \mathrm{Ric}_{\widehat{g}_{s,u}} \ge (m-1)\widehat{g}_{s,u}. \tag{5.24}$$

If $\beta(s,u) = 1$, then Lemma 5.2 gives $\widetilde{g}_{s,u} = A(s,u)^2 g_{S^m}$. In this case $A(s,u)^m = \gamma(s)$, so $\xi(s,u) = 1$ and

$$\widehat{g}_{s,u} = \gamma(s)^{2/m} g_{S^m}. \tag{5.25}$$

In particular, this holds for $u \le -2$.

At $u = 2$,

$$A(s,2) = a(s) \to 1, \qquad \beta(s,2) = \frac{\gamma(s)}{a(s)^m} \to q, \qquad d(s,2) \to 0.$$

Let d_s^{raw}, $\widetilde{d}_s$, and $\widehat{d}_s$ denote the distances of $g_{s,2}^{\mathrm{raw}}$, $\widetilde{g}_{s,2}$, and $\widehat{g}_{s,2}$. By (5.23), $1 \le \xi(s,2)^{-m} \le 1 + C_m d(s,2)/(q\sigma_m)$, so $\xi(s,2) \to 1$. The Gromov–Hausdorff estimate in Lemma 5.2 gives

$$d_{\mathrm{GH}}\big((S^m, \widetilde{d}_s), (S^m, d_s^{\mathrm{raw}})\big) \le A(s,2)\omega_{m,q}(d(s,2)) \longrightarrow 0.$$

Let d_q be the quotient distance of $B_{n,q}$ on the common underlying join quotient, and put

$$c_s = A(s,2), \qquad C_s = \frac{A(s,2)\beta(s,2)}{q} = \frac{\gamma(s)}{q\, a(s)^{m-1}}.$$

For finite s, $0 < c_s < 1 < C_s$, and both c_s and C_s tend to 1. For every absolutely continuous lifted piece used in the quotient-length definition, including pieces contained in either endpoint stratum, the two length integrands F_q and F_s satisfy $c_s F_q \le F_s \le C_s F_q$. Taking infima over the same admissible concatenations gives

$$c_s d_q \le d_s^{\mathrm{raw}} \le C_s d_q.$$

Since every spherical-join distance is at most π, the identity correspondence has distortion at most $\pi \max\{1 - c_s, C_s - 1\}$. Thus

$$d_{\mathrm{GH}}\big((S^m, d_s^{\mathrm{raw}}), B_{n,q}\big) \le \frac{\pi}{2} \max\{1 - c_s, C_s - 1\} \longrightarrow 0.$$

The first two Gromov–Hausdorff estimates bound $\mathrm{diam}(S^m, \widetilde{d}_s)$ for large s. Since $\widehat{d}_s = \xi(s,2)\widetilde{d}_s$,

$$d_{\mathrm{GH}}\big((S^m, \widehat{d}_s), (S^m, \widetilde{d}_s)\big) \le \frac{1}{2}|1 - \xi(s,2)| \,\mathrm{diam}(S^m, \widetilde{d}_s) \longrightarrow 0.$$

Set $h_s = \widehat{g}_{s,2}$. Equation (5.24) gives (5.19)–(5.20), and the three estimates give (5.21).

It remains to make the volume form independent of u. Orient S^m, fix a background metric, and let $\mathscr{G}$ be the Green operator of its Hodge Laplacian on top-degree forms. Define

$$\omega_{s,u} = dv_{\widehat{g}_{s,u}}, \qquad \alpha_{s,u} = \partial_u \omega_{s,u}.$$

Equation (5.24) gives $\int_{S^m} \alpha_{s,u} = 0$. Thus the harmonic projection of $\alpha_{s,u}$ vanishes. Since $\mathscr{G}\alpha_{s,u}$ has top degree, $d\mathscr{G}\alpha_{s,u} = 0$, and the Hodge decomposition yields $d\, d^*\mathscr{G}\alpha_{s,u} = \alpha_{s,u}$. Define

$$\eta_{s,u} = d^*\mathscr{G}\alpha_{s,u}, \qquad \iota_{X_{s,u}}\omega_{s,u} = -\eta_{s,u}.$$

Contraction with the positive volume form is an isomorphism from vector fields to $(m-1)$-forms, so $X_{s,u}$ is uniquely defined and depends smoothly on (s,u). For $-2 \le u \le 2$, let

$$\partial_u \phi_{s,u} = X_{s,u} \circ \phi_{s,u}, \qquad \phi_{s,-2} = \mathrm{id}\,.$$

A smooth time-dependent vector field on the closed manifold S^m has a flow throughout the finite interval $[-2,2]$, and smooth ODE dependence gives joint smoothness in (s,u,x). Cartan's formula gives

$$\frac{d}{du}\left(\phi_{s,u}^*\omega_{s,u}\right) = \phi_{s,u}^*\left(\alpha_{s,u} + d\iota_{X_{s,u}}\omega_{s,u}\right) = 0.$$

The metric $\widehat{g}_{s,u}$ is constant in u on fixed neighbourhoods of $(-\infty,-2]$ and $[2,\infty)$; hence $\alpha_{s,u} = \eta_{s,u} = X_{s,u} = 0$ there. Extend $\phi_{s,u}$ by id for $u \le -2$ and by $\phi_{s,2}$ for $u \ge 2$, and set $g_{s,u} = \phi_{s,u}^*\widehat{g}_{s,u}$. The constant extensions are smooth and satisfy

$$dv_{g_{s,u}} = \phi_{s,u}^*\omega_{s,u} = \omega_{s,-2} = dv_{g_{s,-2}}, \qquad \mathrm{Ric}_{g_{s,u}} \ge (m-1)g_{s,u}.$$

Equation (5.25) and the definition of h_s give the endpoint metrics in the statement.

For $0 \le s \le s_0$, one has $\vartheta(s) = 0$ and $\mathbf{t}^+(s) = \mathbf{t}^-(s)$. Therefore $A(s,u) = \gamma_*^{1/m}$, $\beta(s,u) = 1$, and $\widehat{g}_{s,u} = \gamma_*^{2/m} g_{S^m}$ for every u. It follows that $X_{s,u} = 0$, $\phi_{s,u} = \mathrm{id}$, and $h_s = g_{s,u} = \gamma_*^{2/m} g_{S^m}$. □

6. Spherical deformation

For a tensor field T on S^{n-1}, write $\|T\|_g := \sup_{x\in S^{n-1}} |T(x)|_g$.

Theorem 6.1. *Let $I \subset [0,\infty)$ be a nonempty compact interval. Let $g_{s,u}$, $(s,u) \in I \times \mathbb{R}$, be a smooth family of metrics on S^{n-1} such that*

$$\mathrm{Ric}_{g_{s,u}} \ge (n-2)g_{s,u}, \qquad \partial_u dv_{g_{s,u}} = 0. \tag{6.1}$$

Assume also that

$$\sup_{(s,u)\in I\times\mathbb{R}} \left(\|\partial_u g_{s,u}\|_{g_{s,u}} + \|\partial_u^2 g_{s,u}\|_{g_{s,u}} + \|\nabla^{g_{s,u}}\partial_u g_{s,u}\|_{g_{s,u}}\right) < \infty. \tag{6.2}$$

Assume that $g_{s,u} = g_{s,-2}$ for $u \le -2$ and $g_{s,u} = g_{s,2}$ for $u \ge 2$.

There are constants $\varepsilon_0 = \varepsilon_0(n) > 0$ and $0 < c_n < C_n < \infty$ with the following property. For every $0 < \varepsilon < \varepsilon_0$ there are smooth functions $\chi_\varepsilon : (0,1] \to (0,1]$ and $\psi_\varepsilon : (0,1] \to [-2,2]$, numbers $r_{-,\varepsilon} \in (0,1)$ and $\kappa_\varepsilon \in (0,1/2)$, and a smooth family of metrics on $(0,1] \times S^{n-1}$,

$$G_{s,\varepsilon} := dr^2 + \left(\frac{\sin(\kappa_\varepsilon r)}{\kappa_\varepsilon}\chi_\varepsilon(r)\right)^2 g_{s,\psi_\varepsilon(r)}, \qquad s \in I. \tag{6.3}$$

The resulting data have the following properties:

$$\psi_\varepsilon(r) = -2 \quad (0 < r \le r_{-,\varepsilon}), \qquad \psi_\varepsilon(r) = 2 \quad \textit{for } r \textit{ near } 1, \tag{6.4}$$

$$\lim_{r\downarrow 0}\chi_\varepsilon(r) = 1, \qquad \lim_{r\downarrow 0} r\chi_\varepsilon'(r) = 0, \tag{6.5}$$

$$c_n\varepsilon \le \kappa_\varepsilon \le C_n\varepsilon, \tag{6.6}$$

$$\mathrm{Ric}_{G_{s,\varepsilon}} \ge \left(n-1-C_n\varepsilon^4 - C_n\kappa_\varepsilon^2\right)\kappa_\varepsilon^2 G_{s,\varepsilon}. \tag{6.7}$$

Define

$$\lambda_\varepsilon := \sqrt{\sin^2\varepsilon - 8\varepsilon^4}, \qquad A_\varepsilon := \frac{\lambda_\varepsilon}{\kappa_\varepsilon}, \qquad c_\varepsilon := \frac{\sin\varepsilon}{A_\varepsilon}.$$

These choices satisfy

$$(c_\varepsilon^2 G_{s,\varepsilon})|_{\{1\}\times S^{n-1}} = \sin^2\varepsilon\, g_{s,2}, \tag{6.8}$$

$$\mathrm{II}_{\{1\}\times S^{n-1},\, c_\varepsilon^2 G_{s,\varepsilon}} > \cot\varepsilon\,(c_\varepsilon^2 G_{s,\varepsilon})|_{\{1\}\times S^{n-1}}, \tag{6.9}$$

$$\mathrm{Ric}_{c_\varepsilon^2 G_{s,\varepsilon}} \ge (n-1-C_n\varepsilon^2)c_\varepsilon^2 G_{s,\varepsilon}. \tag{6.10}$$

The map $(s, \varepsilon, r) \mapsto G_{s,\varepsilon}(r)$ is smooth. The functions χ_ε, ψ_ε and the numbers $r_{-,\varepsilon}$, κ_ε, c_ε depend smoothly on ε. Their formulas may involve I and the family $(g_{s,u})_{(s,u)\in I\times\mathbb{R}}$ but do not involve s; the constants ε_0, c_n, C_n depend only on n.

Proposition 6.2. *Let $m \geq 2$, let $A \geq 1$, and let X^m be a closed smooth manifold. Let $(\ell_t)_{t\in\mathbb{R}}$ be a smooth family of metrics on X. Assume that, for every $t \in \mathbb{R}$, the following identities and inequalities hold pointwise on X:*

$$dv_{\ell_t} = dv_{\ell_0}, \qquad \mathrm{Ric}_{\ell_t} \geq (m-1)\ell_t, \tag{6.11}$$

and that

$$|\dot{\ell}_t|_{\ell_t} \leq A, \qquad |\ddot{\ell}_t|_{\ell_t} \leq A, \qquad |\nabla^{\ell_t}\dot{\ell}_t|_{\ell_t} \leq A. \tag{6.12}$$

Here dv_{ℓ_t} denotes the Riemannian volume density and a dot denotes a t-derivative.

Let $J \subset (0,\infty)$ be an interval. Let $f : J \to \mathbb{R}$ and $w : J \to (0,\infty)$ be smooth, put $g(r) := \ell_{f(r)}$, and equip $J \times X$ with $G := dr^2 + w(r)^2 g(r)$. A prime denotes an r-derivative, and $\mathrm{div}_g\, g'$ is the one-form defined by

$$(\mathrm{div}_g\, g')(Z) := \sum_{a=1}^{m} (\nabla^g_{e_a} g')(e_a, Z),$$

where $(e_1, \dots, e_m)$ is any $g(r)$-orthonormal frame. In coordinates, $(\mathrm{div}_g\, g')_i = \nabla^a g'_{ai}$. If $e \in T_xX$ is $g(r)$-unit, then

$$\mathrm{Ric}_G(\partial_r, \partial_r) = -m\frac{w''}{w} - \frac{1}{4}|g'|_g^2, \tag{6.13}$$

$$\mathrm{Ric}_G(\partial_r, w^{-1}e) = \frac{1}{2w}\big(\mathrm{div}_g\, g'\big)(e), \tag{6.14}$$

$$\begin{aligned} \mathrm{Ric}_G(w^{-1}e, w^{-1}e) = w^{-2}\,\mathrm{Ric}_g(e,e) - (m-1)\left(\frac{w'}{w}\right)^2 - \frac{w''}{w} \\ - \frac{m}{2}\frac{w'}{w} g'(e,e) - \frac{1}{2}g''(e,e) + \frac{1}{2}|g'(e,\cdot)|_g^2. \end{aligned} \tag{6.15}$$

In coordinates tangent to X,

$$\begin{aligned} (\mathrm{Ric}_G)_{ij} = (\mathrm{Ric}_g)_{ij} + w^2\Bigg[-\left((m-1)\left(\frac{w'}{w}\right)^2 + \frac{w''}{w}\right) g_{ij} \\ - \frac{m}{2}\frac{w'}{w} g'_{ij} - \frac{1}{2}g''_{ij} + \frac{1}{2}g^{ab} g'_{ai} g'_{bj}\Bigg]. \end{aligned} \tag{6.16}$$

Furthermore,

$$|g'|_g \leq A|f'|, \qquad |\nabla^g g'|_g \leq A|f'|, \qquad |g''|_g \leq A\big(|f''| + |f'|^2\big). \tag{6.17}$$

The mixed component also satisfies

$$\big|\mathrm{Ric}_G(\partial_r, w^{-1}e)\big| \leq \frac{\sqrt{m}\,A}{2w}|f'|. \tag{6.18}$$

Suppose now that $J = (0,1]$ and $0 < \rho < e^{-e}$. Define

$$T(r) := -\log(\rho r), \qquad L(r) := \log T(r),$$

$$h(r) := 1 - \frac{E}{L(r)}, \qquad w(r) := rh(r), \qquad f(r) := -F\log L(r) + c,$$

where $E, F > 0$ and $c \in \mathbb{R}$. Assume that, for every $0 < r \leq 1$,

$$\frac{E}{L(r)} \leq \frac{1}{2}, \qquad \frac{T(r)}{L(r)} > 4, \qquad E \geq 2\sqrt{m}\,AF, \qquad E \geq A^2F^2. \tag{6.19}$$

Then, for every $g(r)$-unit vector e,

$$\mathrm{Ric}_G(\partial_r, \partial_r) \geq \frac{E}{2T(r)L(r)^2 r^2}, \tag{6.20}$$

$$\mathrm{Ric}_G(w^{-1}e, w^{-1}e) \geq \frac{E}{L(r)r^2}, \tag{6.21}$$

$$\left|\mathrm{Ric}_G(\partial_r, w^{-1}e)\right| \leq \frac{\sqrt{m}\,AF}{T(r)L(r)r^2}. \tag{6.22}$$

The Ricci tensor of G is positive definite on $(0,1] \times X$.

Proof. The identity $\partial_r dv_{g(r)} = \frac{1}{2}\operatorname{tr}_{g(r)} g'(r)\, dv_{g(r)}$, together with the first identity in (6.11), gives $\operatorname{tr}_{g(r)} g'(r) = 0$. Differentiating $g^{ab} g'_{ab} = 0$ and using $(g^{ab})' = -g^{ap} g'_{pq} g^{qb}$ gives

$$\operatorname{tr}_{g(r)} g''(r) = |g'(r)|^2_{g(r)}. \tag{6.23}$$

Put $\widehat{g}(r) := w(r)^2 g(r)$. The metric $G = dr^2 + \widehat{g}(r)$ has the form required by [13, Lemma 2.4]: the interval is J, the slice manifold is X, and $r \mapsto \widehat{g}(r)$ is a smooth family of positive-definite metrics. For tangent vectors U, V and a $\widehat{g}(r)$-orthonormal frame $(\widehat{e}_1, \dots, \widehat{e}_m)$, that lemma states

$$\begin{aligned}
\mathrm{Ric}_G(\partial_r, \partial_r) &= -\frac{1}{2}\operatorname{tr}_{\widehat{g}} \widehat{g}'' + \frac{1}{4}|\widehat{g}'|^2_{\widehat{g}}, \\
\mathrm{Ric}_G(U, \partial_r) &= -\frac{1}{2} U\left(\operatorname{tr}_{\widehat{g}} \widehat{g}'\right) + \frac{1}{2}\sum_{j=1}^m (\nabla^{\widehat{g}}_{\widehat{e}_j} \widehat{g}')(U, \widehat{e}_j), \\
\mathrm{Ric}_G(U, V) &= \mathrm{Ric}_{\widehat{g}}(U, V) - \frac{1}{2}\widehat{g}''(U, V) + \frac{1}{2}\sum_{j=1}^m \widehat{g}'(U, \widehat{e}_j)\widehat{g}'(V, \widehat{e}_j) \\
&\quad - \frac{1}{4}\widehat{g}'(U, V)\operatorname{tr}_{\widehat{g}} \widehat{g}'.
\end{aligned}$$

Put $a := w'/w$, let $(e_1, \dots, e_m)$ be $g(r)$-orthonormal, and set $\widehat{e}_j := w^{-1} e_j$. Since w is constant on each slice and $\operatorname{tr}_g g' = 0$,

$$\begin{aligned}
\widehat{g}' &= w^2(2ag + g'), \\
\widehat{g}'' &= w^2\left(2\left(a^2 + \frac{w''}{w}\right) g + 4ag' + g''\right), \\
\operatorname{tr}_{\widehat{g}} \widehat{g}' &= 2ma, \\
|\widehat{g}'|^2_{\widehat{g}} &= 4ma^2 + |g'|^2_g, \\
\operatorname{tr}_{\widehat{g}} \widehat{g}'' &= 2m\left(a^2 + \frac{w''}{w}\right) + |g'|^2_g.
\end{aligned}$$

The final identity uses (6.23). Substitution in the first cylinder formula gives

$$-\frac{1}{2}\operatorname{tr}_{\widehat{g}} \widehat{g}'' + \frac{1}{4}|\widehat{g}'|^2_{\widehat{g}} = -m\frac{w''}{w} - \frac{1}{4}|g'|^2_g,$$

which proves (6.13).

The function $\operatorname{tr}_{\widehat{g}} \widehat{g}' = 2ma$ is constant on each slice. If $\widehat{e} = w^{-1}e$, then

$$\sum_{j=1}^m (\nabla^{\widehat{g}}_{\widehat{e}_j} \widehat{g}')(\widehat{e}, \widehat{e}_j) = \frac{1}{w}\sum_{j=1}^m (\nabla^g_{e_j} g')(e, e_j) = \frac{1}{w}(\operatorname{div}_g g')(e).$$

The last equality follows because g' is symmetric, so $(\nabla^g_Z g')(U, V) = (\nabla^g_Z g')(V, U)$. Substitution in the second cylinder formula proves (6.14).

For tangent coordinate vectors, the three terms in the third cylinder formula which involve r-derivatives are

$$-\frac{1}{2}\widehat{g}''_{ij} = w^2\left[-\left(a^2+\frac{w''}{w}\right)g_{ij} - 2ag'_{ij} - \frac{1}{2}g''_{ij}\right],$$
$$\frac{1}{2}\sum_{\alpha=1}^{m}\widehat{g}'(\partial_i,\widehat{e}_\alpha)\widehat{g}'(\partial_j,\widehat{e}_\alpha) = w^2\left[2a^2g_{ij} + 2ag'_{ij} + \frac{1}{2}g^{ab}g'_{ai}g'_{bj}\right],$$
$$-\frac{1}{4}\widehat{g}'_{ij}\operatorname{tr}_{\widehat{g}}\widehat{g}' = w^2\left[-ma^2g_{ij} - \frac{m}{2}ag'_{ij}\right].$$

For each fixed r, constant rescaling gives $\mathrm{Ric}_{w(r)^2g(r)} = \mathrm{Ric}_{g(r)}$ as symmetric $(0,2)$-tensors. Adding the four terms proves (6.16); evaluation on $w^{-1}e$ proves (6.15).

The chain rule gives

$$g' = \dot{\ell}_{f(r)}f', \qquad g'' = \ddot{\ell}_{f(r)}(f')^2 + \dot{\ell}_{f(r)}f'', \qquad \nabla^g g' = (\nabla^{\ell_{f(r)}}\dot{\ell}_{f(r)})f'.$$

This proves (6.17). If $(e_1,\dots,e_m)$ is a $g(r)$-orthonormal frame, then

$$\left|\left(\operatorname{div}_g g'\right)(e)\right| = \left|\sum_{a=1}^{m}(\nabla_{e_a}g')(e_a,e)\right| \le \sqrt{m}\,|\nabla^g g'|_g.$$

Equation (6.14) gives (6.18).

For the logarithmic functions, $0<\rho<e^{-e}$ and $0<r\le 1$ give $T(r)>e$ and $L(r)>1$. Direct differentiation gives

$$h' = -\frac{E}{rTL^2}, \qquad f' = \frac{F}{rTL}, \tag{6.24}$$
$$w'' = -\frac{E}{rTL^2}\left(1+\frac{1}{T}+\frac{2}{TL}\right). \tag{6.25}$$

Write $x := E/L$ and $\delta := E/(hTL^2) = x/(hTL)$. The first inequality in (6.19) gives $1/2 \le h < 1$. Hence $x/h \le 1$, and $TL>1$ gives $0<\delta<1$. Thus

$$\frac{w'}{w} = \frac{1-\delta}{r}, \qquad 0 < \frac{w'}{w} \le \frac{1}{r}. \tag{6.26}$$

Equations (6.13), (6.17), and (6.25), together with $A^2F^2 \le E$ and $T \ge 1$, give

$$\mathrm{Ric}_G(\partial_r,\partial_r) \ge \frac{1}{r^2TL^2}\left(mE - \frac{A^2F^2}{4T}\right) \ge \frac{E}{2r^2TL^2}.$$

This proves (6.20).

For $0 \le x \le 1/2$,

$$h^{-2} - 1 - 2x = \frac{x^2(3-2x)}{(1-x)^2} \ge 0. \tag{6.27}$$

Moreover, $-w''/w > 0$ by (6.25), and $|g'(e,\cdot)|_g^2 \ge 0$. Since $T/L > 4$ and $L>1$, one has $0 < 1 - T^{-1} - (TL)^{-1} < 1$. Consequently,

$$f'' = -\frac{F}{r^2TL}\left(1-\frac{1}{T}-\frac{1}{TL}\right), \qquad |f''| \le \frac{F}{r^2TL}.$$

Because $0<\delta<1$, equations (6.26) and (6.27) give

$$h^{-2} - (1-\delta)^2 \ge h^{-2} - 1 \ge 2x = \frac{2E}{L}.$$

Using this inequality and (6.17) in (6.15) gives

$$\begin{aligned}\mathrm{Ric}_G(w^{-1}e,w^{-1}e) &\geq (m-1)r^{-2}\big(h^{-2}-(1-\delta)^2\big)-\frac{m}{2r}|g'(e,e)|-\frac{1}{2}|g''(e,e)|\\ &\geq \frac{1}{Lr^2}\left(2(m-1)E-\frac{(m+1)AF}{2T}-\frac{AF^2}{2T^2L}\right)\\ &\geq \frac{1}{Lr^2}\left(2(m-1)E-\frac{(m+1)AF}{2T}-\frac{AF^2}{2TL}\right).\end{aligned}$$

The inequality $E \geq 2\sqrt{m}\,AF$ gives $AF \leq E/(2\sqrt{m})$. Since $A \geq 1$, the inequality $E \geq A^2F^2$ also gives $AF^2 \leq E$. Since $T > 4$ and $L > 1$, the expression in parentheses is at least

$$E\left(2(m-1)-\frac{m+1}{16\sqrt{m}}-\frac{1}{8}\right).$$

For $m \geq 2$,

$$2(m-1)-\frac{m+1}{16\sqrt{m}}-\frac{1}{8} \geq 2(m-1)-\frac{m+1}{16}-\frac{1}{8}=\frac{31m-35}{16}>1.$$

This proves (6.21). Equations (6.18), (6.24), and $h \geq 1/2$ give (6.22).

Define

$$\alpha_r := \frac{E}{2TL^2r^2}, \qquad \gamma_r := \frac{E}{Lr^2}, \qquad b_r := \frac{\sqrt{m}\,AF}{TLr^2}.$$

The tangential restriction of Ric_G is bounded below by $\gamma_r G|_{TX}$, and the mixed one-form $V \mapsto \mathrm{Ric}_G(\partial_r, V)$ has norm at most b_r with respect to $G|_{TX}$. Furthermore,

$$\begin{aligned}\alpha_r\gamma_r - b_r^2 &\geq \frac{E^2}{2TL^3r^4}\left(1-\frac{2mA^2F^2L}{E^2T}\right)\\ &\geq \frac{E^2}{2TL^3r^4}\left(1-\frac{L}{2T}\right)>0.\end{aligned}$$

The second inequality uses $E \geq 2\sqrt{m}\,AF$; the last uses $T/L > 4$. For $s \in \mathbb{R}$ and $V \in T_xX$,

$$\mathrm{Ric}_G(s\partial_r+V, s\partial_r+V) \geq \alpha_r s^2 - 2b_r|s|\,|V|_G + \gamma_r|V|_G^2.$$

Since $\alpha_r > 0$, $\gamma_r > 0$, and $\alpha_r\gamma_r - b_r^2 > 0$, this quadratic form is positive definite. Hence Ric_G is positive definite. ☐

Proposition 6.3. *Let $I \subset [0,\infty)$ be a nonempty compact interval. Let $g_{s,u}$, $(s,u) \in I \times \mathbb{R}$, be a smooth family of metrics on S^{n-1} satisfying* (6.1), (6.2), *and*

$$g_{s,u} = g_{s,-2} \quad (u \leq -2), \qquad g_{s,u} = g_{s,2} \quad (u \geq 2).$$

There are constants $\theta_{\mathrm{tr},n} \in (0,1)$ and $C_{\mathrm{tr},n} < \infty$, depending only on n. There are also a number $L_I \geq 1$ and a smooth nondecreasing map $\rho_I : \mathbb{R} \to [-2,2]$, depending only on I and the family $(g_{s,u})_{(s,u)\in I\times\mathbb{R}}$, such that

$$\rho_I(v) = -2 \quad (v \leq -L_I), \qquad \rho_I(v) = 2 \quad (v \geq L_I).$$

For $0 < \theta < \theta_{\mathrm{tr},n}$, put

$$E := \frac{\theta}{8}, \qquad F := \frac{\theta}{64n}, \qquad \ell_0 := 64n\theta^{-2}, \qquad T_0 := e^{\ell_0}, \qquad r_0(\theta) := e^{-T_0}, \tag{6.28}$$

and, for $0 < r \leq 1$, set

$$\begin{aligned} T_\theta(r) &:= -\log(r_0(\theta)r), & \ell_\theta(r) &:= \log T_\theta(r),\\ c_{\theta,I} &:= L_I + 1 + F\log \ell_0, & \chi_\theta(r) &:= 1-\frac{E}{\ell_\theta(r)},\\ \varphi_\theta(r) &:= -F\log\ell_\theta(r)+c_{\theta,I}, & \psi_\theta(r) &:= \rho_I(\varphi_\theta(r)).\end{aligned}\tag{6.29}$$

The function φ_θ is strictly increasing, $\varphi_\theta(r) \to -\infty$ as $r \downarrow 0$, and $\varphi_\theta(1) = L_I + 1$. Let $r_{-,\theta} \in (0,1)$ be the unique solution of $\varphi_\theta(r_{-,\theta}) = -L_I$, and put

$$b_\theta(r) := r\chi_\theta(r), \qquad g_{s,\theta}(r) := g_{s,\psi_\theta(r)}, \qquad G_{0,s,\theta} := dr^2 + b_\theta(r)^2 g_{s,\theta}(r).$$

For every $s \in I$ and $0 < r \leq 1$,

$$\mathrm{Ric}_{g_{s,\theta}(r)} \geq (n-2) g_{s,\theta}(r), \qquad dv_{g_{s,\theta}(r)} = dv_{g_{s,-2}}. \tag{6.30}$$

Primes below denote r-derivatives. On $(0,1] \times S^{n-1}$,

$$\mathrm{Ric}_{G_{0,s,\theta}} \geq 0, \tag{6.31}$$

$$|1-\chi_\theta| + r\left|\frac{\chi_\theta'}{\chi_\theta}\right| + r|g_{s,\theta}'| + r^2|g_{s,\theta}''| \leq \theta, \tag{6.32}$$

$$r^2 \left|\mathrm{Ric}_{G_{0,s,\theta}}(\partial_r, b_\theta^{-1} e)\right| \leq C_{\mathrm{tr},n}\theta \tag{6.33}$$

for every $g_{s,\theta}(r)$-unit vector e. The tensor norms in (6.32) are pointwise norms with respect to $g_{s,\theta}(r)$. Moreover,

$$g_{s,\theta}(r) = g_{s,-2} \quad (0 < r \leq r_{-,\theta}), \qquad g_{s,\theta}(r) = g_{s,2} \quad \text{for } r \text{ near } 1.$$

At the outer endpoint,

$$|1-\chi_\theta(1)| + |\chi_\theta'(1)| \leq \theta, \tag{6.34}$$

and at the inner endpoint,

$$\lim_{r\downarrow 0} \chi_\theta(r) = 1, \qquad \lim_{r\downarrow 0} r\chi_\theta'(r) = 0. \tag{6.35}$$

The maps $(\theta, r) \mapsto \chi_\theta(r)$, $(\theta, r) \mapsto \varphi_\theta(r)$, and $(\theta, r) \mapsto \psi_\theta(r)$ are smooth on $(0, \theta_{\mathrm{tr},n}) \times (0,1]$, and the maps $\theta \mapsto r_0(\theta)$ and $\theta \mapsto r_{-,\theta}$ are smooth. The functions $r_0, \chi_\theta, \varphi_\theta, \psi_\theta$ and the constants $E, F, \ell_0, T_0, c_{\theta,I}$ depend on n, I, θ, r only through the displayed formulas and do not depend on s. The number L_I and the map ρ_I depend only on I and the family $\{g_{s,u} : (s,u) \in I \times \mathbb{R}\}$.

Proof. Define

$$C_I := \sup_{(s,u)\in I\times\mathbb{R}} \left(\|\partial_u g_{s,u}\|_{g_{s,u}} + \|\partial_u^2 g_{s,u}\|_{g_{s,u}} + \|\nabla^{g_{s,u}}\partial_u g_{s,u}\|_{g_{s,u}}\right).$$

The pointwise norm functions are continuous on $I \times [-2,2] \times S^{n-1}$. Their maxima on S^{n-1} are therefore bounded on $I \times [-2,2]$. Since the family is constant in u outside $I \times [-2,2]$, one has $C_I < \infty$.

Define

$$\eta(t) := \begin{cases} 0, & t \leq 0, \\ e^{-1/t}, & t > 0, \end{cases} \qquad \vartheta_0(t) := \frac{\eta(t)}{\eta(t) + \eta(1-t)}, \qquad \rho_0(v) := -2 + 4\vartheta_0\left(\frac{v+1}{2}\right).$$

For every $j \geq 0$, the derivative $\eta^{(j)}(t)$ on $t > 0$ is $e^{-1/t}$ times a polynomial in t^{-1}; hence $\eta^{(j)}(t) \to 0$ as $t \downarrow 0$. Thus η is smooth on $\mathbb{R}$. The denominator defining ϑ_0 is positive, $\vartheta_0 = 0$ on $(-\infty, 0]$, and $\vartheta_0 = 1$ on $[1, \infty)$. The vanishing jets of η at 0 show that ϑ_0 is smooth at 0 and 1. On $(0,1)$,

$$\vartheta_0'(t) = \frac{\eta'(t)\eta(1-t) + \eta(t)\eta'(1-t)}{(\eta(t) + \eta(1-t))^2} > 0.$$

Hence ρ_0 is smooth and nondecreasing, with $\rho_0(v) = -2$ for $v \leq -1$ and $\rho_0(v) = 2$ for $v \geq 1$.

Set $a_1 := \|\rho_0'\|_{L^\infty(\mathbb{R})}$ and $a_2 := \|\rho_0''\|_{L^\infty(\mathbb{R})}$. For $L \geq 1$, define $k_{s,v}^{(L)} := g_{s,\rho_0(v/L)}$. The chain rule gives

$$\begin{aligned}
\partial_v k_{s,v}^{(L)} &= \frac{\rho_0'(v/L)}{L} (\partial_u g)_{s,\rho_0(v/L)}, \\
\partial_v^2 k_{s,v}^{(L)} &= \frac{\rho_0''(v/L)}{L^2} (\partial_u g)_{s,\rho_0(v/L)} + \frac{\rho_0'(v/L)^2}{L^2} (\partial_u^2 g)_{s,\rho_0(v/L)}, \\
\nabla^{k_{s,v}^{(L)}} \partial_v k_{s,v}^{(L)} &= \frac{\rho_0'(v/L)}{L} (\nabla^{g_{s,u}} \partial_u g_{s,u})\big|_{u=\rho_0(v/L)}.
\end{aligned}$$

Consequently,

$$\|\partial_v k_{s,v}^{(L)}\| + \|\partial_v^2 k_{s,v}^{(L)}\| + \|\nabla\partial_v k_{s,v}^{(L)}\| \le \frac{2a_1 C_I}{L} + \frac{(a_2 + a_1^2)C_I}{L^2}. \tag{6.36}$$

All norms in this estimate are C^0-norms with respect to $k_{s,v}^{(L)}$. Set $L_I := \max\{1, 4a_1 C_I, \sqrt{2(a_2 + a_1^2)C_I}\}$. The right-hand side of (6.36) is then at most one. Define

$$\rho_I(v) := \rho_0(v/L_I), \qquad k_{s,v} := g_{s,\rho_I(v)}.$$

The map ρ_I is smooth and nondecreasing, equals -2 on $(-\infty, -L_I]$, and equals 2 on $[L_I, \infty)$. Moreover,

$$\|\partial_v k_{s,v}\| + \|\partial_v^2 k_{s,v}\| + \|\nabla\partial_v k_{s,v}\| \le 1. \tag{6.37}$$

For every $s \in I$ and $v \in \mathbb{R}$,

$$\partial_v dv_{k_{s,v}} = \rho_I'(v)\, \partial_u dv_{g_{s,u}}\big|_{u=\rho_I(v)} = 0.$$

Thus $dv_{k_{s,v}} = dv_{g_{s,-2}}$ for every $v \in \mathbb{R}$. The first inequality in (6.1), evaluated at $u = \rho_I(v)$, gives $\mathrm{Ric}_{k_{s,v}} \ge (n-2)k_{s,v}$. Also, $k_{s,v} = g_{s,-2}$ for $v \le -L_I$ and $k_{s,v} = g_{s,2}$ for $v \ge L_I$.

Put $m := n - 1$ and $\theta_{\mathrm{tr},n} := 1/4$. For the functions and constants in (6.28)– (6.29), write $T(r) := T_\theta(r)$ and $\ell(r) := \ell_\theta(r)$. Then

$$g_{s,\theta}(r) = g_{s,\rho_I(\varphi_\theta(r))} = k_{s,\varphi_\theta(r)}.$$

Equation (6.30) follows from this identity, the fixed-volume identity for $k_{s,v}$, and $\mathrm{Ric}_{k_{s,v}} \ge (n-2)k_{s,v}$.

Since $n \ge 4$ and $0 < \theta < 1/4$, one has $\ell_0 > 3$ and $0 < r_0(\theta) = e^{-e^{\ell_0}} < e^{-e}$. The function $x \mapsto x/\log x$ is increasing on (e, ∞), and $x \mapsto e^x/x$ is increasing on $(1, \infty)$. Since $T(r) \ge T_0 = e^{\ell_0}$,

$$\frac{E}{\ell(r)} \le \frac{\theta^3}{512n} < \frac{1}{2}, \qquad \frac{T(r)}{\ell(r)} \ge \frac{e^{\ell_0}}{\ell_0} > 4.$$

Moreover,

$$2\sqrt{m}\,F = \frac{\theta\sqrt{n-1}}{32n} \le \frac{\theta}{32} < \frac{\theta}{8} = E, \qquad F^2 = \frac{\theta^2}{4096n^2} < \frac{\theta}{8} = E.$$

Fix $s \in I$. Put

$$A := 1, \qquad X := S^{n-1}, \qquad \widehat{\ell}_v := k_{s,v},$$
$$\rho := r_0(\theta), \qquad c := c_{\theta,I}, \qquad h := \chi_\theta, \qquad w := r\chi_\theta, \qquad f := \varphi_\theta.$$

Here $m \ge 3$, $A \ge 1$, and X is closed and smooth. The family $v \mapsto \widehat{\ell}_v$ is smooth, satisfies $dv_{\widehat{\ell}_v} = dv_{\widehat{\ell}_0}$ and $\mathrm{Ric}_{\widehat{\ell}_v} \ge (m-1)\widehat{\ell}_v$, and (6.37) gives separately

$$\|\dot{\widehat{\ell}}_v\|_{\widehat{\ell}_v} \le A, \qquad \|\ddot{\widehat{\ell}}_v\|_{\widehat{\ell}_v} \le A, \qquad \|\nabla^{\widehat{\ell}_v}\dot{\widehat{\ell}}_v\|_{\widehat{\ell}_v} \le A.$$

Moreover, $J = (0, 1]$, $0 < \rho < e^{-e}$, the functions h, w, f have the logarithmic form required in Proposition 6.2, and the four inequalities in (6.19) are the four inequalities established above. Proposition 6.2 therefore yields, for every $g_{s,\theta}(r)$-unit vector e,

$$\begin{aligned}
\mathrm{Ric}_{G_{0,s,\theta}}(\partial_r, \partial_r) &\ge \frac{E}{2T(r)\ell(r)^2 r^2}, \\
\mathrm{Ric}_{G_{0,s,\theta}}(b_\theta^{-1}e, b_\theta^{-1}e) &\ge \frac{E}{\ell(r)r^2}, \\
\left|\mathrm{Ric}_{G_{0,s,\theta}}(\partial_r, b_\theta^{-1}e)\right| &\le \frac{\sqrt{n-1}\,F}{T(r)\ell(r)r^2}.
\end{aligned} \tag{6.38}$$

The positive-definiteness conclusion of Proposition 6.2 gives $\mathrm{Ric}_{G_{0,s,\theta}} > 0$ and hence (6.31).

Direct differentiation gives

$$\begin{aligned}
\varphi_\theta'(r) &= \frac{F}{rT(r)\ell(r)},\\
\varphi_\theta''(r) &= \frac{F}{r^2T(r)\ell(r)}\left(-1+\frac{1}{T(r)}+\frac{1}{T(r)\ell(r)}\right),\\
\chi_\theta'(r) &= -\frac{E}{rT(r)\ell(r)^2}.
\end{aligned}$$

As $r \downarrow 0$, one has $T(r), \ell(r) \to \infty$; hence (6.35) follows. Since $T(r) \geq e^{\ell_0} > e^3$ and $\ell(r) \geq \ell_0 > 3$,

$$0 < 1 - \frac{1}{T(r)} - \frac{1}{T(r)\ell(r)} < 1.$$

Since $g_{s,\theta}(r) = k_{s,\varphi_\theta(r)}$,

$$g_{s,\theta}' = (\partial_v k)\varphi_\theta', \qquad g_{s,\theta}'' = (\partial_v^2 k)(\varphi_\theta')^2 + (\partial_v k)\varphi_\theta''.$$

Equation (6.37) and $\chi_\theta \geq 1/2$ give

$$\begin{aligned}
|1-\chi_\theta| &\leq \frac{E}{\ell_0}, & r\left|\frac{\chi_\theta'}{\chi_\theta}\right| &\leq \frac{2E}{T_0\ell_0^2},\\
r|g_{s,\theta}'| &\leq \frac{F}{T_0\ell_0}, & r^2|g_{s,\theta}''| &\leq \frac{F}{T_0\ell_0} + \frac{F^2}{T_0^2\ell_0^2}.
\end{aligned}$$

The sum of these four bounds is at most

$$\frac{\theta^3}{512n} + \frac{\theta^5}{16384n^2} + \frac{\theta^3}{2048n^2} + \frac{\theta^6}{16777216n^4}.$$

The ratios of the last three summands to the first are $\theta^2/(32n)$, $1/(4n)$, and $\theta^3/(32768n^3)$, respectively. They are less than one, so the sum is at most $\theta^3/(128n) < \theta$. This proves (6.32). Equation (6.38) gives

$$r^2\left|\mathrm{Ric}_{G_{0,s,\theta}}(\partial_r, b_\theta^{-1}e)\right| \leq \sqrt{n-1}\,F = \frac{\theta\sqrt{n-1}}{64n} < \theta.$$

Thus (6.33) holds with $C_{\mathrm{tr},n} = 1$. At $r = 1$,

$$|1-\chi_\theta(1)| + |\chi_\theta'(1)| \leq \frac{E}{\ell_0} + \frac{E}{T_0\ell_0^2} < \frac{2E}{\ell_0} = \frac{\theta^3}{256n} < \theta,$$

which proves (6.34).

One has $\varphi_\theta(1) = L_I + 1$, $\varphi_\theta(r) \to -\infty$ as $r \downarrow 0$, and $\varphi_\theta'(r) > 0$. Hence there is a unique $r_{-,\theta} \in (0,1)$ satisfying $\varphi_\theta(r_{-,\theta}) = -L_I$. Monotonicity gives $\varphi_\theta(r) \leq -L_I$ for $0 < r \leq r_{-,\theta}$ and $\varphi_\theta(r) > L_I$ near $r = 1$. The defining properties of ρ_I then give the two endpoint identities for $g_{s,\theta}$.

The functions $E, F, \ell_0, T_0, r_0(\theta), c_{\theta,I}, \chi_\theta$, and φ_θ are jointly smooth in their displayed variables for $0 < \theta < \theta_{\mathrm{tr},n}$. Since ρ_I is smooth and chosen before θ, the map $(\theta, r) \mapsto \rho_I(\varphi_\theta(r)) = \psi_\theta(r)$ is smooth on $(0, \theta_{\mathrm{tr},n}) \times (0,1]$. Since $\partial_r\varphi_\theta(r_{-,\theta}) > 0$, the implicit-function theorem gives smooth dependence of $r_{-,\theta}$ on θ. The construction of L_I and ρ_I precedes the choice of θ; both depend only on I and the family $\{g_{s,u} : (s,u) \in I \times \mathbb{R}\}$. Thus the proposition holds with $\theta_{\mathrm{tr},n} = 1/4$ and $C_{\mathrm{tr},n} = 1$. $\square$

Proof of Theorem 6.1. Let $\varepsilon > 0$ satisfy $\varepsilon^4 < \theta_{\mathrm{tr},n}$. The interval I is compact, and the family $(g_{s,u})_{(s,u)\in I\times\mathbb{R}}$ is smooth, equations (6.1) and (6.2) hold, and $g_{s,u} = g_{s,-2}$ for $u \leq -2$ while $g_{s,u} = g_{s,2}$ for $u \geq 2$. Therefore Proposition 6.3 applies with $\theta = \varepsilon^4$. Put

$$\begin{aligned}
\chi &:= \chi_{\varepsilon^4}, & \psi &:= \psi_{\varepsilon^4},\\
r_- &:= r_{-,\varepsilon^4}, & g_s(r) &:= g_{s,\psi(r)},\\
b(r) &:= r\chi(r), & G_{0,s} &:= dr^2 + b(r)^2 g_s(r).
\end{aligned}$$

The endpoint and inner-limit conclusions of that proposition give

$$\psi(r) = -2 \quad (0 < r \le r_-), \qquad \psi(r) = 2 \quad \text{for } r \text{ near } 1, \tag{6.39}$$

$$\lim_{r\downarrow 0} \chi(r) = 1, \qquad \lim_{r\downarrow 0} r\chi'(r) = 0. \tag{6.40}$$

Moreover,

$$\mathrm{Ric}_{G_{0,s}} \ge 0, \qquad r\left|\frac{\chi'}{\chi}\right| + r|g_s'| + r^2|g_s''| \le \varepsilon^4, \tag{6.41}$$

and, for every $g_s(r)$-unit vector e,

$$r^2\left|\mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e)\right| \le C_{\mathrm{tr},n}\varepsilon^4. \tag{6.42}$$

For $0 < \kappa < 1/2$, set

$$s_\kappa(r) := \frac{\sin(\kappa r)}{\kappa}, \qquad a(r) := s_\kappa(r)\chi(r), \qquad G_{\kappa,s} := dr^2 + a(r)^2 g_s(r).$$

Define A_I by

$$A_I := \max\left\{1, \sup_{(s,u)\in I\times\mathbb{R}} \left(\|\partial_u g_{s,u}\|_{g_{s,u}} + \|\partial_u^2 g_{s,u}\|_{g_{s,u}} + \|\nabla^{g_{s,u}}\partial_u g_{s,u}\|_{g_{s,u}}\right)\right\}.$$

This number is finite by (6.2). For fixed $s \in I$, apply the first part of Proposition 6.2 with

$$m = n-1, \qquad A = A_I, \qquad X = S^{n-1},$$
$$J = (0,1], \qquad \ell_t = g_{s,t}, \qquad f = \psi,$$

and successively with $w = a$ and $w = b$. These warping functions are positive because $\chi > 0$, $r > 0$, and $0 < \kappa r < 1/2$. The Ricci and derivative hypotheses are (6.1) and (6.2). The identity $\partial_u dv_{g_{s,u}} = 0$ gives $dv_{g_{s,t}} = dv_{g_{s,-2}}$ for every $t \in \mathbb{R}$. Hence $r \mapsto dv_{g_s(r)}$ is constant and $\mathrm{tr}_{g_s(r)}\, g_s'(r) = 0$. The identities (6.13), (6.14), and (6.15) therefore hold for $G_{\kappa,s}$ and $G_{0,s}$. For either $(w, G_w) = (a, G_{\kappa,s})$ or $(w, G_w) = (b, G_{0,s})$, equation (6.14) gives

$$\mathrm{Ric}_{G_w}(\partial_r, e) = w\,\mathrm{Ric}_{G_w}(\partial_r, w^{-1}e) = \frac{1}{2}(\mathrm{div}_{g_s}\, g_s')(e).$$

In particular, $\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, e) = \mathrm{Ric}_{G_{0,s}}(\partial_r, e)$. Set $\alpha := \chi'/\chi$ and $d_\kappa := \kappa\cot(\kappa r) - r^{-1}$. For $x \in (0, 1/2]$, put

$$F_1(x) := \frac{x\cot x - 1 + x^2/3}{x^4}, \qquad F_2(x) := \frac{(x/\sin x)^2 - 1 - x^2/3}{x^4},$$
$$F_3(x) := \frac{x/\sin x - 1}{x^2}.$$

The expansions

$$x\cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} + O(x^6), \qquad \left(\frac{x}{\sin x}\right)^2 = 1 + \frac{x^2}{3} + \frac{x^4}{15} + O(x^6),$$
$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + O(x^4)$$

give the continuous extensions

$$F_1(0) = -\frac{1}{45}, \qquad F_2(0) = \frac{1}{15}, \qquad F_3(0) = \frac{1}{6}.$$

Let C_{Tay} be the maximum of their absolute values on $[0, 1/2]$. Substitution of $x = \kappa r$ gives

$$d_\kappa = -\frac{\kappa^2 r}{3} + E_1, \qquad |E_1| \le C_{\mathrm{Tay}}\kappa^4 r^3, \tag{6.43}$$

$$s_\kappa(r)^{-2} - r^{-2} = \frac{\kappa^2}{3} + E_2, \qquad |E_2| \le C_{\mathrm{Tay}}\kappa^4 r^2, \tag{6.44}$$

$$\left|\frac{\kappa r}{\sin(\kappa r)} - 1\right| \le C_{\mathrm{Tay}}\kappa^2 r^2. \tag{6.45}$$

The logarithmic derivatives are

$$\frac{a'}{a} = \kappa\cot(\kappa r) + \alpha, \qquad \frac{b'}{b} = \frac{1}{r} + \alpha.$$

Using $w''/w = (w'/w)' + (w'/w)^2$ and $\frac{d}{dr}(\kappa\cot(\kappa r)) = -\kappa^2\csc^2(\kappa r)$ gives $\frac{a''}{a} - \frac{b''}{b} = -\kappa^2 + 2d_\kappa\alpha$. Consequently,

$$\mathrm{Ric}_{G_{\kappa,s}}(\partial_r,\partial_r) - \mathrm{Ric}_{G_{0,s}}(\partial_r,\partial_r) = (n-1)\kappa^2 - 2(n-1)d_\kappa\alpha.$$

Equations (6.41) and (6.43) imply

$$|d_\kappa| \le \left(\frac{1}{3} + C_{\mathrm{Tay}}\right)\kappa^2 r, \qquad |d_\kappa\alpha| \le \left(\frac{1}{3} + C_{\mathrm{Tay}}\right)\varepsilon^4\kappa^2.$$

Hence there is a dimensional constant $C_{n,r} \ge 1$ such that

$$\mathrm{Ric}_{G_{\kappa,s}}(\partial_r,\partial_r) - \mathrm{Ric}_{G_{0,s}}(\partial_r,\partial_r) \ge (n-1-C_{n,r}\varepsilon^4 - C_{n,r}\kappa^2)\kappa^2. \tag{6.46}$$

Put $p_\kappa := a'/a$ and $p_0 := b'/b$. Then

$$p_\kappa - p_0 = d_\kappa, \qquad p_\kappa^2 - p_0^2 = \frac{2d_\kappa}{r} + 2d_\kappa\alpha + d_\kappa^2.$$

Subtracting (6.15) for $w = a$ and $w = b$, the common terms $-\frac{1}{2}g_s''(e,e)$ and $\frac{1}{2}|g_s'(e,\cdot)|^2_{g_s}$ cancel, and one obtains

$$\begin{aligned} D_T :=\ &\mathrm{Ric}_{G_{\kappa,s}}(a^{-1}e, a^{-1}e) - \mathrm{Ric}_{G_{0,s}}(b^{-1}e, b^{-1}e) \\ =\ &(a^{-2} - b^{-2})\,\mathrm{Ric}_{g_s(r)}(e,e) + \kappa^2 - \frac{2(n-2)d_\kappa}{r} \\ &- 2(n-1)d_\kappa\alpha - (n-2)d_\kappa^2 - \frac{n-1}{2}d_\kappa g_s'(e,e). \end{aligned}$$

Since $s_\kappa(r) \le r$, one has $a^{-2} - b^{-2} = \chi^{-2}(s_\kappa^{-2} - r^{-2}) \ge 0$. The link Ricci bound therefore gives

$$(a^{-2} - b^{-2})\,\mathrm{Ric}_{g_s(r)}(e,e) \ge (n-2)\chi^{-2}\big(s_\kappa^{-2} - r^{-2}\big).$$

Equations (6.41), (6.43), and (6.44) imply, for a dimensional constant $C_{n,t} \ge 1$,

$$\begin{aligned} (n-2)\chi^{-2}\big(s_\kappa^{-2} - r^{-2}\big) &\ge \frac{n-2}{3}\kappa^2 - C_{n,t}\kappa^4, \\ -\frac{2(n-2)d_\kappa}{r} &\ge \frac{2(n-2)}{3}\kappa^2 - C_{n,t}\kappa^4, \\ |d_\kappa\alpha| + |d_\kappa g_s'(e,e)| &\le C_{n,t}\varepsilon^4\kappa^2, \\ d_\kappa^2 &\le C_{n,t}\kappa^4. \end{aligned}$$

Here the first estimate uses $s_\kappa(r) \le r$, so that $s_\kappa^{-2} - r^{-2} \ge 0$ and multiplication by $\chi^{-2} \ge 1$ preserves the lower bound. The coefficient of the three leading terms is $(n-2)/3 + 1 + 2(n-2)/3 = n-1$. After enlarging $C_{n,t}$ if necessary,

$$D_T \ge (n-1-C_{n,t}\varepsilon^4 - C_{n,t}\kappa^2)\kappa^2. \tag{6.47}$$

Let e be $g_s(r)$-unit. The coordinate equality above and bilinearity give

$$\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, a^{-1}e) = \frac{1}{a}\,\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, e) = \frac{b}{a}\,\mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e).$$

Consequently,

$$\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, a^{-1}e) - \mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e) = \left(\frac{b}{a} - 1\right)\mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e).$$

Since $b/a = \kappa r/\sin(\kappa r)$, (6.42) and (6.45) give

$$\left|\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, a^{-1}e) - \mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e)\right| \le C_{n,m}\varepsilon^4\kappa^2, \tag{6.48}$$

where $C_{n,m} := C_{\mathrm{tr},n}C_{\mathrm{Tay}}$ is dimensional.

Let

$$V = x\partial_r + ya^{-1}e, \qquad V_0 = x\partial_r + yb^{-1}e, \qquad x^2 + y^2 = 1.$$

Every $G_{\kappa,s}$-unit vector is of this form. The vector V_0 is $G_{0,s}$-unit, and bilinearity gives

$$\begin{aligned}\mathrm{Ric}_{G_{\kappa,s}}(V,V) = {} & \mathrm{Ric}_{G_{0,s}}(V_0,V_0)\\ & + x^2\left[\mathrm{Ric}_{G_{\kappa,s}}(\partial_r,\partial_r) - \mathrm{Ric}_{G_{0,s}}(\partial_r,\partial_r)\right]\\ & + y^2 D_T\\ & + 2xy\left[\mathrm{Ric}_{G_{\kappa,s}}(\partial_r, a^{-1}e) - \mathrm{Ric}_{G_{0,s}}(\partial_r, b^{-1}e)\right].\end{aligned}$$

By (6.41), (6.46), (6.47), and (6.48),

$$\mathrm{Ric}_{G_{\kappa,s}}(V,V) \ge \left(n-1-C_{n,d}\varepsilon^4 - C_{n,d}\kappa^2\right)\kappa^2(x^2+y^2) - 2C_{n,m}\varepsilon^4\kappa^2|xy|,$$

where $C_{n,d} := \max\{C_{n,r}, C_{n,t}\}$. Since $x^2+y^2=1$ and $2|xy| \le x^2+y^2$, this is at least

$$\left(n-1-C_{n,0}\varepsilon^4 - C_{n,0}\kappa^2\right)\kappa^2, \qquad C_{n,0} := C_{n,d} + C_{n,m}.$$

Thus the estimate in (6.7), with $(G_{s,\varepsilon}, \kappa_\varepsilon)$ replaced by $(G_{\kappa,s}, \kappa)$, holds with any final constant $C_n \ge C_{n,0}$ and every $0 < \kappa < 1/2$.

The function $\varepsilon \mapsto (\sin^2\varepsilon - 8\varepsilon^4)/\varepsilon^2$ extends continuously to $\varepsilon = 0$ with value 1. Hence it is positive on some dimensional interval $(0, \varepsilon_{n,\lambda})$. For $0 < \varepsilon < \varepsilon_{n,\lambda}$, put $\lambda_\varepsilon := \sqrt{\sin^2\varepsilon - 8\varepsilon^4}$. The function $\varepsilon \mapsto \lambda_\varepsilon/\varepsilon$ extends continuously to $\varepsilon = 0$ with value 1. Choose a dimensional number $\varepsilon_{n,1} \in (0, \varepsilon_{n,\lambda})$ such that, for $0 < \varepsilon < \varepsilon_{n,1}$,

$$0 < \varepsilon < \min\left\{\frac{1}{8}, \frac{\pi}{4}\right\}, \qquad \frac{1}{2}\varepsilon \le \lambda_\varepsilon \le \varepsilon, \qquad \frac{\varepsilon}{1-\varepsilon^4} < \sin\frac{1}{2}.$$

Equation (6.34) gives $1 - \varepsilon^4 \le \chi(1) \le 1$, and therefore

$$0 < \frac{\lambda_\varepsilon}{\chi(1)} \le \frac{\varepsilon}{1-\varepsilon^4} < \sin\frac{1}{2}.$$

We henceforth assume $0 < \varepsilon < \varepsilon_{n,1}$. The equation

$$\chi(1)\sin\kappa = \lambda_\varepsilon \tag{6.49}$$

has a unique solution $\kappa = \kappa_\varepsilon \in (0, 1/2)$. The tuning equation and the preceding bounds give

$$\frac{\varepsilon}{2} \le \lambda_\varepsilon \le \sin\kappa_\varepsilon = \frac{\lambda_\varepsilon}{\chi(1)} \le \frac{\varepsilon}{1-\varepsilon^4} \le 2\varepsilon.$$

Since $\sin t \le t$ and $\sin t \ge t/2$ on $(0, 1/2)$,

$$\frac{1}{2}\varepsilon \le \kappa_\varepsilon \le 4\varepsilon. \tag{6.50}$$

We shall take $c_n := 1/2$ and choose the final $C_n \ge 4$ below.

Equation (6.49) gives

$$A_\varepsilon := a(1) = \frac{\chi(1)\sin\kappa_\varepsilon}{\kappa_\varepsilon} = \frac{\lambda_\varepsilon}{\kappa_\varepsilon}, \qquad c_\varepsilon := \frac{\sin\varepsilon}{A_\varepsilon}.$$

Moreover,

$$a'(1) = \chi(1)\cos\kappa_\varepsilon + \frac{\sin\kappa_\varepsilon}{\kappa_\varepsilon}\chi'(1), \qquad \chi(1)\cos\kappa_\varepsilon = \sqrt{\chi(1)^2 - \lambda_\varepsilon^2}.$$

Since $1-\varepsilon^4 \le \chi(1) \le 1$, one has $\chi(1)^2-\lambda_\varepsilon^2-\cos^2\varepsilon = \chi(1)^2-1+8\varepsilon^4 \ge 6\varepsilon^4$. Since $0<\varepsilon<\pi/4$, both terms in the denominator below are positive and at most 1. Hence

$$\sqrt{\chi(1)^2-\lambda_\varepsilon^2}-\cos\varepsilon = \frac{\chi(1)^2-\lambda_\varepsilon^2-\cos^2\varepsilon}{\sqrt{\chi(1)^2-\lambda_\varepsilon^2}+\cos\varepsilon} \ge 3\varepsilon^4.$$

Equation (6.34) and $\sin\kappa_\varepsilon/\kappa_\varepsilon \le 1$ give $|(\sin\kappa_\varepsilon/\kappa_\varepsilon)\chi'(1)| \le \varepsilon^4$. Therefore

$$a'(1) > \cos\varepsilon. \tag{6.51}$$

The unscaled boundary metric is $A_\varepsilon^2 g_{s,2}$, and (6.8) follows from $c_\varepsilon^2 A_\varepsilon^2 = \sin^2\varepsilon$. By (6.4), $g_s(r) = g_{s,2}$ on a neighbourhood of $r=1$; hence $g_s'(1)=0$. The inward unit normal at $r=1$ is $-\partial_r$. With $H(r) := a(r)^2 g_s(r)$ and the convention $\mathrm{II}(v,w) = -G(\nabla_v \nu, w)$,

$$\mathrm{II}(v,w) = G(\nabla_v\partial_r, w) = \frac{1}{2}H'(1)(v,w).$$

Since $g_s'(1)=0$,

$$\mathrm{II}_{\{1\}\times S^{n-1},G_{\kappa_\varepsilon,s}} = a(1)a'(1)g_{s,2} = \frac{a'(1)}{A_\varepsilon}G_{\kappa_\varepsilon,s}|_{\{1\}\times S^{n-1}}.$$

For $\bar G = c_\varepsilon^2 G$, one has $\nabla^{\bar G} = \nabla^G$ and $\nu_{\bar G} = c_\varepsilon^{-1}\nu_G$; hence $\mathrm{II}_{\bar G} = c_\varepsilon \mathrm{II}_G$. Thus

$$\mathrm{II}_{\{1\}\times S^{n-1},c_\varepsilon^2 G_{\kappa_\varepsilon,s}} = \frac{a'(1)}{c_\varepsilon A_\varepsilon}(c_\varepsilon^2 G_{\kappa_\varepsilon,s})|_{\{1\}\times S^{n-1}} = \frac{a'(1)}{\sin\varepsilon}(c_\varepsilon^2 G_{\kappa_\varepsilon,s})|_{\{1\}\times S^{n-1}}.$$

Equation (6.51) proves (6.9).

Since the Ricci tensor is unchanged as a $(0,2)$-tensor under constant scaling,

$$\mathrm{Ric}_{c_\varepsilon^2 G_{\kappa_\varepsilon,s}} \ge \left(n-1-C_{n,0}\varepsilon^4 - C_{n,0}\kappa_\varepsilon^2\right)\frac{\kappa_\varepsilon^2}{c_\varepsilon^2}(c_\varepsilon^2 G_{\kappa_\varepsilon,s}).$$

Now

$$\frac{\kappa_\varepsilon^2}{c_\varepsilon^2} = \frac{\lambda_\varepsilon^2}{\sin^2\varepsilon} = 1-\frac{8\varepsilon^4}{\sin^2\varepsilon}.$$

Since $\sin\varepsilon \ge \varepsilon/2$ and (6.50) holds,

$$1-\frac{8\varepsilon^4}{\sin^2\varepsilon} \ge 1-32\varepsilon^2, \qquad n-1-C_{n,0}\varepsilon^4 - C_{n,0}\kappa_\varepsilon^2 \ge n-1-17C_{n,0}\varepsilon^2.$$

Set $C_{n,1} := 17C_{n,0}+32(n-1)$ and $C_n := \max\{4, C_{n,0}, C_{n,1}\}$, and choose $\varepsilon_0>0$ so that

$$\varepsilon_0 < \min\left\{\theta_{\mathrm{tr},n}^{1/4},\ \varepsilon_{n,1},\ \sqrt{\frac{n-1}{34C_{n,0}}}\right\}.$$

For $0<\varepsilon<\varepsilon_0$, the two lower factors $n-1-17C_{n,0}\varepsilon^2$ and $1-32\varepsilon^2$ are positive. Therefore

$$\begin{aligned}
&\left(n-1-C_{n,0}\varepsilon^4 - C_{n,0}\kappa_\varepsilon^2\right)\frac{\lambda_\varepsilon^2}{\sin^2\varepsilon} \\
&\quad\ge (n-1-17C_{n,0}\varepsilon^2)(1-32\varepsilon^2) \\
&\quad= n-1-C_{n,1}\varepsilon^2 + 544C_{n,0}\varepsilon^4 \ge n-1-C_{n,1}\varepsilon^2.
\end{aligned}$$

For every $0<\varepsilon<\varepsilon_0$, equations (6.46)–(6.48) give (6.7) with the final C_n, (6.50) gives (6.6) with $c_n = 1/2$, and the last display gives (6.10).

The functions χ_θ, ψ_θ and the radius $r_{-,\theta}$ in Proposition 6.3 depend smoothly on θ. Define $\mathcal{F}(\varepsilon,\kappa) := \chi_{\varepsilon^4}(1)\sin\kappa - \lambda_\varepsilon$. On $(0,\varepsilon_0)\times(0,1/2)$ this function is smooth, and at every solution of (6.49),

$$\partial_\kappa \mathcal{F}(\varepsilon,\kappa) = \chi_{\varepsilon^4}(1)\cos\kappa > 0.$$

The implicit-function theorem gives smooth dependence of κ_ε on ε. Equations (6.39) and (6.40) prove (6.4) and (6.5). The identities

$$\chi_\varepsilon = \chi_{\varepsilon^4}, \qquad \psi_\varepsilon = \psi_{\varepsilon^4}, \qquad r_{-,\varepsilon} = r_{-,\varepsilon^4}, \qquad c_\varepsilon = \frac{\sin \varepsilon\, \kappa_\varepsilon}{\lambda_\varepsilon} \tag{6.52}$$

give the stated smoothness and parameter dependence. Substitution in (6.3) proves smoothness of $(s, \varepsilon, r) \mapsto G_{s,\varepsilon}(r)$. $\square$

7. Caps with prescribed boundary metrics

Fix a core-admissible Q^n and choose $D_Q, P_Q, \beta_Q, k_Q, \kappa_Q, \mu_Q$ as in Definition 3.1, so that (3.1) holds. Put $\bar\kappa_Q := \min\{1, \kappa_Q\}$.

7.1. **Parametric Ricci gluing.** Throughout this subsection, B is a smooth paracompact manifold, possibly with boundary. For $i = 1, 2$, let M_i be a compact smooth n-manifold, not necessarily connected, with boundary, let $\Sigma_i \subset \partial M_i$ be a compact union of boundary components, and fix diffeomorphisms $\beta_i : \Sigma \to \Sigma_i$ from one compact, possibly disconnected, manifold Σ. Let $b \mapsto h_i(b)$ be smooth families of Riemannian metrics on M_i. Assume

$$\beta_1^*\big(h_1(b)|_{\Sigma_1}\big) = \beta_2^*\big(h_2(b)|_{\Sigma_2}\big) =: h_\Sigma(b). \tag{7.1}$$

Let $\mathrm{II}_{i,b}$ denote the pullback by β_i of the second fundamental form of $\Sigma_i \subset (M_i, h_i(b))$, computed with the inward unit normal. Let $\lambda, \eta, w : B \to \mathbb{R}$ be continuous, with $\eta, w > 0$, and fix $L > 1$. Assume, pointwise on M_i,

$$\mathrm{Ric}_{h_i(b)} > (\lambda(b) + 2\eta(b))h_i(b), \qquad i = 1, 2. \tag{7.2}$$

Assume also, pointwise on Σ,

$$\mathrm{II}_{1,b} + \mathrm{II}_{2,b} > 0. \tag{7.3}$$

Glue Σ_1 to Σ_2 by $\beta_2 \circ \beta_1^{-1}$ and let $M := M_1 \cup_{\beta_2\circ\beta_1^{-1}} M_2$ denote the quotient. Let

$$\mathcal{M}_{\mathrm{top}} := (B \times M_1) \cup_{(b,\beta_1(x))\sim(b,\beta_2(x))} (B \times M_2), \qquad \pi : \mathcal{M}_{\mathrm{top}} \longrightarrow B$$

be the resulting quotient. Let $q : M_1 \sqcup M_2 \to M$ be the quotient map. Since B is locally compact and Hausdorff, $\mathrm{id}_B \times q$ is a quotient map. The quotient universal property then gives mutually inverse continuous maps $[b, x] \mapsto (b, [x])$ and $(b, [x]) \mapsto [b, x]$. Thus $\mathcal{M}_{\mathrm{top}} \cong B \times M$, and π is the product topological bundle with fibre M. Boundary components not contained in Σ_i remain boundary components of M. Since Σ_i is a compact union of boundary components, sufficiently small normal collars of Σ_i are disjoint from $\partial M_i \setminus \Sigma_i$. The interpolation and mollification defined below are supported in these collars and therefore leave $h_i(b)$ unchanged on an open neighbourhood of $\partial M_i \setminus \Sigma_i$.

For $i = 1, 2$, let $\nu_{i,b}$ be the inward unit normal along Σ_i and set

$$\widehat{E}_i(b, r, x) := \exp^{h_i(b)}_{\beta_i(x)}\big(r\nu_{i,b}(\beta_i(x))\big), \qquad x \in \Sigma,$$

for every $r \geq 0$ for which the inward normal geodesic is defined in M_i. Define the simultaneous collar radius by

$$r_{\mathrm{col}}(b) := \sup\Big(\{r > 0 : \widehat{E}_i(b, \cdot, \cdot) \text{ is an embedding on } [0, r] \times \Sigma \text{ for } i = 1, 2\} \cup \{0\}\Big).$$

The tubular-neighbourhood theorem and compactness of Σ_i give $r_{\mathrm{col}}(b) > 0$ for every $b \in B$. For $-r_{\mathrm{col}}(b) < t \leq 0$ and $0 \leq t < r_{\mathrm{col}}(b)$ define

$$E_1(b, t, x) := \widehat{E}_1(b, -t, x), \qquad E_2(b, t, x) := \widehat{E}_2(b, t, x).$$

For $b_0 \in B$ and $0 < c < r_{\mathrm{col}}(b_0)$, smooth dependence of the normal exponential maps and C^1-openness of embeddings on the compact set $[0, c] \times \Sigma$ give a neighbourhood U of b_0 on which both collars are defined and embedded up to c. For such U and c, define

$$\mathcal{E}_{U,c}(b, t, x) := \begin{cases} [b, E_1(b, t, x)], & -c < t \leq 0, \\ [b, E_2(b, t, x)], & 0 < t < c. \end{cases}$$

The preimage of $\mathcal{E}_{U,c}(U \times (-c, c) \times \Sigma)$ in $(B \times M_1) \sqcup (B \times M_2)$ is the saturated union of the two open half-collar images. The quotient map is open on saturated open sets, and the collar inverses agree under the boundary identification. Hence $\mathcal{E}_{U,c}$ is a homeomorphism onto an open subset of $\mathcal{M}_{\mathrm{top}}$. Two signed-collar charts have the identity transition in the coordinates (b, t, x); their transitions to the product charts away from the seam are restrictions of E_i or E_i^{-1}. Together with the product charts away from the seam, these charts endow $\mathcal{M}_{\mathrm{top}}$ with the structure of a smooth manifold with corners, denoted by $\mathcal{M}$. In every such chart, π is the projection to B; hence π is smooth and $d\pi$ is surjective.

For $(b, x) \in U \times \Sigma$, the curves $r \mapsto \widehat{E}_i(b, r, x)$ are unit-speed geodesics orthogonal to Σ_i at $r = 0$. The Gauss lemma and smooth dependence of the exponential map on (b, r, x) give uniquely determined smooth families of metrics $H_{i,b}(t)$ on Σ characterized by

$$
\begin{aligned}
E_1(b)^* h_1(b) &= dt^2 + H_{1,b}(t) \quad (-r_{\mathrm{col}}(b) < t \le 0), \\
E_2(b)^* h_2(b) &= dt^2 + H_{2,b}(t) \quad (0 \le t < r_{\mathrm{col}}(b)).
\end{aligned}
$$

Define $h^{\mathrm{gl}}(b)$ to equal $h_i(b)$ on the two side charts. In a signed-collar chart its pullback is $dt^2 + H_b(t)$, where

$$
H_b(t) = \begin{cases} H_{1,b}(t), & t \le 0, \\ H_{2,b}(t), & t \ge 0. \end{cases}
$$

The collar identities above give agreement on the side-chart overlaps, and (7.1) gives $H_{1,b}(0) = H_{2,b}(0) = h_\Sigma(b)$. Hence h^{gl} is a continuous positive-definite vertical two-tensor on $\mathcal{M}$. Extend tangent vectors along the normal geodesics with vanishing bracket with ∂_r. Since $t = -r$ on the M_1 side and $t = r$ on the M_2 side, metric compatibility, torsion-freeness, and symmetry of the shape operator give $H'_{1,b}(0) = 2\mathrm{II}_{1,b}$ and $H'_{2,b}(0) = -2\mathrm{II}_{2,b}$. Consequently,

$$
H_{1,b}(0) = H_{2,b}(0) = h_\Sigma(b), \qquad H'_{1,b}(0) - H'_{2,b}(0) = 2(\mathrm{II}_{1,b} + \mathrm{II}_{2,b}) > 0. \tag{7.4}
$$

For $b \in B$ and $\delta > 0$ with $2\delta < r_{\mathrm{col}}(b)$, set $H_- := H_b(-\delta)$, $H_+ := H_b(\delta)$, $K_- := \partial_t H_b(-\delta)$, and $K_+ := \partial_t H_b(\delta)$. For $|t| \le \delta$, define

$$
\begin{aligned}
\mathscr{H}_{b,\delta}(t) := {} & \frac{\delta - t}{2\delta} H_- + \frac{\delta + t}{2\delta} H_+ \\
& + \frac{(t-\delta)^2 (t+\delta)}{4\delta^2} \left(K_- - \frac{H_+ - H_-}{2\delta} \right) \\
& + \frac{(t+\delta)^2 (t-\delta)}{4\delta^2} \left(K_+ - \frac{H_+ - H_-}{2\delta} \right).
\end{aligned} \tag{7.5}
$$

Differentiating (7.5) at the two endpoints yields

$$
\begin{aligned}
\mathscr{H}_{b,\delta}(-\delta) &= H_-, & \mathscr{H}_{b,\delta}(\delta) &= H_+, \\
\partial_t \mathscr{H}_{b,\delta}(-\delta) &= K_-, & \partial_t \mathscr{H}_{b,\delta}(\delta) &= K_+.
\end{aligned} \tag{7.6}
$$

Thus (7.5) is the tensor-valued Hermite cubic in [16, equation (1)]. Define $\mathcal{I}_{b,\delta}$ to equal $dt^2 + \mathscr{H}_{b,\delta}(t)$ for $|t| \le \delta$ and $h^{\mathrm{gl}}(b)$ elsewhere on M. Equation (7.6) shows that $\mathcal{I}_{b,\delta}$ is a symmetric C^1 two-tensor, smooth away from $t = \pm\delta$.

Choose even functions $\varrho, \chi \in C_c^\infty((-1, 1))$ with $\varrho \ge 0$, $0 \le \chi \le 1$, $\int_{\mathbb{R}} \varrho = 1$, and $\chi = 1$ on $[-1/2, 1/2]$. For $\epsilon, \nu > 0$, put $\varrho_\epsilon(s) := \epsilon^{-1} \varrho(s/\epsilon)$ and $\epsilon_\nu := \nu^2/(1+\nu)$. Let $J \subset \mathbb{R}$ be open, let $\tau \in J$, and let $A : J \to \Gamma^\infty(\mathrm{Sym}^2 T^*\Sigma)$ be C^1. Suppose that $[\tau - \nu - \epsilon_\nu, \tau + \nu + \epsilon_\nu] \subset J$ and that A is smooth on a neighbourhood of this interval minus $\{\tau\}$. Define

$$
\mathfrak{M}_{\tau,\nu} A(t) := \begin{cases} A(t) + \chi\Big(\dfrac{t-\tau}{\nu}\Big) \left(\displaystyle\int_{\mathbb{R}} \varrho_{\epsilon_\nu}(s) A(t-s)\, ds - A(t) \right), & |t - \tau| < \nu, \\ A(t), & |t - \tau| \ge \nu. \end{cases} \tag{7.7}
$$

For $|t-\tau|<\nu$, every $t-s$ occurring in the integral belongs to J. The integral is computed pointwise in the finite-dimensional fibres of $\mathrm{Sym}^2 T^*\Sigma$. After the change of variables $u=t-s$, each t-derivative falls on the smooth compactly supported kernel $\varrho_{\epsilon_\nu}(t-u)$, while every x-derivative falls on the smooth section $A(u)$. Differentiation under the integral therefore shows that the convolution is smooth in (t,x). Because χ has compact support in $(-1,1)$, the two branches in (7.7) have equal derivatives of every order at $|t-\tau|=\nu$ whenever A is smooth there.

Write

$$\mathcal{I}_{b,\delta}=dt^2+A_{b,\delta}(t)$$

on the signed collar. Assume $0<8\nu<\delta$. If $\tau\in\{-\delta,\delta\}$, $|t-\tau|<\nu$, and $|s|<\epsilon_\nu$, then

$$|t-s|<\delta+2\nu<\frac{5}{4}\delta<2\delta<r_{\mathrm{col}}(b).$$

Thus (7.7) is defined for $A_{b,\delta}$ at both joins. The supports of the two modifications are disjoint. If $A_{b,\delta}(u)$ is positive definite for every $u\in[t-\epsilon_\nu,t+\epsilon_\nu]$, then the identity

$$\mathfrak{M}_{\tau,\nu}A_{b,\delta}(t)=(1-\chi((t-\tau)/\nu))A_{b,\delta}(t)+\chi((t-\tau)/\nu)\int_{\mathbb{R}}\varrho_{\epsilon_\nu}(s)A_{b,\delta}(t-s)\,ds$$

expresses the localized coefficient as a convex combination of positive-definite tensors; it is therefore positive definite. Define $\mathcal{S}_{b,\delta,\nu}$ by replacing $A_{b,\delta}$ with $\mathfrak{M}_{\delta,\nu}(\mathfrak{M}_{-\delta,\nu}A_{b,\delta})$ and leaving dt^2 unchanged. If $|t-\delta|<\nu$ and $|s|<\epsilon_\nu$, then

$$|t-s+\delta|\geq 2\delta-|t-\delta|-|s|>2\delta-2\nu>\nu.$$

Thus the modification centred at $-\delta$ leaves every value sampled by the modification centred at δ unchanged. The same argument with the two centres interchanged proves that the operators commute. On $|t\mp\delta|\leq\nu/2$ the tangential coefficient is a convolution; on the remaining part of either smoothing band the coefficient $A_{b,\delta}$ is smooth. Hence $\mathcal{S}_{b,\delta,\nu}$ is smooth and equals $\mathcal{I}_{b,\delta}$ outside $\{|t+\delta|<\nu\}\cup\{|t-\delta|<\nu\}$. Put $P_b:=dt^2+h_\Sigma(b)$ on the signed collar. Fix $0<2\delta<r_{\mathrm{col}}(b)$. A positive-definite C^1 tensor g, smooth away from finitely many joins, satisfies the *gluing bounds at* (b,δ) if

$$\begin{aligned}\mathrm{Ric}_g&>(\lambda(b)+\eta(b))g &&\text{on every smooth piece,}\\ L^{-2}P_b&<g<L^2P_b &&\text{for } |t|<2\delta.\end{aligned}$$

The Ricci inequality is required to extend strictly to every one-sided closure; the remaining inequalities are pointwise inequalities of quadratic forms. No comparison with P_b is imposed outside the collar $|t|<2\delta$. On the complement of $|t|<2\delta$, both $\mathcal{I}_{b,\delta}$ and $\mathcal{S}_{b,\delta,\nu}$ equal $h^{\mathrm{gl}}(b)$, hence equal $h_1(b)$ on the negative side and $h_2(b)$ on the positive side.

Let $B_1\subset B$ be open. A smooth pair $(\delta_{\mathrm{pr}},\nu_{\mathrm{pr}}):B_1\to(0,\infty)^2$ is *downward admissible* if

$$4\delta_{\mathrm{pr}}<w,\qquad 8\nu_{\mathrm{pr}}<\delta_{\mathrm{pr}},\tag{7.8}$$

and, for every $b\in B_1$,

(a) every $0<\widetilde{\delta}\leq\delta_{\mathrm{pr}}(b)$ satisfies $2\widetilde{\delta}<r_{\mathrm{col}}(b)$, and $\mathcal{I}_{b,\widetilde{\delta}}$ satisfies the gluing bounds at $(b,\widetilde{\delta})$;
(b) every $0<\widetilde{\nu}\leq\nu_{\mathrm{pr}}(b)$ makes $\mathcal{S}_{b,\delta_{\mathrm{pr}}(b),\widetilde{\nu}}$ a smooth metric satisfying the gluing bounds at $(b,\delta_{\mathrm{pr}}(b))$.

Let a finite group G act smoothly on B, let $\sigma:G\to\mathfrak{S}_2$ be a homomorphism, and write $\sigma_\gamma:=\sigma(\gamma)$. Suppose there are diffeomorphisms $\gamma_{i,b}:M_i\to M_{\sigma_\gamma(i)}$, for $\gamma\in G$, $b\in B$, and $i=1,2$, such that each map $(b,x)\mapsto(\gamma b,\gamma_{i,b}(x))$ is smooth and the maps on $(B\times M_1)\sqcup(B\times M_2)$ satisfy

$$(\gamma\tau)_{i,b}=\gamma_{\sigma_\tau(i),\tau b}\circ\tau_{i,b},\qquad e_{i,b}=\mathrm{id}_{M_i}.\tag{7.9}$$

For every $\gamma\in G$ and $b\in B$, suppose that one diffeomorphism $\bar{\gamma}_b:\Sigma\to\Sigma$ satisfies, for both $i=1,2$,

$$\gamma_{i,b}\circ\beta_i=\beta_{\sigma_\gamma(i)}\circ\bar{\gamma}_b.\tag{7.10}$$

The boundary map is therefore $\bar{\gamma}_b = \beta^{-1}_{\sigma_\gamma(i)} \circ \gamma_{i,b} \circ \beta_i$ for either i. Equation (7.9) and the homomorphism identity $\sigma_{\gamma\tau} = \sigma_\gamma \sigma_\tau$ give

$$\overline{\gamma\tau}_b = \bar{\gamma}_{\tau b} \circ \bar{\tau}_b, \qquad \bar{e}_b = \mathrm{id}_\Sigma . \tag{7.11}$$

Require also

$$\gamma^*_{i,b} h_{\sigma_\gamma(i)}(\gamma b) = h_i(b), \qquad (\lambda, \eta, w)(\gamma b) = (\lambda, \eta, w)(b). \tag{7.12}$$

Equation (7.10) preserves the quotient relation defining M. The fibre maps therefore descend to homeomorphisms $\gamma_b : M \to M$ from the fibre over b to the fibre over γb. Equation (7.9) descends to $(\gamma\tau)_b = \gamma_{\tau b} \circ \tau_b$ and $e_b = \mathrm{id}_M$. A diffeomorphism of manifolds with boundary maps the inward half-space to the inward half-space. Hence the isometry $\gamma_{i,b}$ maps the inward unit normal to the inward unit normal. After applying $\gamma_{i,b}$, both geodesics begin at $\beta_{\sigma_\gamma(i)}(\bar{\gamma}_b x)$ with velocity $\nu_{\sigma_\gamma(i),\gamma b}(\beta_{\sigma_\gamma(i)}(\bar{\gamma}_b x))$; uniqueness for the geodesic equation yields

$$\gamma_{i,b}\big(\widehat{E}_i(b, r, x)\big) = \widehat{E}_{\sigma_\gamma(i)}(\gamma b, r, \bar{\gamma}_b x). \tag{7.13}$$

Let $\varepsilon_\gamma = 1$ when σ_γ is the identity and $\varepsilon_\gamma = -1$ when it is the transposition. In signed collar coordinates the descended map is

$$(t, x) \longmapsto (\varepsilon_\gamma t, \bar{\gamma}_b x). \tag{7.14}$$

This formula and its inverse are smooth across the seam; together with the piecewise smoothness away from the seam, it proves that each γ_b is a diffeomorphism. Equations (7.11) and $\varepsilon_{\gamma\tau} = \varepsilon_\gamma \varepsilon_\tau$ show directly that these signed-collar maps satisfy the group law.

Lemma 7.1. *The function $r_{\mathrm{col}} : B \to (0, \infty]$ is lower semicontinuous and admits a smooth positive minorant. Fix a smooth function $\rho : B \to (0, \infty)$ satisfying $0 < \rho < r_{\mathrm{col}}$. For $b \in B$, let $\mathscr{D}_b$ be the set of $d > 0$ such that every $0 < \delta < d$ satisfies $2\delta < \rho(b)$ and makes $\mathcal{I}_{b,\delta}$ satisfy the gluing bounds at (b, δ), and set $d_*(b) := \sup(\mathscr{D}_b \cup \{0\})$. Then $d_* : B \to (0, \infty)$ is lower semicontinuous.*

Let $\delta : B \to (0, \infty)$ be smooth, suppose $2\delta(b) < r_{\mathrm{col}}(b)$, and suppose that $\mathcal{I}_{b,\delta(b)}$ satisfies the gluing bounds at $(b, \delta(b))$ for every b. Let $\mathscr{E}_b$ be the set of $e \in (0, \delta(b)/8)$ such that every $0 < \nu < e$ makes $\mathcal{S}_{b,\delta(b),\nu}$ a smooth metric satisfying the gluing bounds at $(b, \delta(b))$, and set $e_*(b) := \sup(\mathscr{E}_b \cup \{0\})$. Then $e_* : B \to (0, \infty)$ is lower semicontinuous.

Define

$$\begin{aligned} \mathscr{P}_1 &:= \{(b, \delta) : 0 < 2\delta < r_{\mathrm{col}}(b)\}, \\ \mathscr{P}_2 &:= \{(b, \delta, \nu) : 0 < 2\delta < r_{\mathrm{col}}(b),\ 0 < 8\nu < \delta\}. \end{aligned}$$

On $\mathscr{P}_1$, the substitution $t = \delta s$ identifies the three smooth interpolation pieces, with the two one-sided copies of each join retained, with fixed intervals; their tangential coefficients are smooth functions of (b, δ, s, x). On $\mathscr{P}_2$, the substitutions $t = \pm\delta + \nu s$ identify the two smoothing bands with $[-1, 1] \times \Sigma$; the coefficients of $\mathcal{S}_{b,\delta,\nu}$ are smooth functions of (b, δ, ν, s, x).

Proof. If $\Sigma = \emptyset$, then $r_{\mathrm{col}} \equiv \infty$ and $\mathcal{I}_{b,\delta} = \mathcal{S}_{b,\delta,\nu} = h^{\mathrm{gl}}(b)$. Hence

$$\mathscr{D}_b = (0, \rho(b)/2], \qquad d_*(b) = \rho(b)/2,$$

while, for every admissible δ,

$$\mathscr{E}_b = (0, \delta(b)/8), \qquad e_*(b) = \delta(b)/8.$$

The two threshold functions are smooth, and the parameter-dependence assertions follow because both tensors are independent of the width parameters. Assume henceforth that $\Sigma \neq \emptyset$.

Fix $b_0 \in B$. The tubular-neighbourhood theorem gives $r > 0$ for which $\widehat{E}_i(b_0, \cdot, \cdot)$ is an embedding on $[0, r] \times \Sigma$ for $i = 1, 2$. Hence $r_{\mathrm{col}}(b_0) > 0$.

Let $0 < c < r_{\mathrm{col}}(b_0)$ and choose $c < c_1 < r_1 < r_{\mathrm{col}}(b_0)$ such that both normal exponential maps are embeddings on $[0, r_1] \times \Sigma$. Smooth dependence of the inward normals and of the geodesic equation gives

$$\widehat{E}_i(b, \cdot, \cdot) \longrightarrow \widehat{E}_i(b_0, \cdot, \cdot) \quad \text{in } C^1([0, c_1] \times \Sigma, M_i)$$

as $b \to b_0$, after restricting the parameter to a neighbourhood on which the maps are defined up to c_1.

Put $D = [0, c_1] \times \Sigma$ and fix auxiliary distances d_D on D and d_i on M_i. For each i, cover D by finitely many relatively compact coordinate sets $U_{i\ell}$ whose closures lie in larger coordinate sets $\widetilde{U}_{i\ell}$ with convex coordinate images. Choose target charts so that the coordinate representative of $\widehat{E}_i(b_0)|_{\widetilde{U}_{i\ell}}$ is the inclusion. A Lebesgue number for $(U_{i\ell})_\ell$ gives $\varepsilon > 0$ such that any two points of D at d_D-distance less than 2ε belong to one $U_{i\ell}$. For b sufficiently close to b_0, the corresponding coordinate representative $F_{i\ell,b}$ is defined on $\widetilde{U}_{i\ell}$ and satisfies $\|dF_{i\ell,b} - I\| < \frac{1}{2}$. The coordinate segment joining two points of $U_{i\ell}$ lies in $\widetilde{U}_{i\ell}$; integration along that segment gives $|F_{i\ell,b}(z) - F_{i\ell,b}(z')| \geq \frac{1}{2}|z - z'|$. Hence $\widehat{E}_i(b)$ is injective on every $U_{i\ell}$.

The compact set $\{(z, z') \in D^2 : d_D(z, z') \geq \varepsilon\}$ and the injectivity of $\widehat{E}_i(b_0)$ give

$$\mu_i := \min_{d_D(z,z') \geq \varepsilon} d_i\big(\widehat{E}_i(b_0, z), \widehat{E}_i(b_0, z')\big) > 0.$$

After another restriction of the parameter neighbourhood, $\sup_z d_i(\widehat{E}_i(b, z), \widehat{E}_i(b_0, z)) < \mu_i/3$. Thus pairs at distance at least ε have distinct images, while pairs at distance less than ε lie in one $U_{i\ell}$ and are separated by the preceding lower Lipschitz estimate. Each $\widehat{E}_i(b)$ is therefore an injective immersion of the compact manifold D into the Hausdorff manifold M_i, hence an embedding. It follows that $r_{\mathrm{col}}(b) \geq c_1 > c$ near b_0. Therefore every strict superlevel set of r_{col} is open, and r_{col} is lower semicontinuous.

Let $F : B \to (0, \infty]$ be lower semicontinuous. For each $b \in B$, choose $a_b > 0$ and an open neighbourhood U_b such that $4a_b < F$ on U_b. Take a locally finite refinement (U_j) of $(U_b)_{b \in B}$ and choose $a_j > 0$ so that $4a_j < \inf_{U_j} F$. If (ψ_j) is a smooth partition of unity subordinate to (U_j), then

$$f_F := \sum_j a_j \psi_j \tag{7.15}$$

is a pointwise finite smooth sum and

$$0 < f_F(b) < \frac{1}{4} F(b).$$

Applying this construction to r_{col} proves the existence of a smooth positive minorant. For the remainder of the proof, fix an arbitrary smooth function ρ satisfying $0 < \rho < r_{\mathrm{col}}$.

Fix $b_0 \in B$ and set $\kappa_0 := \lambda(b_0) + \frac{3}{2}\eta(b_0)$. Choose a relatively compact coordinate neighbourhood U_0 of b_0 such that

$$\lambda(b) + \eta(b) < \kappa_0 < \lambda(b) + 2\eta(b) \qquad (b \in \overline{U}_0). \tag{7.16}$$

Let $\mathcal{U}_i$ be the unit tangent bundle of the family $h_i(b)$ over $\overline{U}_0 \times M_i$, and let $\mathcal{U}_\Sigma$ be the unit tangent bundle of $h_\Sigma(b)$ over $\overline{U}_0 \times \Sigma$. These spaces are compact. The numbers

$$m_{\mathrm{in}} := \min_{\substack{i \in \{1,2\} \\ (b,x,v) \in \mathcal{U}_i}} (\mathrm{Ric}_{h_i(b)} - \kappa_0 h_i(b))(v, v), \qquad m_{\mathrm{II}} := \min_{(b,x,v) \in \mathcal{U}_\Sigma} (\mathrm{II}_{1,b} + \mathrm{II}_{2,b})(v, v)$$

are positive.

Set $r_c := \frac{1}{2} \inf_{\overline{U}_0} \rho > 0$ and write

$$H_b := h_\Sigma(b), \qquad K_{i,b} := \partial_t H_{i,b}(0), \qquad J_b := K_{1,b} - K_{2,b}.$$

Equation (7.4) gives $J_b = 2(\mathrm{II}_{1,b} + \mathrm{II}_{2,b}) \geq 2m_{\mathrm{II}} H_b$. Fix a background metric on Σ and use its connection to define $C^2(\Sigma)$ norms. All constants in the next estimates are uniform for $b \in \overline{U}_0$. Taylor's formula on the fixed normal collars gives

$$H_{1,b}(-\delta) = H_b - \delta K_{1,b} + O_{C^2}(\delta^2), \qquad \partial_t H_{1,b}(-\delta) = K_{1,b} + O_{C^2}(\delta),$$
$$H_{2,b}(\delta) = H_b + \delta K_{2,b} + O_{C^2}(\delta^2), \qquad \partial_t H_{2,b}(\delta) = K_{2,b} + O_{C^2}(\delta).$$

Let $A_{b,\delta}(t)$ be the central Hermite coefficient. Substitution in (7.5) yields, for $|t| \le \delta$,

$$A_{b,\delta}(t) = H_b - \frac{\delta}{4}J_b + \frac{t}{2}(K_{1,b} + K_{2,b}) - \frac{t^2}{4\delta}J_b + R_{b,\delta}(t), \qquad \sup_{|t|\le\delta} \|\partial_t^j R_{b,\delta}(t)\|_{C^2(\Sigma)} \le C\delta^{2-j} \quad (0 \le j \le 2).$$

Indeed, the errors in the endpoint values are $O_{C^2}(\delta^2)$, and the Hermite terms involving endpoint derivatives have an additional factor 2δ. Each derivative in t contributes a factor $(2\delta)^{-1}$ to the fixed cubic polynomials. Hence

$$A_{b,\delta} = H_b + O_{C^2}(\delta), \qquad \partial_t A_{b,\delta} = O_{C^2}(1), \qquad \partial_t^2 A_{b,\delta} = -\frac{J_b}{2\delta} + O_{C^2}(1).$$

For sufficiently small δ, the inverses of $A_{b,\delta}$ are uniformly bounded, and $A_{b,\delta}^{-1} - H_b^{-1} = O_{C^2}(\delta)$. The cylinder formulas in the proof of Proposition 6.2 therefore give

$$\begin{aligned} \mathrm{Ric}_{\mathcal{I}_{b,\delta}}(\partial_t, \partial_t) &= \frac{\mathrm{tr}_{H_b} J_b}{4\delta} + O(1), \\ \mathrm{Ric}_{\mathcal{I}_{b,\delta}}(\partial_t, \cdot)|_{T\Sigma} &= O(1), \\ \mathrm{Ric}_{\mathcal{I}_{b,\delta}}|_{T\Sigma\times T\Sigma} &= \frac{J_b}{4\delta} + O(1). \end{aligned}$$

Here the errors have uniformly bounded P_b-norm. In the last line, $\mathrm{Ric}_{A_{b,\delta}}$ is bounded because the slice metrics have uniformly bounded tangential C^2 norms and uniformly bounded inverses. The mixed component uses only $A_{b,\delta}$, its first normal derivative, and their first tangential derivatives. The radial leading term follows from $-\frac{1}{2}\,\mathrm{tr}_{A_{b,\delta}}\,\partial_t^2 A_{b,\delta}$; replacing $A_{b,\delta}^{-1}$ by H_b^{-1} changes it by $O(1)$. Since $\dim\Sigma = n - 1 \ge 1$, these estimates imply

$$\mathrm{Ric}_{\mathcal{I}_{b,\delta}} \ge \left(\frac{m_{\mathrm{II}}}{2\delta} - C\right) P_b \qquad (|t| \le \delta).$$

Also $\mathcal{I}_{b,\delta} \le 2P_b$ for sufficiently small δ. Choose $d_{\mathrm{Ric}} > 0$ so that $2d_{\mathrm{Ric}} < \inf_{\overline{U}_0} \rho$ and, for $0 < \delta < d_{\mathrm{Ric}}$,

$$\frac{m_{\mathrm{II}}}{2\delta} - C - 2|\kappa_0| > 0.$$

It follows that $\mathrm{Ric}_{\mathcal{I}_{b,\delta}} > \kappa_0 \mathcal{I}_{b,\delta}$ on the central piece, including its one-sided endpoint values. On the other two pieces and their one-sided closures, the tensor is $h_i(b)$ and $\mathrm{Ric}_{h_i(b)} - \kappa_0 h_i(b) \ge m_{\mathrm{in}} h_i(b) > 0$.

There is $C_0 < \infty$ such that, for $b \in \overline{U}_0$ and every t in the signed domain of $H_{i,b}$ with $|t| \le 2d_{\mathrm{Ric}}$,

$$\|H_{i,b}(t) - h_\Sigma(b)\|_{h_\Sigma(b)} \le C_0|t|, \qquad \|\partial_t H_{i,b}(t)\|_{h_\Sigma(b)} \le C_0.$$

The Hermite formula and the mean-value estimate for $(H_+ - H_-)/(2\delta)$ give a constant C_{prod} such that

$$\sup_{|t|\le 2\delta} \|\mathcal{I}_{b,\delta} - P_b\|_{P_b} \le C_{\mathrm{prod}}\delta \qquad (b \in \overline{U}_0,\ 0 < \delta < d_{\mathrm{Ric}}).$$

Define the positive number

$$\vartheta_L := \min\{1 - L^{-2}, L^2 - 1\} > 0.$$

Choose $d_0 \in (0, d_{\mathrm{Ric}})$ so that $C_{\mathrm{prod}} d_0 < \vartheta_L$. Then, for $b \in \overline{U}_0$ and $0 < \delta < d_0$,

$$L^{-2} P_b < \mathcal{I}_{b,\delta} < L^2 P_b \qquad (|t| \le 2\delta).$$

By (7.16), these are the gluing bounds at (b, δ). Since b_0 was arbitrary, $d_* > 0$ on B. The definition also gives $d_*(b) \le \rho(b)/2 < \infty$.

After setting $t = \delta s$, the Hermite formula is smooth in (b, δ, s) for $\delta > 0$; the outer pieces are restrictions of the smooth collar families. Thus the interpolation depends smoothly on (b, δ) on each fixed rescaled smooth piece. The Ricci tensor is a smooth expression in the inverse metric and the derivatives of the metric through order two, hence is continuous from positive-definite C^2 metrics to continuous symmetric two-tensors.

Let $0 < c < d_*(b_0)$. By the definition of the supremum, there is $d^0 \in \mathscr{D}_{b_0}$ with $d^0 > c$; fix c' with $c < c' < d^0$. Shrink a neighbourhood U' of b_0 so that $2c' < \rho(b)$ on U'. Set

$$d_1 := \min\{c'/2, d_0\}, \qquad U_1 := U' \cap U_0.$$

Then every $b \in U_1$ and every $0 < \delta < d_1$ satisfy the gluing bounds. For each $\delta_0 \in [d_1, c']$, choose

$$\delta_0 < \widehat{\delta_0} < \min\{2\delta_0, d^0\}.$$

The product bound for the admissible width $\widehat{\delta_0}$ applies at $t = \pm 2\delta_0$, where both interpolations equal the original collar metric. Hence the product inequalities for $\mathcal{I}_{b_0,\delta_0}$ extend strictly to the closed interval $|t| \leq 2\delta_0$. Identify its three closed smooth pieces by $t = \delta_0 s$, with $s \in [-2,-1]$, $[-1,1]$, and $[1,2]$, and represent each join by its two one-sided copies. Let $\mathscr{K}_{\delta_0}$ be the disjoint union of these three compact pieces with two copies of each join. On its P_{b_0}-unit sphere bundle, the three continuous functions

$$\begin{aligned}
&(\mathrm{Ric}_{\mathcal{I}_{b_0,\delta_0}} - (\lambda(b_0) + \eta(b_0))\mathcal{I}_{b_0,\delta_0})(v,v),\\
&(\mathcal{I}_{b_0,\delta_0} - L^{-2}P_{b_0})(v,v),\\
&(L^2 P_{b_0} - \mathcal{I}_{b_0,\delta_0})(v,v)
\end{aligned}$$

have positive minima. Continuity in (b,δ) on the fixed rescaled pieces gives open sets $U_{\delta_0} \ni b_0$ and $I_{\delta_0} \ni \delta_0$ on which all three displayed functions remain positive. Choose $\delta_1, \dots, \delta_m \in [d_1, c']$ such that $[d_1, c'] \subset \bigcup_{j=1}^m I_{\delta_j}$ and put

$$U := U_1 \cap U' \cap \bigcap_{j=1}^{m} U_{\delta_j}.$$

Then every $b \in U$ and every $0 < \delta \leq c'$ are admissible. Hence $d_*(b) \geq c' > c$ on U, proving lower semicontinuity of d_*.

Fix now a smooth δ satisfying the hypotheses of the lemma. For $b_0 \in B$, choose

$$2\delta(b_0) < r_0 < r_{\mathrm{col}}(b_0).$$

Lower semicontinuity of r_{col} and continuity of δ give a relatively compact neighbourhood U_0 of b_0 such that

$$2\delta(b) < r_0 < r_{\mathrm{col}}(b) \qquad (b \in \overline{U}_0).$$

Thus both signed collar families are defined on one fixed collar over $\overline{U}_0$. Since δ is positive and continuous, put $\delta_{\min} := \min_{b \in \overline{U}_0} \delta(b) > 0$.

Fix a finite coordinate atlas on Σ and a background connection on $\mathrm{Sym}^2 T^*\Sigma$; use the resulting coordinate norms for mixed derivatives and $\|\cdot\|_{P_b}$ at order zero.

For $0 < \nu < 1/2$, one has $\epsilon_\nu < \nu/2$. Near a join $t = \tau$, the cutoff in (7.7) equals one on $|t - \tau| \leq \nu/2$. Since the first normal derivative of the interpolation is continuous across the join, tensor-valued distributional differentiation gives

$$\partial_t^2(\varrho_{\epsilon_\nu} * A) = \varrho_{\epsilon_\nu} * (\partial_t^2 A). \tag{7.17}$$

Put

$$T_{\tau,\nu} := [\tau - \nu, \tau - \nu/2] \cup [\tau + \nu/2, \tau + \nu].$$

On $T_{\tau,\nu}$, the convolution interval lies on one smooth side of the join. In the following estimate the C^2 norm is the maximum of all mixed normal and tangential derivatives of total order at most two on $T_{\tau,\nu} \times \Sigma$, in the fixed charts and background connection. Let $K \Subset \mathscr{P}_1$ and choose $0 < \nu_K < \min\{1/2, \frac{1}{8}\min_K \delta\}$. Evenness of ϱ, Taylor's formula, and the maximum on K of the required one-sided derivatives give a constant C_K such that

$$\|\mathfrak{M}_{\tau,\nu} A - A\|_{C^2(T_{\tau,\nu}\times\Sigma)} \leq C_K \epsilon_\nu^2 (1 + \nu^{-1} + \nu^{-2}) \longrightarrow 0 \tag{7.18}$$

for $(b,\delta)\in K$ and $0<\nu<\nu_K$. On the central region, convolution converges to A in C^1 in the normal variable and in every fixed tangential C^r norm. Uniformly for $b\in\overline{U}_0$, there is a modulus $\omega^{(0)}_{U_0}(\nu)\to 0$ such that

$$\|\mathcal{S}_{b,\delta(b),\nu}-\mathcal{I}_{b,\delta(b)}\|_{P_b}\le \omega^{(0)}_{U_0}(\nu)$$

on the smoothing bands for $b\in\overline{U}_0$. The two smoothing regions are disjoint because $8\nu<\delta(b)$. For a join $\tau=\pm\delta$, put $A^{\tau}_{b,\delta}(r):=A_{b,\delta}(\tau+r)$, and denote its restrictions to $r\le 0$ and $r\ge 0$ by $A^{\tau,-}_{b,\delta}$ and $A^{\tau,+}_{b,\delta}$. The explicit collar and Hermite formulas extend both one-sided coefficients smoothly to a fixed neighbourhood of $r=0$. On every fixed compact r-interval, $A^{\tau}_{b,\delta}$ is therefore the distributional family

$$\mathbf{1}_{(-\infty,0)}A^{\tau,-}_{b,\delta}+\mathbf{1}_{[0,\infty)}A^{\tau,+}_{b,\delta},$$

whose coefficients depend smoothly on (b,δ). After the change of variables $s=\epsilon_\nu z$, the convolution on the normalized band $t=\tau+\nu q$ is

$$\int_{-1}^{1}\varrho(z)A^{\tau}_{b,\delta}(\nu q-\epsilon_\nu z)\,dz.$$

Translation, positive dilation, and convolution by the fixed smooth kernel preserve smooth parameter dependence. Thus the displayed integral is smooth in (b,δ,ν,q,x) for $\delta,\nu>0$. Together with the formula $t=\delta q$ on the interpolation pieces, this proves the parameter-smoothness assertions in the statement.

For $G=dt^2+A(t)$, write a prime for ∂_t and compute tangential covariant derivatives with the slice metric $A(t)$. Apart from symmetry in the lower indices, the nonzero Christoffel symbols are

$$\Gamma^t_{ij}=-\frac{1}{2}A'_{ij},\qquad \Gamma^k_{ti}=\frac{1}{2}A^{k\ell}A'_{\ell i},\qquad \Gamma^k_{ij}=\Gamma(A)^k_{ij}.$$

Direct substitution gives

$$\begin{aligned}\mathrm{Ric}_G(\partial_t,\partial_t)&=-\frac{1}{2}\operatorname{tr}_A A''+\frac{1}{4}\operatorname{tr}(A^{-1}A'A^{-1}A'),\\ \mathrm{Ric}_G(\partial_t,\partial_i)&=\frac{1}{2}\left(\nabla^j A'_{ij}-\nabla_i\operatorname{tr}_A A'\right),\qquad (7.19)\\ (\mathrm{Ric}_G)_{ij}&=(\mathrm{Ric}_A)_{ij}-\frac{1}{2}A''_{ij}-\frac{1}{4}(\operatorname{tr}_A A')A'_{ij}+\frac{1}{2}(A'A^{-1}A')_{ij}.\qquad (7.20)\end{aligned}$$

Each component is affine in A'' and smooth in the remaining displayed jets. Since the collars are fixed over $\overline{U}_0$ and $\delta\ge\delta_{\min}$, there is $C_2<\infty$ such that $\|A''\|\le C_2$ on every one-sided smoothing region. If $B_\nu:=\varrho_{\epsilon_\nu}*A$ on the central region, then (7.17) and the convergences

$$B_\nu\to A,\qquad \partial_t B_\nu\to\partial_t A,$$

together with the analogous convergence of the tangential derivatives appearing in (7.19)–(7.20), are bounded by moduli independent of $b\in\overline{U}_0$. The bound on A'' therefore gives a modulus $\omega^{(2)}_{U_0}(\nu)\to 0$ such that

$$\left\|\mathrm{Ric}_{dt^2+B_\nu}(t)-\int_{\mathbb{R}}\varrho_{\epsilon_\nu}(s)\,\mathrm{Ric}_{dt^2+A}(t-s)\,ds\right\|_{P_b}\le\omega^{(2)}_{U_0}(\nu).\qquad (7.21)$$

Set $\delta_0:=\delta(b_0)$ and $\mathscr{J}:=[-5/4,-1]\sqcup[-1,-3/4]\sqcup[3/4,1]\sqcup[1,5/4]$, where the copies of -1 and 1 are distinct. The substitution $t=\delta_0 s$ identifies $\mathscr{J}\times\Sigma$ with the compact disjoint union of the four one-sided smooth neighbourhoods of the joins. This union is contained in $|t|<2\delta_0$. On its P_{b_0}-unit sphere bundle, define

$$m_{\mathrm{Ric}}:=\min\left(\mathrm{Ric}_{\mathcal{I}_{b_0,\delta_0}}-(\lambda(b_0)+\eta(b_0))\mathcal{I}_{b_0,\delta_0}\right)(v,v)>0$$

and let $m_L>0$ be the minimum of the two functions

$$(\mathcal{I}_{b_0,\delta_0}-L^{-2}P_{b_0})(v,v),\qquad (L^2P_{b_0}-\mathcal{I}_{b_0,\delta_0})(v,v).$$

Using $t = \delta(b)s$ on the four fixed intervals in $\mathscr{J}$, choose a relatively compact neighbourhood $U_{\mathrm{sm}} \Subset U_0$ such that, for $b \in \overline{U}_{\mathrm{sm}}$, every P_b-unit vector on these one-sided pieces satisfies

$$\begin{aligned}&(\mathrm{Ric}_{\mathcal{I}_{b,\delta(b)}} - (\lambda(b) + \eta(b))\mathcal{I}_{b,\delta(b)})(v,v) \geq m_{\mathrm{Ric}}/2,\\&(\mathcal{I}_{b,\delta(b)} - L^{-2}P_b)(v,v) \geq m_L/2,\\&(L^2P_b - \mathcal{I}_{b,\delta(b)})(v,v) \geq m_L/2.\end{aligned}$$

On the transition parts of the smoothing bands the reference interpolation is smooth. Equation (7.18), continuity of $g \mapsto \mathrm{Ric}_g$ in the C^2 topology, and boundedness of $\lambda + \eta$ on $\overline{U}_{\mathrm{sm}}$ give a modulus $\omega^{(\mathrm{tr})}_{U_{\mathrm{sm}}}(\nu) \to 0$ such that

$$\left\| \begin{aligned}&\mathrm{Ric}_{\mathcal{S}_{b,\delta(b),\nu}} - (\lambda(b) + \eta(b))\mathcal{S}_{b,\delta(b),\nu}\\&\quad - \left(\mathrm{Ric}_{\mathcal{I}_{b,\delta(b)}} - (\lambda(b) + \eta(b))\mathcal{I}_{b,\delta(b)}\right)\end{aligned} \right\|_{P_b} \leq \omega^{(\mathrm{tr})}_{U_{\mathrm{sm}}}(\nu) \tag{7.22}$$

on those transition parts for every $b \in \overline{U}_{\mathrm{sm}}$, with one modulus $\omega^{(\mathrm{tr})}_{U_{\mathrm{sm}}}$.

Choose $e_0 > 0$ such that

$$\begin{aligned}&e_0 < \frac{1}{2}, \qquad 8e_0 < \delta_{\min},\\&\sup_{0<\nu\leq e_0} \omega^{(2)}_{U_0}(\nu) < \frac{1}{4}m_{\mathrm{Ric}}, \qquad \sup_{0<\nu\leq e_0} \omega^{(\mathrm{tr})}_{U_{\mathrm{sm}}}(\nu) < \frac{1}{4}m_{\mathrm{Ric}},\\&\sup_{0<\nu\leq e_0} \omega^{(0)}_{U_0}(\nu) < \frac{1}{4}m_L.\end{aligned}$$

If $|t \mp \delta(b)| < \nu$ and $|s| < \epsilon_\nu$, then

$$|t - s \mp \delta(b)| < \nu + \epsilon_\nu < \frac{3}{2}\nu < \frac{3}{16}\delta(b) < \frac{1}{4}\delta(b).$$

Thus every tensor sampled by the central convolution lies in the compact one-sided regions on which the margins are at least $m_{\mathrm{Ric}}/2$ and $m_L/2$. Write $c_b := \lambda(b) + \eta(b)$, let $A_b(t)$ be the tangential coefficient of $\mathcal{I}_{b,\delta(b)}$, and put

$$\widetilde{A}_{b,\nu}(t) := \int_{\mathbb{R}} \varrho_{\epsilon_\nu}(s)A_b(t-s)\,ds.$$

The one-sided inequalities give

$$\int_{\mathbb{R}} \varrho_{\epsilon_\nu}(s)\left(\mathrm{Ric}_{dt^2+A_b}(t-s) - c_b(dt^2 + A_b(t-s))\right)ds \geq \frac{1}{2}m_{\mathrm{Ric}}P_b.$$

Since $dt^2 + \widetilde{A}_{b,\nu}$ is the convolution of $dt^2 + A_b$ in the tangential coefficient, (7.21) yields

$$\mathrm{Ric}_{dt^2+\widetilde{A}_{b,\nu}} - c_b(dt^2 + \widetilde{A}_{b,\nu}) \geq \frac{1}{4}m_{\mathrm{Ric}}P_b$$

on the central parts. On the transition parts, (7.22) gives

$$\mathrm{Ric}_{\mathcal{S}_{b,\delta(b),\nu}} - (\lambda(b) + \eta(b))\mathcal{S}_{b,\delta(b),\nu} \geq \frac{1}{4}m_{\mathrm{Ric}}P_b.$$

The P_b-operator-norm estimate and the product margin give

$$\mathcal{S}_{b,\delta(b),\nu} - L^{-2}P_b \geq \frac{1}{4}m_LP_b, \qquad L^2P_b - \mathcal{S}_{b,\delta(b),\nu} \geq \frac{1}{4}m_LP_b.$$

Outside the smoothing bands the tensor is unchanged. Hence $\mathcal{S}_{b,\delta(b),\nu}$ satisfies the gluing bounds for $b \in \overline{U}_{\mathrm{sm}}$ and $0 < \nu < e_0$. Thus $e_*(b_0) > 0$. Since $\mathscr{E}_b \subset (0, \delta(b)/8)$, one has $e_*(b) \leq \delta(b)/8 < \infty$ for every $b \in B$.

Let $0 < c < e_*(b_0)$. There is a threshold $e^0 > c$ in $\mathscr{E}_{b_0}$; choose c' with $c < c' < e^0$. Shrink a neighbourhood U' of b_0 so that $8c' < \delta(b)$ on U'. Set

$$e_1 := \min\{c'/2, e_0\}, \qquad U_1 := U' \cap U_{\mathrm{sm}}.$$

Then every $b \in U_1$ and every $0 < \nu < e_1$ are admissible. Fix $\nu_0 \in [e_1, c']$. On the two smoothing bands, use the fixed coordinates

$$t = \pm\delta(b) + \nu s, \qquad -1 \le s \le 1.$$

At (b_0, ν_0), the three continuous functions on the compact P_{b_0}-unit sphere bundle of these normalized bands,

$$\begin{aligned}
&(\mathrm{Ric}_{\mathcal{S}_{b,\delta(b),\nu}} - (\lambda(b) + \eta(b))\mathcal{S}_{b,\delta(b),\nu})(v,v),\\
&(\mathcal{S}_{b,\delta(b),\nu} - L^{-2}P_b)(v,v),\\
&(L^2 P_b - \mathcal{S}_{b,\delta(b),\nu})(v,v)
\end{aligned}$$

have positive minima. Their continuity in (b,ν) gives open sets $U_{\nu_0} \ni b_0$ and $I_{\nu_0} \ni \nu_0$ with $U_{\nu_0} \times I_{\nu_0} \subset \{(b,\nu) \in U' \times (0,\infty) : 8\nu < \delta(b)\}$ on which all three functions remain positive. Outside the smoothing bands the tensor is $\mathcal{I}_{b,\delta(b)}$, which satisfies the gluing bounds by the hypothesis on δ. Choose $\nu_1, \dots, \nu_r \in [e_1, c']$ with $[e_1, c'] \subset \bigcup_{j=1}^r I_{\nu_j}$ and put

$$U := U_1 \cap \bigcap_{j=1}^{r} U_{\nu_j}.$$

Then every $b \in U$ and every $0 < \nu \le c'$ are admissible. Hence $c' \in \mathscr{E}_b$ and $e_*(b) \ge c' > c$ on U. Therefore e_* is lower semicontinuous. The lower semicontinuity of r_{col} makes both strict parameter subgraphs in the statement open. The fixed-coordinate calculations above prove the asserted smooth dependence on those subgraphs. □

The lower semicontinuity of r_{col} makes

$$\mathscr{C} := \{(b,t,x) \in B \times \mathbb{R} \times \Sigma : |t| < r_{\mathrm{col}}(b)\}$$

open. The local signed-collar maps combine to a smooth map

$$\mathcal{E} : \mathscr{C} \longrightarrow \mathcal{M}, \qquad \mathcal{E}(b,t,x) = \begin{cases} [b, E_1(b,t,x)], & t \le 0,\\ [b, E_2(b,t,x)], & t > 0. \end{cases}$$

It is a homeomorphism onto the open signed-collar neighbourhood and satisfies $\pi \circ \mathcal{E} = \mathrm{pr}_B$. All parameter-dependent metrics below are smooth vertical metrics on this fixed smooth total space with corners; their restriction to the fibre over b is denoted by $g(b)$.

Proposition 7.2. *Let B be a smooth paracompact manifold, possibly with boundary. For $i = 1, 2$, let M_i be a compact smooth n-manifold, not necessarily connected, with boundary; let $\Sigma_i \subset \partial M_i$ be a compact union of boundary components; and let $\beta_i : \Sigma \to \Sigma_i$ be diffeomorphisms from one compact, possibly disconnected, manifold Σ. Let $b \mapsto h_i(b)$ be smooth metric families satisfying*

$$\beta_1^*(h_1(b)|_{\Sigma_1}) = \beta_2^*(h_2(b)|_{\Sigma_2}) =: h_\Sigma(b).$$

Let $\mathrm{II}_{i,b}$ be the pullback by β_i of the second fundamental form computed with the inward unit normal. Let $\lambda, \eta, w : B \to \mathbb{R}$ be continuous, with $\eta, w > 0$, and let $L > 1$. Suppose

$$\mathrm{Ric}_{h_i(b)} > (\lambda(b) + 2\eta(b))h_i(b) \quad (i = 1,2), \qquad \mathrm{II}_{1,b} + \mathrm{II}_{2,b} > 0.$$

Let $\pi : \mathcal{M} \to B$ be the smooth bundle of manifolds with boundary obtained by gluing the two families along $\beta_2 \circ \beta_1^{-1}$ and using the signed-collar smooth structure above; its total space is a manifold with corners. There are smooth functions $\delta, \nu : B \to (0,\infty)$ and a smooth positive-definite vertical metric

$$g \in \Gamma^\infty\big(\mathrm{Sym}^2(\ker d\pi)^*\big)$$

such that, for every $b \in B$,

$$0 < 2\delta(b) < r_{\mathrm{col}}(b), \qquad 4\delta(b) < w(b), \qquad 0 < 8\nu(b) < \delta(b),$$
$$\mathrm{Ric}_{g(b)} > (\lambda(b) + \eta(b))g(b), \qquad g(b) = \mathcal{S}_{b,\delta(b),\nu(b)} \quad \textit{in the signed-collar chart } \mathcal{E}.$$

Put $\ell_b := 2\delta(b)$. Then, for $i = 1, 2$,

$$g(b) = h_i(b) \quad \textit{on } M_i \setminus \widehat{E}_i\big(b, [0, \ell_b) \times \Sigma\big), \qquad 2\ell_b < w(b),$$

and, on $(-\ell_b, \ell_b) \times \Sigma$,

$$L^{-2}(dt^2 + h_\Sigma(b)) \le g(b) \le L^2(dt^2 + h_\Sigma(b)). \tag{7.23}$$

Proof. Choose a smooth function ρ with $0 < \rho < r_{\mathrm{col}}$, whose existence is part of Lemma 7.1, and let d_* be the interpolation threshold associated with ρ. The function $F_1 := \min\{d_*, w/4\}$ is positive. For every $c \in \mathbb{R}$, its strict superlevel set is $\{d_* > c\} \cap \{w > 4c\}$; hence F_1 is lower semicontinuous. The partition-of-unity construction (7.15) gives a smooth function δ satisfying

$$0 < \delta < \frac{1}{4}F_1 \le \min\{d_*/4, w/16\}.$$

Fix $b \in B$. Since $\delta(b) < d_*(b)$, there is $d_b \in \mathscr{D}_b$ with $\delta(b) < d_b$. The downward definition of $\mathscr{D}_b$ gives

$$2\delta(b) < \rho(b) < r_{\mathrm{col}}(b)$$

and shows that $\mathcal{I}_{b,\delta(b)}$ satisfies the gluing bounds at $(b, \delta(b))$.

Apply Lemma 7.1 to this smooth interpolation width and write e_* for the resulting smoothing threshold. The function $F_2 := \min\{e_*, \delta/8\}$ is positive. Its strict superlevel sets are intersections of strict superlevel sets of e_* and $\delta/8$, so F_2 is lower semicontinuous. A second application of (7.15) gives a smooth function ν with

$$0 < \nu < \frac{1}{4}F_2 \le \min\{e_*/4, \delta/32\}.$$

For each b, the inequality $\nu(b) < e_*(b)$ yields an $e_b \in \mathscr{E}_b$ with $\nu(b) < e_b$. The downward definition of $\mathscr{E}_b$ shows that $\mathcal{S}_{b,\delta(b),\nu(b)}$ is a smooth positive-definite metric satisfying the gluing bounds at $(b, \delta(b))$.

Define $g(b) := \mathcal{S}_{b,\delta(b),\nu(b)}$. On the three normalized interpolation pieces, the first parameter-smoothness assertion of Lemma 7.1 and the smoothness of δ make the coefficients smooth in (b, t, x). On the two normalized smoothing bands, the second parameter-smoothness assertion and the smoothness of δ, ν give the same conclusion. Outside the collar, g equals the original metric. The cutoff in (7.7) has compact support in $(-1, 1)$, so these descriptions agree on open neighbourhoods of their common boundaries. Thus g is a smooth vertical metric on $\mathcal{M}$.

The interpolation equals $h^{\mathrm{gl}}(b)$ for $|t| \ge \delta(b)$, and the mollification changes it only when $|t \mp \delta(b)| < \nu(b)$. Therefore $g(b) = h_i(b)$ on the corresponding side whenever $|t| \ge \delta(b) + \nu(b)$. Since

$$\delta(b) + \nu(b) < 2\delta(b) = \ell_b < \rho(b) < r_{\mathrm{col}}(b), \qquad 2\ell_b = 4\delta(b) < w(b),$$

the support assertion follows. The Ricci and product inequalities are the gluing bounds for $\mathcal{S}_{b,\delta(b),\nu(b)}$; replacing the strict product inequalities by weak ones gives (7.23). □

Proposition 7.3. *Assume the hypotheses and retain the notation of Proposition 7.2. Let $B_1 \subset B$ be open, let $B_0 \subset B_1$ be closed in B, and let $(\delta_{\mathrm{pr}}, \nu_{\mathrm{pr}})$ be a downward-admissible pair on B_1. There exist smooth functions $\delta, \nu : B \to (0, \infty)$ and a smooth positive-definite vertical metric $g \in \Gamma^\infty(\mathrm{Sym}^2(\ker d\pi)^*)$ such that, for every $b \in B$,*

$$0 < 2\delta(b) < r_{\mathrm{col}}(b), \qquad 4\delta(b) < w(b), \qquad 0 < 8\nu(b) < \delta(b), \qquad \mathrm{Ric}_{g(b)} > (\lambda(b) + \eta(b))g(b),$$

and, in the signed-collar chart, $g(b) = \mathcal{S}_{b,\delta(b),\nu(b)}$. Put $\ell_b := 2\delta(b)$. Then $0 < \ell_b < r_{\mathrm{col}}(b)$ and $2\ell_b < w(b)$; moreover, for $i = 1, 2$,

$$g(b) = h_i(b) \quad \textit{on } M_i \setminus \widehat{E}_i\big(b, [0, \ell_b) \times \Sigma\big),$$

and, on $(-\ell_b, \ell_b) \times \Sigma$,

$$L^{-2}(dt^2 + h_\Sigma(b)) \le g(b) \le L^2(dt^2 + h_\Sigma(b)).$$

The choices retain the prescribed widths on B_0*:*

$$(\delta, \nu) = (\delta_{\mathrm{pr}}, \nu_{\mathrm{pr}}) \qquad \text{on } B_0. \tag{7.24}$$

The corresponding metric satisfies

$$g(b) = \mathcal{S}_{b,\delta_{\mathrm{pr}}(b),\nu_{\mathrm{pr}}(b)} \qquad (b \in B_0). \tag{7.25}$$

Proof. Choose a smooth function $\rho : B \to (0, \infty)$ with $0 < \rho < r_{\mathrm{col}}$, and let d_* be the interpolation threshold associated with ρ in Lemma 7.1. Set $F_\delta := \min\{d_*, w/4\}$. For $c \ge 0$,

$$\{F_\delta > c\} = \{d_* > c\} \cap \{w > 4c\};$$

hence F_δ is positive and lower semicontinuous. The smooth-minorant construction (7.15) gives a smooth $\delta_0 > 0$ satisfying $\delta_0 < F_\delta/4$. In particular, $\delta_0 < d_*$ and $4\delta_0 < w$. For each $b \in B$, choose $d_b \in \mathcal{D}_b$ with $\delta_0(b) < d_b$. If $0 < d \le \delta_0(b)$, then $d < d_b$; thus $2d < \rho(b) < r_{\mathrm{col}}(b)$ and $\mathcal{I}_{b,d}$ satisfies the gluing bounds at (b, d).

The paracompact Hausdorff manifold B is normal. Choose an open set U such that

$$B_0 \subset U \subset \overline{U} \subset B_1.$$

Smooth partitions of unity subordinate to $\{B_1, B \setminus \overline{U}\}$ and $\{U, B \setminus B_0\}$ give smooth functions $\chi_\delta, \chi_\nu : B \to [0, 1]$ such that

$$\operatorname{supp} \chi_\delta \subset B_1, \qquad \chi_\delta = 1 \text{ on } \overline{U}, \qquad \operatorname{supp} \chi_\nu \subset U, \qquad \chi_\nu = 1 \text{ on } B_0.$$

The functions

$$\overline{\delta}_{\mathrm{pr}}(b) := \begin{cases} \chi_\delta(b)\delta_{\mathrm{pr}}(b), & b \in B_1, \\ 0, & b \notin B_1, \end{cases} \qquad \overline{\nu}_{\mathrm{pr}}(b) := \begin{cases} \chi_\nu(b)\nu_{\mathrm{pr}}(b), & b \in B_1, \\ 0, & b \notin B_1 \end{cases}$$

are smooth because each cutoff vanishes on a neighbourhood of $B \setminus B_1$.

Set $\delta := \overline{\delta}_{\mathrm{pr}} + (1 - \chi_\delta)\delta_0$. Then $\delta = \delta_{\mathrm{pr}}$ on $\overline{U}$. Fix $b \in B_1$. If $\delta_{\mathrm{pr}}(b) \ge \delta_0(b)$, then $0 < \delta(b) \le \delta_{\mathrm{pr}}(b)$, and downward admissibility gives $2\delta(b) < r_{\mathrm{col}}(b)$ and the gluing bounds for $\mathcal{I}_{b,\delta(b)}$. If $\delta_{\mathrm{pr}}(b) < \delta_0(b)$, then $0 < \delta(b) \le \delta_0(b) < d_b$, so the defining property of $\mathcal{D}_b$ gives the same conclusions. For $b \notin B_1$ one has $\delta(b) = \delta_0(b) < d_b$, so the defining property of $\mathcal{D}_b$ gives the same conclusions. Thus, for every $b \in B$,

$$0 < 2\delta(b) < r_{\mathrm{col}}(b), \qquad \mathcal{I}_{b,\delta(b)} \text{ satisfies the gluing bounds at } (b, \delta(b)).$$

On B_1, the number $\delta(b)$ is a convex combination of $\delta_{\mathrm{pr}}(b)$ and $\delta_0(b)$; since both satisfy $4d < w(b)$, so does $\delta(b)$. The same inequality follows from $\delta = \delta_0$ outside B_1. Hence $4\delta < w$ on B.

Apply Lemma 7.1 to the smooth interpolation width δ, and let e_* be the resulting smoothing threshold. A smooth minorant ν_0 of e_* may be chosen so that

$$0 < \nu_0 < \frac{1}{4} e_* \le \frac{1}{32}\delta.$$

For each $b \in B$, choose $e_b \in \mathcal{E}_b$ with $\nu_0(b) < e_b$. Every $0 < v \le \nu_0(b)$ then satisfies $8v < \delta(b)$ and makes $\mathcal{S}_{b,\delta(b),v}$ a smooth metric satisfying the gluing bounds.

Set $\nu := \overline{\nu}_{\mathrm{pr}} + (1 - \chi_\nu)\nu_0$. On $\operatorname{supp} \chi_\nu \subset U$ one has $\delta = \delta_{\mathrm{pr}}$. If $\nu_{\mathrm{pr}}(b) \ge \nu_0(b)$ there, then $0 < \nu(b) \le \nu_{\mathrm{pr}}(b)$, and downward admissibility applies. If $\nu_{\mathrm{pr}}(b) < \nu_0(b)$, then $0 < \nu(b) \le \nu_0(b) < e_b$, and the defining property of $\mathcal{E}_b$ applies. Outside $\operatorname{supp} \chi_\nu$ one has $\nu = \nu_0 < e_b$, so the defining property of $\mathcal{E}_b$ gives the same conclusions. Consequently, for every $b \in B$,

$$0 < 8\nu(b) < \delta(b), \qquad \mathcal{S}_{b,\delta(b),\nu(b)} \text{ satisfies the gluing bounds at } (b, \delta(b)).$$

Define $g(b)$ by $\mathcal{S}_{b,\delta(b),\nu(b)}$ in the signed collar and by $h_i(b)$ outside the distinguished collars. The set $\{(b,t,x) : |t| < 2\delta(b)\}$ is open and lies in the signed-collar domain. The parameter-smoothness assertions of Lemma 7.1, composed with the smooth functions δ and ν, give smooth coefficients on the normalized interpolation pieces and smoothing bands. On the open overlap

$$\delta(b) + \nu(b) < |t| < 2\delta(b),$$

the smoothed interpolation equals the original collar metric. The two descriptions therefore define a smooth positive-definite vertical metric.

The gluing bounds give the asserted Ricci and bilipschitz inequalities. The metric differs from $h_i(b)$ only for $|t| < \delta(b) + \nu(b) < 2\delta(b) = \ell_b$; hence the support assertion holds. The inequalities already proved give $0 < \ell_b < r_{\mathrm{col}}(b)$ and $2\ell_b < w(b)$. On B_0 one has $\chi_\delta = \chi_\nu = 1$, so (7.24) follows. Substitution in the definition of g gives (7.25). $\square$

Proposition 7.4. *Assume the hypotheses and retain the notation of Proposition 7.2. Suppose in addition that a finite group G acts smoothly on B and that the maps $\gamma_{i,b}$ and $\bar{\gamma}_b$ satisfy* (7.9), (7.10), *and* (7.12). *There are smooth functions $\delta, \nu : B \to (0,\infty)$ and a smooth positive-definite vertical metric $g \in \Gamma^\infty(\mathrm{Sym}^2(\ker d\pi)^*)$ such that, for every $b \in B$,*

$$0 < 2\delta(b) < r_{\mathrm{col}}(b), \qquad 4\delta(b) < w(b), \qquad 0 < 8\nu(b) < \delta(b), \tag{7.26}$$

$$\mathrm{Ric}_{g(b)} > (\lambda(b) + \eta(b))g(b), \tag{7.27}$$

$$g(b) = \mathcal{S}_{b,\delta(b),\nu(b)} \quad \text{in the signed-collar chart.} \tag{7.28}$$

Put $\ell_b := 2\delta(b)$. Then $0 < \ell_b < r_{\mathrm{col}}(b)$ and $2\ell_b < w(b)$; moreover, for $i = 1,2$,

$$g(b) = h_i(b) \quad \text{on } M_i \setminus \widehat{E}_i\big(b, [0,\ell_b) \times \Sigma\big), \tag{7.29}$$

and, on $(-\ell_b, \ell_b) \times \Sigma$,

$$L^{-2}(dt^2 + h_\Sigma(b)) \le g(b) \le L^2(dt^2 + h_\Sigma(b)). \tag{7.30}$$

The pair (δ,ν) is downward admissible on B in the sense of (7.8) *and* (a)–(b). *For every $\gamma \in G$ and $b \in B$,*

$$\delta(\gamma b) = \delta(b), \qquad \nu(\gamma b) = \nu(b), \qquad \gamma_b^* g(\gamma b) = g(b). \tag{7.31}$$

Let $B_1 \subset B$ be open and G-invariant, and let $B_0 \subset B_1$ be closed in B and G-invariant. Suppose that the smooth downward-admissible pair $(\delta_{\mathrm{pr}}, \nu_{\mathrm{pr}})$ on B_1 satisfies

$$\delta_{\mathrm{pr}}(\gamma b) = \delta_{\mathrm{pr}}(b), \qquad \nu_{\mathrm{pr}}(\gamma b) = \nu_{\mathrm{pr}}(b) \qquad (\gamma \in G,\ b \in B_1). \tag{7.32}$$

Under these additional hypotheses, the preceding choices may be made so that

$$(\delta,\nu) = (\delta_{\mathrm{pr}}, \nu_{\mathrm{pr}}) \quad \text{on } B_0, \tag{7.33}$$

$$g(b) = \mathcal{S}_{b,\delta_{\mathrm{pr}}(b),\nu_{\mathrm{pr}}(b)} \qquad (b \in B_0). \tag{7.34}$$

Proof. Fix $\gamma \in G$. Equations (7.12), (7.13), and (7.14) give the signed-collar map $\Gamma_{\gamma,b}(t,x) := (\varepsilon_\gamma t, \bar{\gamma}_b(x))$ and the tangential identity

$$\bar{\gamma}_b^* H_{\gamma b}(\varepsilon_\gamma t) = H_b(t). \tag{7.35}$$

If $0 < r < r_{\mathrm{col}}(b)$, choose $r' \in (r, r_{\mathrm{col}}(b))$ for which both normal exponential maps over b are embeddings on $[0,r'] \times \Sigma$. Equation (7.13) and the diffeomorphisms $\gamma_{i,b}$ and $\bar{\gamma}_b$ imply that both normal exponential maps over γb are embeddings on $[0,r] \times \Sigma$. Thus $r_{\mathrm{col}}(\gamma b) \ge r_{\mathrm{col}}(b)$. Applying this inequality to γ^{-1} at γb yields

$$r_{\mathrm{col}}(\gamma b) = r_{\mathrm{col}}(b). \tag{7.36}$$

Choose a smooth function $\rho_0 : B \to (0,\infty)$ satisfying $0 < \rho_0 < r_{\mathrm{col}}$, whose existence is part of Lemma 7.1, and set

$$\rho(b) := \frac{1}{|G|} \sum_{\tau \in G} \rho_0(\tau^{-1} b). \tag{7.37}$$

The function ρ is smooth and positive. Reindexing the finite sum shows that $\rho(\gamma b) = \rho(b)$. By (7.36), every summand in (7.37) is smaller than $r_{\mathrm{col}}(b)$; hence $0 < \rho(b) < r_{\mathrm{col}}(b)$.

Fix $0 < 2d < \rho(b)$. Put

$$J_{\gamma,b,d}(t) := \bar{\gamma}_b^* \mathscr{H}_{\gamma b,d}(\varepsilon_\gamma t).$$

Differentiating (7.35) gives

$$\varepsilon_\gamma \bar{\gamma}_b^* H'_{\gamma b}(\varepsilon_\gamma t) = H'_b(t).$$

The endpoint identities (7.6) therefore give

$$\partial_t^j J_{\gamma,b,d}(\pm d) = \partial_t^j H_b(\pm d) \qquad (j = 0, 1).$$

The difference between $J_{\gamma,b,d}$ and $\mathscr{H}_{b,d}$ is a tensor-valued polynomial of degree at most three with a double zero at each of $-d$ and d. It therefore vanishes identically. Consequently,

$$\Gamma_{\gamma,b}^* \mathcal{I}_{\gamma b,d} = \mathcal{I}_{b,d}. \tag{7.38}$$

Equations (7.1), (7.10), and (7.12) give

$$\bar{\gamma}_b^* h_\Sigma(\gamma b) = h_\Sigma(b), \qquad \Gamma_{\gamma,b}^* P_{\gamma b} = P_b. \tag{7.39}$$

For every smooth metric q one has $\mathrm{Ric}_{\Gamma_{\gamma,b}^* q} = \Gamma_{\gamma,b}^* \,\mathrm{Ric}_q$. Together with $\lambda(\gamma b) = \lambda(b)$, $\eta(\gamma b) = \eta(b)$, $\rho(\gamma b) = \rho(b)$, (7.38), and (7.39), pullback carries the gluing bounds at $(\gamma b, d)$ to the gluing bounds at (b, d). Applying the same implication to γ^{-1} gives $\mathscr{D}_{\gamma b} = \mathscr{D}_b$ and $d_*(\gamma b) = d_*(b)$.

Let $A_b(t)$ be a tangential tensor family satisfying $A_b(t) = \bar{\gamma}_b^* A_{\gamma b}(\varepsilon_\gamma t)$. Linearity and continuity of pullback on smooth symmetric tensors allow it to pass through the convolution integral. The change of variable $s = \varepsilon_\gamma u$, together with evenness of ϱ, gives

$$\bar{\gamma}_b^* \left(\int_{\mathbb{R}} \varrho_{\epsilon_\nu}(s) A_{\gamma b}(\varepsilon_\gamma t - s)\, ds \right) = \int_{\mathbb{R}} \varrho_{\epsilon_\nu}(u) A_b(t-u)\, du.$$

Evenness of χ gives

$$\chi\left(\frac{\varepsilon_\gamma (t-\tau)}{\nu} \right) = \chi\left(\frac{t-\tau}{\nu} \right).$$

Substitution in (7.7) yields, for $\tau \in \{-d, d\}$,

$$\bar{\gamma}_b^* \big(\mathfrak{M}_{\varepsilon_\gamma \tau,\nu} A_{\gamma b} \big)(\varepsilon_\gamma t) = \big(\mathfrak{M}_{\tau,\nu} A_b \big)(t). \tag{7.40}$$

If $8\nu < d$, the two enlarged smoothing supports have radius less than 2ν and centres separated by $2d$; hence they are disjoint. Each localized operator is the identity on every tensor value sampled by the other, so the operators commute. Equations (7.38) and (7.40) imply

$$\Gamma_{\gamma,b}^* \mathcal{S}_{\gamma b,d,\nu} = \mathcal{S}_{b,d,\nu}. \tag{7.41}$$

Suppose that $d : B \to (0,\infty)$ is smooth, $d(\gamma b) = d(b)$, and $\mathcal{I}_{b,d(b)}$ satisfies the gluing bounds for every b. Equations (7.41) and (7.39), together with the equalities for λ and η, show that a smoothing width belongs to $\mathscr{E}_{\gamma b}$ if and only if it belongs to $\mathscr{E}_b$. Hence

$$\mathscr{E}_{\gamma b} = \mathscr{E}_b, \qquad e_*(\gamma b) = e_*(b). \tag{7.42}$$

Let $F : B \to (0,\infty]$ be positive, lower semicontinuous, and G-invariant. If f is smooth and $0 < f < F$, define

$$\mathcal{A}_G f(b) := \frac{1}{|G|} \sum_{\tau \in G} f(\tau^{-1} b). \tag{7.43}$$

Reindexing proves $\mathcal{A}_G f(\gamma b) = \mathcal{A}_G f(b)$. Moreover, $f(\tau^{-1} b) < F(\tau^{-1} b) = F(b)$ for every τ, so $0 < \mathcal{A}_G f < F$.

By Lemma 7.1, d_* is positive and lower semicontinuous. Equation (7.12) and the invariance of d_* show that $F_1 := \min\{d_*, w/4\}$ is positive, lower semicontinuous, and G-invariant. Apply (7.15) to F_1 and average the obtained function by (7.43); call it δ_0. Then

$$0 < \delta_0 < d_*, \qquad 4\delta_0 < w, \qquad \delta_0(\gamma b) = \delta_0(b). \tag{7.44}$$

For each b, choose $d_b \in \mathscr{D}_b$ with $\delta_0(b) < d_b$. If $0 < \widetilde{\delta} \le \delta_0(b)$, then $\widetilde{\delta} < d_b$; the definition of $\mathscr{D}_b$ gives $2\widetilde{\delta} < \rho(b) < r_{\mathrm{col}}(b)$ and shows that $\mathcal{I}_{b,\widetilde{\delta}}$ satisfies the gluing bounds at $(b, \widetilde{\delta})$. Apply Lemma 7.1 to δ_0. Equation (7.42) gives $e_*(\gamma b) = e_*(b)$ for every $\gamma \in G$ and $b \in B$. By Lemma 7.1, e_* is positive and lower semicontinuous. Equation (7.42) and the invariance of δ_0 show that $F_2 := \min\{e_*, \delta_0/8\}$ is positive, lower semicontinuous, and G-invariant. Apply (7.15) and (7.43) to F_2; call this function ν_0. It satisfies

$$0 < \nu_0 < e_*, \qquad 8\nu_0 < \delta_0, \qquad \nu_0(\gamma b) = \nu_0(b). \tag{7.45}$$

For each b, choose $e_b \in \mathscr{E}_b$ with $\nu_0(b) < e_b$. If $0 < \widetilde{\nu} \le \nu_0(b)$, then $\widetilde{\nu} < e_b < \delta_0(b)/8$; the definition of $\mathscr{E}_b$ shows that $\mathcal{S}_{b,\delta_0(b),\widetilde{\nu}}$ is a smooth metric satisfying the gluing bounds at $(b, \delta_0(b))$. Hence (δ_0, ν_0) is downward admissible on B. Define $g_0(b)$ by $\mathcal{S}_{b,\delta_0(b),\nu_0(b)}$ in the signed collar and by $h_i(b)$ on the complement of the modified collar in M_i. The downward interpolation property gives $0 < 2\delta_0(b) < \rho(b) < r_{\mathrm{col}}(b)$. Together with (7.44) and (7.45), this proves (7.26); for $\ell_b = 2\delta_0(b)$ it also gives $0 < \ell_b < r_{\mathrm{col}}(b)$ and $2\ell_b < w(b)$. The definition of g_0 gives (7.28). Since $\nu_0(b) < e_b$ and $e_b \in \mathscr{E}_b$, the metric $\mathcal{S}_{b,\delta_0(b),\nu_0(b)}$ satisfies the gluing bounds at $(b, \delta_0(b))$. Its Ricci inequality gives (7.27), and its product comparison gives (7.30). The interpolation equals $h^{\mathrm{gl}}(b)$ for $|t| \ge \delta_0(b)$, and the smoothing changes it only where $|t \mp \delta_0(b)| < \nu_0(b)$. Hence $g_0(b) = h_i(b)$ on each side for $|t| \ge \delta_0(b) + \nu_0(b)$; since $\delta_0(b) + \nu_0(b) < 2\delta_0(b) = \ell_b$, this proves (7.29). Smooth dependence on b follows from the last assertion of Lemma 7.1 in the fixed signed-collar charts and from equality with $h_i(b)$ on the open overlap $\delta_0(b) + \nu_0(b) < |t| < 2\delta_0(b)$. On the collar, (7.41) gives $\gamma_b^* g_0(\gamma b) = g_0(b)$; off the collar this equality is (7.12). Together with $\delta_0(\gamma b) = \delta_0(b)$ and $\nu_0(\gamma b) = \nu_0(b)$, these identities prove that (δ_0, ν_0, g_0) satisfies (7.31). Thus $(\delta, \nu, g) := (\delta_0, \nu_0, g_0)$ proves the first assertion.

For the relative assertion, assume the hypotheses following (7.32). Since the smooth paracompact manifold B is normal and $B_0 \subset B_1$, there is an open set $U_0 \subset B$ with $B_0 \subset U_0$ and $\overline{U_0} \subset B_1$. Put $U := \bigcap_{\gamma\in G} \gamma U_0$. The finite intersection is open. Since $\gamma B_0 = B_0$ for every $\gamma \in G$, one has $B_0 \subset U$, while $\overline{U} \subset \bigcap_{\gamma\in G} \gamma \overline{U_0} \subset B_1$. Reindexing the intersection gives $\gamma U = U$. A smooth partition of unity subordinate to $\{B_1, B \setminus \overline{U}\}$ gives $\chi_\delta^0 : B \to [0,1]$ with $\operatorname{supp} \chi_\delta^0 \subset B_1$ and $\chi_\delta^0|_{\overline{U}} = 1$. A smooth partition of unity subordinate to $\{U, B \setminus B_0\}$ gives $\chi_\nu^0 : B \to [0,1]$ with $\operatorname{supp} \chi_\nu^0 \subset U$ and $\chi_\nu^0|_{B_0} = 1$. Set

$$\chi_\delta(b) := \frac{1}{|G|} \sum_{\gamma\in G} \chi_\delta^0(\gamma^{-1} b), \qquad \chi_\nu(b) := \frac{1}{|G|} \sum_{\gamma\in G} \chi_\nu^0(\gamma^{-1} b).$$

Reindexing each finite sum gives $\chi_\delta(\gamma b) = \chi_\delta(b)$ and $\chi_\nu(\gamma b) = \chi_\nu(b)$. The identities $\gamma B_1 = B_1$, $\gamma U = U$, $\gamma \overline{U} = \overline{U}$, and $\gamma B_0 = B_0$ give

$$\operatorname{supp} \chi_\delta \subset B_1, \quad \chi_\delta|_{\overline{U}} = 1, \quad \operatorname{supp} \chi_\nu \subset U, \quad \chi_\nu|_{B_0} = 1. \tag{7.46}$$

The function $\chi_\delta \delta_{\mathrm{pr}}$ on B_1, extended by zero to B, is smooth because $\operatorname{supp} \chi_\delta \subset B_1$. Define

$$\delta := \chi_\delta \delta_{\mathrm{pr}} + (1 - \chi_\delta)\delta_0. \tag{7.47}$$

It is smooth and positive, satisfies $\delta(\gamma b) = \delta(b)$, and equals δ_{pr} on $\overline{U}$. Fix $b \in B$ and $0 < \widetilde{\delta} \le \delta(b)$. If $b \in B_1$ and $\delta_{\mathrm{pr}}(b) \ge \delta_0(b)$, then $\widetilde{\delta} \le \delta_{\mathrm{pr}}(b)$. Downward interpolation admissibility of $(\delta_{\mathrm{pr}}, \nu_{\mathrm{pr}})$ gives $2\widetilde{\delta} < r_{\mathrm{col}}(b)$ and shows that $\mathcal{I}_{b,\widetilde{\delta}}$ satisfies the gluing bounds at $(b, \widetilde{\delta})$. If $b \in B_1$ and $\delta_0(b) > \delta_{\mathrm{pr}}(b)$, choose $d_b \in \mathscr{D}_b$ with $\delta_0(b) < d_b$. Then $\widetilde{\delta} \le \delta(b) \le \delta_0(b) < d_b$, so the definition of $\mathscr{D}_b$ gives $2\widetilde{\delta} < \rho(b) < r_{\mathrm{col}}(b)$ and the gluing bounds for $\mathcal{I}_{b,\widetilde{\delta}}$ at $(b, \widetilde{\delta})$. If $b \notin B_1$, then $\delta(b) = \delta_0(b)$; choosing $d_b \in \mathscr{D}_b$ with $\delta_0(b) < d_b$ gives $2\widetilde{\delta} < \rho(b) < r_{\mathrm{col}}(b)$ and the gluing bounds for $\mathcal{I}_{b,\widetilde{\delta}}$ at $(b, \widetilde{\delta})$.

On B_1, the function δ is a convex combination of δ_{pr} and δ_0, and both satisfy $4\delta < w$; on $B \setminus B_1$ one has $\delta = \delta_0$. Therefore

$$2\delta < r_{\mathrm{col}}, \qquad 4\delta < w. \tag{7.48}$$

Taking $\widetilde{\delta} = \delta(b)$ in the preceding cases shows that $\mathcal{I}_{b,\delta(b)}$ satisfies the gluing bounds at $(b, \delta(b))$ for every b.

Apply Lemma 7.1 to this δ. Equation (7.42) gives $e_*(\gamma b) = e_*(b)$ for its smoothing threshold. Apply (7.15) and (7.43) to $\min\{e_*, \delta/8\}$ and call this function ν_{ref}. It satisfies

$$0 < \nu_{\mathrm{ref}} < e_*, \qquad 8\nu_{\mathrm{ref}} < \delta, \qquad \nu_{\mathrm{ref}}(\gamma b) = \nu_{\mathrm{ref}}(b).$$

For each b, choose $e_b \in \mathscr{E}_b$ with $\nu_{\mathrm{ref}}(b) < e_b$. The function $\chi_\nu \nu_{\mathrm{pr}}$ on B_1, extended by zero to B, is smooth. Define

$$\nu := \chi_\nu \nu_{\mathrm{pr}} + (1 - \chi_\nu)\nu_{\mathrm{ref}}. \tag{7.49}$$

This function is smooth and positive and satisfies $\nu(\gamma b) = \nu(b)$. On $\operatorname{supp}\chi_\nu$ one has $\delta = \delta_{\mathrm{pr}}$ by (7.46). Fix $b \in B$ and $0 < \widetilde{\nu} \le \nu(b)$. If $b \in \operatorname{supp}\chi_\nu$ and $\nu_{\mathrm{pr}}(b) \ge \nu_{\mathrm{ref}}(b)$, then $\widetilde{\nu} \le \nu_{\mathrm{pr}}(b)$, so downward smoothing admissibility gives the gluing bounds and $8\widetilde{\nu} < \delta(b)$. If $b \in \operatorname{supp}\chi_\nu$ and $\nu_{\mathrm{ref}}(b) > \nu_{\mathrm{pr}}(b)$, then $\widetilde{\nu} \le \nu(b) \le \nu_{\mathrm{ref}}(b) < e_b$. Since $e_b \in \mathscr{E}_b$, one has $8\widetilde{\nu} < \delta(b)$, and $\mathcal{S}_{b,\delta(b),\widetilde{\nu}}$ is a smooth metric satisfying the gluing bounds at $(b, \delta(b))$. If $b \notin \operatorname{supp}\chi_\nu$, then $\nu(b) = \nu_{\mathrm{ref}}(b) < e_b$. Since $e_b \in \mathscr{E}_b$, one has $8\widetilde{\nu} < \delta(b)$, and $\mathcal{S}_{b,\delta(b),\widetilde{\nu}}$ is a smooth metric satisfying the gluing bounds at $(b, \delta(b))$. Hence (δ, ν) is downward admissible on B and

$$8\nu < \delta, \qquad \mathcal{S}_{b,\delta(b),\nu(b)} \text{ satisfies the gluing bounds for every } b. \tag{7.50}$$

Define $g(b)$ by $\mathcal{S}_{b,\delta(b),\nu(b)}$ in the signed collar and by $h_i(b)$ off the modified collar in M_i. The last assertion of Lemma 7.1 and equality with $h_i(b)$ on the open collar overlap give a smooth vertical metric. Equations (7.48) and (7.50) give (7.26); with $\ell_b = 2\delta(b)$ they also give $0 < \ell_b < r_{\mathrm{col}}(b)$ and $2\ell_b < w(b)$. The definition of g gives (7.28). The gluing bounds in (7.50) give (7.27) and (7.30). The interpolation equals $h^{\mathrm{gl}}(b)$ for $|t| \ge \delta(b)$, and the smoothing changes it only where $|t \mp \delta(b)| < \nu(b)$. Hence $g(b) = h_i(b)$ on each side for $|t| \ge \delta(b) + \nu(b)$; the inequality $\delta(b) + \nu(b) < 2\delta(b) = \ell_b$ proves (7.29). On B_0, equations (7.46), (7.47), and (7.49) give (7.33); substitution in the definition of g gives (7.34). Finally, (7.41) on the signed collar and (7.12) off it give $\gamma_b^* g(\gamma b) = g(b)$. Together with $\delta(\gamma b) = \delta(b)$ and $\nu(\gamma b) = \nu(b)$, this proves (7.31). □

Smooth identifications with a fixed manifold. The vertical metrics above can be regarded as smooth metric families on a fixed glued manifold. We record the identifications, including the relative and equivariant choices needed below. Choose auxiliary smooth metrics k_i on the fixed pieces M_i, and use their signed normal collars to give the quotient M a fixed smooth structure, denoted by M^{ref}. For $b \in B$ and $0 \le \tau \le 1$, set

$$k_{i,b,\tau} := (1 - \tau)k_i + \tau h_i(b).$$

These tensors are positive definite. No curvature condition is imposed on this auxiliary path. Let $E_{i,b,\tau}(r, x)$ be its inward unit-speed normal collar, with $E_{i,b,\tau}(0, x) = \beta_i(x)$.

For each side, prescribe an open set $\mathcal{O}_i \subset B \times M_i$ containing $B \times \Sigma_i$, disjoint from the unglued boundary components and from any prescribed protected regions away from the seam. Its fibre is denoted by $\mathcal{O}_i(b)$. There is a smooth function $a : B \to (0, \infty)$ such that, for both sides and every $\tau \in [0, 1]$, the map $E_{i,b,\tau}$ is an embedding on $[0, 4a(b)] \times \Sigma$ and its image is contained in $\mathcal{O}_i(b)$. Indeed, the compactness of $[0, 1] \times \Sigma$ and smooth dependence of the normal exponential maps give such a radius locally in b. Embeddings of a compact collar and containment of its image in the prescribed open set persist under small parameter changes, as in the proof of Lemma 7.1. The supremum $a_*(b)$ of the admissible radii is therefore positive and lower semicontinuous. The admissible radii form a downward interval; hence any smooth positive minorant of a_*, supplied by (7.15), has the required property. No uniform positive radius on B is needed.

Choose $\zeta \in C^\infty([0,\infty),[0,1])$ with $\zeta = 1$ on $[0,1]$ and $\operatorname{supp}\zeta \subset [0,2)$. Define a time-dependent vector field on M_i by

$$V_{i,b,\tau}\big(E_{i,b,\tau}(r,x)\big) = \zeta\left(\frac{r}{a(b)}\right)\partial_\tau E_{i,b,\tau}(r,x), \qquad 0 \le r < 4a(b),$$

and extend it by zero. The collar inverses and the cutoff show that this is smooth in (b,τ), including at the edge of its support. Since $E_{i,b,\tau}(0,x) = \beta_i(x)$ is independent of τ, the vector field vanishes on Σ_i. It vanishes near every unglued boundary component as well. Compactness of M_i gives a flow $F_{i,b,\tau}$ throughout $[0,1]$, with $F_{i,b,0} = \mathrm{id}$, smooth in all parameters. The flow fixes the boundary and is the identity outside $\mathcal{O}_i(b)$. For $0 \le r \le a(b)$, the curve $\tau \mapsto E_{i,b,\tau}(r,x)$ solves this flow equation, so uniqueness gives

$$F_{i,b,\tau}\big(E_{i,b,0}(r,x)\big) = E_{i,b,\tau}(r,x).$$

The two maps $F_{i,b,1}$ consequently descend to a fibre-preserving bijection

$$\mathcal{T} : B \times M^{\mathrm{ref}} \longrightarrow \mathcal{M}.$$

In the reference and target signed-collar charts its expression is $(b,t,x) \mapsto (b,t,x)$ for $|t| < a(b)$. Away from the seam it is the smooth side flow. The inverse has the same local descriptions, with the inverse side flows. Thus $\mathcal{T}$ is a diffeomorphism, and the vertical pullback $\mathcal{T}_b^* g(b)$ is a smooth metric family on M^{ref}.

The choices can retain a previously fixed reference family. More precisely, suppose that on a closed set $B_0 \subset B$ the input metrics and prescribed collar neighbourhoods agree with those of a reference family defined on an open neighbourhood B_1 of B_0. Use the same k_i and ζ for both constructions, and let a_{ref} be the reference radius, chosen strictly below its admissibility threshold. Then $a_{\mathrm{ref}}(b) < a_*(b)$ for $b \in B_0$. Lower semicontinuity of a_* and continuity of a_{ref} give an open neighbourhood $U \subset B_1$ of B_0 on which this strict inequality holds. Choose a smooth function $\theta : B \to [0,1]$ with $\theta = 1$ on B_0 and $\operatorname{supp}\theta \subset U$. If a_0 is any smooth positive minorant of a_*, set

$$a = (1-\theta)a_0 + \theta a_{\mathrm{ref}},$$

where the last product is extended by zero outside B_1. On U both radii are strictly below a_*, and outside U one has $a = a_0$. Consequently $0 < a < a_*$ everywhere and $a = a_{\mathrm{ref}}$ on B_0. On B_0 the auxiliary metric paths, vector fields, and initial conditions are identical; uniqueness gives $F_{i,b,\tau} = F^{\mathrm{ref}}_{i,b,\tau}$ there. Thus equality of the glued vertical metrics on B_0 becomes literal equality of their pullbacks on the fixed manifold. For a family independent of one parameter, make all choices on the reduced parameter space and pull them back.

For equivariance, suppose that the fibre maps in Proposition 7.4 are independent of b, as are all the copy and reflection maps used below; write γ_i and $\bar\gamma$ for these fixed maps. Average the auxiliary metric on $M_1 \sqcup M_2$ over this finite group, and take the prescribed neighbourhoods to be invariant. The path $k_{i,b,\tau}$ is then equivariant. Naturality of the normal exponential map gives

$$\gamma_i E_{i,b,\tau}(r,x) = E_{\sigma_\gamma(i),\gamma b,\tau}(r,\bar\gamma x).$$

In particular, a_* is invariant. Averaging a smooth positive minorant of a_* gives an invariant one. In the relative case take B_0, B_1 and the reference data, including a_{ref}, invariant; average a_0 and θ before forming a. This preserves the prescribed radius. The displayed normal-collar identity and the common radial cutoff imply

$$(\gamma_i)_* V_{i,b,\tau} = V_{\sigma_\gamma(i),\gamma b,\tau}.$$

Uniqueness of the flow proves equivariance of $F_{i,b,\tau}$ and hence of $\mathcal{T}$. If the group interchanges the two sides, both signed collar structures carry the same map $(t,x) \mapsto (-t,\bar\gamma x)$, so the conclusion also holds across the seam. This assertion concerns fixed fibre maps; no parameter-independent action is asserted for the general fibre maps of Proposition 7.4.

For a selected gluing width $\delta(b)$, the sets $\mathcal{O}_i(b)$ may be chosen inside the side normal collars of radius $\delta(b)/2$. Thus the identification introduces no change outside the original gluing collar. At successive seams we apply the construction in order, retaining the fixed side manifolds and auxiliary data already chosen for the reference family. In the applications below, fixed-manifold identifications include these maps; metrics, geometric regions, and collar coordinates are transported by pullback. On the protected regions the maps are the identity. Curvature inequalities, intrinsic diameters, volumes, and boundary tensors are preserved by these fibrewise isometries.

7.2. Caps over compact parameter sets.

Theorem 7.5. *Let $I \subset [0,\infty)$ be a nonempty compact interval. Let h_s, $s \in I$, be a smooth family of metrics on S^{n-1} satisfying $\mathrm{Ric}_{h_s} \geq (n-2)h_s$. Let $\widehat{g}_{s,u}$, $(s,u) \in I \times [-2,2]$, be a smooth family of metrics. Assume that the family is constant in the u-variable on relative open neighbourhoods of $I \times \{-2\}$ and $I \times \{2\}$ and that*

$$\widehat{g}_{s,-2} = \tau_s^2 g_{S^{n-1}}, \qquad \widehat{g}_{s,2} = h_s,$$
$$\mathrm{Ric}_{\widehat{g}_{s,u}} \geq (n-2)\widehat{g}_{s,u}, \qquad \mathrm{Vol}(S^{n-1}, \widehat{g}_{s,u}) = \mathrm{Vol}(S^{n-1}, h_s),$$
$$\tau_s := \left(\frac{\mathrm{Vol}(S^{n-1}, h_s)}{\sigma_{n-1}}\right)^{1/(n-1)}, \qquad \sup_{s\in I} \tau_s < \frac{1}{4}\bar{\kappa}_Q.$$

There are constants $\varepsilon_I > 0$ and $C_I, D_I, V_I > 0$, and a smooth family of metrics $(s,\varepsilon) \mapsto K_{s,\varepsilon}$ on P_Q, defined for $(s,\varepsilon) \in I \times (0,\varepsilon_I)$, such that

$$\beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}) = \sin^2 \varepsilon\, h_s, \tag{7.51}$$
$$\beta_Q^* \mathrm{II}_{\partial P_Q, K_{s,\varepsilon}} > \cot \varepsilon\, \beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}), \tag{7.52}$$
$$\mathrm{Ric}_{K_{s,\varepsilon}} \geq (n-1-C_I\varepsilon^2)K_{s,\varepsilon}, \tag{7.53}$$
$$\mathrm{diam}(P_Q, K_{s,\varepsilon}) \leq D_I\varepsilon, \tag{7.54}$$
$$\mathrm{Vol}(P_Q, K_{s,\varepsilon}) \leq V_I\varepsilon^n. \tag{7.55}$$

If, for some $\tau > 0$, $\widehat{g}_{s,u} \equiv \tau^2 g_{S^{n-1}}$ on $I \times [-2,2]$, then there are a smooth family K_ε on P_Q and, for $r = 1,2$, smooth functions $(\delta_r, \nu_r) : (0,\varepsilon_I) \to (0,\infty)^2$ such that $K_{s,\varepsilon} = K_\varepsilon$ for every $s \in I$, and the interpolation and mollification at seam r use the widths $\delta_r(\varepsilon)$ and $\nu_r(\varepsilon)$.

Let $P_Q^\pm$ be oppositely oriented copies of P_Q, with their core metrics denoted by $k_Q^\pm$. Let $\iota : P_Q^+ \to P_Q^-$ be the copy diffeomorphism, and let $\beta_Q^\pm : S^{n-1} \to \partial P_Q^\pm$ satisfy $\iota \circ \beta_Q^+ = \beta_Q^-$. Suppose two input pairs satisfy the preceding hypotheses and, under the identity of their model spheres,

$$h_s^+ = h_s^-, \qquad \widehat{g}^+_{s,u} = \widehat{g}^-_{s,u}, \qquad \iota^* k_Q^- = k_Q^+.$$

After replacing ε_I by a common smaller value, the cap families and the smooth seam-width functions $(\delta_r^\pm, \nu_r^\pm) : I \times (0,\varepsilon_I) \to (0,\infty)^2$, $r = 1,2$, may be chosen so that

$$\iota^* K^-_{s,\varepsilon} = K^+_{s,\varepsilon}, \qquad \delta_r^+ = \delta_r^-, \qquad \nu_r^+ = \nu_r^- \quad (r = 1,2).$$

Proof. Orient S^{n-1} and put $\omega_{s,u} = dv_{\widehat{g}_{s,u}}$. Fix a background metric, and denote by $\mathcal{H}$, $\mathcal{G}$, and d^* its harmonic projection, Green operator, and codifferential. Set $\alpha_{s,u} := \partial_u \omega_{s,u}$. The volume hypothesis gives

$$\int_{S^{n-1}} \alpha_{s,u} = \partial_u \mathrm{Vol}(S^{n-1}, \widehat{g}_{s,u}) = 0.$$

Since S^{n-1} is connected, its harmonic top-degree forms are the constant multiples of the background volume form; hence $\mathcal{H}\alpha_{s,u} = 0$. The Hodge decomposition is

$$\alpha_{s,u} = \mathcal{H}\alpha_{s,u} + dd^*\mathcal{G}\alpha_{s,u} + d^*d\mathcal{G}\alpha_{s,u}.$$

The form $\mathcal{G}\alpha_{s,u}$ has top degree, so $d\mathcal{G}\alpha_{s,u} = 0$. Therefore

$$\alpha_{s,u} = dd^*\mathcal{G}\alpha_{s,u}. \tag{7.56}$$

Put $\eta_{s,u} := d^*\mathcal{G}\alpha_{s,u}$. Contraction with the positive form $\omega_{s,u}$ is a vector-bundle isomorphism from vector fields to $(n-2)$-forms; it determines a unique vector field $X_{s,u}$ by $\iota_{X_{s,u}}\omega_{s,u} = -\eta_{s,u}$. The operators above are continuous linear operators on smooth forms, and $(s,u) \mapsto \widehat{g}_{s,u}$ is smooth; hence $X_{s,u}$ is smooth in (s,u). It vanishes wherever $u \mapsto \widehat{g}_{s,u}$ is constant. Let

$$\partial_u \phi_{s,u} = X_{s,u} \circ \phi_{s,u}, \qquad \phi_{s,-2} = \mathrm{id}\,.$$

The flow exists for $u \in [-2,2]$ because S^{n-1} is compact, and smooth parameter-dependent ODE theory gives smooth dependence on (s,u). By (7.56) and Cartan's formula,

$$\frac{d}{du}\left(\phi_{s,u}^*\omega_{s,u}\right) = \phi_{s,u}^*(\alpha_{s,u} - d\eta_{s,u}) = 0.$$

Thus, setting $g_{s,u} := \phi_{s,u}^*\widehat{g}_{s,u}$, one has

$$dv_{g_{s,u}} = dv_{g_{s,-2}}, \qquad \mathrm{Ric}_{g_{s,u}} \geq (n-2)g_{s,u}.$$

The map $u \mapsto g_{s,u}$ is constant on neighbourhoods of the two endpoint slices. Extension by its endpoint values therefore gives a smooth family on $I \times \mathbb{R}$. Each norm function in (6.2) is continuous on the compact set $I \times [-2,2] \times S^{n-1}$ and vanishes outside this set. The maximum of the sum of these three functions is finite, which proves (6.2).

Let ε_I initially be the constant ε_0 in Theorem 6.1. Apply that theorem and write $\chi = \chi_\varepsilon$, $\psi = \psi_\varepsilon$, $\kappa = \kappa_\varepsilon$, $r_- := r_{-,\varepsilon}$, and $G_{\kappa,s} = G_{s,\varepsilon}$. For $0 < r \leq r_-$,

$$G_{\kappa,s} = dr^2 + \left(\tau_s \frac{\sin(\kappa r)}{\kappa}\chi(r)\right)^2 g_{S^{n-1}}. \tag{7.57}$$

Choose C_κ such that $\kappa \leq C_\kappa \varepsilon$, and choose $C_0 > C_n(C_\kappa^2 + 1)$. Decrease ε_I so that $\varepsilon_I < 1$ and $n - 1 - C_n\varepsilon^4 - C_n\kappa^2 > 0$ for $0 < \varepsilon < \varepsilon_I$, and put

$$R_{s,\varepsilon} := (n-1-C_n\varepsilon^4 - C_n\kappa^2)\kappa^2, \qquad \lambda_{s,\varepsilon}^{\mathrm{cap}} := (n-1-2C_0\varepsilon^2)\kappa^2,$$
$$m_{s,\varepsilon} := R_{s,\varepsilon} - \lambda_{s,\varepsilon}^{\mathrm{cap}}.$$

These definitions give

$$\begin{aligned} m_{s,\varepsilon} - C_0\varepsilon^2\kappa^2 &= \left(C_0\varepsilon^2 - C_n\varepsilon^4 - C_n\kappa^2\right)\kappa^2 \\ &\geq \left(C_0 - C_n - C_nC_\kappa^2\right)\varepsilon^2\kappa^2 > 0. \end{aligned} \tag{7.58}$$

Consequently $m_{s,\varepsilon} \geq C_0\varepsilon^2\kappa^2 > 0$. For the next construction, set $d := \bar{\kappa}_Q/2$ and $e := 3\bar{\kappa}_Q/4$. For $0 < r < r_-$, define

$$b_{s,\varepsilon}(r) := \tau_s \frac{\sin(\kappa r)}{\kappa}\chi(r),$$
$$c_{s,\varepsilon}(r) := \tau_s\left(\cos(\kappa r)\chi(r) + \frac{\sin(\kappa r)}{\kappa}\chi'(r)\right).$$

These functions extend to $r = 0$ by $b_{s,\varepsilon}(0) = 0$ and $c_{s,\varepsilon}(0) = \tau_s$. Equation (6.52) gives $\chi_\varepsilon = \chi_{\varepsilon^4}$. Apply (6.28) and (6.29) with $\theta = \varepsilon^4$, and write $T_\varepsilon(r) := T_{\varepsilon^4}(r)$ and $\ell_\varepsilon(r) := \ell_{\varepsilon^4}(r)$. These definitions give

$$\chi_\varepsilon(r) = 1 - \frac{\varepsilon^4}{8\ell_\varepsilon(r)}, \qquad r\chi_\varepsilon'(r) = -\frac{\varepsilon^4}{8T_\varepsilon(r)\ell_\varepsilon(r)^2}.$$

For every compact $J \Subset (0, \varepsilon_I)$, both error terms tend to zero as $r \downarrow 0$, uniformly for $\varepsilon \in J$. Hence the displayed extensions of $b_{s,\varepsilon}$ and $c_{s,\varepsilon}$ are jointly continuous in (s, ε, r). For $z = (s, \varepsilon)$, let $\rho(z)$ be the supremum of the numbers $\rho \in (0, r_-(z))$ such that every $r \in [0, \rho]$ satisfies

$$c_{s,\varepsilon}(r) < d,$$
$$R_{s,\varepsilon} b_{s,\varepsilon}(r)^2 < (n-1)(1-d^2),$$
$$R_{s,\varepsilon}(1-e^2)b_{s,\varepsilon}(r)^2 < \mu_Q(1-d^2).$$

The identities $b_{s,\varepsilon}(0) = 0$ and $c_{s,\varepsilon}(0) = \tau_s$, together with $\sup_I \tau_s < \bar{\kappa}_Q/4 < d$, give $\rho(z) > 0$. The set of radii occurring in the definition is downward closed. If $0 < \widehat{r}_0 < \rho(z_0)$, choose radii $\widehat{r}_1, \widehat{r}_2$ such that

$$\widehat{r}_0 < \widehat{r}_2 < \widehat{r}_1 < \rho(z_0).$$

Then $\widehat{r}_1 < r_-(z_0)$ and the three strict inequalities hold on $[0, \widehat{r}_1]$. Smoothness of r_- and compactness of $[0, \widehat{r}_2]$ give a neighbourhood of z_0 on which $\widehat{r}_2 < r_-(z)$ and on which the three inequalities retain positive minimum margins on $[0, \widehat{r}_2]$. Hence $\widehat{r}_2$ is admissible for every parameter in this neighbourhood, so $\rho(z) \geq \widehat{r}_2 > \widehat{r}_0$. Thus $\{z : \rho(z) > \widehat{r}_0\}$ is open, and ρ is lower semicontinuous. The smooth-minorant construction (7.15) gives a smooth function r_c satisfying $0 < r_c(s, \varepsilon) < \rho(s, \varepsilon)$. Set $b := b_{s,\varepsilon}(r_c)$ and $c := c_{s,\varepsilon}(r_c)$. Since $0 < \kappa r_c < \pi$ and $0 < \chi \leq 1$,

$$0 < b \leq \tau_s r_c \leq \tau_s. \tag{7.59}$$

The choice of r_c gives

$$c < d, \qquad (n-1)\frac{1-d^2}{b^2} > R_{s,\varepsilon}, \qquad \frac{\mu_Q(1-d^2)}{(1-e^2)b^2} > R_{s,\varepsilon}. \tag{7.60}$$

Define $\Lambda^{\mathrm{rd}} := \sqrt{1-d^2}/b$ and $T_d := \arccos(d)/\Lambda^{\mathrm{rd}}$. On the round n-ball use

$$dt^2 + (\Lambda^{\mathrm{rd}})^{-2}\sin^2(\Lambda^{\mathrm{rd}} t) g_{S^{n-1}}, \qquad 0 \leq t \leq T_d.$$

Its boundary metric is $b^2 g_{S^{n-1}}$, and its Ricci tensor equals $(n-1)(\Lambda^{\mathrm{rd}})^2 g$. Identify this boundary with the slice $r = r_c$ of (7.57) by the identity of S^{n-1}. The inward normal of the transition cylinder at $r = r_c$ is $+\partial_r$, whereas the inward normal of the round ball at $t = T_d$ is $-\partial_t$. Hence

$$\mathrm{II}_{\mathrm{tr}} = -\frac{c}{b}\, b^2 g_{S^{n-1}}, \qquad \mathrm{II}_{\mathrm{rd}} = \frac{d}{b}\, b^2 g_{S^{n-1}},$$

and therefore

$$\mathrm{II}_{\mathrm{tr}} + \mathrm{II}_{\mathrm{rd}} = \frac{d-c}{b}\, b^2 g_{S^{n-1}} > 0. \tag{7.61}$$

Define the inner cutting parameters by $T_e := \arccos(e)/\Lambda^{\mathrm{rd}}$ and $s_c := \sqrt{1-e^2}/\Lambda^{\mathrm{rd}}$. Then $0 < T_e < T_d$. Let $B := I \times (0, \varepsilon_I)$, let $C = [0,1] \times S^{n-1}$, and let D^n be the closed unit ball. Define the radial diffeomorphisms by

$$\begin{aligned} F^{\mathrm{tr}}_{s,\varepsilon}(t, x) &= \big(r_c + (1-r_c)t, x\big), \\ F^{\mathrm{rd}}_{s,\varepsilon}(\rho x) &= \big(T_d \rho, x\big), \qquad 0 \leq \rho \leq 1. \end{aligned}$$

The second formula uses geodesic polar coordinates and sends 0 to the round centre. Both maps are smooth radial diffeomorphisms onto the transition cylinder $[r_c, 1] \times S^{n-1}$ and the round ball $0 \leq t \leq T_d$. Their pullbacks define smooth metric families h_1 and h_2 on the fixed manifolds $M_1 = C$ and $M_2 = D^n$. Take $\Sigma = S^{n-1}$, $\Sigma_1 = \{0\} \times S^{n-1}$, and $\Sigma_2 = \partial D^n$, with $\beta_1(x) = (0, x)$ and $\beta_2(x) = x$. The common boundary metric is $\beta_1^*(h_1|_{\partial M_1}) = \beta_2^*(h_2|_{\partial M_2}) = b^2 g_{S^{n-1}}$. On B, set $\lambda = \lambda^{\mathrm{cap}}_{s,\varepsilon}$, $\eta = m_{s,\varepsilon}/4$, $w = (T_d - T_e)/4$, and $L = 2$. The functions η and w are smooth and positive. Equations (6.7) and (7.60) give

$$\mathrm{Ric}_{h_1} \geq R_{s,\varepsilon} h_1 = \big(\lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}\big) h_1, \qquad \mathrm{Ric}_{h_2} > R_{s,\varepsilon} h_2.$$

Both lower coefficients exceed $\lambda + 2\eta = \lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}/2$. Equation (7.61) is precisely the positive-definite sum of the two inward second fundamental forms under the displayed boundary identification. The support width w is smaller than the available round annulus width $T_d - T_e$. The boundary equality, the two Ricci inequalities, (7.61), and the width inequality satisfy the hypotheses of Proposition 7.2 for the first seam. Denote the resulting smooth glued metric by $G^{(1)}_{s,\varepsilon}$. Its Ricci estimate is

$$\mathrm{Ric}_{G^{(1)}_{s,\varepsilon}} > \left(\lambda^{\mathrm{cap}}_{s,\varepsilon} + \frac{1}{4} m_{s,\varepsilon}\right) G^{(1)}_{s,\varepsilon}. \tag{7.62}$$

The support conclusion of that proposition confines the smoothing on the round side to $T_d - (T_d - T_e)/4 < t \le T_d$. It leaves both the round region $0 \le t \le T_e$ and the outer boundary $r = 1$ of the transition cylinder unchanged. Use the collar-straightening construction at each seam, with support inside its gluing collar. At the first seam keep a neighbourhood of $0 \le t \le T_e$ fixed, and at both seams keep a neighbourhood of the outer boundary $r = 1$ fixed. The resulting smooth identifications with fixed manifolds are used in the following pullbacks.

Delete $0 \le t < T_e$. The affine map

$$F^{\mathrm{ann}}_{s,\varepsilon}(v, x) = \big(T_e + (T_d - T_e)v, x\big), \qquad (v, x) \in [0, 1] \times S^{n-1},$$

identifies the retained round annulus with a fixed cylinder. Together with $F^{\mathrm{tr}}_{s,\varepsilon}$ and the fixed first-seam gluing, it identifies the first glued piece after deletion with one fixed compact manifold with boundary; the pullback metric family on this fixed manifold is smooth. Its inner boundary has metric $s_c^2 g_{S^{n-1}}$. The inward normal of the retained annulus at $t = T_e$ is $+\partial_t$, so $\mathrm{II}_{\mathrm{ann}} = -(e/s_c)s_c^2 g_{S^{n-1}}$. Identify this boundary with ∂P_Q by β_Q. Constant scaling gives

$$\beta_Q^* \mathrm{II}_{\partial P_Q, s_c^2 k_Q} \ge \frac{\kappa_Q}{s_c}\, s_c^2 g_{S^{n-1}}, \qquad \mathrm{Ric}_{s_c^2 k_Q} \ge \frac{\mu_Q}{s_c^2}(s_c^2 k_Q).$$

Since $s_c^2 = (1 - e^2)b^2/(1 - d^2)$, (7.60) gives

$$\frac{\mu_Q}{s_c^2} > R_{s,\varepsilon}. \tag{7.63}$$

It also gives

$$\mathrm{II}_{\mathrm{ann}} + \beta_Q^* \mathrm{II}_{\partial P_Q, s_c^2 k_Q} \ge \frac{\kappa_Q - e}{s_c}\, s_c^2 g_{S^{n-1}} > 0. \tag{7.64}$$

For the second seam, let $M_1^{(2)}$ be the first glued piece after removing $0 \le t < T_e$, and let $M_2^{(2)} = P_Q$. The map $F^{\mathrm{ann}}_{s,\varepsilon}$ above identifies $M_1^{(2)}$ with a fixed compact manifold with boundary. Take the model seam $\Sigma = S^{n-1}$, let $\Sigma_1^{(2)}$ be the inner boundary $t = T_e$ of $M_1^{(2)}$, and let $\Sigma_2^{(2)} = \partial P_Q$; use the identity parametrization on $\Sigma_1^{(2)}$ and β_Q on $\Sigma_2^{(2)}$. The common boundary metric is $s_c^2 g_{S^{n-1}}$. For this seam, choose $\lambda = \lambda^{\mathrm{cap}}_{s,\varepsilon}$, $\eta = m_{s,\varepsilon}/16$, $w = \min\{T_e, T_d - T_e\}/4$, and $L = 2$. The functions η and w are smooth and positive, since T_e and $T_d - T_e$ are fixed positive multiples of b. By (7.62), the metric on $M_1^{(2)}$ has Ricci coefficient greater than $\lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}/4$. By (7.63), the core has Ricci coefficient greater than $R_{s,\varepsilon} = \lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}$. Both coefficients exceed $\lambda + 2\eta = \lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}/8$. Equation (7.64) is the positive-definite sum of the two inward second fundamental forms under the displayed boundary identification. The inequalities $w \le T_e/4$ and $w \le (T_d - T_e)/4$ place the modification inside the chosen collars and separate it from the first-seam support. The boundary equality, the Ricci inequalities, (7.64), and these width inequalities satisfy the hypotheses of Proposition 7.2 for the second seam. Its conclusion gives a smooth metric $\widetilde{K}_{s,\varepsilon}$ satisfying

$$\mathrm{Ric}_{\widetilde{K}_{s,\varepsilon}} > \left(\lambda^{\mathrm{cap}}_{s,\varepsilon} + \frac{1}{16} m_{s,\varepsilon}\right) \widetilde{K}_{s,\varepsilon}.$$

On the round side, the first smoothing support is contained in $\{t > T_d - w_1\}$, where $w_1 = (T_d - T_e)/4$, and the second is contained in $\{t < T_e + w_2\}$, where $w_2 = \min\{T_e, T_d - T_e\}/4$. Since

$$T_e + w_2 \le T_e + \frac{1}{4}(T_d - T_e) < T_d - \frac{1}{4}(T_d - T_e) = T_d - w_1,$$

the supports are disjoint. The support conclusions of the two gluing applications also leave the outer boundary $r = 1$ unchanged. The transition cylinder and the retained round annulus form a compact cylinder attached to ∂P_Q through β_Q. Fix a collar $[0, 2) \times \partial P_Q \hookrightarrow P_Q$, with $\{0\} \times \partial P_Q$ mapped onto the boundary. After identifying the attached cylinder with $[-2, 0] \times \partial P_Q$, choose a smooth increasing diffeomorphism $[-2, 2) \to [0, 2)$ which equals the identity on $[1, 2)$. Together with the identity outside the fixed collar, it gives a diffeomorphism of the glued manifold onto P_Q; choose the cylinder parametrization so that its outer boundary identification is $\beta_Q : S^{n-1} \to \partial P_Q$. The maps $F^{\mathrm{tr}}_{s,\varepsilon}$ and $F^{\mathrm{ann}}_{s,\varepsilon}$ identify the metric pieces with fixed manifolds; the smooth collar identifications just chosen therefore make $(s,\varepsilon) \mapsto \widetilde{K}_{s,\varepsilon}$ a smooth metric family on the fixed manifold P_Q.

Scale by c_ε^2, where

$$c_\varepsilon = \frac{\sin \varepsilon\, \kappa}{\lambda_\varepsilon}, \qquad \frac{\lambda_\varepsilon^2}{\sin^2 \varepsilon} = 1 - \frac{8\varepsilon^4}{\sin^2 \varepsilon}.$$

Since the Ricci tensor is unchanged as a $(0,2)$-tensor under constant scaling,

$$\begin{aligned}&\frac{\lambda^{\mathrm{cap}}_{s,\varepsilon} + m_{s,\varepsilon}/16}{c_\varepsilon^2}\\ &\quad= \left(n - 1 - 2C_0\varepsilon^2 + \frac{1}{16}(2C_0\varepsilon^2 - C_n\varepsilon^4 - C_n\kappa^2)\right)\frac{\lambda_\varepsilon^2}{\sin^2 \varepsilon}.\end{aligned}$$

For $0 < \varepsilon < 1$ and $\kappa^2 \le C_\kappa^2 \varepsilon^2$, the first factor on the right is at least $n - 1 - A_0\varepsilon^2$, where $A_0 := 15C_0/8 + C_n(1 + C_\kappa^2)/16$. Since $\sin \varepsilon \ge \varepsilon/2$,

$$\frac{\lambda_\varepsilon^2}{\sin^2 \varepsilon} = 1 - \frac{8\varepsilon^4}{\sin^2 \varepsilon} \ge 1 - 32\varepsilon^2.$$

After decreasing ε_I so that both factors are positive, their product is at least $n-1-(A_0+32(n-1))\varepsilon^2$. Choose $C_I \ge A_0 + 32(n-1)$. Then

$$\mathrm{Ric}_{c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}} \ge (n - 1 - C_I\varepsilon^2) c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}. \tag{7.65}$$

Because both smoothing supports are disjoint from the outer boundary, the boundary metric and inward second fundamental form after constant scaling satisfy

$$\beta_Q^*(c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}|_{\partial P_Q}) = \sin^2 \varepsilon\, \phi_{s,2}^* h_s, \tag{7.66}$$

$$\beta_Q^* \mathrm{II}_{\partial P_Q, c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}} > \cot \varepsilon\, \sin^2 \varepsilon\, \phi_{s,2}^* h_s. \tag{7.67}$$

For $u \in [-2,2]$, put $\widehat{\phi}_{s,u} := \beta_Q \circ \phi_{s,u} \circ \beta_Q^{-1}$ on ∂P_Q. Choose a fixed collar $[0,1] \times \partial P_Q$, with $t = 0$ at the outer boundary, and a smooth function $\lambda_0 : [0,1] \to [0,1]$ such that $\lambda_0 = 1$ near 0 and $\lambda_0 = 0$ on $[1/2,1]$. On the collar define $F_s(t,x) = (t, \widehat{\phi}^{-1}_{s,-2+4\lambda_0(t)}(x))$, and extend it by the identity. The inverse is obtained by replacing $\widehat{\phi}^{-1}$ by $\widehat{\phi}$, so F_s is a smooth family of diffeomorphisms. It equals the identity near the inner end of the collar, and its boundary restriction is $\widehat{\phi}^{-1}_{s,2}$. Since $\lambda_0 = 1$ on a neighbourhood of $t = 0$, the map has product form there. As an isometry from the pullback metric to the original metric, it sends the inward unit normal to the inward unit normal. Set $K_{s,\varepsilon} := F_s^*(c_\varepsilon^2 \widetilde{K}_{s,\varepsilon})$. Using (7.66),

$$\begin{aligned}\beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}) &= (\phi_{s,2}^{-1})^* \beta_Q^*(c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}|_{\partial P_Q})\\ &= (\phi_{s,2}^{-1})^*(\sin^2 \varepsilon\, \phi_{s,2}^* h_s) = \sin^2 \varepsilon\, h_s.\end{aligned}$$

The definition $\mathrm{II}(v,w) = -\langle \nabla_v \nu, w\rangle$ and the preceding normal identity give

$$\begin{aligned}\beta_Q^* \mathrm{II}_{\partial P_Q, K_{s,\varepsilon}} &= (\phi_{s,2}^{-1})^* \beta_Q^* \mathrm{II}_{\partial P_Q, c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}}\\ &> \cot \varepsilon\, \sin^2 \varepsilon\, h_s.\end{aligned}$$

Thus (7.51) and (7.52) hold. The pullback identity for the Ricci tensor and (7.65) give

$$\mathrm{Ric}_{K_{s,\varepsilon}} = F_s^*\, \mathrm{Ric}_{c_\varepsilon^2 \widetilde{K}_{s,\varepsilon}} \ge (n - 1 - C_I\varepsilon^2) K_{s,\varepsilon},$$

which is (7.53). The map F_s is an isometry from $(P_Q, K_{s,\varepsilon})$ to $(P_Q, c_\varepsilon^2 \widetilde{K}_{s,\varepsilon})$; hence diameter and volume are unchanged.

For each application of Proposition 7.2, choose inside the region where (7.23) holds a closed signed collar that contains the smoothing support, has boundary disjoint from that support, and has total signed-coordinate width at most the chosen function w. Denote these collars by $C^{(1)}_{s,\varepsilon}$ and $C^{(2)}_{s,\varepsilon}$. Let $\mathcal{T}_{s,\varepsilon}$, $\mathcal{B}_{s,\varepsilon}$, and $\mathcal{P}_{s,\varepsilon}$ be the closures of the remaining parts of the transition cylinder, the round annulus, and the scaled core, respectively. These five compact sets cover the cap and can be ordered so that consecutive sets intersect.

Fix a background metric g_0 on S^{n-1}. Smoothness on the compact set $I \times [-2,2] \times S^{n-1}$ gives $A_I \geq 1$ such that $A_I^{-1} g_0 \leq g_{s,u} \leq A_I g_0$ for all $(s,u) \in I \times [-2,2]$. Define

$$D_g := \sup_{(s,u)\in I\times[-2,2]} \operatorname{diam}(S^{n-1}, g_{s,u}), \qquad V_g := \sup_{s\in I} \operatorname{Vol}(S^{n-1}, h_s).$$

Then $D_g \leq A_I^{1/2} \operatorname{diam}(S^{n-1}, g_0) < \infty$, and smoothness of $s \mapsto h_s$ on the nonempty compact interval I gives $V_g < \infty$. On $\mathcal{T}_{s,\varepsilon}$ the metric equals $G_{s,\varepsilon}$. Its warping coefficient satisfies $0 < (\sin(\kappa r)/\kappa)\chi(r) \leq r \leq 1$. Joining two points by two radial segments and one curve in a slice gives $\operatorname{diam} \mathcal{T}_{s,\varepsilon} \leq 2 + D_g$. Since $\operatorname{Vol}(S^{n-1}, g_{s,u}) = \operatorname{Vol}(S^{n-1}, h_s)$ for every (s,u),

$$\operatorname{Vol} \mathcal{T}_{s,\varepsilon} \leq \int_{r_c}^{1} \left(\frac{\sin(\kappa r)}{\kappa} \chi(r) \right)^{n-1} \operatorname{Vol}(S^{n-1}, g_{s,\psi(r)})\, dr \leq V_g.$$

Therefore

$$\operatorname{diam} \mathcal{T}_{s,\varepsilon} \leq 2 + D_g, \qquad \operatorname{Vol} \mathcal{T}_{s,\varepsilon} \leq V_g. \tag{7.68}$$

Define

$$C_d := \frac{\arccos d}{\sqrt{1-d^2}}, \qquad C_{d,e} := \sqrt{\frac{1-e^2}{1-d^2}}, \qquad C_B := 2C_d + \frac{\pi}{\sqrt{1-d^2}}.$$

These definitions give $T_d = C_d b$, $s_c = C_{d,e} b$, and $0 < b \leq \sup_I \tau_s$. Two points of $\mathcal{B}_{s,\varepsilon}$ can be joined by radial segments of total length at most $2T_d$ and an arc in one round slice of length at most $\pi/\Lambda^{\mathrm{rd}}$. Hence its diameter is at most $C_B b$. Since $\sin(\Lambda^{\mathrm{rd}} t)/\Lambda^{\mathrm{rd}} \leq t$, its volume is at most the volume of the full round ball of radius T_d estimated by the Euclidean radial integral. Thus

$$\operatorname{diam} \mathcal{B}_{s,\varepsilon} \leq C_B b, \qquad \operatorname{Vol} \mathcal{B}_{s,\varepsilon} \leq \frac{\sigma_{n-1}}{n} C_d^n b^n, \tag{7.69}$$

$$\operatorname{Vol} \mathcal{P}_{s,\varepsilon} \leq s_c^n \operatorname{Vol}(P_Q, k_Q). \tag{7.70}$$

For a smoothing collar C, write $\operatorname{diam}_{\mathrm{int}} C$ for the diameter of the intrinsic length metric induced by the final glued metric. The product metric $P = dt^2 + h_\Sigma$ and (7.23) with $L = 2$ give

$$\operatorname{diam}_{\mathrm{int}} C \leq 2\big(w + \operatorname{diam}(\Sigma, h_\Sigma)\big), \qquad \operatorname{Vol} C \leq 2^n w \operatorname{Vol}(\Sigma, h_\Sigma).$$

At the first seam, $h_\Sigma = b^2 g_{S^{n-1}}$ and $w_1 = (T_d - T_e)/4$; hence

$$\operatorname{diam}_{\mathrm{int}} C^{(1)}_{s,\varepsilon} \leq 2(w_1 + \pi b), \qquad \operatorname{Vol} C^{(1)}_{s,\varepsilon} \leq 2^n w_1 \sigma_{n-1} b^{n-1}. \tag{7.71}$$

At the second seam, $h_\Sigma = s_c^2 g_{S^{n-1}}$ and $w_2 = \min\{T_e, T_d - T_e\}/4$; hence

$$\operatorname{diam}_{\mathrm{int}} C^{(2)}_{s,\varepsilon} \leq 2(w_2 + \pi s_c), \qquad \operatorname{Vol} C^{(2)}_{s,\varepsilon} \leq 2^n w_2 \sigma_{n-1} s_c^{n-1}. \tag{7.72}$$

For $x, y \in \mathcal{P}_{s,\varepsilon}$, let γ be a minimizing curve from x to y for the intrinsic length metric of $(P_Q, s_c^2 k_Q)$. Outside $C^{(2)}_{s,\varepsilon}$ the final metric equals $s_c^2 k_Q$. If γ meets the collar, let p and q be its first and last intersection points. For every $\eta > 0$, the definition of the intrinsic distance gives a curve in $C^{(2)}_{s,\varepsilon}$ from p to q of length at most $\operatorname{diam}_{\mathrm{int}} C^{(2)}_{s,\varepsilon} + \eta$. Replacing the segment of γ from p to q and letting $\eta \downarrow 0$ yields

$$\operatorname{diam} \mathcal{P}_{s,\varepsilon} \leq s_c \operatorname{diam}(P_Q, k_Q) + 2(w_2 + \pi s_c). \tag{7.73}$$

By (7.59), $0 < b \leq \sup_I \tau_s$. In the order

$$\mathcal{T}_{s,\varepsilon}, \quad C^{(1)}_{s,\varepsilon}, \quad \mathcal{B}_{s,\varepsilon}, \quad C^{(2)}_{s,\varepsilon}, \quad \mathcal{P}_{s,\varepsilon},$$

consecutive sets intersect and their union is the cap. The ambient diameter of either collar is at most its intrinsic diameter. Hence the diameter of the union is at most the sum of the displayed diameter bounds, while its volume is at most the sum of the displayed volume bounds. Equations (7.68), (7.69), (7.70), (7.73), (7.71), and (7.72) give constants $D_I^0, V_I^0 > 0$ such that

$$\operatorname{diam}(P_Q, \widetilde{K}_{s,\varepsilon}) \le D_I^0, \qquad \operatorname{Vol}(P_Q, \widetilde{K}_{s,\varepsilon}) \le V_I^0. \tag{7.74}$$

After decreasing ε_I, $\lambda_\varepsilon^2 \ge \sin^2 \varepsilon/4$, and hence

$$c_\varepsilon = \frac{\sin \varepsilon\ \kappa}{\lambda_\varepsilon} \le 2C_\kappa \varepsilon. \tag{7.75}$$

Constant scaling multiplies diameters by c_ε and volumes by c_ε^n. Set $D_I := 2C_\kappa D_I^0$ and $V_I := (2C_\kappa)^n V_I^0$. Equations (7.74) and (7.75) prove (7.54) and (7.55).

The functions $\tau_s, \chi_\varepsilon, \psi_\varepsilon, \kappa_\varepsilon, R_{s,\varepsilon}, \lambda_{s,\varepsilon}^{\mathrm{cap}}, m_{s,\varepsilon}, r_c, b, c, \Lambda^{\mathrm{rd}}, T_d, T_e, s_c$, and c_ε are smooth in (s, ε). The maps $F_{s,\varepsilon}^{\mathrm{tr}}$, $F_{s,\varepsilon}^{\mathrm{rd}}$, and $F_{s,\varepsilon}^{\mathrm{ann}}$ identify all pieces with fixed reference manifolds. The smooth-parameter conclusions of the two applications of Proposition 7.2 give smooth collar metrics on those fixed manifolds. The formulas defining c_ε and F_s are smooth in their parameters, so $(s, \varepsilon) \mapsto K_{s,\varepsilon}$ is smooth.

Suppose that $\widehat{g}_{s,u} \equiv \tau^2 g_{S^{n-1}}$. The endpoint identities give $h_s = \tau^2 g_{S^{n-1}}$ and $\tau_s = \tau$. Hence $\omega_{s,u}$ does not depend on u, so $\alpha_{s,u} = \eta_{s,u} = 0$. Contraction with the volume form gives $X_{s,u} = 0$, and the initial-value problem defining $\phi_{s,u}$ gives $\phi_{s,u} = \mathrm{id}$. Thus $g_{s,u} = \tau^2 g_{S^{n-1}}$ and the terminal collar map F_s is the identity.

Let $\pi : I \times (0, \varepsilon_I) \to (0, \varepsilon_I)$ be given by $\pi(s, \varepsilon) = \varepsilon$. Before the first gluing, the transition metric, the round metric, the core metric, their boundary metrics and inward second fundamental forms, and the functions λ, η, w are pullbacks by π of objects on $(0, \varepsilon_I)$. Choose the admissible cut radius r_c on this reduced parameter space by the smooth-minorant construction used above, and define $b, c, \Lambda^{\mathrm{rd}}, T_d, T_e$, and s_c there. Apply Proposition 7.2 to the first seam over $(0, \varepsilon_I)$ and pull its widths and glued metric back by π. The first glued metric is the pullback by π of a metric over $(0, \varepsilon_I)$. The second seam is obtained by deleting the region $0 \le t < T_e(\varepsilon)$ in every fibre; its boundary metric, second fundamental form, Ricci margin, and support bound are again pullbacks by π. A second application of Proposition 7.2 over $(0, \varepsilon_I)$ gives second-seam widths and a final glued metric pulled back by π. The terminal collar diffeomorphism is the identity, so there is a metric K_ε with $K_{s,\varepsilon} = K_\varepsilon$ for every $s \in I$.

For the reflection assertion, give the two model spheres the same orientation and use the same background metric in the Hodge operators. Under their identity, the forms $\alpha_{s,u}$ and $\eta_{s,u}$, the Moser vector fields, and their flows are equal on the two copies. Use the same spherical-deformation functions on both copies. Their cut-radius threshold functions are equal; choose one smooth minorant r_c and use it in both constructions. Choose a collar on P_Q^+, transport it to P_Q^- by ι, and use the same collar cutoff in the two terminal pullbacks.

For either seam type, take $M_i = M_i^+ \sqcup M_i^-$ and $\Sigma = \Sigma^+ \sqcup \Sigma^-$. Let $\mathbb{Z}/2$ act trivially on the parameter space and exchange the two components by the copy maps. Denote the boundary maps by $\beta_i^\pm : \Sigma^\pm \to \partial M_i^\pm$. The copy maps $\iota_i : M_i^+ \to M_i^-$ and $\bar{\iota} : \Sigma^+ \to \Sigma^-$ act as the nontrivial element on the plus components, and their inverses act on the minus components. They satisfy

$$\iota_i \circ \beta_i^+ = \beta_i^- \circ \bar{\iota}, \qquad \iota_i^* h_i^-(s, \varepsilon) = h_i^+(s, \varepsilon).$$

The functions λ, η, and w have equal restrictions to the two components. These identities give the boundary compatibility, metric invariance, and cocycle hypotheses of Proposition 7.4 for the $\mathbb{Z}/2$-action interchanging the copies. Denote the resulting first-seam metrics by $g_1^\pm$. The proposition gives $\delta_1^+ = \delta_1^-$ and $\nu_1^+ = \nu_1^-$, while the descended copy map pulls g_1^- back to g_1^+.

The equality $T_e^+ = T_e^-$ places the two second seams at equal round coordinate values. Their smoothing neighbourhoods are disjoint from the first-seam supports. The involution of the first glued metric maps one inward unit normal to the other; therefore

$$\bar{\iota}^* h_\Sigma^- = h_\Sigma^+, \qquad \bar{\iota}^*(\mathrm{II}_1^- + \mathrm{II}_2^-) = \mathrm{II}_1^+ + \mathrm{II}_2^+.$$

The identities $\lambda^+ = \lambda^-$, $\eta^+ = \eta^-$, and $w^+ = w^-$ hold on the paired components. A second application of Proposition 7.4 gives $\delta_2^+ = \delta_2^-$, $\nu_2^+ = \nu_2^-$, and $\iota^*\widetilde{K}^-_{s,\varepsilon} = \widetilde{K}^+_{s,\varepsilon}$. Constant scaling commutes with ι. The synchronized Moser flows, transported collars, and common cutoff give $F_s^- \circ \iota = \iota \circ F_s^+$, so the terminal collar pullbacks commute with ι. Consequently $\iota^* K^-_{s,\varepsilon} = K^+_{s,\varepsilon}$, proving the final assertion. $\square$

7.3. **A cap family on the half-line.**

Proposition 7.6. *Let h_s, $s \geq 0$, be a smooth family of metrics on S^{n-1}, and let $\widehat{g}_{s,u}$, $(s,u) \in [0,\infty)\times[-2,2]$, be a smooth family satisfying*

$$\widehat{g}_{s,-2} = \tau_s^2 g_{S^{n-1}}, \qquad \widehat{g}_{s,2} = h_s,$$
$$\mathrm{Ric}_{\widehat{g}_{s,u}} \geq (n-2)\widehat{g}_{s,u}, \qquad \mathrm{Vol}(S^{n-1}, \widehat{g}_{s,u}) = \mathrm{Vol}(S^{n-1}, h_s).$$

The scale is

$$\tau_s = \left(\frac{\mathrm{Vol}(S^{n-1}, h_s)}{\sigma_{n-1}}\right)^{1/(n-1)} < \frac{1}{4}\bar{\kappa}_Q \qquad (s \geq 0).$$

Assume that there are open sets U_-, U_+ in $[0,\infty)\times[-2,2]$ such that

$$[0,\infty)\times\{-2\} \subset U_-, \qquad [0,\infty)\times\{2\} \subset U_+,$$
$$\widehat{g}_{s,u} = \widehat{g}_{s,-2} \quad ((s,u) \in U_-), \qquad \widehat{g}_{s,u} = \widehat{g}_{s,2} \quad ((s,u) \in U_+).$$

There are constants $\varepsilon_0 > 0$ and $C, D, V > 0$, and a smooth family of metrics $(s,\varepsilon) \mapsto K_{s,\varepsilon}$ on P_Q, for $s \geq 0$ and $0 < \varepsilon < \varepsilon_0$, satisfying

$$\beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}) = \sin^2\varepsilon\, h_s, \tag{7.76}$$
$$\beta_Q^* \mathrm{II}_{\partial P_Q, K_{s,\varepsilon}} > \cot\varepsilon\, \beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}), \tag{7.77}$$
$$\mathrm{Ric}_{K_{s,\varepsilon}} \geq (n-1-C\varepsilon^2) K_{s,\varepsilon}, \tag{7.78}$$
$$\mathrm{diam}(P_Q, K_{s,\varepsilon}) \leq D\varepsilon, \qquad \mathrm{Vol}(P_Q, K_{s,\varepsilon}) \leq V\varepsilon^n. \tag{7.79}$$

The following choices can be made simultaneously with these conclusions.

(a) *If*

$$h_s = h_0, \qquad \widehat{g}_{s,u} = \widehat{g}_{0,u} \quad (s \geq 0,\ -2 \leq u \leq 2),$$

then $K_{s,\varepsilon} = K^*_\varepsilon$ can be chosen for every $s \geq 0$. For each of the two gluing seams, its interpolation and mollification widths

$$(\delta_r^*, \nu_r^*) : (0,\varepsilon_0) \longrightarrow (0,\infty)^2, \qquad r = 1, 2,$$

can be chosen smooth and downward admissible in the sense of (7.8) *and* (a)–(b).

(b) *Let $s_0 > 0$. If*

$$h_s = h_0, \qquad \widehat{g}_{s,u} = \widehat{g}_{0,u} \quad (0 \leq s \leq 3s_0,\ -2 \leq u \leq 2),$$

fix the reference family K^*_ε and the four width functions in (a). *The family on $[0,\infty)$ may be chosen so that*

$$K_{s,\varepsilon} = K^*_\varepsilon \qquad (0 \leq s \leq 2s_0,\ 0 < \varepsilon < \varepsilon_0).$$

(c) *Suppose the construction is performed on copies $P_Q^\pm$, equipped with copied core metrics $k_Q^\pm$, with copy diffeomorphism $\iota : P_Q^+ \to P_Q^-$ and boundary parametrizations $\beta_Q^\pm$ satisfying*

$$\iota \circ \beta_Q^+ = \beta_Q^-, \qquad \iota^* k_Q^- = k_Q^+.$$

If, under the fixed model-sphere identification,

$$h_s^+ = h_s^-, \qquad \widehat{g}^+_{s,u} = \widehat{g}^-_{s,u} \quad (s \geq 0,\ -2 \leq u \leq 2),$$

then the width pair at seam r in the plus copy can be chosen equal to the width pair at seam r in the minus copy for $r = 1, 2$. Under (b), *these equalities can be imposed together with*

$$(\delta_r^\pm, \nu_r^\pm) = (\delta_r^*, \nu_r^*) \quad \text{on } [0, 2s_0] \times (0, \varepsilon_0), \qquad r = 1, 2.$$

Proof. Put $m = n-1$. Orient S^m and fix a background metric. Let $\mathcal{G}$ and d^* denote its Green operator and codifferential. Set

$$\omega_{s,u} := dv_{\widehat{g}_{s,u}}, \qquad \alpha_{s,u} := \partial_u \omega_{s,u}, \qquad \eta_{s,u} := d^*\mathcal{G}\alpha_{s,u}.$$

For every s and u, one has $\int_{S^m} \omega_{s,u} = \mathrm{Vol}(S^m, h_s)$. Differentiation with respect to u gives $\int_{S^m} \alpha_{s,u} = 0$. The harmonic projection of the top-degree form $\alpha_{s,u}$ is therefore zero. Since $\mathcal{G}\alpha_{s,u}$ has degree m, one has $d\mathcal{G}\alpha_{s,u} = 0$. The Hodge identity gives

$$d\eta_{s,u} = dd^*\mathcal{G}\alpha_{s,u} = \alpha_{s,u}. \tag{7.80}$$

Define $X_{s,u}$ by $\iota_{X_{s,u}}\omega_{s,u} = -\eta_{s,u}$. The vector field is smooth in (s,u). On $U_- \cup U_+$ one has $\alpha_{s,u} = \eta_{s,u} = 0$, and therefore $X_{s,u} = 0$. Let

$$\partial_u \phi_{s,u} = X_{s,u} \circ \phi_{s,u}, \qquad \phi_{s,-2} = \mathrm{id}\,.$$

The flow exists on $[-2,2]$ because S^m is compact; parameter-dependent ODE existence gives smooth dependence on (s,u). By (7.80) and Cartan's formula, $\frac{d}{du}(\phi^*_{s,u}\omega_{s,u}) = \phi^*_{s,u}(\alpha_{s,u} - d\eta_{s,u}) = 0$. Define $g_{s,u} := \phi^*_{s,u}\widehat{g}_{s,u}$ and extend it to $u \in \mathbb{R}$ by its endpoint values. Then

$$\begin{aligned} dv_{g_{s,u}} &= dv_{g_{s,-2}}, \qquad \mathrm{Ric}_{g_{s,u}} \geq (m-1)g_{s,u}, \\ g_{s,-2} &= \tau_s^2 g_{S^{n-1}}, \qquad g_{s,2} = \phi^*_{s,2}h_s. \end{aligned} \tag{7.81}$$

For $s \geq 0$, let

$$B(s) := 1 + \max_{(u,x)\in[-2,2]\times S^m} \left(|\partial_u g_{s,u}|_{g_{s,u}}(x) + |\partial_u^2 g_{s,u}|_{g_{s,u}}(x) + |\nabla^{g_{s,u}}\partial_u g_{s,u}|_{g_{s,u}}(x) \right).$$

The function under the maximum is continuous on $[0,\infty)\times[-2,2]\times S^m$. Its restriction to $\{s\}\times[-2,2]\times S^m$ has a finite maximum, and the maximum varies continuously with s. Choose $\zeta_0 \in C_c^\infty([0,\infty))$ with $0 \leq \zeta_0 \leq 1$, $\zeta_0 = 1$ on $[0,1]$, and $\operatorname{supp}\zeta_0 \subset [0,2)$. For $j \geq 1$, choose $\zeta_j \in C_c^\infty([0,\infty))$ with $0 \leq \zeta_j \leq 1$, $\zeta_j = 1$ on $[j, j+1]$, and $\operatorname{supp}\zeta_j \subset (j-1, j+2)$. For every integer $j \geq 0$ put

$$b_j := \max_{s\in[\max\{0,j-1\},j+2]} B(s), \qquad A_{\mathrm{maj}}(s) := 1 + \sum_{j=0}^\infty b_j\zeta_j(s).$$

The sum is locally finite. For every $s \geq 0$, choose j with $s \in [j, j+1]$; then $\zeta_j(s) = 1$, and the interval $[\max\{0, j-1\}, j+2]$ used to define b_j contains s. Hence $A_{\mathrm{maj}}(s) \geq 1 + B(s) > B(s)$. Choose a smooth nondecreasing map $\rho_0 : \mathbb{R} \to [-2,2]$ satisfying

$$\rho_0(v) = -2 \quad (v \leq -1), \qquad \rho_0(v) = 2 \quad (v \geq 1).$$

Set $a_1 := \|\rho_0'\|_{L^\infty(\mathbb{R})}$ and $a_2 := \|\rho_0''\|_{L^\infty(\mathbb{R})}$, and choose $c_\rho \geq \max\{1, 4a_1, \sqrt{2(a_2 + a_1^2)}\}$. Put $L(s) := c_\rho(1 + A_{\mathrm{maj}}(s))^2$, $\rho_s(v) := \rho_0(v/L(s))$, and $k_{s,v} := g_{s,\rho_s(v)}$. All norms below are taken with respect to $k_{s,v}$. The chain rule and the fact that $\rho_s(v)$ is independent of the sphere variable give

$$\begin{aligned} \partial_v k_{s,v} &= \frac{\rho_0'(v/L(s))}{L(s)}\partial_u g_{s,\rho_s(v)}, \\ \partial_v^2 k_{s,v} &= \frac{\rho_0''(v/L(s))}{L(s)^2}\partial_u g_{s,\rho_s(v)} + \frac{\rho_0'(v/L(s))^2}{L(s)^2}\partial_u^2 g_{s,\rho_s(v)}, \\ \nabla^{k_{s,v}}\partial_v k_{s,v} &= \frac{\rho_0'(v/L(s))}{L(s)}\nabla^{g_{s,\rho_s(v)}}\partial_u g_{s,\rho_s(v)}. \end{aligned}$$

Since $B(s) < A_{\mathrm{maj}}(s)$,

$$\begin{aligned} &\|\partial_v k_{s,v}\| + \|\partial_v^2 k_{s,v}\| + \|\nabla^{k_{s,v}}\partial_v k_{s,v}\| \\ &\qquad \leq \frac{2a_1B(s)}{L(s)} + \frac{(a_2 + a_1^2)B(s)}{L(s)^2} \leq \frac{2a_1}{c_\rho} + \frac{a_2 + a_1^2}{c_\rho^2} \leq 1. \end{aligned}$$

Consequently,

$$\|\partial_v k_{s,v}\|_{k_{s,v}} + \|\partial_v^2 k_{s,v}\|_{k_{s,v}} + \|\nabla^{k_{s,v}} \partial_v k_{s,v}\|_{k_{s,v}} \le 1, \tag{7.82}$$

$$dv_{k_{s,v}} = dv_{g_{s,-2}}, \qquad \mathrm{Ric}_{k_{s,v}} \ge (n-2)k_{s,v}. \tag{7.83}$$

Moreover, $k_{s,v}$ equals $g_{s,-2}$ for $v \le -L(s)$ and $g_{s,2}$ for $v \ge L(s)$.

Let $C_{\mathrm{sph}} = C_{\mathrm{sph}}(n) \ge 1$ be chosen so that the estimates (6.43)–(6.48) hold with $C_n \le C_{\mathrm{sph}}$. For $0 < \varepsilon < 1$, set

$$\begin{aligned} \theta &:= \varepsilon^4, & E &:= \frac{\theta}{8}, & F &:= \frac{\theta}{64n}, \\ \ell_0 &:= 64n\theta^{-2}, & T_0 &:= e^{\ell_0}, & r_0 &:= e^{-T_0}. \end{aligned} \tag{7.84}$$

Since $n \ge 4$ and $0 < \varepsilon < 1$, one has $\ell_0 > 1$, whence $T_0 > e$ and

$$0 < r_0 = e^{-T_0} < e^{-e}. \tag{7.85}$$

Choose $\varepsilon_0 \in (0, 1/10)$ so that $\varepsilon_0^4 < 1/4$. Then, for $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \frac{E}{\ell_0} &< \frac{1}{2}, & \frac{T_0}{\ell_0} &> 4, \\ E &\ge 2\sqrt{m}\,F, & E &\ge F^2. \end{aligned} \tag{7.86}$$

For $0 < r \le 1$, put

$$T(r) := -\log(r_0 r), \qquad \ell(r) := \log T(r),$$

$$\chi_\varepsilon(r) := 1 - \frac{E}{\ell(r)}, \qquad \varphi_{s,\varepsilon}(r) := -F \log \ell(r) + L(s) + 1 + F \log \ell_0,$$

$$\psi_{s,\varepsilon} := \rho_s \circ \varphi_{s,\varepsilon}.$$

Then $\partial_r \varphi_{s,\varepsilon} = F/(rT\ell) > 0$, $\varphi_{s,\varepsilon}(r) \to -\infty$ as $r \downarrow 0$, and $\varphi_{s,\varepsilon}(1) = L(s)+1$. The equation $\varphi_{s,\varepsilon}(r_-(s,\varepsilon)) = -L(s)$ has one solution in $(0,1)$; the implicit-function theorem makes r_- smooth. Put $G^0_{s,\varepsilon} := dr^2 + (r\chi_\varepsilon(r))^2 g_{s,\psi_{s,\varepsilon}(r)}$. Fix $s \ge 0$ and apply Proposition 6.2 with

$$m = n-1, \quad A = 1, \quad X = S^{n-1}, \quad \ell_v = k_{s,v},$$

$$\rho = r_0, \quad E = \theta/8, \quad F = \theta/(64n), \quad c = L(s) + 1 + F\log \ell_0,$$

$$h = \chi_\varepsilon, \quad w = r\chi_\varepsilon, \quad f = \varphi_{s,\varepsilon}.$$

Equation (7.82) bounds each of the three family derivatives by one, and (7.83) supplies the fixed volume density and the lower Ricci bound. Equations (7.85) and (7.86) verify the four conditions in (6.19); the first two remain valid for $0 < r \le 1$ because $T(r) \ge T_0$ and $T/\log T$ is increasing on (e, ∞). Hence, for every $g_{s,\psi_{s,\varepsilon}(r)}$-unit vector e,

$$\mathrm{Ric}_{G^0_{s,\varepsilon}}(\partial_r, \partial_r) \ge \frac{E}{2\ell(r)^2 T(r) r^2},$$

$$\mathrm{Ric}_{G^0_{s,\varepsilon}}((r\chi_\varepsilon)^{-1}e, (r\chi_\varepsilon)^{-1}e) \ge \frac{E}{\ell(r) r^2},$$

$$\left|\mathrm{Ric}_{G^0_{s,\varepsilon}}(\partial_r, (r\chi_\varepsilon)^{-1}e)\right| \le \frac{\sqrt{n-1}\,F}{\ell(r)T(r)r^2}.$$

The Schur-complement conclusion of Proposition 6.2 gives

$$\mathrm{Ric}_{G^0_{s,\varepsilon}} > 0. \tag{7.87}$$

Writing $g(r) := g_{s,\psi_{s,\varepsilon}(r)} = k_{s,\varphi_{s,\varepsilon}(r)}$, the derivatives and chain rules are

$$\chi_\varepsilon'(r) = -\frac{E}{rT(r)\ell(r)^2}, \qquad \varphi_{s,\varepsilon}'(r) = \frac{F}{rT(r)\ell(r)},$$
$$\varphi_{s,\varepsilon}''(r) = \frac{F}{r^2T(r)\ell(r)}\left(-1 + \frac{1}{T(r)} + \frac{1}{T(r)\ell(r)}\right),$$
$$g'(r) = (\partial_v k)_{s,\varphi_{s,\varepsilon}(r)}\, \varphi_{s,\varepsilon}'(r),$$
$$g''(r) = (\partial_v^2 k)_{s,\varphi_{s,\varepsilon}(r)}\, \varphi_{s,\varepsilon}'(r)^2 + (\partial_v k)_{s,\varphi_{s,\varepsilon}(r)}\, \varphi_{s,\varepsilon}''(r).$$

Consequently (7.82), $\chi_\varepsilon \geq 1/2$, $T(r) \geq T_0$, and $\ell(r) \geq \ell_0$ imply

$$|1-\chi_\varepsilon| + r\left|\frac{\chi_\varepsilon'}{\chi_\varepsilon}\right| \leq \frac{E}{\ell_0} + \frac{2E}{T_0\ell_0^2},$$
$$r|\partial_r g_{s,\psi_{s,\varepsilon}}| + r^2|\partial_r^2 g_{s,\psi_{s,\varepsilon}}| \leq \frac{3F}{T_0\ell_0} + \frac{F^2}{T_0^2\ell_0^2}.$$

The explicit values in (7.84) give

$$\frac{E}{\ell_0} + \frac{2E}{T_0\ell_0^2} + \frac{3F}{T_0\ell_0} + \frac{F^2}{T_0^2\ell_0^2} \leq \frac{\theta^3}{512n} + \frac{\theta^5}{16384n^2} + \frac{3\theta^3}{4096n^2} + \frac{\theta^6}{16777216n^4} < \theta = \varepsilon^4.$$

Consequently,

$$|1-\chi_\varepsilon| + r\left|\frac{\chi_\varepsilon'}{\chi_\varepsilon}\right| + r|\partial_r g_{s,\psi_{s,\varepsilon}}| + r^2|\partial_r^2 g_{s,\psi_{s,\varepsilon}}| \leq \varepsilon^4, \tag{7.88}$$
$$r^2\left|\mathrm{Ric}_{G^0_{s,\varepsilon}}(\partial_r, (r\chi_\varepsilon)^{-1}e)\right| \leq \sqrt{n-1}\,F \leq \varepsilon^4. \tag{7.89}$$

The numerical constants in these estimates depend on n and not on s.

Define $\lambda_\varepsilon := \sqrt{\sin^2\varepsilon - 8\varepsilon^4}$. After decreasing ε_0,

$$\lambda_\varepsilon > 0, \qquad \frac{\lambda_\varepsilon}{\chi_\varepsilon(1)} < \sin\frac{1}{2}, \qquad \frac{1}{2}\varepsilon \leq \lambda_\varepsilon \leq \varepsilon.$$

Consequently, the equation

$$\chi_\varepsilon(1)\sin\kappa_\varepsilon = \lambda_\varepsilon \tag{7.90}$$

has a unique solution $\kappa_\varepsilon \in (0,1/2)$, depending smoothly on ε, and

$$\frac{1}{2}\varepsilon \leq \kappa_\varepsilon \leq 4\varepsilon. \tag{7.91}$$

Define

$$G_{s,\varepsilon} := dr^2 + \left(\frac{\sin(\kappa_\varepsilon r)}{\kappa_\varepsilon}\chi_\varepsilon(r)\right)^2 g_{s,\psi_{s,\varepsilon}(r)}.$$

The identities (6.13)–(6.48), together with (7.87)–(7.89), give

$$\mathrm{Ric}_{G_{s,\varepsilon}} \geq \mathcal{R}_\varepsilon G_{s,\varepsilon}, \qquad \mathcal{R}_\varepsilon := (n-1-C_{\mathrm{sph}}\varepsilon^4 - C_{\mathrm{sph}}\kappa_\varepsilon^2)\kappa_\varepsilon^2.$$

Decrease ε_0 so that $\mathcal{R}_\varepsilon > 0$. For $0 < r \leq r_-(s,\varepsilon)$,

$$G_{s,\varepsilon} = dr^2 + \left(\tau_s\frac{\sin(\kappa_\varepsilon r)}{\kappa_\varepsilon}\chi_\varepsilon(r)\right)^2 g_{S^{n-1}}.$$

Define

$$c_\varepsilon := \frac{\sin\varepsilon\,\kappa_\varepsilon}{\lambda_\varepsilon}. \tag{7.92}$$

Let $a_{s,\varepsilon}(r) := \kappa_\varepsilon^{-1}\sin(\kappa_\varepsilon r)\chi_\varepsilon(r)$. Equation (7.90) gives

$$a_{s,\varepsilon}(1) = \frac{\lambda_\varepsilon}{\kappa_\varepsilon}, \qquad c_\varepsilon^2 a_{s,\varepsilon}(1)^2 = \sin^2\varepsilon.$$

Since (7.88) gives $1-\varepsilon^4 \le \chi_\varepsilon(1) \le 1$ and $|\chi_\varepsilon'(1)| \le \varepsilon^4$,

$$\begin{aligned}\chi_\varepsilon(1)^2 - \lambda_\varepsilon^2 - \cos^2\varepsilon &= \chi_\varepsilon(1)^2 - 1 + 8\varepsilon^4 \ge 6\varepsilon^4,\\ \sqrt{\chi_\varepsilon(1)^2-\lambda_\varepsilon^2} - \cos\varepsilon &\ge 3\varepsilon^4.\end{aligned}$$

Here both terms in the denominator of the rationalized difference are at most one. Moreover

$$\left|\frac{\sin\kappa_\varepsilon}{\kappa_\varepsilon}\chi_\varepsilon'(1)\right| \le \varepsilon^4.$$

As

$$a_{s,\varepsilon}'(1) = \sqrt{\chi_\varepsilon(1)^2-\lambda_\varepsilon^2} + \frac{\sin\kappa_\varepsilon}{\kappa_\varepsilon}\chi_\varepsilon'(1),$$

one obtains $a_{s,\varepsilon}'(1) > \cos\varepsilon$. Since $\varphi_{s,\varepsilon}(1) = L(s)+1$, continuity gives $\varphi_{s,\varepsilon}(r) > L(s)$ for r in a neighbourhood of 1. Thus $\psi_{s,\varepsilon}(r) = 2$ and $\partial_r g_{s,\psi_{s,\varepsilon}(r)} = 0$ on that neighbourhood. The inward unit normal at $r=1$ is $-\partial_r$, and therefore

$$\mathrm{II}_{\{1\}\times S^{n-1},G_{s,\varepsilon}} = \frac{a_{s,\varepsilon}'(1)}{a_{s,\varepsilon}(1)} G_{s,\varepsilon}|_{\{1\}\times S^{n-1}}.$$

Under constant scaling by c_ε^2 the second fundamental form is multiplied by c_ε. Therefore

$$(c_\varepsilon^2 G_{s,\varepsilon})|_{\{1\}\times S^{n-1}} = \sin^2\varepsilon\, g_{s,2}, \tag{7.93}$$

$$\begin{aligned}\mathrm{II}_{\{1\}\times S^{n-1},c_\varepsilon^2 G_{s,\varepsilon}} &= \frac{a_{s,\varepsilon}'(1)}{\sin\varepsilon}(c_\varepsilon^2 G_{s,\varepsilon})|_{\{1\}\times S^{n-1}}\\ &> \cot\varepsilon\,(c_\varepsilon^2 G_{s,\varepsilon})|_{\{1\}\times S^{n-1}}.\end{aligned} \tag{7.94}$$

Choose $C_{\mathrm{mar}} > 17C_{\mathrm{sph}}$ and decrease ε_0 so that $\lambda_\varepsilon^{\mathrm{cap}} := (n-1-2C_{\mathrm{mar}}\varepsilon^2)\kappa_\varepsilon^2 > 0$. Define the positive margin by

$$m_\varepsilon := \mathcal{R}_\varepsilon - \lambda_\varepsilon^{\mathrm{cap}} = \left(2C_{\mathrm{mar}}\varepsilon^2 - C_{\mathrm{sph}}\varepsilon^4 - C_{\mathrm{sph}}\kappa_\varepsilon^2\right)\kappa_\varepsilon^2 \ge C_{\mathrm{mar}}\varepsilon^2\kappa_\varepsilon^2 > 0.$$

Define

$$d := \frac{1}{2}\bar\kappa_Q, \qquad e := \frac{3}{4}\bar\kappa_Q,$$

$$Q_1(s,\varepsilon) := \frac{(n-1)(1-d^2)}{\tau_s^2\mathcal{R}_\varepsilon}, \qquad Q_2(s,\varepsilon) := \frac{\mu_Q(1-d^2)}{(1-e^2)\tau_s^2\mathcal{R}_\varepsilon}.$$

Set the cutting radius by

$$r_c(s,\varepsilon) := \frac{1}{4}\left(1 + r_-(s,\varepsilon)^{-2} + Q_1(s,\varepsilon)^{-1} + Q_2(s,\varepsilon)^{-1}\right)^{-1/2}. \tag{7.95}$$

The defining formula gives

$$r_c^2 < \frac{1}{16}r_-^2, \qquad r_c^2 < \frac{1}{16}Q_i \quad (i=1,2).$$

In particular $0 < r_c < r_-$ and $r_c^2 < Q_i$. Put

$$\begin{aligned} b &:= \tau_s\frac{\sin(\kappa_\varepsilon r_c)}{\kappa_\varepsilon}\chi_\varepsilon(r_c),\\ c &:= \tau_s\left(\cos(\kappa_\varepsilon r_c)\chi_\varepsilon(r_c) + \frac{\sin(\kappa_\varepsilon r_c)}{\kappa_\varepsilon}\chi_\varepsilon'(r_c)\right).\end{aligned}$$

Since $\chi_\varepsilon' \le 0$, $0 < \chi_\varepsilon \le 1$, and $\sin(\kappa r)/\kappa \le r$,

$$b \le \tau_s r_c, \qquad c \le \tau_s < \bar{\kappa}_Q/4 < d.$$

The inequalities $r_c^2 < Q_i$ give

$$(n-1)\frac{1-d^2}{b^2} > \mathcal{R}_\varepsilon, \qquad \frac{\mu_Q(1-d^2)}{(1-e^2)b^2} > \mathcal{R}_\varepsilon. \tag{7.96}$$

Define

$$\Lambda^{\mathrm{rd}} := \frac{\sqrt{1-d^2}}{b}, \qquad T_d := \frac{\arccos d}{\Lambda^{\mathrm{rd}}},$$
$$T_e := \frac{\arccos e}{\Lambda^{\mathrm{rd}}}, \qquad s_c := \frac{\sqrt{1-e^2}}{\Lambda^{\mathrm{rd}}}.$$

Let $C = [0,1] \times S^{n-1}$ and let D^n be the closed unit ball. Pull back the transition cylinder $[r_c, 1] \times S^{n-1}$ to C by $(t,x) \mapsto (r_c + (1-r_c)t, x)$, and pull back the round ball

$$dt^2 + (\Lambda^{\mathrm{rd}})^{-2}\sin^2(\Lambda^{\mathrm{rd}} t) g_{S^{n-1}}, \qquad 0 \le t \le T_d,$$

to D^n by radial dilation. On the parameter manifold $\mathcal{B} := [0,\infty) \times (0,\varepsilon_0)$ use the first-seam pieces $M_1^{(1)} = C$, $M_2^{(1)} = D^n$, and model seam $\Sigma^{(1)} = S^{n-1}$. Set $\Sigma_1^{(1)} := \{0\} \times S^{n-1} \subset \partial C$, $\Sigma_2^{(1)} := \partial D^n$, $\beta_1^{(1)}(x) := (0,x)$, and $\beta_2^{(1)}(x) := x$. Their pulled-back boundary metrics are both $b^2 g_{S^{n-1}}$. The inward second fundamental forms have principal curvatures $-c/b$ and d/b, respectively; hence

$$\mathrm{II}_1^{(1)} + \mathrm{II}_2^{(1)} = \frac{d-c}{b}\, b^2 g_{S^{n-1}} > 0. \tag{7.97}$$

For the first seam, choose

$$\lambda^{(1)} = \lambda_\varepsilon^{\mathrm{cap}}, \qquad \eta^{(1)} = \frac{1}{4}m_\varepsilon,$$
$$w^{(1)} = \frac{1}{4}(T_d - T_e), \qquad L^{(1)} = 2.$$

These functions are smooth on $\mathcal{B}$, and $\eta^{(1)}, w^{(1)} > 0$. The transition metric has Ricci coefficient $\mathcal{R}_\varepsilon = \lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon$. The round metric has Ricci coefficient $(n-1)(\Lambda^{\mathrm{rd}})^2 > \mathcal{R}_\varepsilon$ by (7.96). Thus both coefficients exceed $\lambda^{(1)} + 2\eta^{(1)} = \lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/2$. Together with the common boundary metric and (7.97), these are (7.1)– (7.3) for the displayed pieces. Apply Proposition 7.2 with these objects. Denote its widths by δ_1, ν_1 and its glued metric by $G_{s,\varepsilon}^{(1)}$. Then

$$\mathrm{Ric}_{G_{s,\varepsilon}^{(1)}} > (\lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/4) G_{s,\varepsilon}^{(1)}. \tag{7.98}$$

If $\ell_1 = 2\delta_1$, then $2\ell_1 < w^{(1)} = (T_d - T_e)/4$. The support conclusion of Proposition 7.2 shows that the first smoothing collar is disjoint from $0 \le t \le T_e$; its intersection with the round piece lies in $t > T_e$. Hence $G_{s,\varepsilon}^{(1)}$ equals the round metric on a neighbourhood of $0 \le t \le T_e$. Apply the collar-straightening construction at the two seams in order. Its supports lie inside the corresponding gluing collars, so the first identification fixes a neighbourhood of the round region $0 \le t \le T_e$, and both identifications fix a neighbourhood of the outer boundary $r = 1$. These identifications are included when the glued metrics are pulled back to the fixed manifolds below.

Under the radial dilation of the round ball to D^n, the slice $t = T_e$ corresponds to the fixed Euclidean radius

$$\rho_e := \frac{T_e}{T_d} = \frac{\arccos e}{\arccos d} \in (0,1).$$

Delete the fixed subball $\{|x| < \rho_e\} \subset D^n$. The complement of this subball in the first glued manifold is a fixed compact manifold $M_1^{(2)}$. On its round annulus use the fixed coordinate $v = (|x| - \rho_e)/(1-\rho_e) \in [0,1]$; in physical round coordinates the map is $(v,x) \mapsto (T_e + (T_d - T_e)v, x)$. The pullback of $G_{s,\varepsilon}^{(1)}$ to this fixed manifold is a smooth vertical metric. Let $M_2^{(2)} = P_Q$ and use the model seam $\Sigma^{(2)} = S^{n-1}$. Set $\Sigma_1^{(2)} := \{v = 0\} \times S^{n-1} \subset \partial M_1^{(2)}$, $\Sigma_2^{(2)} := \partial P_Q$, $\beta_1^{(2)}(x) := (0, x)$, and $\beta_2^{(2)} := \beta_Q$. Since the first smoothing support is disjoint from $t = T_e$, the pulled-back boundary metric on $M_1^{(2)}$ is $s_c^2 g_{S^{n-1}}$ and its inward principal curvature is $-e/s_c$. Give $M_2^{(2)}$ the metric $s_c^2 k_Q$. Its pulled-back boundary metric is also $s_c^2 g_{S^{n-1}}$, its inward principal curvatures are at least κ_Q/s_c, and its Ricci coefficient is μ_Q/s_c^2. Since $s_c^2 = (1-e^2)b^2/(1-d^2)$, (7.96) gives $\mu_Q/s_c^2 > \mathcal{R}_\varepsilon$, and

$$\mathrm{II}_1^{(2)} + \mathrm{II}_2^{(2)} \geq \frac{\kappa_Q - e}{s_c} s_c^2 g_{S^{n-1}} > 0. \tag{7.99}$$

For the second seam, choose

$$\lambda^{(2)} = \lambda_\varepsilon^{\mathrm{cap}}, \qquad \eta^{(2)} = \frac{1}{16} m_\varepsilon,$$
$$w^{(2)} = \frac{1}{4}\min\{T_e, T_d - T_e\}, \qquad L^{(2)} = 2.$$

The function $\eta^{(2)}$ is positive and smooth. Since T_e and $T_d - T_e$ are fixed positive multiples of the smooth positive function b, the same is true of $w^{(2)}$. By (7.98), the first piece has Ricci coefficient greater than $\lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/4$. The core coefficient is greater than $\mathcal{R}_\varepsilon = \lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon$. Both exceed $\lambda^{(2)} + 2\eta^{(2)} = \lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/8$. The common boundary metric and (7.99) complete the hypotheses (7.1)– (7.3). Apply Proposition 7.2 to the second seam. Denote its widths by δ_2, ν_2 and the twice-glued metric by $\widetilde{K}_{s,\varepsilon}$. If $\ell_2 = 2\delta_2$, then $2\ell_2 < w^{(2)} \leq (T_d - T_e)/4$. The first and second smoothing supports are consequently disjoint. The Ricci conclusion is

$$\mathrm{Ric}_{\widetilde{K}_{s,\varepsilon}} > (\lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/16)\widetilde{K}_{s,\varepsilon}.$$

The retained round annulus and the transition cylinder form one compact cylinder attached to ∂P_Q through β_Q. Fix a collar $[0,2) \times \partial P_Q \hookrightarrow P_Q$ whose slice $\{0\} \times \partial P_Q$ is the boundary, and identify the attached cylinder with $[-2, 0] \times \partial P_Q$. A smooth increasing diffeomorphism $[-2, 2) \to [0, 2)$ which is the identity on $[1, 2)$, together with the identity outside the fixed collar, identifies

$$P_Q \cup_{S^{n-1}} ([T_e, T_d] \times S^{n-1}) \cup_{S^{n-1}} ([r_c, 1] \times S^{n-1})$$

with P_Q.

Scale by c_ε^2. Since the Ricci tensor is unchanged as a $(0,2)$-tensor under constant scaling,

$$\frac{\lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/16}{c_\varepsilon^2} = \left(n - 1 - \frac{15}{8}C_{\mathrm{mar}}\varepsilon^2 - \frac{C_{\mathrm{sph}}}{16}(\varepsilon^4 + \kappa_\varepsilon^2)\right)\frac{\lambda_\varepsilon^2}{\sin^2\varepsilon}.$$

Define $C_{\mathrm{sc}} := 15C_{\mathrm{mar}}/8 + 17C_{\mathrm{sph}}/16$. For $0 < \varepsilon < 1$, (7.91) gives

$$n - 1 - \frac{15}{8}C_{\mathrm{mar}}\varepsilon^2 - \frac{C_{\mathrm{sph}}}{16}(\varepsilon^4 + \kappa_\varepsilon^2) \geq n - 1 - C_{\mathrm{sc}}\varepsilon^2.$$

After decreasing ε_0, $\sin\varepsilon \geq \varepsilon/2$ and hence

$$\frac{\lambda_\varepsilon^2}{\sin^2\varepsilon} = 1 - \frac{8\varepsilon^4}{\sin^2\varepsilon} \geq 1 - 32\varepsilon^2 > 0.$$

Therefore

$$\frac{\lambda_\varepsilon^{\mathrm{cap}} + m_\varepsilon/16}{c_\varepsilon^2} \geq n - 1 - \big(C_{\mathrm{sc}} + 32(n-1)\big)\varepsilon^2.$$

With $C := C_{\mathrm{sc}} + 32(n-1)$,

$$\mathrm{Ric}_{c_\varepsilon^2\widetilde{K}_{s,\varepsilon}} \geq (n - 1 - C\varepsilon^2)c_\varepsilon^2\widetilde{K}_{s,\varepsilon}. \tag{7.100}$$

At the outer boundary, (7.93) gives the metric $\sin^2\varepsilon\, g_{s,2}$ and (7.94) gives its strict inward second-fundamental-form inequality.

For $u \in [-2,2]$, set $\widehat{\phi}_{s,u} := \beta_Q \circ \phi_{s,u} \circ \beta_Q^{-1}$. Fix a collar $[0,1] \times \partial P_Q$, with $t = 0$ at the outer boundary, and a smooth function $\vartheta_0 : [0,1] \to [0,1]$ satisfying $\vartheta_0 = 1$ on $[0,1/4]$ and $\vartheta_0 = 0$ on $[1/2,1]$. Define

$$\mathcal{F}_s(t,x) := \left(t, \widehat{\phi}^{-1}_{s,-2+4\vartheta_0(t)}(x)\right)$$

on the collar and extend it by the identity. Its inverse is obtained by replacing $\widehat{\phi}^{-1}_{s,-2+4\vartheta_0(t)}$ with $\widehat{\phi}_{s,-2+4\vartheta_0(t)}$, so $\mathcal{F}_s$ is a smooth family of diffeomorphisms. Put $K_{s,\varepsilon} := \mathcal{F}_s^*(c_\varepsilon^2 \widetilde{K}_{s,\varepsilon})$. Its boundary restriction satisfies $\mathcal{F}_s \circ \beta_Q = \beta_Q \circ \phi_{s,2}^{-1}$. Consequently,

$$\beta_Q^*(K_{s,\varepsilon}|_{\partial P_Q}) = (\phi_{s,2}^{-1})^*(\sin^2\varepsilon\, \phi_{s,2}^* h_s) = \sin^2\varepsilon\, h_s.$$

The map $\mathcal{F}_s$ is an isometry from $(P_Q, K_{s,\varepsilon})$ to $(P_Q, c_\varepsilon^2\widetilde{K}_{s,\varepsilon})$, so $\mathrm{Ric}_{K_{s,\varepsilon}} = \mathcal{F}_s^*\,\mathrm{Ric}_{c_\varepsilon^2\widetilde{K}_{s,\varepsilon}}$, and (7.100) gives (7.78). The differential of $\mathcal{F}_s$ maps the inward unit normal of $K_{s,\varepsilon}$ to the inward unit normal of the scaled metric. Therefore

$$\beta_Q^*\mathrm{II}_{\partial P_Q, K_{s,\varepsilon}} = (\phi_{s,2}^{-1})^*\beta_Q^*\mathrm{II}_{\partial P_Q, c_\varepsilon^2\widetilde{K}_{s,\varepsilon}},$$

and (7.94) gives (7.77).

Bonnet–Myers and (7.81) give

$$\operatorname{diam}(S^{n-1}, g_{s,u}) \le \pi, \qquad \operatorname{Vol}(S^{n-1}, g_{s,u}) = \tau_s^{n-1}\sigma_{n-1} \le 4^{-(n-1)}\bar{\kappa}_Q^{\,n-1}\sigma_{n-1}.$$

Choose, for each seam, a closed signed collar containing the support of the metric modification, having boundary disjoint from that support, and having total signed-coordinate width smaller than $w^{(r)}$ for seam r, $r = 1,2$. Denote these collars by $\mathcal{C}^{(1)}_{s,\varepsilon}$ and $\mathcal{C}^{(2)}_{s,\varepsilon}$. Let $\mathcal{T}_{s,\varepsilon}$, $\mathcal{B}_{s,\varepsilon}$, and $\mathcal{P}_{s,\varepsilon}$ be the closures of the remaining portions of the transition cylinder, the round annulus, and the scaled core. In the order

$$\mathcal{T}_{s,\varepsilon}, \quad \mathcal{C}^{(1)}_{s,\varepsilon}, \quad \mathcal{B}_{s,\varepsilon}, \quad \mathcal{C}^{(2)}_{s,\varepsilon}, \quad \mathcal{P}_{s,\varepsilon},$$

consecutive sets intersect and their union is the cap.

The warping coefficient on the transition cylinder satisfies

$$0 < \frac{\sin(\kappa_\varepsilon r)}{\kappa_\varepsilon}\chi_\varepsilon(r) \le r \le 1.$$

Joining two points by radial segments and a curve in the slice $r = 1$ gives $\operatorname{diam}\mathcal{T}_{s,\varepsilon} \le 2 + \pi$. The transition volume is bounded by

$$\operatorname{Vol}\mathcal{T}_{s,\varepsilon} \le \int_{r_c}^1 \left(\frac{\sin(\kappa_\varepsilon r)}{\kappa_\varepsilon}\chi_\varepsilon(r)\right)^{n-1} \operatorname{Vol}(S^{n-1}, g_{s,\psi_{s,\varepsilon}(r)})\, dr \le 4^{-(n-1)}\bar{\kappa}_Q^{\,n-1}\sigma_{n-1}.$$

Define

$$C_d := \frac{\arccos d}{\sqrt{1-d^2}}, \qquad C_{d,e} := \sqrt{\frac{1-e^2}{1-d^2}},$$
$$W_1 := \frac{\arccos d - \arccos e}{4\sqrt{1-d^2}}, \qquad W_2 := \frac{\min\{\arccos e, \arccos d - \arccos e\}}{4\sqrt{1-d^2}}.$$

These constants satisfy

$$T_d = C_d b, \qquad s_c = C_{d,e} b,$$

and the two support-width bounds are $W_1 b$ and $W_2 b$. Radial segments and a round-slice arc give

$$\operatorname{diam}\mathcal{B}_{s,\varepsilon} \le \left(2C_d + \frac{\pi}{\sqrt{1-d^2}}\right) b, \qquad \operatorname{Vol}\mathcal{B}_{s,\varepsilon} \le \frac{\sigma_{n-1}}{n} C_d^n b^n.$$

Moreover, $\operatorname{Vol}\mathcal{P}_{s,\varepsilon} \le C_{d,e}^n b^n \operatorname{Vol}(P_Q, k_Q)$. For a smoothing collar $\mathcal{C}$, let $\operatorname{diam}_{\mathrm{int}}\mathcal{C}$ denote the diameter of its intrinsic length metric. Equation (7.23) with $L = 2$ gives

$$\operatorname{diam}_{\mathrm{int}}\mathcal{C} \le 2\big(w + \operatorname{diam}(\Sigma, h_\Sigma)\big), \qquad \operatorname{Vol}\mathcal{C} \le 2^n w \operatorname{Vol}(\Sigma, h_\Sigma).$$

At the first seam $h_\Sigma = b^2 g_{S^{n-1}}$ and $w = W_1 b$; at the second seam $h_\Sigma = s_c^2 g_{S^{n-1}}$ and $w = W_2 b$. Hence

$$\begin{aligned} \operatorname{diam}_{\mathrm{int}} C^{(1)}_{s,\varepsilon} &\le 2(W_1 + \pi) b, & \operatorname{Vol} C^{(1)}_{s,\varepsilon} &\le 2^n W_1 \sigma_{n-1} b^n, \\ \operatorname{diam}_{\mathrm{int}} C^{(2)}_{s,\varepsilon} &\le 2(W_2 + \pi C_{d,e}) b, & \operatorname{Vol} C^{(2)}_{s,\varepsilon} &\le 2^n W_2 C_{d,e}^{n-1} \sigma_{n-1} b^n. \end{aligned}$$

For $x, y \in \mathcal{P}_{s,\varepsilon}$, let γ be a minimizing curve from x to y in $(P_Q, s_c^2 k_Q)$. Outside $C^{(2)}_{s,\varepsilon}$ the metric $\widetilde{K}_{s,\varepsilon}$ equals $s_c^2 k_Q$. If γ meets the collar, let z_- and z_+ be its first and last collar intersections. For every $\eta > 0$, the definition of the intrinsic length metric gives a curve in $C^{(2)}_{s,\varepsilon}$ from z_- to z_+ of length at most $\operatorname{diam}_{\mathrm{int}} C^{(2)}_{s,\varepsilon} + \eta$. Replacing the segment of γ between z_- and z_+ and then letting $\eta \downarrow 0$ gives

$$\operatorname{diam} \mathcal{P}_{s,\varepsilon} \le s_c \operatorname{diam}(P_Q, k_Q) + 2(W_2 + \pi C_{d,e}) b \le \big(C_{d,e} \operatorname{diam}(P_Q, k_Q) + 2W_2 + 2\pi C_{d,e}\big) b. \tag{7.101}$$

Since $0 < b < 1$, the sum of the five diameter bounds, using (7.101), and the sum of the five volume bounds define constants $D_0, V_0 > 0$, depending only on $n, \bar{\kappa}_Q, \mu_Q, \operatorname{diam}(P_Q, k_Q)$, and $\operatorname{Vol}(P_Q, k_Q)$, such that

$$\operatorname{diam}(P_Q, \widetilde{K}_{s,\varepsilon}) \le D_0, \qquad \operatorname{Vol}(P_Q, \widetilde{K}_{s,\varepsilon}) \le V_0.$$

After decreasing ε_0, one has $\lambda_\varepsilon^2 \ge \sin^2 \varepsilon/4$ and hence $c_\varepsilon \le 2\kappa_\varepsilon \le 8\varepsilon$. Constant scaling and pullback preserve the preceding bounds up to the factors c_ε and c_ε^n. Thus (7.79) holds with $D := 8D_0$ and $V := 8^n V_0$.

The derivative $\partial_r \varphi_{s,\varepsilon}$ is positive at $r = r_-(s, \varepsilon)$, so the implicit-function theorem makes r_- smooth. The derivative with respect to κ in (7.90) is $\chi_\varepsilon(1) \cos \kappa_\varepsilon > 0$, so κ_ε is smooth. Equations (7.92) and (7.95) then make $c_\varepsilon, r_c, b, c, \Lambda^{\mathrm{rd}}, T_d, T_e$, and s_c smooth in their parameters. The pullback metrics on C, D^n, the retained round annulus, and P_Q are smooth vertical metrics on fixed bundles over $\mathcal{B}$. The two applications of Proposition 7.2 give smooth functions δ_r, ν_r and smooth glued vertical metrics, $r = 1, 2$. The fixed collar identification, constant scaling, and the smooth map $(s, t, x) \mapsto \mathcal{F}_s(t, x)$ preserve smooth parameter dependence. Thus $(s, \varepsilon) \mapsto K_{s,\varepsilon}$ is smooth.

Suppose

$$h_s = h_0, \qquad \widehat{g}_{s,u} = \widehat{g}_{0,u} \quad (s \ge 0,\ -2 \le u \le 2).$$

Choose one background metric and its Green operator for all s, use the s-constant Moser flow defined by these choices, and choose $A_{\mathrm{maj}}(s) = A_*$ with $A_* > B(0)$. The formulas defining the scalar functions and the metrics on the four fixed pieces then contain ε but no s. At each seam apply Proposition 7.4 with the trivial group and parameter manifold $(0, \varepsilon_0)$. Its ordinary conclusion produces a smooth downward-admissible pair (δ_r^*, ν_r^*), $r = 1, 2$, and a glued metric depending only on ε. Constant scaling and the collar pullback also depend only on ε; call the final cap K_ε^*. Pulling this family back by $[0, \infty) \times (0, \varepsilon_0) \to (0, \varepsilon_0)$ proves (a).

Suppose

$$h_s = h_0, \qquad \widehat{g}_{s,u} = \widehat{g}_{0,u} \quad (0 \le s \le 3s_0,\ -2 \le u \le 2).$$

Then $B(s) = B(0)$ on this interval. Let A_* be the majorant used to construct the reference family K_ε^*, chosen so that

$$A_* > B(s) \qquad (0 \le s \le 3s_0).$$

Continuity of B gives $\eta_0 > 0$ such that $B(s) < A_*$ on $[0, 3s_0 + \eta_0]$. Let $A_1 := A_{\mathrm{maj}}$ and choose $v \in C^\infty([0, \infty), [0, 1])$ with

$$v = 0 \quad \text{on } [0, 3s_0], \qquad v = 1 \quad \text{on } [3s_0 + \eta_0, \infty).$$

Define $\widetilde{A}(s) := A_* + v(s) A_1(s)$. On $[0, 3s_0 + \eta_0]$ one has $\widetilde{A} \ge A_* > B$; on the complement one has $\widetilde{A} = A_* + A_1 > B$. Replacing A_{maj} by $\widetilde{A}$ makes $L(s)$, ρ_s, and $k_{s,v}$ equal to the values used for K_ε^* on $0 \le s \le 3s_0$. Equations (7.84)– (7.95) then give equality there of $\chi_\varepsilon, \varphi_{s,\varepsilon}, \psi_{s,\varepsilon}, r_-, G^0_{s,\varepsilon}, G_{s,\varepsilon}, \kappa_\varepsilon, c_\varepsilon, r_c, b, c, \Lambda^{\mathrm{rd}}, T_d, T_e$, and s_c with the functions used for K_ε^*.

Use the reference cap family K_ε^* from (a). Let $\delta_i^*(\varepsilon), \nu_i^*(\varepsilon)$, $i = 1, 2$, be its interpolation and mollification widths at the two seams. They are smooth on $(0, \varepsilon_0)$ and are downward admissible for seam i in the construction of K^*_ε, $i = 1, 2$. At each seam retain the auxiliary background metrics and cutoff used for this reference family. After selecting the seam widths below, choose the straightening radius by the relative construction on the corresponding set $B_0^{(i)}$. The input metrics and the gluing widths agree with the reference data there, so its radius is admissible there. The resulting identifications agree with the reference identifications on $B_0^{(i)}$; thus the following metric equalities hold on the fixed manifolds. For the paired copies all these choices are transported by the copy maps, so the identifications are equivariant as well.

For the first seam set

$$B_0^{(1)} = [0, \tfrac{5}{2}s_0] \times (0, \varepsilon_0), \qquad B_1^{(1)} = [0, 3s_0) \times (0, \varepsilon_0),$$

and define on $B_1^{(1)}$

$$\delta^{(1)}_{\mathrm{pr}}(s, \varepsilon) := \delta^*_1(\varepsilon), \qquad \nu^{(1)}_{\mathrm{pr}}(s, \varepsilon) := \nu^*_1(\varepsilon).$$

On $B_1^{(1)}$, the transition and round metrics, the two boundary maps, $\lambda^{(1)}, \eta^{(1)}, w^{(1)}, L^{(1)}$, and both normal-collar tensors coincide with those used to construct K^*_ε. Hence $(\delta^{(1)}_{\mathrm{pr}}, \nu^{(1)}_{\mathrm{pr}})$ is downward admissible there. The set $B_1^{(1)}$ is open in $\mathcal{B} = [0, \infty) \times (0, \varepsilon_0)$, whereas $B_0^{(1)}$ is closed in $\mathcal{B}$ and $B_0^{(1)} \subset B_1^{(1)}$. Write $G^{(1),*}_\varepsilon$ for the first glued metric in the construction of K^*_ε. Apply Proposition 7.3. If $\delta^{(1)}, \nu^{(1)}$ are the selected widths and $G^{(1)}_{s,\varepsilon}$ is the first glued metric, then

$$(\delta^{(1)}, \nu^{(1)}) = (\delta^{(1)}_{\mathrm{pr}}, \nu^{(1)}_{\mathrm{pr}}), \qquad G^{(1)}_{s,\varepsilon} = G^{(1),*}_\varepsilon \quad \text{on } B_0^{(1)}. \tag{7.102}$$

The first smoothing support is disjoint from the round region $0 \le t \le T_e$. Since

$$B_1^{(2)} = [0, \tfrac{5}{2}s_0) \times (0, \varepsilon_0) \subset B_0^{(1)},$$

(7.102) and the equality of T_d, T_e, s_c imply that the first glued metric near $t = T_e$, the retained round annulus, its boundary map, the scaled core metric, and $\lambda^{(2)}, \eta^{(2)}, w^{(2)}, L^{(2)}$ coincide with those used to construct K^*_ε on $B_1^{(2)}$. Define there

$$\delta^{(2)}_{\mathrm{pr}}(s, \varepsilon) := \delta^*_2(\varepsilon), \qquad \nu^{(2)}_{\mathrm{pr}}(s, \varepsilon) := \nu^*_2(\varepsilon).$$

This pair is downward admissible for the second seam. Put $B_0^{(2)} = [0, 2s_0] \times (0, \varepsilon_0)$. The set $B_1^{(2)}$ is open in $\mathcal{B}$, the set $B_0^{(2)}$ is closed in $\mathcal{B}$, and $B_0^{(2)} \subset B_1^{(2)}$. Write $G^{(2),*}_\varepsilon$ for the twice-glued metric in the construction of K^*_ε. A second application of Proposition 7.3 gives, for the selected widths $\delta^{(2)}, \nu^{(2)}$ and the twice-glued metric $G^{(2)}_{s,\varepsilon}$,

$$(\delta^{(2)}, \nu^{(2)}) = (\delta^{(2)}_{\mathrm{pr}}, \nu^{(2)}_{\mathrm{pr}}), \qquad G^{(2)}_{s,\varepsilon} = G^{(2),*}_\varepsilon \quad \text{on } B_0^{(2)}. \tag{7.103}$$

Choose once the collar diffeomorphism identifying the twice-glued manifold with P_Q and use it in both constructions. On $B_0^{(2)}$, the scaling factor c_ε is the same in the reference and global constructions, and $\widehat{g}_{s,u} = \widehat{g}_{0,u}$ for $-2 \le u \le 2$. The background metric, Green operator, and initial condition $\phi_{s,-2} = \mathrm{id}$ are fixed. Thus

$$(\omega_{s,u}, \alpha_{s,u}, \eta_{s,u}, X_{s,u}) = (\omega_{0,u}, \alpha_{0,u}, \eta_{0,u}, X_{0,u}) \quad ((s, \varepsilon) \in B_0^{(2)}).$$

Uniqueness for the flow equation gives $\phi_{s,u} = \phi_{0,u}$ there, and the fixed collar cutoff gives $\mathcal{F}_s = \mathcal{F}_0$. Equation (7.103) therefore yields

$$K_{s,\varepsilon} = K^*_\varepsilon \qquad ((s, \varepsilon) \in B_0^{(2)}). \tag{7.104}$$

For the paired-copy assertion, make the background Hodge data and all scalar auxiliary choices identical under the fixed model-sphere identification, and transport the radial and core collars through the fixed copy diffeomorphism. Then every pre-gluing piece and seam datum is carried to its counterpart by the copy map. Fix one seam type and denote the pieces on its two sides by M_i^+, M_i^-, $i = 1, 2$, their metrics by h_i^+, h_i^-, the seam hypersurfaces by Σ^+, Σ^-, and the boundary maps by $\beta_i^\pm : \Sigma^\pm \to \partial M_i^\pm$.

Denote the common pulled-back boundary metric and the two pulled-back second fundamental forms by

$$h_\Sigma^\pm := (\beta_1^\pm)^*(h_1^\pm|_{\partial M_1^\pm}) = (\beta_2^\pm)^*(h_2^\pm|_{\partial M_2^\pm}),$$
$$\mathrm{II}_i^\pm := (\beta_i^\pm)^*\mathrm{II}_{\partial M_i^\pm, h_i^\pm} \qquad (i = 1, 2).$$

Let $\iota_i : M_i^+ \to M_i^-$ and $\bar\iota : \Sigma^+ \to \Sigma^-$ be the copy identifications. They satisfy, for every (s, ε),

$$\iota_i \circ \beta_i^+ = \beta_i^- \circ \bar\iota, \qquad \iota_i^* h_i^-(s, \varepsilon) = h_i^+(s, \varepsilon). \tag{7.105}$$

Define $M_i := M_i^+ \sqcup M_i^-$ and $\Sigma := \Sigma^+ \sqcup \Sigma^-$. The component-exchanging diffeomorphisms are

$$\gamma_i|_{M_i^+} := \iota_i, \qquad \gamma_i|_{M_i^-} := \iota_i^{-1}, \qquad \bar\gamma|_{\Sigma^+} := \bar\iota, \qquad \bar\gamma|_{\Sigma^-} := \bar\iota^{-1}.$$

The nontrivial element γ of $\mathbb{Z}/2$ fixes $\mathcal{B}$ and acts by these maps. For every $b \in \mathcal{B}$, set $\gamma_{i,b} := \gamma_i$ and $\bar\gamma_b := \bar\gamma$. Since $\gamma_{i,b}^2 = \mathrm{id}_{M_i}$ and $\bar\gamma_b^2 = \mathrm{id}_\Sigma$, (7.9) and (7.11) hold with $\sigma_\gamma = \mathrm{id}$. Equation (7.105) is (7.10) for these maps. The isometries in (7.105) map inward unit normals to inward unit normals. The definition of II therefore gives

$$\bar\iota^* h_\Sigma^- = h_\Sigma^+, \qquad \bar\iota^*\mathrm{II}_i^- = \mathrm{II}_i^+ \quad (i = 1, 2).$$

The functions λ, η, w have equal restrictions to the two seam components. When the widths from (b) are used, the sets $B_j^{(r)}$ are invariant and both components carry the same prescribed pair (δ_r^*, ν_r^*); thus $\gamma B_j^{(r)} = B_j^{(r)}$ for $j = 0, 1$ and $r = 1, 2$.

Apply the first assertion of Proposition 7.4 to the first seam pair when no width pair is fixed on a subset. When the width pair from (b) is fixed, apply (7.33)– (7.34). In either case,

$$\delta_1^+ = \delta_1^-, \qquad \nu_1^+ = \nu_1^-, \qquad \gamma^* G_{s,\varepsilon}^{(1),-} = G_{s,\varepsilon}^{(1),+}.$$

In the second case (7.102) also holds. The second seam in each copy is the slice $t = T_e$. Since $T_e^+ = T_e^-$ and these slices are disjoint from the first smoothing supports, the descended involution interchanges the second seams and is an isometry of the first glued metric. It maps the two inward normals to one another, so

$$\bar\gamma^* h_\Sigma^- = h_\Sigma^+, \qquad \bar\gamma^*(\mathrm{II}_1^- + \mathrm{II}_2^-) = \mathrm{II}_1^+ + \mathrm{II}_2^+.$$

For the second seam,

$$(\lambda^+, \eta^+, w^+) = (\lambda^-, \eta^-, w^-), \qquad \gamma B_j^{(2)} = B_j^{(2)} \quad (j = 0, 1),$$

and, when the reference widths are fixed, both components carry the prescribed pair (δ_2^*, ν_2^*). If no width pair is fixed on a subset, apply the first assertion of Proposition 7.4; otherwise apply (7.33)–(7.34). Equation (7.31) gives

$$\delta_2^+ = \delta_2^-, \qquad \nu_2^+ = \nu_2^-.$$

In the second case, (7.34) gives (7.103). Thus the width pair at seam r agrees in the plus and minus copies for $r = 1, 2$. In the relative case, the common scaling, fixed collar identification, synchronized Moser flows, and common collar cutoff also reproduce (7.104) on both copies. Hence (b) and (c) hold simultaneously. □

8. Fixed-volume degeneration and Ricci closability

For $\beta > 0$, put $d_n(\beta) := (n-1)\big[1 - (1 + \beta/2)^{-2/n}\big]$. Then $0 < d_n(\beta) < n - 1$.

Theorem 8.1. *There is $q_0 = q_0(n, Q) > 0$ such that, for every $q \in (0, q_0)$, M_Q carries a smooth family g_s, $s \geq 0$, satisfying*

$$\mathrm{Ric}_{g_s} \geq (n-1)g_s, \tag{8.1}$$
$$\mathrm{Vol}(M_Q, g_s) = q\sigma_n, \tag{8.2}$$
$$(M_Q, g_s) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}. \tag{8.3}$$

There is $s_* > 0$ such that $g_s = g_0$ for $0 \le s \le s_*$. The metric g_0 is Ricci closable, and $dv_{g_s} = dv_{g_0}$ for every $s \ge 0$.

Proof. Choose $\gamma_* \in (0,1)$ such that $\gamma_*^{1/(n-1)} < \bar\kappa_Q/4$, and put $h_* := \gamma_*^{2/(n-1)} g_{S^{n-1}}$. Since $\mathrm{Ric}_{g_{S^{n-1}}} = (n-2)g_{S^{n-1}}$ and $\gamma_*^{2/(n-1)} < 1$,

$$\mathrm{Ric}_{h_*} = (n-2)g_{S^{n-1}} \ge (n-2)h_*, \qquad \left(\frac{\mathrm{Vol}(S^{n-1}, h_*)}{\sigma_{n-1}}\right)^{1/(n-1)} = \gamma_*^{1/(n-1)} < \frac{1}{4}\bar\kappa_Q.$$

For $s \ge 0$ and $-2 \le u \le 2$, set $h_s^{\mathrm{ref}} := h_*$ and $\widehat{g}^{\mathrm{ref}}_{s,u} := h_*$. With $\tau_* := \gamma_*^{1/(n-1)} < \bar\kappa_Q/4$, one has

$$\begin{aligned} \widehat{g}^{\mathrm{ref}}_{s,-2} &= \tau_*^2 g_{S^{n-1}}, & \widehat{g}^{\mathrm{ref}}_{s,2} &= h^{\mathrm{ref}}_s, \\ \mathrm{Ric}_{\widehat{g}^{\mathrm{ref}}_{s,u}} &\ge (n-2)\widehat{g}^{\mathrm{ref}}_{s,u}, & \mathrm{Vol}(S^{n-1}, \widehat{g}^{\mathrm{ref}}_{s,u}) &= \mathrm{Vol}(S^{n-1}, h^{\mathrm{ref}}_s) = \gamma_*\sigma_{n-1}. \end{aligned}$$

Define the endpoint neighbourhoods by $U_- = U_+ := [0,\infty) \times [-2,2]$, with the relative topology. These sets are open and contain the two endpoint slices. The constant family is smooth, its endpoint metrics are the two metrics displayed above, its Ricci coefficient is at least $n-2$, its volume is independent of u, and its scale τ_* satisfies $\tau_* < \bar\kappa_Q/4$. Hence Proposition 7.6, together with (a), gives $\varepsilon_{\mathrm{cap}} > 0$ and finite constants $C_{\mathrm{cap}}, D_{\mathrm{cap}}, V_{\mathrm{cap}}$, a metric K^*_ε on P_Q for every $0 < \varepsilon < \varepsilon_{\mathrm{cap}}$, and smooth downward-admissible width pairs (δ^*_r, ν^*_r), $r = 1, 2$. The family obtained by pulling K^*_ε back under $[0,\infty) \times (0, \varepsilon_{\mathrm{cap}}) \to (0, \varepsilon_{\mathrm{cap}})$ satisfies (7.76)–(7.79). Fix the reference cylinder $C := [0,1] \times S^{n-1}$ and, for $0 < \varepsilon < \pi/2$, the diffeomorphism

$$\Theta_\varepsilon(r, x) := (\varepsilon + (\pi - 2\varepsilon)r, x) : C \longrightarrow [\varepsilon, \pi - \varepsilon] \times S^{n-1}.$$

Equip C with the pullback by Θ_ε of $dt^2 + \sin^2 t\, h_*$. Attach oppositely oriented copies of K^*_ε to its two boundary components, using β_Q. The fixed glued manifold

$$P_Q \cup_{S^{n-1}} C \cup_{S^{n-1}} (-P_Q) \cong Q\#(-Q) = M_Q$$

is M_Q for every ε. After this identification, t denotes the physical suspension coordinate supplied by Θ_ε. After pullback to S^{n-1} by the boundary identifications, the two boundary metrics at either seam are $\sin^2\varepsilon\, h_*$. The inward normal of the suspension cylinder is ∂_t at $t = \varepsilon$ and $-\partial_t$ at $t = \pi - \varepsilon$; with the convention of Section 2, both suspension boundaries satisfy $\mathrm{II}_{\mathrm{susp}} = -\cot\varepsilon\, \sin^2\varepsilon\, h_*$. Consequently, $\mathrm{II}_{\mathrm{cap}} + \mathrm{II}_{\mathrm{susp}} > 0$. Choose $0 < \varepsilon_1 < \min\{\varepsilon_{\mathrm{cap}}, \pi/8\}$ so that $n - 1 - (C_{\mathrm{cap}} + 1)\varepsilon^2 > 0$ for $0 < \varepsilon < \varepsilon_1$, and put $\mathcal{B}_{\mathrm{init}} := (0, \varepsilon_1)$. Over $\mathcal{B}_{\mathrm{init}}$ take

$$M_1 := P_Q \sqcup (-P_Q), \qquad M_2 := C, \qquad \Sigma := S^{n-1} \sqcup S^{n-1},$$

with vertical metrics

$$h_1(\varepsilon) := K^*_\varepsilon \sqcup K^*_\varepsilon, \qquad h_2(\varepsilon) := \Theta^*_\varepsilon(dt^2 + \sin^2 t\, h_*).$$

Write $\Sigma = \Sigma^- \sqcup \Sigma^+$, with each component identified with S^{n-1}. Define the boundary maps by

$$\begin{aligned} \beta_1|_{\Sigma^-} &:= \beta_Q : \Sigma^- \to \partial P_Q, & \beta_1|_{\Sigma^+} &:= \beta_Q : \Sigma^+ \to \partial(-P_Q), \\ \beta_2(x, -) &:= (0, x), & \beta_2(x, +) &:= (1, x). \end{aligned}$$

Their common pullback metric is $h_\Sigma(\varepsilon) = \sin^2\varepsilon\, h_*$. Let $\mathfrak{r}$ be the nontrivial element of $\mathbb{Z}/2$. It fixes ε, interchanges the two components of M_1, acts on M_2 by $(r, x) \mapsto (1 - r, x)$, and interchanges the components of Σ. The homomorphism $\sigma : \mathbb{Z}/2 \to \mathfrak{S}_2$ is trivial because $\mathfrak{r}$ preserves the cap side M_1 and the cylinder side M_2. The boundary maps satisfy (7.10); the vertical metrics satisfy (7.12); and the signed normal coordinates at the two seams are carried to one another without sign reversal. Define

$$\begin{aligned} \lambda_{\mathrm{init}}(\varepsilon) &:= n - 1 - (C_{\mathrm{cap}} + 2)\varepsilon^2, & \eta_{\mathrm{init}}(\varepsilon) &:= \frac{1}{2}\varepsilon^2, \\ w_{\mathrm{init}}(\varepsilon) &:= \varepsilon, & L &:= 2. \end{aligned}$$

In the standing gluing notation on $\mathcal{B}_{\mathrm{init}}$ use $(\lambda, \eta, w) := (\lambda_{\mathrm{init}}, \eta_{\mathrm{init}}, w_{\mathrm{init}})$. For the suspension metric, the warped-product formulas give

$$\mathrm{Ric}(\partial_t, \partial_t) = n-1, \qquad \mathrm{Ric}((\sin t)^{-1}v, (\sin t)^{-1}v) = 1 + \frac{\mathrm{Ric}_{h_*}(v,v) - (n-2)\cos^2 t}{\sin^2 t} \geq n-1$$

for every h_*-unit vector v. Hence the suspension has Ricci coefficient at least $n-1$, while

$$n - 1 - C_{\mathrm{cap}}\varepsilon^2 > \lambda_{\mathrm{init}} + 2\eta_{\mathrm{init}} = n - 1 - (C_{\mathrm{cap}} + 1)\varepsilon^2.$$

The common boundary identity above is (7.1), and $\beta_Q^*\mathrm{II}_{\mathrm{cap}} + \mathrm{II}_{\mathrm{susp}} > 0$ is (7.3). The cap and cylinder Ricci estimates are strictly greater than $\lambda_{\mathrm{init}} + 2\eta_{\mathrm{init}}$, so (7.2) holds on both components of Σ. The $\mathbb{Z}/2$ maps satisfy (7.9)– (7.12). Apply the ordinary assertion of Proposition 7.4 on $\mathcal{B}_{\mathrm{init}}$. Denote the equal interpolation and mollification widths at the two reflected seams by $\delta^*_{\mathrm{out}}(\varepsilon)$ and $\nu^*_{\mathrm{out}}(\varepsilon)$. Their downward admissibility gives (a) for every $0 < \widetilde{\delta} \leq \delta^*_{\mathrm{out}}(\varepsilon)$ and (b) for every $0 < \widetilde{\nu} \leq \nu^*_{\mathrm{out}}(\varepsilon)$. The metric $\widehat{G}_\varepsilon$ on M_Q satisfies $\mathfrak{r}^*\widehat{G}_\varepsilon = \widehat{G}_\varepsilon$ and

$$\mathrm{Ric}_{\widehat{G}_\varepsilon} > \left(n - 1 - (C_{\mathrm{cap}} + \tfrac{3}{2})\varepsilon^2\right)\widehat{G}_\varepsilon.$$

Use the equivariant collar identifications to regard this as a smooth family on the fixed M_Q. Choose their supports inside the two outer gluing collars. They are the identity on a neighbourhood of $[2\varepsilon, \pi - 2\varepsilon] \times S^{n-1}$, in particular near the equator. Retain these auxiliary data and straightening radii for the reference family used below. Let E^-_ε be the union of the lower cap, its smoothing collar, and the suspension strip up to $t = 2\varepsilon$; let E^+_ε be its reflected copy. For such a connected polar piece E, write $\mathrm{diam}_{\mathrm{int}}\, E$ for the diameter of its intrinsic length metric. Let $S(h_*)$ denote the complete suspension of (S^{n-1}, h_*), with metric $g_{S(h_*)}$. Its volume is $\gamma_*\sigma_{n-1}\int_0^\pi \sin^{n-1} t\, dt = \gamma_*\sigma_n$. Denote the two smoothing collars by $C^\pm_\varepsilon$. Their signed normal length is less than ε. Equation (7.30), with $L = 2$, gives

$$\mathrm{Vol}(C^\pm_\varepsilon) \leq 2^n\gamma_*\sigma_{n-1}\varepsilon^n, \qquad \mathrm{diam}_{\mathrm{int}}(C^\pm_\varepsilon) \leq (2 + 2\pi)\varepsilon.$$

The suspension polar region satisfies

$$\mathrm{Vol}\big([0, 2\varepsilon] \times S^{n-1}, dt^2 + \sin^2 t\, h_*\big) \leq \frac{2^n\gamma_*\sigma_{n-1}}{n}\varepsilon^n.$$

Consequently,

$$\mathrm{Vol}(E^\pm_\varepsilon) \leq \left(V_{\mathrm{cap}} + 2^n\gamma_*\sigma_{n-1} + \frac{2^n\gamma_*\sigma_{n-1}}{n}\right)\varepsilon^n.$$

Let $P^-_\varepsilon \subset S(h_*)$ and $P^+_\varepsilon \subset S(h_*)$ be the closed polar regions with suspension coordinate in $[0, 2\varepsilon]$ and $[\pi - 2\varepsilon, \pi]$. Suspension coordinates induce a Riemannian isometry

$$(M_Q \setminus \mathrm{int}(E^-_\varepsilon \cup E^+_\varepsilon), \widehat{G}_\varepsilon) \longrightarrow (S(h_*) \setminus \mathrm{int}(P^-_\varepsilon \cup P^+_\varepsilon), g_{S(h_*)})$$

which is the identity on the two boundary slices. Therefore the absolute volume difference is bounded by the sum of the volumes of the four removed regions, and

$$\left|\mathrm{Vol}(M_Q, \widehat{G}_\varepsilon) - \gamma_*\sigma_n\right| \leq C_*\varepsilon^n, \qquad \mathrm{diam}_{\mathrm{int}}\, E^\pm_\varepsilon \leq D_*\varepsilon, \qquad \mathrm{diam}\, E^\pm_\varepsilon \leq D_*\varepsilon, \tag{8.4}$$

where one may take

$$C_* := 2V_{\mathrm{cap}} + 2^{n+1}\gamma_*\sigma_{n-1} + \frac{2^{n+2}\gamma_*\sigma_{n-1}}{n}, \qquad D_* := 2D_{\mathrm{cap}} + 8 + 6\pi.$$

Let $\eta > 0$. A point in the retained cap can be joined to its original boundary in (P_Q, K^*_ε) by a curve of length at most $D_{\mathrm{cap}}\varepsilon + \eta$. Stop this curve at its first intersection with C^-_ε. The retained part has unchanged metric. From that intersection, join to the suspension-side boundary of C^-_ε by a curve in the collar of length at most $(2 + 2\pi)\varepsilon + \eta$, and then follow a radial segment to $t = 2\varepsilon$, of length at most 2ε. For a point already in the collar omit the first part; for a point in the remaining suspension strip use only the radial segment. All these curves lie in E_ε^-. Thus the intrinsic distance of every point of E_ε^- to the slice $t = 2\varepsilon$ is at most $(D_{\mathrm{cap}} + 4 + 2\pi)\varepsilon$, after letting $\eta \downarrow 0$. The intrinsic diameter of that slice is at most $2\pi\varepsilon$. Joining two points through the slice gives $\operatorname{diam}_{\mathrm{int}} E_\varepsilon^- \le (2D_{\mathrm{cap}} + 8 + 6\pi)\varepsilon$; reflection gives the same bound for E_ε^+. The ambient diameter is no greater than the intrinsic one. Fix $0 < \varepsilon_* < \varepsilon_1$ so small that, for every $0 < \varepsilon \le \varepsilon_*$,

$$\mathrm{Ric}_{\widehat{G}_\varepsilon} \ge \frac{n-1}{4}\widehat{G}_\varepsilon, \qquad \left|\mathrm{Vol}(M_Q, \widehat{G}_\varepsilon) - \gamma_*\sigma_n\right| \le C_*\varepsilon^n, \qquad C_*\varepsilon_*^n < \frac{1}{8}\gamma_*\sigma_n. \tag{8.5}$$

Define the reference constants by

$$\lambda_* := \frac{n-1}{4}, \qquad \widehat{G}_* := \widehat{G}_{\varepsilon_*}, \qquad V_* := \mathrm{Vol}(M_Q, \widehat{G}_*).$$

The identity $\mathfrak{r}^*\widehat{G}_* = \widehat{G}_*$ and $\mathfrak{r}^2 = \mathrm{id}$ define an isometric $\mathbb{Z}/2$-action on the closed manifold M_Q. Its fixed-point set is the nonempty embedded hypersurface $Y_* := \{\pi/2\} \times S^{n-1}$. The vector field ∂_t is a nowhere-vanishing normal field along Y_*, so the normal bundle is trivial. Since $\mathrm{Ric}_{\widehat{G}_*} \ge \lambda_*\widehat{G}_* > 0$, all hypotheses of [13, Proposition 3.2] hold. More precisely, the construction in the proof provides a compact smooth $(n+1)$-manifold with positive Ricci curvature and strictly convex boundary isometric to the prescribed metric. If its preliminary normalization replaces $\widehat{G}_*$ by $s^2\widehat{G}_*$, multiply the filling metric by s^{-2}. The induced boundary metric is then $\widehat{G}_*$, while positive Ricci curvature and strict convexity are preserved. Retain the connected component containing the connected boundary M_Q, discarding any closed components. Denote the resulting filling by (W_Q, H_Q) and its boundary identification by $\beta_{\mathrm{fill}} : M_Q \longrightarrow \partial W_Q$. Thus

$$\beta_{\mathrm{fill}}^*(H_Q|_{\partial W_Q}) = \widehat{G}_*, \qquad \mathrm{Ric}_{H_Q} > 0, \qquad \beta_{\mathrm{fill}}^*\mathrm{II}_{\partial W_Q, H_Q} > 0.$$

Here the second fundamental form has the convention of Section 2. Compactness gives $\kappa_{\mathrm{fill}} > 0$ such that

$$\beta_{\mathrm{fill}}^*\mathrm{II}_{\partial W_Q, H_Q} \ge \kappa_{\mathrm{fill}}\widehat{G}_*.$$

Choose $c_Q > 0$ satisfying

$$2c_Q < \kappa_{\mathrm{fill}}, \qquad 4(n-1)c_Q^2 < \lambda_*.$$

Fix $\epsilon \in (0,1)$ and put $a_\epsilon = c_Q\sqrt{1-\epsilon}$. Define

$$\chi(t) = \begin{cases} \exp\left(-\dfrac{t}{1-t}\right), & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \qquad f_\epsilon(t) = 1 + a_\epsilon \int_0^t (1 + \chi(s))\, ds.$$

For $t < 1$, one has $\chi(t) = \exp(1)\exp(-(1-t)^{-1})$; hence every derivative tends to zero as $t \uparrow 1$. Thus χ and f_ϵ are smooth on $[0,\infty)$. Moreover,

$$f_\epsilon(0) = 1, \qquad f_\epsilon'(0) = 2a_\epsilon, \qquad a_\epsilon \le f_\epsilon' \le 2a_\epsilon,$$
$$f_\epsilon'' < 0 \quad (0 \le t < 1), \qquad f_\epsilon'' = 0 \quad (t \ge 1).$$

On $[0,\infty) \times M_Q$, consider

$$P_\epsilon = dt^2 + f_\epsilon(t)^2\widehat{G}_*.$$

The warped-product Ricci formulas give

$$\mathrm{Ric}_{P_\epsilon}(\partial_t, \partial_t) = -n\frac{f_\epsilon''}{f_\epsilon} \ge 0, \qquad \mathrm{Ric}_{P_\epsilon}(\partial_t, f_\epsilon^{-1}v) = 0,$$

and, for every $\widehat{G}_*$-unit vector v,

$$\begin{aligned}\mathrm{Ric}_{P_\epsilon}(f_\epsilon^{-1}v, f_\epsilon^{-1}v) &= \frac{\mathrm{Ric}_{\widehat{G}_*}(v,v) - (n-1)(f_\epsilon')^2}{f_\epsilon^2} - \frac{f_\epsilon''}{f_\epsilon} \\ &\ge \frac{\lambda_* - 4(n-1)c_Q^2}{f_\epsilon^2} > 0.\end{aligned}$$

Thus P_ϵ has nonnegative Ricci curvature everywhere and positive Ricci curvature on $[0,1)\times M_Q$. At $t=0$, its boundary metric is $\widehat{G}_*$ and its inward-convention second fundamental form is $\mathrm{II}_{\mathrm{cyl}} = -2a_\epsilon\widehat{G}_*$. Consequently,

$$\beta^*_{\mathrm{fill}}\mathrm{II}_{\partial W_Q,H_Q} + \mathrm{II}_{\mathrm{cyl}} \geq (\kappa_{\mathrm{fill}} - 2a_\epsilon)\widehat{G}_* > 0.$$

Apply [16, Theorem A(i)], in ambient dimension $n+1$ with $k=n$, to (W_Q,H_Q) and $([0,1)\times M_Q, P_\epsilon)$. Both metrics have positive Ricci curvature, their compact boundaries are isometric, and the sum of their second fundamental forms is positive definite. Choose the smoothing supported in a sufficiently small neighbourhood of the seam, disjoint from $\{t\geq 1/4\}$. Extend by the unchanged metric P_ϵ on the remainder of the cylinder. The two descriptions agree on an open overlap and give a connected smooth manifold without boundary, with nonnegative Ricci curvature.

On this manifold define $u=0$ on the filling and on the cylinder segment $t\leq 2$, and $u=t-2$ for $t\geq 2$. Near $t=2$, the metric is unchanged and $u=\max\{t-2,0\}$, so u is locally 1-Lipschitz. Integration along curves makes it globally 1-Lipschitz. Its sublevel sets are compact, since they consist of the compact filling and bounded closed cylinder segments. Every closed ball centred in the filling is therefore compact; any other closed bounded ball is a closed subset of such a ball. The manifold is proper and complete.

For $t\geq 1$,

$$f_\epsilon(t) = a_\epsilon t + B_\epsilon, \qquad B_\epsilon = 1 + a_\epsilon\int_0^1 \chi(s)\,ds > 0.$$

Put $A_\epsilon = 2 + B_\epsilon/a_\epsilon$, and let K_ϵ be the compact region on the filling side of the slice $t=2$. Scale the entire metric by A_ϵ^{-2}. On $t\geq 2$, the coordinate

$$R = \frac{t + B_\epsilon/a_\epsilon}{A_\epsilon}$$

ranges over $[1,\infty)$, and the scaled metric is

$$dR^2 + R^2 a_\epsilon^2\widehat{G}_* = dR^2 + R^2(1-\epsilon)c_Q^2\widehat{G}_*.$$

Choose a basepoint in the interior of the filling. These data satisfy Definition 2.2. Since $\epsilon\in(0,1)$ was arbitrary, $c_Q^2\widehat{G}_*$ is Ricci closable.

One has

$$n-1-d_n\left(\frac{\gamma_*}{q}-1\right) = (n-1)\left(\frac{2q}{\gamma_*+q}\right)^{2/n} \longrightarrow 0 \quad (q\downarrow 0).$$

Moreover, $(q\sigma_n/V_*)^{2/n}\to 0$ as $q\downarrow 0$. If $q<\gamma_*/2$, then

$$\frac{1}{2}(\gamma_*-q)\sigma_n > \frac{1}{4}\gamma_*\sigma_n > C_*\varepsilon_*^n$$

by (8.5). Hence there exists $q_0\in(0,\gamma_*/2)$ such that, whenever $0<q<q_0$,

$$n-1-d_n\left(\frac{\gamma_*}{q}-1\right) < \lambda_*, \tag{8.6}$$

$$\left(\frac{q\sigma_n}{V_*}\right)^{2/n} \leq c_Q^2, \tag{8.7}$$

$$\frac{1}{2}(\gamma_*-q)\sigma_n > C_*\varepsilon_*^n. \tag{8.8}$$

Fix such a q and apply Theorem 5.3; the inequalities $0<q<\gamma_*<1$ are among its hypotheses. Thus

$$\mathrm{Ric}_{h_s}\geq (n-2)h_s, \qquad \mathrm{Vol}(S^{n-1},h_s) = \gamma(s)\sigma_{n-1},$$

$$(S^{n-1},h_s)\xrightarrow[s\to\infty]{\mathrm{GH}} B_{n,q}.$$

Moreover, $\tau_s = \gamma(s)^{1/(n-1)} \le \gamma_*^{1/(n-1)} < \bar\kappa_Q/4$. Let $g_{s,u}$ and $\phi_{s,u}$ be the path and isotopy in Theorem 5.3, and let $s_{\mathrm{pl}} > 0$ be the length of the initial plateau supplied by that theorem. Define $\widehat{g}^{\mathrm{cap}}_{s,u} := (\phi^{-1}_{s,u})^* g_{s,u}$. Pullback by $\phi^{-1}_{s,u}$ gives

$$\widehat{g}^{\mathrm{cap}}_{s,-2} = \gamma(s)^{2/(n-1)} g_{S^{n-1}}, \qquad \widehat{g}^{\mathrm{cap}}_{s,2} = h_s,$$
$$\mathrm{Ric}_{\widehat{g}^{\mathrm{cap}}_{s,u}} \ge (n-2)\widehat{g}^{\mathrm{cap}}_{s,u}, \qquad \mathrm{Vol}(S^{n-1}, \widehat{g}^{\mathrm{cap}}_{s,u}) = \gamma(s)\sigma_{n-1}.$$

Let $\delta_{\mathrm{pl}} > 0$ be the fixed endpoint-plateau width in Theorem 5.3. The endpoint identities for $g_{s,u}$ and $\phi_{s,u}$ imply

$$\widehat{g}^{\mathrm{cap}}_{s,u} = \widehat{g}^{\mathrm{cap}}_{s,-2} \quad (-2 \le u \le -2+\delta_{\mathrm{pl}}), \qquad \widehat{g}^{\mathrm{cap}}_{s,u} = \widehat{g}^{\mathrm{cap}}_{s,2} \quad (2-\delta_{\mathrm{pl}} \le u \le 2).$$

Choose $s_0 \in (0, s_{\mathrm{pl}}/3]$. Then

$$h_s = h_0, \qquad g_{s,u} = g_{0,u}, \qquad \phi_{s,u} = \phi_{0,u} \quad (0 \le s \le 3s_0,\ -2 \le u \le 2).$$

These identities imply

$$h_s = h_0, \qquad \widehat{g}^{\mathrm{cap}}_{s,u} = \widehat{g}^{\mathrm{cap}}_{0,u} \quad (0 \le s \le 3s_0,\ -2 \le u \le 2).$$

The scale satisfies $\tau_s = \gamma(s)^{1/(n-1)} < \bar\kappa_Q/4$. The families h_s and $\widehat{g}^{\mathrm{cap}}_{s,u}$ are smooth. Their endpoint metrics are $\gamma(s)^{2/(n-1)} g_{S^{n-1}}$ and h_s; their Ricci coefficients are at least $n-2$; their total volumes equal $\gamma(s)\sigma_{n-1}$; and $\tau_s < \bar\kappa_Q/4$. The relative open sets

$$U_- = [0,\infty) \times [-2, -2+\delta_{\mathrm{pl}}), \qquad U_+ = [0,\infty) \times (2-\delta_{\mathrm{pl}}, 2]$$

contain the endpoint slices and carry the required u-plateaux. Apply Proposition 7.6 using K^*_ε and the four downward-admissible widths from (a), and use (b) with the number s_0. This gives $\varepsilon_0 > 0$, finite constants C, D, V, and a smooth cap family $K_{s,\varepsilon}$, $s \ge 0$, $0 < \varepsilon < \varepsilon_0$, satisfying

$$K_{s,\varepsilon} = K^*_\varepsilon \qquad (0 \le s \le 2s_0,\ 0 < \varepsilon < \varepsilon_0).$$

Decrease ε_0, if necessary, so that $\varepsilon_0 \le \varepsilon_1$, and increase C, if necessary, so that $C \ge C_{\mathrm{cap}}$. Choose $0 < \varepsilon_{\mathrm{join}} < \frac{1}{4}\min\{\varepsilon_0, \varepsilon_*\}$. The family satisfies

$$\mathrm{Ric}_{K_{s,\varepsilon}} \ge (n-1-C\varepsilon^2) K_{s,\varepsilon},$$
$$\operatorname{diam} K_{s,\varepsilon} \le D\varepsilon, \qquad \operatorname{Vol} K_{s,\varepsilon} \le V\varepsilon^n.$$

Let the global parameter space be $\mathcal{B} = [0,\infty) \times (0, \varepsilon_0)$. For the two outer seams, put

$$M_1 := P_Q \sqcup (-P_Q), \qquad M_2 := \mathcal{C}, \qquad \Sigma := S^{n-1} \sqcup S^{n-1},$$

and define the vertical metrics on $\mathcal{B}$ by

$$h_1(s,\varepsilon) := K_{s,\varepsilon} \sqcup K_{s,\varepsilon}, \qquad h_2(s,\varepsilon) := \Theta^*_\varepsilon(dt^2 + \sin^2 t\, h_s).$$

The boundary maps are the two copies of β_Q and the restrictions of Θ_ε to $r = 0, 1$; the common pullback metric is $h_\Sigma(s,\varepsilon) = \sin^2 \varepsilon\, h_s$. Let $\mathfrak{r}$ act trivially on $\mathcal{B}$, interchange the components of M_1, act on M_2 by $(r,x) \mapsto (1-r, x)$, and interchange the components of Σ. Again $\sigma : \mathbb{Z}/2 \to \mathfrak{S}_2$ is trivial. The boundary maps satisfy (7.10); the vertical metrics satisfy (7.12); and the restrictions of λ, η, w to the paired seam components are equal. Set

$$\lambda(s,\varepsilon) = n-1-(C+2)\varepsilon^2, \qquad \eta(s,\varepsilon) = \frac{1}{2}\varepsilon^2,$$
$$w(s,\varepsilon) = \varepsilon, \qquad L = 2.$$

For the relative choice put

$$\mathcal{B}_0 = [0, s_0] \times (0, \varepsilon_0), \qquad \mathcal{B}_1 = [0, 2s_0) \times (0, \varepsilon_0).$$

The first set is closed in $\mathcal{B}$, and the second is open. On $\mathcal{B}_1$ the two metric families, boundary identifications, second fundamental forms, and the functions η, w equal those used to construct $\widehat{G}_\varepsilon$. The reference curvature function is $\lambda_{\mathrm{init}}(\varepsilon) = n-1-(C_{\mathrm{cap}}+2)\varepsilon^2$, and $C \geq C_{\mathrm{cap}}$ gives $\lambda(s,\varepsilon) \leq \lambda_{\mathrm{init}}(\varepsilon)$. Prescribe $\delta^*_{\mathrm{out}}(\varepsilon)$ and $\nu^*_{\mathrm{out}}(\varepsilon)$ on $\mathcal{B}_1$, using identical functions on the two reflected seams. Their reference downward admissibility gives, for every smaller interpolation width and, at the selected interpolation width, every smaller mollification width, the reference gluing bounds, including

$$\mathrm{Ric} > \big(\lambda_{\mathrm{init}} + \eta\big) g = \big(n-1-(C_{\mathrm{cap}} + \tfrac{3}{2})\varepsilon^2\big) g.$$

This Ricci bound implies the corresponding bound with λ in place of λ_{init}, and the bilipschitz condition is unchanged. Hence the prescribed pair is downward admissible for the present data. In particular, its smoothed metric satisfies

$$\mathrm{Ric} > \big(n-1-(C+\tfrac{3}{2})\varepsilon^2\big) g = (\lambda+\eta) g.$$

Moreover,

$$\lambda + 2\eta = n-1-(C+1)\varepsilon^2 < n-1-C\varepsilon^2.$$

For the suspension cylinder, the warped-product Ricci formulas and $\mathrm{Ric}_{h_s} \geq (n-2)h_s$ give

$$\mathrm{Ric}_{\Theta^*_\varepsilon(dt^2+\sin^2 t\, h_s)} \geq (n-1)\Theta^*_\varepsilon(dt^2+\sin^2 t\, h_s) > (\lambda+2\eta)\Theta^*_\varepsilon(dt^2+\sin^2 t\, h_s).$$

The cylinder inward second fundamental form at either seam is $\mathrm{II}_{\mathrm{susp}} = -\cot\varepsilon\, h_\Sigma(s,\varepsilon)$, whereas (7.77) gives $\beta^*_Q \mathrm{II}_{\mathrm{cap}} > \cot\varepsilon\, h_\Sigma(s,\varepsilon)$. Thus their sum is positive definite. The cap and cylinder Ricci estimates are strictly greater than $\lambda + 2\eta$, so (7.2) holds. The reflected fibre maps satisfy (7.9)– (7.12); moreover $\mathfrak{r}\mathcal{B}_j = \mathcal{B}_j$ for $j = 0, 1$, and

$$\delta^*_{\mathrm{out}}(\mathfrak{r}b) = \delta^*_{\mathrm{out}}(b), \qquad \nu^*_{\mathrm{out}}(\mathfrak{r}b) = \nu^*_{\mathrm{out}}(b) \quad (b \in \mathcal{B}_1).$$

Apply the relative assertion (7.33)– (7.34) of Proposition 7.4, and denote the metric produced on M_Q by $\widehat{G}(s,\varepsilon)$. Choose the fixed-manifold identifications by the relative equivariant collar-straightening construction, retaining those of $\widehat{G}_\varepsilon$ on $\mathcal{B}_0$. Their supports are inside the outer gluing collars, and the suspension coordinates on $[2\varepsilon, \pi-2\varepsilon] \times S^{n-1}$ are unchanged. In particular, comparison with the reference family on $\mathcal{B}_0$ is an equality of metrics on the same fixed M_Q.

$$\mathrm{Ric}_{\widehat{G}(s,\varepsilon)} > \big(n-1-(C+\tfrac{3}{2})\varepsilon^2\big)\widehat{G}(s,\varepsilon). \tag{8.9}$$

The common outer boundary has metric $\sin^2\varepsilon\, h_s$. Since the link dimension is $n-1$ and $\mathrm{Ric}_{h_s} \geq (n-2)h_s$, the Bonnet–Myers diameter estimate gives $\mathrm{diam}(S^{n-1}, h_s) \leq \pi$. Let $C^-_{s,\varepsilon}$ and $C^+_{s,\varepsilon}$ be the two outer smoothing collars. The signed normal length of each collar is less than ε. Equation (7.30), with $L = 2$, gives, for either sign,

$$\begin{aligned} \mathrm{diam}_{\mathrm{int}}\, C^\pm_{s,\varepsilon} &\leq 2\varepsilon + 2\sin\varepsilon\ \mathrm{diam}(S^{n-1}, h_s) \leq (2+2\pi)\varepsilon, \\ \mathrm{Vol}\, C^\pm_{s,\varepsilon} &\leq 2^n\varepsilon\ \sin^{n-1}\varepsilon\ \mathrm{Vol}(S^{n-1}, h_s) \leq 2^n \gamma_* \sigma_{n-1}\varepsilon^n. \end{aligned}$$

For either pole, $\mathrm{Vol}([0,2\varepsilon] \times S^{n-1}, dt^2 + \sin^2 t\, h_s) \leq 2^n\gamma_*\sigma_{n-1}\varepsilon^n/n$. Let $E^-_{s,\varepsilon}$ be the union of the lower cap, the collar $C^-_{s,\varepsilon}$, and the remaining suspension strip up to $t = 2\varepsilon$; define $E^+_{s,\varepsilon}$ by reflection. For $\eta > 0$, join a point in the retained cap to its original boundary in $(P_Q, K_{s,\varepsilon})$ by a curve of length at most $D\varepsilon + \eta$, and retain only the segment preceding its first intersection with $C^-_{s,\varepsilon}$. Its metric is unchanged. Join the last point to the suspension-side boundary inside the collar with length at most $(2+2\pi)\varepsilon + \eta$, and then radially to $t = 2\varepsilon$ with length at most 2ε. Points in the collar or the remaining strip require only the corresponding final parts of this curve. Letting $\eta \downarrow 0$ gives an intrinsic distance to the slice at most $(D+4+2\pi)\varepsilon$. The intrinsic diameter of the slice is at most $2\pi\varepsilon$. Joining two points through that slice, and applying reflection, gives

$$\mathrm{diam}_{\mathrm{int}}\, E^\pm_{s,\varepsilon} \leq (2D+8+6\pi)\varepsilon, \qquad \mathrm{diam}\, E^\pm_{s,\varepsilon} \leq \mathrm{diam}_{\mathrm{int}}\, E^\pm_{s,\varepsilon}.$$

The volume of either $E^{\pm}_{s,\varepsilon}$ is bounded by

$$\left(V + 2^n \gamma_* \sigma_{n-1} + \frac{2^n \gamma_* \sigma_{n-1}}{n} \right) \varepsilon^n.$$

Suspension coordinates give a Riemannian isometry between the complements of $E^-_{s,\varepsilon} \cup E^+_{s,\varepsilon}$ in $(M_Q, \widehat{G}(s,\varepsilon))$ and of the two closed polar regions in the complete suspension of h_s. The absolute volume difference is therefore bounded by the sum of the volumes of the four removed regions. Consequently

$$\left| \mathrm{Vol}(M_Q, \widehat{G}(s,\varepsilon)) - \gamma(s)\sigma_n \right| \le A_{\mathrm{vol}} \varepsilon^n, \tag{8.10}$$

$$\mathrm{diam}_{\mathrm{int}}\, E^{\pm}_{s,\varepsilon} \le A_{\mathrm{diam}} \varepsilon, \qquad \mathrm{diam}\, E^{\pm}_{s,\varepsilon} \le A_{\mathrm{diam}} \varepsilon. \tag{8.11}$$

where one may take

$$A_{\mathrm{vol}} = 2V + 2^{n+1} \gamma_* \sigma_{n-1} + \frac{2^{n+2} \gamma_* \sigma_{n-1}}{n}, \qquad A_{\mathrm{diam}} = 2D + 8 + 6\pi.$$

Set $A_{\mathrm{Ric}} = C + 2$ and define the relative volume excess by $\beta(s) = \gamma(s)/q - 1 > 0$. For $s \ge 0$ define

$$\rho(s) := \min \left\{ \varepsilon_0,\ \varepsilon_*,\ \frac{1}{1+s},\ \sqrt{\frac{d_n(\beta(s))}{2A_{\mathrm{Ric}}}},\ \left(\frac{q\sigma_n \beta(s)}{2A_{\mathrm{vol}}} \right)^{1/n},\ \frac{1}{(1+s)A_{\mathrm{diam}}} \right\}.$$

Every entry in the minimum defining ρ is positive and continuous; hence ρ is positive and continuous. Choose a locally finite cover (U_j) of $[0,\infty)$ by intervals of length < 2 with compact closures, a smooth partition of unity (φ_j) subordinate to this cover, and constants $r_j > 0$ satisfying

$$4r_j < \inf_{\overline{U}_j} \rho, \qquad r_j < \frac{1}{2}\varepsilon_*.$$

Then $\varepsilon_\infty(s) := \sum_j \varphi_j(s) r_j$ satisfies $0 < \varepsilon_\infty(s) < \rho(s)/4$. If $\varphi_j(s) > 0$, then $s \in U_j$. Hence

$$r_j < \frac{1}{4} \inf_{t \in U_j} \rho(t) \le \frac{1}{4}\rho(s) \le \frac{1}{4(1+s)}.$$

Moreover $\varepsilon_\infty(s) < \varepsilon_*/2$ and $\varepsilon_\infty(s) \le [4(1+s)]^{-1} \to 0$. Choose $\theta, \chi \in C^\infty([0,\infty),[0,1])$ such that

$$\theta = 0 \text{ on } [0, s_0/4], \qquad \theta = 1 \text{ on } [3s_0/8, \infty), \qquad \chi = 0 \text{ on } [0, 5s_0/8], \qquad \chi = 1 \text{ on } [s_0, \infty).$$

Define the two cap scales by

$$\varepsilon_{\mathrm{seed}}(s) := (1-\theta(s))\varepsilon_* + \theta(s)\varepsilon_{\mathrm{join}}, \qquad \varepsilon_{\mathrm{glob}}(s) := (1-\chi(s))\varepsilon_{\mathrm{join}} + \chi(s)\varepsilon_\infty(s).$$

Define

$$\widehat{G}_s := \begin{cases} \widehat{G}_{\varepsilon_{\mathrm{seed}}(s)}, & 0 \le s \le s_0/2, \\ \widehat{G}(s, \varepsilon_{\mathrm{glob}}(s)), & s \ge s_0/2. \end{cases}$$

For $s \le s_0$ and $0 < \varepsilon < \varepsilon_0$, equation (7.104) identifies the two caps with the fixed caps. Equation (7.33) identifies the outer gluing widths, and (7.34) identifies the smoothing collar at each of the two outer seams. The intervening suspension cylinder and the boundary identifications are unchanged. Therefore

$$\widehat{G}(s,\varepsilon) = \widehat{G}_\varepsilon. \tag{8.12}$$

On a neighbourhood of $s_0/2$, both definitions of $\widehat{G}_s$ are therefore equal to $\widehat{G}_{\varepsilon_{\mathrm{join}}}$. Hence $s \mapsto \widehat{G}_s$ is smooth. It is constant on $[0, s_0/4]$ and satisfies $\widehat{G}_0 = \widehat{G}_*$. For $0 \le s \le s_0$, one has

$$0 < \varepsilon_{\mathrm{seed}}(s) \le \varepsilon_*, \qquad 0 < \varepsilon_{\mathrm{glob}}(s) < \varepsilon_*,$$

where the first inequality is used on $[0, s_0/2]$ and the second on $[s_0/2, s_0]$. On $[0, s_0]$ one has $h_s = h_0$, $g_{s,u} = g_{0,u}$, and $\phi_{s,u} = \phi_{0,u}$; moreover (8.12) applies on the second interval. Moreover

$$\gamma(s) = \gamma_*, \qquad \beta(s) = \frac{\gamma_*}{q} - 1, \qquad \frac{1}{2} q\sigma_n \beta(s) = \frac{1}{2}(\gamma_* - q)\sigma_n.$$

Thus (8.5) and (8.6) give

$$\mathrm{Ric}_{\widehat{G}_s} \geq \lambda_* \widehat{G}_s > \big(n - 1 - d_n(\beta(s))\big) \widehat{G}_s,$$

while (8.4) and (8.8) give

$$\Big|\mathrm{Vol}(M_Q, \widehat{G}_s) - \gamma(s)\sigma_n\Big| \leq C_* \varepsilon_*^n < \frac{1}{2} q\sigma_n \beta(s).$$

Consequently

$$\mathrm{Ric}_{\widehat{G}_s} \geq (n - 1 - d_n(\beta(s)))\widehat{G}_s, \qquad \Big|\mathrm{Vol}(M_Q, \widehat{G}_s) - \gamma(s)\sigma_n\Big| \leq \frac{1}{2} q\sigma_n \beta(s).$$

For $s \geq s_0$, one has $\varepsilon_{\mathrm{glob}}(s) = \varepsilon_\infty(s) < \rho(s)/4$. Hence

$$A_{\mathrm{Ric}} \varepsilon_{\mathrm{glob}}(s)^2 < \frac{A_{\mathrm{Ric}}}{16} \rho(s)^2 \leq \frac{1}{32} d_n(\beta(s)) < d_n(\beta(s)),$$

and (8.9) yields

$$\mathrm{Ric}_{\widehat{G}_s} > \big(n - 1 - A_{\mathrm{Ric}} \varepsilon_{\mathrm{glob}}(s)^2\big) \widehat{G}_s > \big(n - 1 - d_n(\beta(s))\big) \widehat{G}_s.$$

Likewise

$$A_{\mathrm{vol}} \varepsilon_{\mathrm{glob}}(s)^n < \frac{A_{\mathrm{vol}}}{4^n} \rho(s)^n \leq \frac{q\sigma_n \beta(s)}{2\, 4^n} < \frac{1}{2} q\sigma_n \beta(s),$$

and (8.10) gives

$$\Big|\mathrm{Vol}(M_Q, \widehat{G}_s) - \gamma(s)\sigma_n\Big| < \frac{1}{2} q\sigma_n \beta(s).$$

Therefore, for all $s \geq 0$,

$$\mathrm{Ric}_{\widehat{G}_s} \geq (n - 1 - d_n(\beta(s)))\widehat{G}_s, \tag{8.13}$$

$$\Big|\mathrm{Vol}(M_Q, \widehat{G}_s) - \gamma(s)\sigma_n\Big| \leq \frac{1}{2} q\sigma_n \beta(s). \tag{8.14}$$

For $s \geq s_0$, put $E_s^\pm := E^\pm_{s,\varepsilon_{\mathrm{glob}}(s)}$. This definition gives

$$\mathrm{diam}_{\mathrm{int}}\, E_s^\pm \leq A_{\mathrm{diam}} \varepsilon_\infty(s) < \frac{A_{\mathrm{diam}}}{4} \rho(s) \leq \frac{1}{4(1+s)}.$$

Consequently, $\mathrm{diam}_{\mathrm{int}}\, E_s^\pm \to 0$ and $\mathrm{diam}\, E_s^\pm \to 0$.

Let $S(h_s)$ denote the complete suspension of (S^{n-1}, h_s), and let $P^\pm_{s,\varepsilon} \subset S(h_s)$ be its two polar regions $t \in [0, 2\varepsilon]$ and $t \in [\pi - 2\varepsilon, \pi]$. Their intrinsic diameters are at most $(4 + 2\pi)\varepsilon$. Put

$$C_s^M := M_Q \setminus \mathrm{int}(E_s^- \cup E_s^+), \qquad C_s^S := S(h_s) \setminus \mathrm{int}\big(P^-_{s,\varepsilon_{\mathrm{glob}}(s)} \cup P^+_{s,\varepsilon_{\mathrm{glob}}(s)}\big).$$

The suspension coordinate and the identity on S^{n-1} define a Riemannian isometry $I_s : C_s^M \to C_s^S$ for the intrinsic length metrics; it maps each boundary slice to the corresponding boundary slice. Identify the two Riemannian manifolds through I_s and denote the resulting common region by C_s. Choose points $e_s^\pm \in E_s^\pm$ and denote the suspension poles by $o_s^\pm$. Add to this graph all pairs

$$(x, o_s^\pm), \quad x \in E_s^\pm, \qquad (e_s^\pm, y), \quad y \in P^\pm_{s,\varepsilon_{\mathrm{glob}}(s)}.$$

The relation C_s defined by these pairs projects onto both spaces. Put

$$\Delta_s^\pm := \mathrm{diam}_{\mathrm{int}}\, E_s^\pm + \mathrm{diam}_{\mathrm{int}}\, P^\pm_{s,\varepsilon_{\mathrm{glob}}(s)}.$$

Let $B_s^\pm$ be the two boundary slices of C_s. For this fixed s, write

$$P_s^\pm := P^\pm_{s,\varepsilon_{\mathrm{glob}}(s)}, \qquad d_M := d_{\widehat{G}_s}, \qquad d_S := d_{S(h_s)}.$$

Let $z, z' \in C_s$, let γ be a minimizing curve from z to z' in (M_Q, d_M), and fix $\eta > 0$. Perform the replacements in the order $-$, then $+$. If γ meets E_s^-, its first and last points in E_s^- lie in B_s^-: the endpoints of γ lie in C_s, and $E_s^- \cap C_s = B_s^-$. Replace the intervening subcurve by a curve in P_s^- with the same endpoints and length at most $\operatorname{diam}_{\mathrm{int}} P_s^- + \eta$. If the first and last points coincide, use the constant curve. If E_s^- is not met, leave the curve unchanged. The resulting curve lies in

$$P_s^- \cup_{B_s^-} (C_s \cup E_s^+).$$

Lengths on its retained subarcs are those of the original metric, and lengths on the inserted subarc are those of P_s^-. Its total length is at most $d_M(z,z') + \operatorname{diam}_{\mathrm{int}} P_s^- + \eta$.

If the resulting curve meets E_s^+, its first and last points in E_s^+ lie in B_s^+, since E_s^+ is disjoint from P_s^-. Replace the intervening subcurve by a curve in P_s^+ with the same endpoints and length at most $\operatorname{diam}_{\mathrm{int}} P_s^+ + \eta$, using a constant curve when the endpoints coincide. If E_s^+ is not met, leave the curve unchanged. This second replacement may delete part of the preceding curve, including the first inserted subarc, but it adds at most the displayed length. The final curve lies in $C_s \cup P_s^- \cup P_s^+ = S(h_s)$; its retained subarcs in C_s have the same lengths for both metrics. Therefore

$$d_S(z,z') \le d_M(z,z') + \operatorname{diam}_{\mathrm{int}} P_s^- + \operatorname{diam}_{\mathrm{int}} P_s^+ + 2\eta.$$

Starting with a minimizing curve in $S(h_s)$ and performing the same two replacements, now replacing $P_s^\pm$ by $E_s^\pm$, gives

$$d_M(z,z') \le d_S(z,z') + \operatorname{diam}_{\mathrm{int}} E_s^- + \operatorname{diam}_{\mathrm{int}} E_s^+ + 2\eta.$$

Letting $\eta \downarrow 0$ yields

$$\left| d_{\widehat{G}_s}(z,z') - d_{S(h_s)}(z,z') \right| \le \Delta_s^- + \Delta_s^+. \tag{8.15}$$

Choose $z_s^\pm \in B_s^\pm$, using the same points in the identified boundary slices. For each sign $\sigma \in \{-,+\}$ and each polar pair $(x,y) \in E_s^\sigma \times P_s^\sigma$, the intrinsic diameter bounds give

$$d_M(x, z_s^\sigma) + d_S(y, z_s^\sigma) \le \Delta_s^\sigma.$$

For a pair (z,z) in the graph of I_s, use the anchor z instead, with zero error. Two pairs in the same polar region have distortion at most Δ_s^σ, since each of their two distances is bounded by the corresponding intrinsic diameter. For all other pairs, the sum of the two anchor errors is at most $\Delta_s^- + \Delta_s^+$. The triangle inequality and (8.15), applied between the anchors in C_s, give

$$\left| d_{\widehat{G}_s}(x,x') - d_{S(h_s)}(y,y') \right| \le 2\Delta_s^- + 2\Delta_s^+$$

for every two pairs $(x,y), (x',y') \in \mathcal{C}_s$. Since $\operatorname{diam}_{\mathrm{int}} P^\pm_{s,\varepsilon_{\mathrm{glob}}(s)} \le (4+2\pi)\varepsilon_{\mathrm{glob}}(s)$,

$$\operatorname{dis}(\mathcal{C}_s) \le 2\operatorname{diam}_{\mathrm{int}} E_s^- + 2\operatorname{diam}_{\mathrm{int}} E_s^+ + 4(4+2\pi)\varepsilon_{\mathrm{glob}}(s). \tag{8.16}$$

Consequently, $d_{\mathrm{GH}}((M_Q, \widehat{G}_s), S(h_s)) \to 0$. Let $\mathcal{R}_s$ be a correspondence between (S^{n-1}, h_s) and $B_{n,q}$ with $\operatorname{dis}(\mathcal{R}_s) \to 0$. Relate (t,x) to (t,y) whenever $(x,y) \in \mathcal{R}_s$. The spherical suspension distance is determined by $\cos d = \cos t \cos t' + \sin t \sin t' \cos(d_{\mathrm{link}} \wedge \pi)$. The displayed right-hand side and the function $\arccos : [-1,1] \to [0,\pi]$ are uniformly continuous on their compact domains. Hence the relation $((t,x),(t,y))$ defined by $(x,y) \in \mathcal{R}_s$ has distortion tending to zero. Thus $S(h_s) \xrightarrow[s\to\infty]{\mathrm{GH}} S(B_{n,q}) = L_{n,q}$. Together with (8.16), this gives

$$(M_Q, \widehat{G}_s) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}. \tag{8.17}$$

Put $V_q := q\sigma_n$ and define the positive scale

$$b_s := \left(\frac{V_q}{\mathrm{Vol}(M_Q, \widehat{G}_s)}\right)^{1/n} > 0, \qquad g_s^0 := b_s^2 \widehat{G}_s.$$

The functions b_s and g_s^0 depend smoothly on s, and $\mathrm{Vol}(M_Q, g_s^0) = V_q$. Since $\gamma(s) = q(1+\beta(s))$, equation (8.14) gives

$$\mathrm{Vol}(M_Q, \widehat{G}_s) \geq V_q \left(1 + \frac{\beta(s)}{2}\right), \qquad b_s^{-2} \geq \left(1 + \frac{\beta(s)}{2}\right)^{2/n}.$$

Together with $n - 1 - d_n(\beta) = (n-1)(1+\beta/2)^{-2/n}$, this yields

$$\mathrm{Ric}_{g_s^0} = \mathrm{Ric}_{\widehat{G}_s} \geq \frac{n-1-d_n(\beta(s))}{b_s^2} g_s^0 \geq (n-1) g_s^0.$$

Moreover $\beta(s) \to 0$ and (8.14) imply $b_s \to 1$. Since the spaces in (8.17) converge to the compact space $L_{n,q}$, there are $S_1, D_1 > 0$ such that $\mathrm{diam}(M_Q, \widehat{G}_s) \leq D_1$ for $s \geq S_1$. The identity correspondence between $(M_Q, \widehat{G}_s)$ and $(M_Q, b_s^2 \widehat{G}_s)$ has distortion at most $|b_s - 1| \,\mathrm{diam}(M_Q, \widehat{G}_s)$. Hence (8.17) gives

$$(M_Q, g_s^0) \xrightarrow[s\to\infty]{\mathrm{GH}} L_{n,q}. \tag{8.18}$$

At $s = 0$ one has $\widehat{G}_0 = \widehat{G}_*$ and $b_0^2 = (q\sigma_n/V_*)^{2/n} \leq c_Q^2$ by (8.7). If g is Ricci closable and $0 < a \leq 1$, then $a^2 g$ is Ricci closable. Indeed, for $0 < \eta < 1$ put $\delta = 1 - a^2(1-\eta) \in (0,1)$. A closing whose conical end has cross-section $(1-\delta)g = (1-\eta)a^2 g$ is a closing for $a^2 g$. Apply this observation to $g = c_Q^2 \widehat{G}_*$ and $a = b_0/c_Q \leq 1$. Since $g_0^0 = b_0^2 \widehat{G}_*$, the metric g_0^0 is Ricci closable.

Let $\omega_s = dv_{g_s^0}$. Since $\int_{M_Q} \omega_s = V_q$, the top form $\dot{\omega}_s = \partial_s \omega_s$ has integral zero. Fix a background metric on the closed connected oriented manifold M_Q, let $\mathcal{G}$ be its Green operator on top forms, and set

$$\eta_s = d^* \mathcal{G} \dot{\omega}_s, \qquad \iota_{Y_s} \omega_s = -\eta_s.$$

The harmonic top forms are the constant multiples of the background volume form; hence the harmonic projection of $\dot{\omega}_s$ vanishes. Since $\mathcal{G}\dot{\omega}_s$ has top degree, $d\mathcal{G}\dot{\omega}_s = 0$. The Hodge identity gives $d\eta_s = dd^*\mathcal{G}\dot{\omega}_s = \dot{\omega}_s$. The vector field Y_s depends smoothly on s. For every $S > 0$ it has a unique flow on $[0, S]$ because M_Q is compact; uniqueness makes these finite-time flows agree on overlapping intervals. Uniqueness defines one flow ψ_s on $[0,\infty)$ with $\psi_0 = \mathrm{id}$. Cartan's formula gives

$$\frac{d}{ds}(\psi_s^* \omega_s) = \psi_s^*(\dot{\omega}_s + d\iota_{Y_s}\omega_s) = 0,$$

so $\psi_s^* \omega_s = \omega_0$. On $[0, s_0/4]$ one has $\widehat{G}_s = \widehat{G}_*$ and $b_s = b_0$; hence $g_s^0 = g_0^0$, $Y_s = 0$, and $\psi_s = \mathrm{id}$. Define $g_s := \psi_s^* g_s^0$. Then $dv_{g_s} = \psi_s^* \omega_s = \omega_0 = dv_{g_0}$ for every $s \geq 0$, and $g_s = g_0$ for $0 \leq s \leq s_0/4$. Since pullback commutes with the Ricci tensor,

$$\mathrm{Ric}_{g_s} = \psi_s^* \mathrm{Ric}_{g_s^0} \geq (n-1)\psi_s^* g_s^0 = (n-1) g_s.$$

Also

$$\mathrm{Vol}(M_Q, g_s) = \int_{M_Q} dv_{g_s} = \int_{M_Q} \omega_0 = q\sigma_n.$$

The diffeomorphism ψ_s is an isometry from (M_Q, g_s) to (M_Q, g_s^0), so (8.18) gives (8.3). Finally $g_0 = g_0^0$ is Ricci closable. The theorem follows with $s_* = s_0/4$. □

9. One-ended realization

9.1. Dimension-dependent radial warping.

Lemma 9.1. *For every integer $n \geq 2$ there are smooth functions*

$$h : (0,\infty) \to (0,1), \qquad f : (0,\infty) \to \mathbb{R},$$

depending only on n, with $f' > 0$, such that the following holds. Let X^n be a closed smooth manifold and let $(\ell_t)_{t\in\mathbb{R}}$ be a smooth family of metrics such that, for every $t \in \mathbb{R}$,

$$dv_{\ell_t} = dv_{\ell_0}, \qquad \mathrm{Ric}_{\ell_t} \geq (n-1)\ell_t, \qquad \ell_t = \ell_0 \quad (t \geq 0),$$

and, pointwise on X,

$$|\partial_t \ell_t|_{\ell_t} + |\partial_t^2 \ell_t|_{\ell_t} + |\nabla^{\ell_t}\partial_t \ell_t|_{\ell_t} \leq 1.$$

Then $dr^2 + r^2 h(r)^2 \ell_{f(r)}$ has nonnegative Ricci curvature. Moreover, for $\varphi(r) = rh(r)$,

$$h(r) \to 1, \qquad f(r) \to -\infty,$$
$$rf'(r) \to 0, \qquad \varphi'(r) \to 1 \quad (r \downarrow 0),$$

and

$$0 < \varphi'(r) < 1, \qquad \varphi''(r) \leq 0 \qquad (r > 0).$$

There is $r_0 = r_0(n) > 0$ such that

$$0 < \varphi'(r) < 1, \qquad \varphi''(r) < 0 \qquad (0 < r \leq r_0). \tag{9.1}$$

Proof. Define the dimensional constants by

$$F_n := \frac{1}{128(n+1)}, \qquad E_n := \frac{1}{16}, \qquad L_n := 64(n+1),$$
$$T_n := e^{L_n}, \qquad \rho_n := e^{-T_n}.$$

Since $L_n \geq 192$, one has $T_n = e^{L_n} > e$ and $0 < \rho_n = e^{-T_n} < e^{-e}$. Put $c_n^0 := F_n \log L_n$. For $0 < r \leq 1$, write $T(r) := -\log(\rho_n r) = T_n - \log r$ and $L(r) := \log T(r)$, and define

$$h_0(r) = 1 - \frac{E_n}{L(r)}, \qquad f_0(r) = -F_n \log L(r) + c_n^0. \tag{9.2}$$

Then $T(r) \geq T_n$, $L(r) \geq L_n$, and $f_0(1) = 0$. Direct differentiation gives

$$h_0'(r) = -\frac{E_n}{rT(r)L(r)^2}, \qquad f_0'(r) = \frac{F_n}{rT(r)L(r)}, \tag{9.3}$$

$$\varphi_0'(r) = 1 - \frac{E_n}{L(r)} - \frac{E_n}{T(r)L(r)^2}, \qquad \varphi_0''(r) = -\frac{E_n}{rT(r)L(r)^2} - \frac{E_n}{rT(r)^2L(r)^2} - \frac{2E_n}{rT(r)^2L(r)^3}, \tag{9.4}$$

where $\varphi_0(r) = rh_0(r)$. Since $n \geq 2$ and $L(r) \geq L_n \geq 192$, $E_n/L(r) + E_n/(T(r)L(r)^2) \leq 1/(16L_n) + 1/(16L_n^2) < 1/2$. Consequently,

$$\frac{1}{2} < \varphi_0'(r) < 1, \qquad \varphi_0''(r) < 0 \qquad (0 < r \leq 1).$$

Put $g(r) = \ell_{f_0(r)}$ and $G = dr^2 + \varphi_0(r)^2 g(r)$. The function $s \mapsto s/\log s$ is increasing for $s > e$. Hence, for $0 < r \leq 1$,

$$\frac{E_n}{L(r)} \leq \frac{1}{16L_n} < \frac{1}{2}, \qquad \frac{T(r)}{L(r)} \geq \frac{T_n}{L_n} > 4,$$
$$E_n \geq 2\sqrt{n}\,F_n, \qquad E_n \geq F_n^2, \tag{9.5}$$

because $2\sqrt{n}\,F_n/E_n = \sqrt{n}/(4(n+1)) \le 1$ and $F_n^2/E_n = 1/(1024(n+1)^2) \le 1$. For Proposition 6.2, take

$$\begin{gathered} m = n, \quad A = 1, \quad J = (0,1], \quad \rho = \rho_n, \\ E = E_n, \quad F = F_n, \quad c = c_n^0, \\ h = h_0, \quad w = \varphi_0, \quad f = f_0. \end{gathered}$$

The identities $dv_{\ell_t} = dv_{\ell_0}$ and $\mathrm{Ric}_{\ell_t} \ge (n-1)\ell_t$ give (6.11). The three nonnegative summands in the pointwise derivative bound are each at most $A = 1$, giving (6.12). The inequality $0 < \rho_n < e^{-e}$ and (9.5) give all four inequalities in (6.19). Therefore, for every $g(r)$-unit vector v,

$$\begin{aligned} \mathrm{Ric}_G(\partial_r, \partial_r) &\ge \frac{E_n}{2L(r)^2T(r)r^2}, \\ \mathrm{Ric}_G(\varphi_0(r)^{-1}v, \varphi_0(r)^{-1}v) &\ge \frac{E_n}{L(r)r^2}, \\ \left|\mathrm{Ric}_G(\partial_r, \varphi_0(r)^{-1}v)\right| &\le \frac{\sqrt{n}\,F_n}{L(r)T(r)r^2}. \end{aligned}$$

The Schur-complement conclusion of Proposition 6.2 gives $\mathrm{Ric}_G > 0$ on $(0,1] \times X$.

Since $T_n - \log 2 > e$, the formulas in (9.2) define smooth functions on $(0,2)$. Put $\delta_0 := 1$ and $w_0(r) := rh_0(r)$ on this interval. Formula (9.4) gives $0 < w_0'(1) < 1$ and $w_0'' < 0$ on a neighbourhood of 1. Choose $0 < \delta < \delta_0/2$ such that $w_0'' < 0$ on $[1, 1+2\delta]$ and $\int_1^{1+2\delta}(-w_0''(t))\,dt < w_0'(1)$. Choose $\chi \in C^\infty([1,\infty),[0,1])$ with $\chi = 1$ on $[1,1+\delta]$ and $\operatorname{supp}\chi \subset [1, 1+2\delta)$. Define the smooth function $\kappa : [1,\infty) \to [0,\infty)$ by

$$\kappa(r) = \begin{cases} \chi(r)(-w_0''(r)), & 1 \le r < 1+2\delta, \\ 0, & r \ge 1+2\delta. \end{cases}$$

Because $1 + 2\delta < 2$, the first branch is defined. Since $\operatorname{supp}\chi$ is closed in $[1,\infty)$ and contained in $[1, 1+2\delta)$, there is $\epsilon_\chi > 0$ such that $\chi = 0$ on $[1+2\delta-\epsilon_\chi, \infty)$. The two branches therefore agree on a neighbourhood of $1+2\delta$, so κ is smooth. For $r \ge 1$, define

$$\widehat{w}'(r) = w_0'(1) - \int_1^r \kappa(t)\,dt, \qquad \widehat{w}(r) = w_0(1) + \int_1^r \widehat{w}'(t)\,dt.$$

On $[1, 1+\delta]$ one has $\kappa = -w_0''$, and therefore $\widehat{w}'' = w_0''$. Since $\widehat{w}(1) = w_0(1)$ and $\widehat{w}'(1) = w_0'(1)$, integration gives $\widehat{w} = w_0$ on $[1, 1+\delta]$. Furthermore

$$0 < \widehat{w}'(r) \le w_0'(1) < 1, \qquad \widehat{w}''(r) = -\kappa(r) \le 0,$$

and $\widehat{w}$ is linear for $r \ge 1+2\delta$. Define $w = w_0$ on $(0,1]$, $w = \widehat{w}$ on $[1,\infty)$, and $h = w/r$. Since $w(1) = w_0(1) > 0$ and $w' > 0$, while $w(1) < 1$ and $(w-r)' = w' - 1 < 0$ on $[1,\infty)$, one has $0 < w(r) < r$ for $r \ge 1$. Thus $0 < h < 1$ on $(0,\infty)$.

Choose $0 < \delta_1 < \delta_0/3$ and $\beta \in C^\infty([1,\infty),[0,1])$ with $\beta = 0$ on $[1, 1+\delta_1]$ and $\beta = 1$ on $[1+2\delta_1, \infty)$. For $1 \le r \le 1+2\delta_1$, put $a(r) = (1-\beta(r))f_0'(r) + \beta(r)$, and put $a(r) = 1$ for $r \ge 1+2\delta_1$. Then a is smooth and positive. Define

$$f(r) = f_0(r) \quad (0 < r \le 1), \qquad f(r) = \int_1^r a(t)\,dt \quad (r \ge 1).$$

Since $f_0(1) = 0$ and $a = f_0'$ near 1, the two definitions agree on a neighbourhood of 1. Thus f is smooth and strictly increasing, with $f(r) > 0$ for $r > 1$.

Define the global metric and slice family by

$$\overline{G} := dr^2 + w(r)^2\ell_{f(r)} = dr^2 + r^2h(r)^2\ell_{f(r)}, \qquad \overline{g}(r) := \ell_{f(r)}.$$

For $0 < r \le 1$, one has $\overline{G} = G$. For $r \ge 1$, one has $\overline{g}(r) = \ell_0$. Hence $\overline{g}' = \overline{g}'' = 0$ and $\operatorname{div}_{\overline{g}} \overline{g}' = 0$. Equations (6.13), (6.14), and (6.15) give, for every ℓ_0-unit vector v,

$$\begin{aligned}
\operatorname{Ric}_{\overline{G}}(\partial_r, \partial_r) &= -n\frac{w''}{w} \ge 0,\\
\operatorname{Ric}_{\overline{G}}(\partial_r, w^{-1}v) &= 0,\\
\operatorname{Ric}_{\overline{G}}(w^{-1}v, w^{-1}v) &\ge w^{-2}\big((n-1) - ww'' - (n-1)(w')^2\big) \ge 0.
\end{aligned}$$

Thus $\operatorname{Ric}_{\overline{G}} \ge 0$ on $[1,\infty) \times X$. The metric $\overline{G}$ is smooth at $r = 1$: the functions w, f agree with w_0, f_0 on a neighbourhood of 1. Since $t \mapsto \ell_t$ is smooth and equals ℓ_0 for $t \ge 0$, one has $\partial_t^j \ell_t|_{t=0} = 0$ for every $j \ge 1$. Repeated differentiation of $r \mapsto \ell_{f(r)}$ shows that all one-sided derivatives at $r = 1$ agree with those of the constant family ℓ_0. Together with the inner estimate, this proves $\operatorname{Ric}_{\overline{G}} \ge 0$ on $(0,\infty) \times X$. All choices in the extension depend only on w_0, f_0, hence only on n.

Finally, (9.3) and (9.4) give

$$h(r) \to 1, \qquad f(r) \to -\infty, \qquad rf'(r) = \frac{F_n}{T(r)L(r)} \to 0, \qquad \varphi'(r) \to 1 \quad (r \downarrow 0).$$

The inequalities for φ_0 on $(0,1]$ and for $\widehat{w}$ on $[1,\infty)$ give $0 < \varphi' < 1$ and $\varphi'' \le 0$ on $(0,\infty)$. Equation (9.1) holds with $r_0 = 1$. □

9.2. **Radial completion and smooth approximation.** Ricci closability in the following proposition means the conical-end condition of Definition 2.2. The realization is proved directly under this condition: no equivalence with a distance-annulus formulation is assumed, and no closing domain is required to be a metric ball.

Proposition 9.2. *Let $n \ge 2$, let X^n be a closed connected smooth manifold, and let $(g_t)_{t\in\mathbb{R}}$ be a smooth family of Riemannian metrics on X. Assume that g_0 is Ricci closable and that there is a compact metric space (Σ, d_Σ) such that*

$$\begin{aligned}
dv_{g_t} &= dv_{g_0}, &\qquad \operatorname{Ric}_{g_t} &\ge (n-1)g_t,\\
g_t &= g_0\ (t \ge 0), &\qquad (X, g_t) &\xrightarrow[t\to-\infty]{\mathrm{GH}} (\Sigma, d_\Sigma).
\end{aligned}$$

Then there exist a complete pointed classical noncollapsed Ricci limit (Y^{n+1}, d, p) and a number $r_0 > 0$ such that

$$\operatorname{Tan}(Y, p) = \{\mathrm{C}(\Sigma)\}, \qquad (B_{r_0}(p), p) \cong \big(\mathrm{C}_{\mathrm{top},r_0}(X), o\big).$$

Moreover $Y \setminus \{p\}$ is the smooth manifold $(0,\infty) \times X$, and Y is realized by complete connected smooth $(n+1)$-manifolds without boundary, with nonnegative Ricci curvature and a common positive lower bound for the volumes of their based unit balls.

Proof. Slow reparametrization. For $j \ge 0$ set

$$A_j := 1 + \sup_{t\in[-j-2,-j+1]} \left(\|\partial_t g_t\|_{g_t} + \|\partial_t^2 g_t\|_{g_t} + \|\nabla^{g_t}\partial_t g_t\|_{g_t}\right).$$

The tensor norms in this definition are supremum norms on X. The smoothness of $(t,x) \mapsto g_t(x)$ and the compactness of $[-j-2,-j+1] \times X$ imply $A_j < \infty$. Choose $\vartheta, \xi \in C^\infty([0,1],[0,1])$, constant on neighbourhoods of 0 and 1, with

$$\vartheta(0) = \xi(0) = 0, \qquad \vartheta(1) = \xi(1) = 1, \qquad \vartheta' \ge 0.$$

Fix $C_0 \ge 1$ such that

$$\|\vartheta'\|_\infty + \|\vartheta''\|_\infty + \|\xi'\|_\infty + \|\xi''\|_\infty \le C_0.$$

Define C_{cut} and η_0, and, for each j, choose L_j by

$$C_{\mathrm{cut}} := 3(1+2C_0) + (1+2C_0)^2 + 2C_0, \qquad \eta_0 := \min\left\{\frac{1}{2}, \frac{1}{2C_{\mathrm{cut}}}\right\},$$
$$L_j \geq 1, \qquad \frac{2C_0A_j}{L_j} + \frac{(C_0+C_0^2)A_j}{L_j^2} \leq \eta_0.$$

Set $T_0 = 0$ and $T_{j+1} = T_j - L_j$. Since $L_j \geq 1$, one has $T_j \leq -j$; hence the intervals $[T_{j+1}, T_j]$ cover $(-\infty, 0]$. Define $\psi(t) = 0$ for $t \geq 0$ and, for $t \in [T_{j+1}, T_j]$, put

$$\psi(t) = -(j+1) + \vartheta\left(\frac{t - T_{j+1}}{L_j}\right), \qquad \widetilde{g}_t := g_{\psi(t)}.$$

The endpoint constancy of ϑ makes ψ smooth. It is nondecreasing, maps $[T_{j+1}, T_j]$ onto $[-j-1, -j]$, and tends to $-\infty$ as $t \to -\infty$. On $[T_{j+1}, T_j]$ the chain rule gives

$$\begin{aligned} \|\partial_t \widetilde{g}_t\|_{\widetilde{g}_t} &\leq \frac{C_0A_j}{L_j}, \\ \|\partial_t^2 \widetilde{g}_t\|_{\widetilde{g}_t} &\leq \frac{(C_0+C_0^2)A_j}{L_j^2}, \\ \|\nabla^{\widetilde{g}_t} \partial_t \widetilde{g}_t\|_{\widetilde{g}_t} &\leq \frac{C_0A_j}{L_j}. \end{aligned}$$

Consequently

$$\|\partial_t \widetilde{g}_t\|_{\widetilde{g}_t} + \|\partial_t^2 \widetilde{g}_t\|_{\widetilde{g}_t} + \|\nabla^{\widetilde{g}_t} \partial_t \widetilde{g}_t\|_{\widetilde{g}_t} \leq \eta_0. \tag{9.6}$$

$dv_{\widetilde{g}_t} = dv_{g_0}$ for every t, $\mathrm{Ric}_{\widetilde{g}_t} \geq (n-1)\widetilde{g}_t$, and $\widetilde{g}_t = g_0$ for $t \geq 0$. Since $\psi(t) \to -\infty$ as $t \to -\infty$, the hypothesis on (g_t) gives

$$(X, \widetilde{g}_t) \xrightarrow[t \to -\infty]{\mathrm{GH}} (\Sigma, d_\Sigma). \tag{9.7}$$

The identities $dv_{\widetilde{g}_t} = dv_{g_0}$, $\mathrm{Ric}_{\widetilde{g}_t} \geq (n-1)\widetilde{g}_t$, and $\widetilde{g}_t = g_0$ for $t \geq 0$, together with (9.6) and $\eta_0 \leq 1$, instantiate Lemma 9.1 with X and $(\widetilde{g}_t)$. Let h, f be its dimension-only functions. Then

$$\bar{g} = dr^2 + r^2 h(r)^2 \widetilde{g}_{f(r)}, \qquad \mathrm{Ric}_{\bar{g}} \geq 0. \tag{9.8}$$

The associated limits are

$$h(r) \to 1, \qquad f(r) \to -\infty, \qquad rf'(r) \to 0 \quad (r \downarrow 0). \tag{9.9}$$

Every curve whose radial coordinate is unbounded has infinite length, because $|dr| \leq ds_{\bar{g}}$. Bonnet–Myers gives $\mathrm{diam}(X, \widetilde{g}_t) \leq \pi$. If $0 < r \leq s$ and $x, y \in X$, a radial segment followed by a curve in the slice $\{r\} \times X$ gives

$$d_{\bar{g}}((r,x),(s,y)) \leq s - r + \pi r h(r). \tag{9.10}$$

Thus every sequence with radial coordinate tending to zero is Cauchy, and two such sequences have mutual distance tending to zero. Let (r_i, x_i) be Cauchy. The radial coordinates are Cauchy because the radial function is 1-Lipschitz. If their limit is positive, choose $0 < a < b < \infty$ so that the sequence is eventually contained in $[a, b] \times X$. This cylinder is compact in the product topology. The coefficients of (9.8) and their inverse are bounded on it, so its Riemannian distance induces the product topology. A subsequence converges in the product topology; the Cauchy property then gives convergence of the entire sequence. If the radial limit is zero, (9.10) identifies the sequence with the common zero-radius class. An unbounded radial sequence is not Cauchy. The metric completion Y consequently adds exactly one point p. For $(r, x) \in (0, \infty) \times X$,

$$d_Y(p, (r,x)) = r. \tag{9.11}$$

The radial segment gives the upper bound, and the 1-Lipschitz radial function gives the lower bound.

For $R > 0$, the map

$$\overline{\Theta}_R : \overline{\mathrm{C}}_{\mathrm{top},R}(X) \longrightarrow \overline{B}^Y_R(p), \qquad \overline{\Theta}_R([r,x]) = (r,x), \qquad \overline{\Theta}_R(o) = p,$$

is a continuous bijection: continuity at o follows from (9.11), and away from o both spaces have the product topology. Its domain is compact and its target is Hausdorff; hence it is a homeomorphism. Thus every closed bounded ball in Y is compact, so Y is proper and complete. For every $r_0 > 0$, restriction gives the pointed homeomorphism $\mathrm{C}_{\mathrm{top},r_0}(X) \to B_{r_0}(p)$, and $Y \setminus \{p\} = (0,\infty) \times X$ as smooth manifolds.

The space Y is a length space. For two points in the punctured manifold, this follows from the isometric inclusion into the completion and the definition of $d_{\bar{g}}$. If one endpoint is p, a radial segment has length equal to the distance by (9.11). To obtain a minimizing curve between distinct $y, z \in Y$, choose curves γ_j joining them with $L(\gamma_j) \le d_Y(y,z) + j^{-1}$ and parametrize them with constant speed on $[0,1]$. Their images lie in the compact ball $\overline{B}_{d_Y(y,z)+1}(y)$, and their Lipschitz constants are bounded by $d_Y(y,z)+1$. Arzelà–Ascoli gives a uniformly convergent subsequence with limit γ. For each partition $0 = t_0 < \cdots < t_k = 1$,

$$\sum_{a=1}^{k} d_Y(\gamma(t_{a-1}), \gamma(t_a)) = \lim_j \sum_{a=1}^{k} d_Y(\gamma_j(t_{a-1}), \gamma_j(t_a)) \le d_Y(y,z).$$

Taking the supremum over partitions gives $L(\gamma) \le d_Y(y,z)$; the reverse inequality follows from the endpoints. Thus Y is geodesic. For $y = z$, the constant curve is minimizing.

The tangent cone. Let $\rho_i \downarrow 0$ be arbitrary, and fix $0 < \alpha < b < \infty$. Under $r = \rho_i s$, the rescaled metric on $[\alpha/2, 2b] \times X$ is $ds^2 + s^2 h(\rho_i s)^2 \widetilde{g}_{f(\rho_i s)}$. Since $r f'(r) \to 0$,

$$\begin{aligned} \sup_{\alpha/2 \le s \le 2b} |f(\rho_i s) - f(\rho_i)| &\le \max\{|\log(\alpha/2)|, |\log(2b)|\} \\ &\quad \cdot \sup_{\min\{\alpha/2,1\}\rho_i \le r \le \max\{2b,1\}\rho_i} |r f'(r)| \longrightarrow 0. \end{aligned} \tag{9.12}$$

For a nonzero tangent vector $v \in T_x X$,

$$\left| \frac{d}{dt} \log \widetilde{g}_t(v,v) \right| \le \|\partial_t \widetilde{g}_t\|_{\widetilde{g}_t} \le 1.$$

Integration between $f(\rho_i)$ and $f(\rho_i s)$, followed by (9.12), gives numbers $\omega_i(\alpha, b) \downarrow 0$ such that

$$e^{-\omega_i} \widetilde{g}_{f(\rho_i)} \le \widetilde{g}_{f(\rho_i s)} \le e^{\omega_i} \widetilde{g}_{f(\rho_i)} \qquad (\alpha/2 \le s \le 2b).$$

Moreover $h(\rho_i s) \to 1$ uniformly on this interval. Since $f(\rho_i) \to -\infty$, (9.7) gives

$$(X, \widetilde{g}_{f(\rho_i)}) \xrightarrow[i\to\infty]{\mathrm{GH}} (\Sigma, d_\Sigma).$$

Put $g_i = \widetilde{g}_{f(\rho_i)}$. In what follows, distances for $ds^2 + s^2 g_i$ are the ambient distances in the completed metric cone $\mathrm{C}(X, g_i)$. These cones are geodesic. Indeed, $\operatorname{diam}(X, g_i) \le \pi$ by Bonnet–Myers. If $\theta = d_{g_i}(x, x') \in (0, \pi]$, a minimizing geodesic from x to x' is an isometric copy of $[0, \theta]$. Its cone, by the cone-distance formula, is an isometric Euclidean sector of angle θ. A straight segment in that sector minimizes between any two specified points on its boundary rays. For $\theta = 0$, or when an endpoint is the vertex, a radial segment suffices. Define

$$\delta_i = \omega_i(\alpha, b) + 2 \sup_{\alpha/2 \le s \le 2b} |\log h(\rho_i s)|, \qquad \eta_i = e^{\delta_i/2} - 1.$$

Then $\delta_i \to 0$ and

$$e^{-\delta_i}(ds^2 + s^2 g_i) \le ds^2 + s^2 h(\rho_i s)^2 \widetilde{g}_{f(\rho_i s)} \le e^{\delta_i}(ds^2 + s^2 g_i). \tag{9.13}$$

Let z, z' have radial coordinates $r_z, r_{z'} \in [\alpha, b]$, and put $S = r_z + r_{z'}$. In both metrics the distance between z and z' is at most $S \le 2b$, by joining the endpoints through the vertex. A minimizing curve cannot meet $\{s > 2b\}$, since its total radial variation would then be strictly larger than $2b$. Any curve meeting $\{s < \alpha/2\}$ has length at least $S - \alpha$. If minimizing curves for both metrics avoid this inner region, (9.13) compares their lengths in both directions by the factor $1 + \eta_i$. If a minimizing curve for one metric meets the inner region, its distance is at least $S - \alpha$, whereas the other distance is at most S; in the opposite direction, either the other minimizer also meets the inner region or it avoids it and its length is controlled by the tensor comparison. In all cases,

$$\left| d_{\rho_i^{-2}\bar{g}}(z, z') - d_{ds^2+s^2 g_i}(z, z') \right| \le 2\alpha + 2b\,\eta_i \tag{9.14}$$

for all such endpoints.

Choose correspondences $\mathcal{R}_i \subset X \times \Sigma$ with $\varepsilon_i := \operatorname{dis}(\mathcal{R}_i) \to 0$. Bonnet–Myers and Gromov–Hausdorff convergence give $\operatorname{diam}(X, g_i) \le \pi$ and $\operatorname{diam}\Sigma \le \pi$. The relation

$$\mathcal{A}_i^{\text{cone}} := \{((s, x), (s, y)) : \alpha \le s \le b,\ (x, y) \in \mathcal{R}_i\}$$

is a correspondence from $[\alpha, b] \times X$, endowed with $ds^2 + s^2 g_i$, to $[\alpha, b] \times \Sigma$, endowed with its cone metric. The cone distance formula and the 1-Lipschitz property of cosine give

$$\operatorname{dis}(\mathcal{A}_i^{\text{cone}}) \le \sqrt{2}\, b\, \sqrt{\varepsilon_i}.$$

Let $\mathcal{A}_i$ be the relation between the annulus in $(Y, \rho_i^{-1} d)$ and the annulus in $\mathrm{C}(\Sigma)$ obtained by composing the identity relation in (9.14) with $\mathcal{A}_i^{\text{cone}}$. Then

$$\operatorname{dis}(\mathcal{A}_i) \le 2\alpha + 2b\,\eta_i + \sqrt{2}\, b\, \sqrt{\varepsilon_i}.$$

Let $o_i = p$ in $(Y, \rho_i^{-1} d)$ and let o be the vertex of $\mathrm{C}(\Sigma)$. Set

$$\begin{aligned} \mathcal{B}_i := \mathcal{A}_i &\cup \{(z, o) : z \in \overline{B}_\alpha(o_i)\} \\ &\cup \{(o_i, z) : z \in \overline{B}_\alpha(o)\}. \end{aligned}$$

This is a correspondence between the closed radius-b balls. Two inner pairs have distortion at most 2α; an inner pair and an annular pair have distortion at most α; and two annular pairs satisfy the displayed bound. Consequently

$$\operatorname{dis}\mathcal{B}_i \le 2\alpha + 2b\,\eta_i + \sqrt{2}\, b\, \sqrt{\varepsilon_i}.$$

Letting $i \to \infty$ and then $\alpha \downarrow 0$ proves $(Y, \rho_i^{-1} d, p) \to (\mathrm{C}(\Sigma), o)$ in pointed Gromov–Hausdorff topology. The sequence (ρ_i) was arbitrary; hence $\operatorname{Tan}(Y, p) = \{\mathrm{C}(\Sigma)\}$.

Smooth approximations. For $m \ge 2$ put $a_m = 1 - m^{-1}$ and define $c_m : \mathbb{R} \to \mathbb{R}$ by

$$c_m(t) = \begin{cases} 0, & t \le -2m, \\ t\,\xi\!\left(\dfrac{t + 2m}{m}\right), & -2m \le t \le -m, \\ t, & t \ge -m. \end{cases}$$

The endpoint constancy of ξ makes c_m smooth and gives $\|c_m'\|_\infty \le 1 + 2C_0$ and $\|c_m''\|_\infty \le 2C_0$. Put $k_t^{(m)} := a_m \widetilde{g}_{c_m(t)}$. For every t,

$$dv_{k_t^{(m)}} = a_m^{n/2}\, dv_{g_0}. \tag{9.15}$$

Constant scaling leaves the Ricci tensor unchanged as a $(0,2)$-tensor. Consequently,

$$\operatorname{Ric}_{k_t^{(m)}} = \operatorname{Ric}_{\widetilde{g}_{c_m(t)}} \ge (n-1)\widetilde{g}_{c_m(t)} \ge (n-1) k_t^{(m)}.$$

The connection is unchanged by constant scaling. For a symmetric 2-tensor T and a covariant 3-tensor S,

$$\|a_m T\|_{a_m \widetilde{g}} = \|T\|_{\widetilde{g}}, \qquad \|a_m S\|_{a_m \widetilde{g}} = a_m^{-1/2} \|S\|_{\widetilde{g}}.$$

Since $a_m \geq 1/2$, the chain rule and (9.6) give

$$\begin{aligned}\|\partial_t k_t^{(m)}\|_{k_t^{(m)}} + \|\partial_t^2 k_t^{(m)}\|_{k_t^{(m)}} + \|\nabla^{k_t^{(m)}} \partial_t k_t^{(m)}\|_{k_t^{(m)}} \\ \leq \eta_0\big((1+\sqrt{2})\|c_m'\|_\infty + \|c_m'\|_\infty^2 + \|c_m''\|_\infty\big) \\ \leq \eta_0\big(3\|c_m'\|_\infty + \|c_m'\|_\infty^2 + \|c_m''\|_\infty\big) \leq 1.\end{aligned}$$

Moreover

$$k_t^{(m)} = a_m g_0 \quad (t \leq -2m \text{ or } t \geq 0),$$

and $k_t^{(m)} \to \widetilde{g}_t$ smoothly on compact t-intervals. The common volume form in (9.15), the displayed Ricci inequality, the pointwise derivative estimate, and $k_t^{(m)} = a_m g_0$ for $t \geq 0$ instantiate Lemma 9.1 with the family $(k_t^{(m)})$. For the same dimension-only functions h, f, the metric

$$dr^2 + r^2 h(r)^2 k_{f(r)}^{(m)} \tag{9.16}$$

has nonnegative Ricci curvature.

Let $r_{\mathrm{CN}} > 0$ be the number in (9.1) and put $\varphi(r) = rh(r)$ for $r > 0$. That lemma gives

$$0 < \varphi'(r) < 1, \qquad \varphi''(r) \leq 0 \qquad (r > 0),$$

with $\varphi''(r) < 0$ for $0 < r \leq r_{\mathrm{CN}}$, and

$$\frac{\varphi(r)}{r} \to 1, \qquad \varphi'(r) \to 1 \quad (r \downarrow 0).$$

For every sufficiently small $R \in (0, r_{\mathrm{CN}})$ there are $d_R \in (0,1)$, $\rho_R \in (0,R)$, and a smooth function $\varphi_R : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}\varphi_R(r) = d_R r \quad (0 \leq r \leq \rho_R), \qquad \varphi_R(r) = \varphi(r) \quad (r \geq R), \\ 0 < \varphi_R' \leq 1, \qquad \varphi_R'' \leq 0, \qquad d_R \to 1 \quad (R \downarrow 0).\end{aligned} \tag{9.17}$$

Indeed, put $\kappa = -\varphi''$. Then $\kappa > 0$ on $(0,R)$. For $0 < \epsilon < R$, $\int_\epsilon^R \kappa(r)\,dr = \varphi'(\epsilon) - \varphi'(R)$. Letting $\epsilon \downarrow 0$ and using $\varphi'(\epsilon) \to 1$ gives $\kappa \in L^1(0,R)$ and $\int_0^R \kappa(r)\,dr = 1 - \varphi'(R)$. Choose $0 < \alpha_0 < \beta_0 < \gamma_0 < R$ and $\chi \in C^\infty([0,R],[0,1])$ such that $\chi = 1$ near 0 and $\operatorname{supp}\chi \subset [0,\alpha_0)$. Put

$$M_0 = \int_0^R \chi\kappa, \qquad M_1 = \int_0^R r\chi(r)\kappa(r)\,dr.$$

Since $\kappa > 0$ and $\chi = 1$ on a nonempty interval adjoining 0, $M_0 > 0$ and $M_1 > 0$. Choose a nonnegative smooth function ζ_R supported in (β_0, γ_0) and satisfying $\int_0^R r\zeta_R(r)\,dr = M_1$. Since $\operatorname{supp}\zeta_R \subset (\beta_0,\gamma_0)$, $\beta_0 \int_0^R \zeta_R(r)\,dr \leq M_1$. Moreover $M_1 < \alpha_0 M_0 < \beta_0 M_0$. Hence $\int_0^R \zeta_R < M_0$. Set

$$\kappa_R = (1-\chi)\kappa + \zeta_R, \qquad d_R = 1 + \int_0^R (\kappa_R - \kappa), \qquad p_R(r) = d_R - \int_0^r \kappa_R(t)\,dt.$$

Define $\varphi_R(r) = \int_0^r p_R(t)\,dt$ for $r \leq R$ and $\varphi_R = \varphi$ for $r \geq R$. Since $\varphi(r) \to 0$ and $\varphi'(r) \to 1$ as $r \downarrow 0$, integration of $\varphi'' = -\kappa$, with $\kappa \in L^1(0,R)$, gives

$$\varphi'(R) = 1 - \int_0^R \kappa(t)\,dt, \qquad \varphi(R) = R - \int_0^R (R-t)\kappa(t)\,dt.$$

Consequently,

$$\begin{aligned}p_R(R) - \varphi'(R) &= d_R - 1 - \int_0^R (\kappa_R - \kappa) = 0, \\ \varphi_R(R) - \varphi(R) &= R(d_R - 1) - \int_0^R (R-t)(\kappa_R - \kappa)(t)\,dt \\ &= \int_0^R t(\kappa_R - \kappa)(t)\,dt = 0.\end{aligned}$$

The last equality is the prescribed first-moment identity for ζ_R. Since $\kappa_R = \kappa$ near R, all higher jets agree there. Since $\kappa_R = 0$ on a neighbourhood of 0, there is $\rho_R \in (0, R)$ such that $p_R \equiv d_R$ on $[0, \rho_R]$; hence $\varphi_R(r) = d_R r$ there. The inequalities

$$0 < d_R < 1, \qquad p_R(r) \geq p_R(R) = \varphi'(R) > 0$$

hold for all sufficiently small R, and

$$0 < 1 - d_R \leq \int_0^R \kappa = 1 - \varphi'(R) \to 0.$$

On $[0, R]$, nonnegativity of κ_R gives $0 < \varphi_R' \leq d_R < 1$ and $\varphi_R'' \leq 0$. On $[R, \infty)$, the same inequalities, with upper bound 1 for the derivative, follow from $\varphi_R = \varphi$ and the global inequalities in Lemma 9.1. Thus (9.17) holds.

Choose $R_m \downarrow 0$, within the range of this assertion, so that

$$f(R_m) < -2m, \qquad R_m < m^{-1}.$$

Put $d_m = d_{R_m}$, $\varphi_m = \varphi_{R_m}$, and $\rho_m = \rho_{R_m}$. Since f is increasing, $k^{(m)}_{f(r)} = a_m g_0$ for $0 < r \leq R_m$. The strict inequality $f(R_m) < -2m$ and continuity of f give $\epsilon_m^{\text{join}} > 0$ such that

$$k^{(m)}_{f(r)} = a_m g_0 \qquad (R_m - \epsilon_m^{\text{join}} < r < R_m + \epsilon_m^{\text{join}}).$$

Define $\bar{g}_m := dr^2 + \varphi_m(r)^2 k^{(m)}_{f(r)}$. This metric is smooth across $r = R_m$: the slice metric $k^{(m)}_{f(r)}$ is constant on a neighbourhood of R_m, while φ_m agrees with φ to every order at R_m. For $r \geq R_m$ this is (9.16). For $r \leq R_m$ the cross-sectional metric is $a_m g_0$. The warped-product formulas give

$$\operatorname{Ric}_{\bar{g}_m}(\partial_r, \partial_r) = -n \frac{\varphi_m''}{\varphi_m} \geq 0$$

and, for every $(a_m g_0)$-unit vector v,

$$\operatorname{Ric}_{\bar{g}_m}(\varphi_m^{-1} v, \varphi_m^{-1} v) \geq \varphi_m^{-2} \left(\frac{n-1}{a_m} - \varphi_m \varphi_m'' - (n-1)(\varphi_m')^2 \right) \geq 0.$$

Consequently, $\operatorname{Ric}_{\bar{g}_m} \geq 0$. For $0 < r \leq \rho_m$,

$$\bar{g}_m = dr^2 + r^2 b_m g_0, \qquad b_m := a_m d_m^2 \in (0,1), \qquad b_m \to 1. \tag{9.18}$$

On every compact subset of $(0, \infty) \times X$,

$$\bar{g}_m \longrightarrow \bar{g} \quad \text{smoothly}. \tag{9.19}$$

Indeed, for a compact radial interval and all sufficiently large m, R_m lies below the interval and $c_m(f(r)) = f(r)$ there; then $\bar{g}_m = dr^2 + r^2 h(r)^2 a_m \widetilde{g}_{f(r)}$.

Put $\epsilon_m = 1 - b_m$. Then $\epsilon_m \in (0, 1)$ and $(1 - \epsilon_m) g_0 = b_m g_0$. By Ricci closability of g_0, choose the data

$$(N_m, H_m, q_m), \qquad K_m, \qquad \Phi_m : [1, \infty) \times X \longrightarrow N_m \setminus \operatorname{Int}_{N_m} K_m$$

of Definition 2.2 for $\epsilon = \epsilon_m$. Define

$$W_m = K_m \cup \Phi_m([1, 2] \times X).$$

This is a compact connected smooth manifold with boundary, and $q_m \in \operatorname{Int}_{N_m} W_m$. Its boundary and a neighbourhood of that boundary satisfy

$$\partial W_m = \Phi_m(\{2\} \times X), \qquad \Phi_m^* H_m = dR^2 + R^2 b_m g_0 \quad (3/2 < R < 5/2). \tag{9.20}$$

Let d_m^{int} be the intrinsic length metric induced by H_m on W_m, and put

$$D_m = \operatorname{diam}(W_m, d_m^{\text{int}}) < \infty.$$

Indeed, comparison with Euclidean metrics in interior charts and boundary half-charts shows that the intrinsic distance induces the topology of W_m; connectedness and compactness then give finite diameter. Choose

$$0 < r_m < \min\left\{\rho_m, \frac{1}{m}, \frac{2}{m(1+D_m)}\right\},$$

and define

$$\mathcal{F}_m = \left(W_m, \left(\frac{r_m}{2}\right)^2 H_m\right), \qquad \delta_m = \frac{r_m}{2}D_m, \qquad e_m = r_m + \delta_m.$$

Then $0 \le \delta_m < 1/m$ and $0 < e_m < 2/m$.

On the conical collar of $\mathcal{F}_m$, use the coordinate $r = (r_m/2)R$. The boundary $R = 2$ becomes $r = r_m$, and (9.20) gives the collar metric $dr^2 + r^2 b_m g_0$. Define

$$\beta_m : X \longrightarrow \partial\mathcal{F}_m, \qquad \beta_m(x) = \Phi_m(2, x).$$

Identify $\beta_m(x)$ with (r_m, x) in $([r_m, \infty) \times X, \bar{g}_m)$. Choose $0 < \omega_m < \min\{r_m/4, \rho_m - r_m\}$. Since $r_m < \rho_m$, (9.18) shows that on the resulting collar $r_m - \omega_m < r < r_m + \omega_m$ both metrics are restrictions of the single smooth tensor $dr^2 + r^2 b_m g_0$. All normal jets therefore agree. The quotient is a connected smooth manifold without boundary (M_m, G_m, p_m), with $p_m = q_m$, and

$$\mathrm{Ric}_{G_m} \ge 0. \tag{9.21}$$

The inserted region has intrinsic diameter δ_m. Define $\mathfrak{r}_m : M_m \to [r_m, \infty)$ by setting $\mathfrak{r}_m = r_m$ on $\mathcal{F}_m$ and $\mathfrak{r}_m(r, x) = r$ on the exterior. This function is continuous and 1-Lipschitz. Indeed, its radial differential has norm one on the exterior, it is constant on the filling, and on a collar crossing the seam it is the maximum of the collar coordinate and r_m. Integration along curves gives the global Lipschitz inequality. For every exterior point (r, x), this inequality gives the lower bound below. A curve in $\mathcal{F}_m$ from p_m to $\beta_m(x)$, followed by the outward radial segment, gives the upper bound by taking the infimum of its length:

$$\begin{gathered} r - r_m \le d_{G_m}(p_m, (r, x)) \le r - r_m + \delta_m, \\ \left|d_{G_m}(p_m, (r, x)) - r\right| \le e_m. \end{gathered} \tag{9.22}$$

Moreover, $d_{G_m}(p_m, z) \le \delta_m$ for $z \in \mathcal{F}_m$. For every $L > 0$, the closed based ball of radius L is contained in

$$\mathcal{F}_m \cup ([r_m, L + r_m] \times X),$$

which is compact. The Riemannian distance induces the manifold topology, so that closed ball is compact. Any other closed bounded ball is a closed subset of a based ball. Thus (M_m, G_m) is proper and complete.

Pointed convergence and noncollapse. Fix $R > 0$ and $0 < \alpha < R$. For all sufficiently large m, $r_m < \alpha/2$ and $\delta_m < R$. Smooth convergence in (9.19) gives numbers $\eta_m = \eta_m(\alpha, R) \to 0$, with $0 < \eta_m < 1$, satisfying

$$(1 - \eta_m)\bar{g} \le \bar{g}_m \le (1 + \eta_m)\bar{g} \quad \text{on } [\alpha/2, 3R] \times X.$$

Define

$$\lambda_m(\alpha, R) = \max\left\{\sqrt{1 + \eta_m} - 1, (1 - \eta_m)^{-1/2} - 1\right\}.$$

Then $\lambda_m(\alpha, R) \to 0$. Let z, z' be exterior points with radial coordinates $r_z, r_{z'} \in [\alpha, R]$, and put $S = r_z + r_{z'}$. Joining the corresponding points through the vertex in Y gives $d_Y(z, z') \le S$. Joining them by radial segments and a curve in $\mathcal{F}_m$ gives

$$d_{G_m}(z, z') \le S - 2r_m + \delta_m \le S + \delta_m < 3R.$$

The manifold M_m is complete and hence geodesic, and the completion argument above proves that Y is geodesic. Thus minimizing curves exist in both spaces. They cannot reach radial coordinate

$3R$, since their total radial variation would be at least $6R - S \geq 4R$, contrary to the preceding upper bounds.

A joining curve meeting $\mathcal{F}_m \cup ([r_m, \alpha/2] \times X)$ in M_m, or $\overline{B}^Y_{\alpha/2}(p)$ in Y, has length at least $S - \alpha$. A minimizing curve avoiding the corresponding inner region lies in the common exterior where the tensor comparison applies. Applying this distinction first to a minimizing curve in Y and then to one in M_m gives, respectively,

$$d_{G_m}(z, z') - d_Y(z, z') \leq \max\{2R\lambda_m, \alpha + \delta_m\},$$
$$d_Y(z, z') - d_{G_m}(z, z') \leq \max\{3R\lambda_m, \alpha\}.$$

Consequently,

$$|d_{G_m}(z, z') - d_Y(z, z')| \leq \alpha + \delta_m + 3R\lambda_m(\alpha, R). \tag{9.23}$$

For $L \geq r_m$, define

$$\mathcal{D}_m(L) = \mathcal{F}_m \cup ([r_m, L] \times X),$$

endowed with the distance restricted from M_m. Define a correspondence between $\mathcal{D}_m(R)$ and $\overline{B}^Y_R(p)$ by

$$\begin{aligned} C_m(\alpha, R) := &\{(z, p) : z \in \mathcal{D}_m(\alpha)\} \cup \{(p_m, y) : y \in \overline{B}^Y_\alpha(p)\} \\ &\cup \{((r, x), (r, x)) : \alpha \leq r \leq R,\ x \in X\}. \end{aligned}$$

Both projections are surjective, and (p_m, p) belongs to the correspondence. For two outer pairs, (9.23) applies. Two points of $\mathcal{D}_m(\alpha)$ can be joined through the filling with length at most $2\alpha + \delta_m$, whereas $\operatorname{diam} \overline{B}^Y_\alpha(p) \leq 2\alpha$. Also $d_{G_m}(p_m, z) \leq \alpha + \delta_m$ for $z \in \mathcal{D}_m(\alpha)$. Thus two inner pairs have distortion at most $2\alpha + \delta_m$, including when they belong to different inner parts of the correspondence.

For a pair (z, p) with $z \in \mathcal{D}_m(\alpha)$ and an outer pair represented by $w = (r, x)$, radial variation and a path traversing the filling once give

$$r - \alpha \leq d_{G_m}(z, w) \leq r + \alpha + \delta_m.$$

Since $d_Y(p, w) = r$, their distortion is at most $\alpha + \delta_m$. For a pair (p_m, y) with $y \in \overline{B}^Y_\alpha(p)$ and the same outer pair, (9.22) and the triangle inequality give

$$|d_{G_m}(p_m, w) - r| \leq e_m, \qquad |d_Y(y, w) - r| \leq \alpha.$$

Their distortion is at most $\alpha + e_m$. Hence

$$\operatorname{dis} C_m(\alpha, R) \leq 2\alpha + e_m + 3R\lambda_m(\alpha, R).$$

It remains to compare $\mathcal{D}_m(R)$ with the actual based ball. Equation (9.22) gives

$$\mathcal{D}_m(R) \subset \overline{B}^{M_m}_{R+e_m}(p_m).$$

If a point of $\mathcal{D}_m(R)$ lies outside $\overline{B}^{M_m}_R(p_m)$, truncate a minimizing geodesic from p_m to that point at distance R. The remaining length is at most e_m. Conversely, an exterior point of $\overline{B}^{M_m}_R(p_m)$ has radial coordinate at most $R + r_m$, by the lower bound in (9.22). If that coordinate exceeds R, the radial segment to $r = R$ has length at most r_m. Points in the filling already belong to $\mathcal{D}_m(R)$. Therefore, in the ambient metric of M_m,

$$d^{M_m}_H\big(\mathcal{D}_m(R), \overline{B}^{M_m}_R(p_m)\big) \leq e_m.$$

Define

$$\mathcal{E}^R_m = \Big\{(b, z) \in \overline{B}^{M_m}_R(p_m) \times \mathcal{D}_m(R) : d_{G_m}(b, z) \leq e_m\Big\}.$$

Compactness and the Hausdorff estimate show that both projections are surjective. Moreover, $(p_m, p_m) \in \mathcal{E}^R_m$. Compose the two correspondences by setting

$$\mathcal{S}_m(\alpha, R) = \Big\{(b, y) : \text{there is } z \text{ with } (b, z) \in \mathcal{E}^R_m,\ (z, y) \in C_m(\alpha, R)\Big\}.$$

This is a correspondence between the actual closed based balls, and it contains (p_m, p). For two pairs with witnesses z, z', the triangle inequality gives

$$\begin{aligned}|d_{G_m}(b, b') - d_Y(y, y')| &\leq d_{G_m}(b, z) + |d_{G_m}(z, z') - d_Y(y, y')| + d_{G_m}(z', b') \\ &\leq 2e_m + \operatorname{dis} C_m(\alpha, R).\end{aligned}$$

Thus

$$\operatorname{dis} \mathcal{S}_m(\alpha, R) \leq 2\alpha + 3e_m + 3R\lambda_m(\alpha, R).$$

The metric realization of a correspondence places each related pair at distance at most half the displayed bound. Consequently the closed based balls admit a common metric realization in which both their Hausdorff distance and the distance between p_m and p are at most

$$\alpha + \frac{3}{2}e_m + \frac{3}{2}R\lambda_m(\alpha, R).$$

First let $m \to \infty$ and then $\alpha \downarrow 0$. For every $R > 0$, the closed based balls converge in pointed Gromov–Hausdorff topology. Hence

$$(M_m, d_{G_m}, p_m) \xrightarrow{\text{pGH}} (Y, d, p). \tag{9.24}$$

For all sufficiently large m, the annulus

$$\mathcal{A} := \{(r, x) : 1/4 \leq r \leq 1/2\}$$

is contained in $B_1^{G_m}(p_m)$ by (9.22). Smooth convergence gives

$$\operatorname{Vol}_{G_m}(\mathcal{A}) \longrightarrow \operatorname{Vol}_{\bar{g}}(\mathcal{A}) > 0.$$

Thus $\operatorname{Vol}_{G_m}(B_1(p_m))$ has a positive lower bound for all sufficiently large m. Taking the minimum with the finitely many remaining positive unit-ball volumes gives

$$\operatorname{Vol}_{G_m}(B_1(p_m)) \geq v > 0 \qquad (m \geq 2). \tag{9.25}$$

Together with (9.21) and (9.24), this proves that (Y, d, p) is a classical noncollapsed Ricci limit with the asserted realizing sequence. □

10. The realization theorem

Proof of Theorem 3.2. Choose $D_Q, P_Q, \beta_Q, k_Q, \kappa_Q, \mu_Q$ satisfying (3.1). Let $q_0 > 0$ be the constant supplied by Theorem 8.1 for $D_Q, P_Q, \beta_Q, k_Q, \kappa_Q, \mu_Q$, and replace it by $\min\{q_0, 1\}$. Fix $q \in (0, q_0)$, let $(g_s)_{s\geq 0}$ be the family supplied by that theorem, and define

$$\ell_t = \begin{cases} g_{-t}, & t \leq 0, \\ g_0, & t \geq 0. \end{cases}$$

Let $s_* > 0$ be the constant in Theorem 8.1 and put $\delta := s_*/2$. Then $g_s = g_0$ for $0 \leq s \leq \delta$, and hence $\partial_s^j g_s|_{s=0} = 0$ for every integer $j \geq 1$. The left and right t-derivatives of the displayed piecewise family agree at $t = 0$; therefore $(\ell_t)_{t\in\mathbb{R}}$ is smooth and $\ell_t = g_0$ for $t > -\delta$. For every $t \in \mathbb{R}$,

$$dv_{\ell_t} = dv_{g_0}, \qquad \operatorname{Ric}_{\ell_t} \geq (n-1)\ell_t,$$
$$\ell_t = g_0 \quad (t \geq 0), \qquad (M_Q, \ell_t) \xrightarrow[t\to-\infty]{\text{GH}} L_{n,q}.$$

The metric g_0 is Ricci closable. The manifold M_Q is closed, connected, and smooth of dimension n, and $L_{n,q}$ is compact. Apply Proposition 9.2 with

$$X = M_Q, \qquad \Sigma = L_{n,q}, \qquad g_t = \ell_t.$$

The five hypotheses are

$$dv_{\ell_t} = dv_{g_0}, \qquad \mathrm{Ric}_{\ell_t} \geq (n-1)\ell_t,$$
$$\ell_t = g_0 \ (t \geq 0), \qquad (M_Q, \ell_t) \xrightarrow[t\to-\infty]{\mathrm{GH}} L_{n,q},$$

together with Ricci closability of g_0. Proposition 9.2 therefore gives a complete pointed classical noncollapsed Ricci limit $(Y_{Q,q}, d, p)$, a number $r_0 > 0$, and complete connected smooth realizers without boundary having nonnegative Ricci curvature and a common positive unit-ball volume lower bound, such that

$$\mathrm{Tan}(Y_{Q,q}, p) = \{\mathrm{C}(L_{n,q})\}, \qquad (B_{r_0}(p), p) \cong \big(\mathrm{C}_{\mathrm{top},r_0}(M_Q), o\big).$$

Proposition 9.2 also gives $Y_{Q,q} \setminus \{p\} \cong_{\mathrm{diff}} (0,\infty) \times M_Q$. By (5.3), $\mathrm{Tan}(Y_{Q,q}, p) = \{\mathbb{R}^{n-1} \times \mathrm{C}(S^1_{2\pi q})\}$. Thus (i) and (ii) hold; the displayed diffeomorphism and the stated properties of the realizing sequence also hold.

Set $U := B_{r_0}(p)$, $U^\times := U \setminus \{p\}$, and $Z := Y_{Q,q} \setminus U$. Then $\overline{Z} = Z \subset Y_{Q,q} \setminus \{p\} = \mathrm{Int}_{Y_{Q,q}}(Y_{Q,q} \setminus \{p\})$. Excision, with Z removed from the pair $(Y_{Q,q}, Y_{Q,q} \setminus \{p\})$, gives

$$H_j(Y_{Q,q}, Y_{Q,q} \setminus \{p\}; A) \cong H_j(U, U^\times; A).$$

The pointed homeomorphism in (ii) identifies U with $\mathrm{C}_{\mathrm{top},r_0}(M_Q)$ and $U^\times$ with $(0, r_0) \times M_Q$. The maps

$$[r, x] \longmapsto [(1-\tau)r, x] \qquad (0 \leq \tau \leq 1)$$

contract U to its vertex. If $r_* := r_0/2$, the maps

$$(r, x) \longmapsto \big((1-\tau)r + \tau r_*, x\big) \qquad (0 \leq \tau \leq 1)$$

deformation retract $U^\times$ onto $\{r_*\}\times M_Q$. The contraction shows that U is path connected. The connected smooth manifold M_Q is locally path connected and hence path connected; the displayed deformation retraction therefore shows that $U^\times$ is path connected. For $j \geq 1$, the homology sequence of the pair and the contractibility of U give

$$H_j(U, U^\times; A) \cong \widetilde{H}_{j-1}(U^\times; A) \cong \widetilde{H}_{j-1}(M_Q; A).$$

For every nonempty path-connected space W, the augmentation identifies $H_0(W; A)$ with A. Under these identifications, the inclusion $U^\times \hookrightarrow U$ acts as the identity of A. The degree-zero part of the homology sequence therefore gives

$$H_0(U, U^\times; A) = 0 = \widetilde{H}_{-1}(M_Q; A),$$

where the last equality is the convention in Section 2. This proves (iii).

Density. In the construction of Proposition 9.2, let ψ be the slow reparametrization and put $\widetilde{\ell}_t := \ell_{\psi(t)}$. Equations (9.8) and (9.9), applied with $g_t = \ell_t$, give on $Y_{Q,q} \setminus \{p\}$ the metric

$$d\rho^2 + \rho^2 h(\rho)^2 \widetilde{\ell}_{f(\rho)}, \qquad h(\rho) \longrightarrow 1 \quad (\rho \downarrow 0).$$

Every $\widetilde{\ell}_t$ has volume $q\sigma_n$. The Riemannian volume form is $\rho^n h(\rho)^n \, d\rho \, dv_{\widetilde{\ell}_{f(\rho)}}$, and (9.11) gives $B_r(p) \setminus \{p\} = (0, r) \times M_Q$. The inclusion of the punctured Riemannian manifold into its metric completion is isometric, so $\mathcal{H}^{n+1}$ restricts there to Riemannian volume. A singleton has zero $(n+1)$-dimensional Hausdorff measure. Therefore

$$\mathcal{H}^{n+1}(B_r(p)) = q\sigma_n \int_0^r \rho^n h(\rho)^n \, d\rho.$$

The limit $h(\rho) \to 1$ implies

$$\left| \int_0^r \rho^n \big(h(\rho)^n - 1\big) \, d\rho \right| \leq \sup_{0<\rho<r} |h(\rho)^n - 1| \frac{r^{n+1}}{n+1} = o(r^{n+1}).$$

Since $\sigma_n = (n+1)\omega_{n+1}$,

$$\mathcal{H}^{n+1}(B_r(p)) = q\omega_{n+1}r^{n+1} + o(r^{n+1}).$$

Thus $\Theta^{n+1}(Y_{Q,q}, p) = q$, proving (iv).

For every $m \geq 2$, (M_m, G_m) is a complete connected smooth $(n+1)$-manifold without boundary and (9.21) gives $\mathrm{Ric}_{G_m} \geq 0$. Hence $(M_m, d_{G_m}, \mathcal{H}^{n+1}_{d_{G_m}})$ is a noncollapsed RCD$(0, n+1)$ space. Equations (9.24) and (9.25) give

$$(M_m, d_{G_m}, p_m) \xrightarrow{\mathrm{pGH}} (Y_{Q,q}, d, p), \qquad \mathcal{H}^{n+1}_{d_{G_m}}(B_1(p_m)) \geq v > 0 \quad (m \geq 2).$$

Apply [10, Theorem 1.10] with $(X_m, d_m, p_m) := (M_m, d_{G_m}, p_m)$, $K = 0$, and $N = n+1$. The sequence consists of noncollapsed RCD$(0, n+1)$ spaces; (9.24) is the required pointed Gromov–Hausdorff convergence; (9.25) supplies the displayed uniform lower volume bound; and each (M_m, d_{G_m}) is a topological $(n+1)$-manifold without boundary. Theorem 1.10 of [10] therefore yields a noncollapsed RCD$(0, n+1)$ structure $(Y_{Q,q}, d, \mathcal{H}^{n+1})$ with $\partial Y_{Q,q} = \emptyset$. Thus $\partial Y_{Q,q} = \emptyset$, which is (v).

By Lemma 5.1, the unique tangent at p splits $\mathbb{R}^{n-1}$ but not $\mathbb{R}^n$. Hence $p \in \mathcal{S}^{n-1}(Y_{Q,q}) \setminus \mathcal{S}^{n-2}(Y_{Q,q})$. The manifold M_Q is closed, connected, and oriented, so $\widetilde{H}_0(M_Q; \mathbb{Z}) = 0$ and $H_n(M_Q; \mathbb{Z}) \cong \mathbb{Z}$. By [14, Corollary 6.7 and Theorem 3.5], M_Q admits a Morse function and has the homotopy type of a CW complex with one cell of dimension equal to the index of each critical point. The critical set is a compact discrete subset of M_Q, hence finite, and every critical index lies in $\{0, \dots, n\}$. Consequently

$$H_j(M_Q; \mathbb{Z}) = 0 \qquad (j > n).$$

Thus the integral homology of M_Q can differ from that of S^n only in degrees $1, \dots, n-1$. If M_Q is not an integral homology n-sphere, there exists $i \in \{1, \dots, n-1\}$ such that $\widetilde{H}_i(M_Q; \mathbb{Z}) \neq 0$. Assertion (iii) gives $H_{i+1}(Y_{Q,q}, Y_{Q,q} \setminus \{p\}; \mathbb{Z}) \neq 0$ in degree $i+1 < n+1$. Suppose that p were topologically regular. Choose a pointed homeomorphism $\chi : (O, p) \to (B_1^{\mathbb{R}^{n+1}}(0), 0)$ from an open neighbourhood O of p, and put $W := \chi^{-1}(B_{1/2}^{\mathbb{R}^{n+1}}(0))$. The set $Y_{Q,q} \setminus W$ is closed and is contained in $Y_{Q,q} \setminus \{p\} = \mathrm{Int}_{Y_{Q,q}}(Y_{Q,q} \setminus \{p\})$. Excision and the chart give

$$H_j(Y_{Q,q}, Y_{Q,q} \setminus \{p\}; \mathbb{Z}) \cong H_j(W, W \setminus \{p\}; \mathbb{Z}) \cong H_j(B_{1/2}^{\mathbb{R}^{n+1}}(0), B_{1/2}^{\mathbb{R}^{n+1}}(0) \setminus \{0\}; \mathbb{Z}).$$

The ball is contractible and its punctured ball deformation retracts onto S^n. The homology sequence of this pair therefore gives

$$H_j(Y_{Q,q}, Y_{Q,q} \setminus \{p\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = n+1, \\ 0, & j \neq n+1. \end{cases}$$

This contradicts the nonzero group in degree $i+1 < n+1$. Hence $p \in \mathrm{Stop}(Y_{Q,q})$.

Let $\varphi : L_{n,q} \to S^n$ be the homeomorphism in (5.1). Define

$$F([r, z]) = r\varphi(z), \qquad G(0) = o,$$
$$G(x) = [|x|, \varphi^{-1}(x/|x|)] \quad (x \neq 0),$$

where $F : \mathrm{C}(L_{n,q}) \to \mathbb{R}^{n+1}$ and $G : \mathbb{R}^{n+1} \to \mathrm{C}(L_{n,q})$. The cone formula gives $d(o, [r, z]) = r$. On $(0, \infty) \times L_{n,q}$ it induces the product topology: convergence in the cone metric forces convergence of the radii and, by the cosine term, convergence in $L_{n,q}$; substituting convergent radii and link points in the cone-distance formula gives the converse implication. Thus F and G are continuous away from the vertices, and the radial identities make them continuous at the vertices. They are inverse maps. Therefore $\mathrm{C}(L_{n,q}) \cong_{\mathrm{top}} \mathbb{R}^{n+1}$. The isometry in (5.3) preserves distance from the cone vertex and carries it to the distinguished point $(0, o)$. Hence that point is a topological $(n+1)$-manifold point. ☐

11. The Cheeger–Colding counterexample

Proof of Corollary 3.4. Fix an integer $N \geq 5$ and put $n := N-1$, so that $n \geq 4$. The proof of Corollary 3.3 constructs a core-admissible n-manifold Q and a number q satisfying $0 < q < \min\{q_0(n,Q),1\}$. For $M_Q := Q\#(-Q)$, that proof also gives

$$H_2(M_Q;\mathbb{Z}) \cong \mathbb{Z}^2. \tag{11.1}$$

Apply Theorem 3.2 to (n,Q,q), and denote the limit by (Y,d,p). Then

$$\begin{gathered}\partial Y = \emptyset, \qquad \operatorname{Tan}(Y,p) = \left\{\mathbb{R}^{N-2}\times C(S^1_{2\pi q})\right\}, \\ Y\setminus\{p\} \cong_{\mathrm{diff}} (0,\infty)\times M_Q.\end{gathered} \tag{11.2}$$

The conclusion of Theorem 3.2 supplies complete connected smooth realizing N-manifolds without boundary, with nonnegative Ricci curvature and a common positive lower bound for their based unit-ball volumes. The proof of Proposition 9.2 identifies $d|_{Y\setminus\{p\}}$ with the Riemannian distance of the smooth metric $g := \bar{g}$ in (9.8). Let $y \in Y\setminus\{p\}$ and let $r_i \downarrow 0$. Choose $\rho > 0$ such that 4ρ is smaller than the convexity radius at y and $\overline{B}_{4\rho}(y) \subset Y\setminus\{p\}$. The exponential map is a diffeomorphism from $B^{T_yY}_{4\rho}(0)$ onto $B_{4\rho}(y)$, and this ball is strongly convex. Identify T_yY with $\mathbb{R}^N$ by an orthonormal basis. In the normal coordinates defined by $\exp_y$ and this basis, $g_{ab}(z) = \delta_{ab} + O(|z|^2)$. Put $d_i := r_i^{-1}d$ and fix $R > 0$. For all sufficiently large i, $4Rr_i < 4\rho$, and

$$\Phi_i : \overline{B}^{\mathbb{R}^N}_{4R}(0) \longrightarrow Y, \qquad \Phi_i(z) := \exp_y(r_i z),$$

is defined. There is $\varepsilon_i(R) \downarrow 0$ such that, on $\overline{B}^{\mathbb{R}^N}_{4R}(0)$,

$$(1-\varepsilon_i(R))g_{\mathrm{eucl}} \leq r_i^{-2}\Phi_i^* g \leq (1+\varepsilon_i(R))g_{\mathrm{eucl}}. \tag{11.3}$$

For every $0 < L \leq 4R$ and every $z \in \overline{B}^{\mathbb{R}^N}_L(0)$, the radial geodesic gives $d_i(y,\Phi_i(z)) = |z|$. Conversely, strong convexity gives, for every $x \in \overline{B}^{(Y,d_i)}_L(y)$, a minimizing radial geodesic from y to x whose initial vector has g_y-length at most Lr_i. Hence

$$\Phi_i\big(\overline{B}^{\mathbb{R}^N}_L(0)\big) = \overline{B}^{(Y,d_i)}_L(y) \qquad (0 < L \leq 4R). \tag{11.4}$$

Let $z_0, z_1 \in \overline{B}^{\mathbb{R}^N}_R(0)$. The Euclidean segment joining them and the upper inequality in (11.3) give

$$d_i\big(\Phi_i(z_0),\Phi_i(z_1)\big) \leq \sqrt{1+\varepsilon_i(R)}\,|z_0 - z_1|.$$

Let γ_i be the minimizing geodesic between $\Phi_i(z_0)$ and $\Phi_i(z_1)$. For $w \in \gamma_i$ and all sufficiently large i,

$$\begin{aligned} d_i(y,w) &\leq d_i\big(y,\Phi_i(z_0)\big) + d_i\big(\Phi_i(z_0),w\big) \\ &\leq R + d_i\big(\Phi_i(z_0),\Phi_i(z_1)\big) \\ &\leq R + 2R\sqrt{1+\varepsilon_i(R)} < 4R. \end{aligned}$$

Equation (11.4) with $L = 4R$ places γ_i inside $\Phi_i(B^{\mathbb{R}^N}_{4R}(0))$; put $\widetilde{\gamma}_i := \Phi_i^{-1}\circ\gamma_i$. The lower inequality in (11.3) gives

$$\begin{aligned} d_i\big(\Phi_i(z_0),\Phi_i(z_1)\big) &= L_{r_i^{-2}\Phi_i^* g}(\widetilde{\gamma}_i) \\ &\geq \sqrt{1-\varepsilon_i(R)}\,L_{g_{\mathrm{eucl}}}(\widetilde{\gamma}_i) \\ &\geq \sqrt{1-\varepsilon_i(R)}\,|z_0 - z_1|. \end{aligned}$$

Consequently, the closed balls satisfy

$$d_{\mathrm{GH}}\left(\overline{B}^{(Y,d_i)}_R(y), \overline{B}^{\mathbb{R}^N}_R(0)\right) \longrightarrow 0.$$

Since this holds for every $R < \infty$ and the sequence (r_i) was arbitrary,

$$\mathrm{Tan}(Y, y) = \{\mathbb{R}^N\} \qquad (y \neq p). \tag{11.5}$$

The normal-coordinate diffeomorphism makes every $y \neq p$ topologically regular. Since $\mathbb{R}^N$ splits $\mathbb{R}^{N-2}$, (11.5) gives $y \notin \mathcal{S}^{N-3}(Y)$ for every $y \neq p$. The tangent in (11.2) splits $\mathbb{R}^{N-2}$, so $p \notin \mathcal{S}^{N-3}(Y)$. Consequently

$$\mathcal{S}^{N-3}(Y) = \emptyset. \tag{11.6}$$

Since $N - 1 \geq 4$, an integral homology $(N-1)$-sphere has vanishing second integral homology. Equation (11.1) therefore shows that M_Q is not an integral homology sphere, and Theorem 3.2 gives $p \in \mathrm{Stop}(Y)$. The normal-coordinate diffeomorphism at every point of $Y \setminus \{p\}$ shows that $\mathrm{Stop}(Y) \subset \{p\}$; hence $\mathrm{Stop}(Y) = \{p\}$. The strata are nested: a splitting by $\mathbb{R}^{N-2}$ induces one by $\mathbb{R}^{N-3}$, so $\mathcal{S}^{N-4}(Y) \subset \mathcal{S}^{N-3}(Y)$. Equation (11.6) gives $\mathcal{S}^{N-4}(Y) = \emptyset$. Since $D_{N-4}(Y) = \mathcal{S}^{N-4}(Y)$ and $\mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = D_{N-4}(Y) \setminus \mathcal{R}_{\mathrm{met}}(Y)$, substitution gives

$$\mathcal{S}^{N-4}(Y) = \emptyset, \qquad \mathcal{S}^{\mathrm{CC}}_{N-4}(Y) = \emptyset. \tag{11.7}$$

Therefore

$$\mathrm{Int}_Y\big(Y \setminus \mathcal{S}^{N-4}(Y)\big) = \mathrm{Int}_Y\big(Y \setminus \mathcal{S}^{\mathrm{CC}}_{N-4}(Y)\big) = Y.$$

Assertion (iii) of Theorem 3.2 and (11.1) give

$$H_3(Y, Y \setminus \{p\}; \mathbb{Z}) \cong \mathbb{Z}^2. \tag{11.8}$$

Suppose that p had a neighbourhood U_0 homeomorphic to an open subset $O \subset \mathbb{R}^d$ for some integer $d \geq 0$, and let $z_0 \in O$ correspond to p. Choose $\rho > 0$ such that $\overline{B}^{\mathbb{R}^d}_{2\rho}(z_0) \subset O$, and let $U \subset U_0$ be the inverse image of $B^{\mathbb{R}^d}_{2\rho}(z_0)$. The set $Y \setminus U$ is closed and is contained in the open set $Y \setminus \{p\}$. Excision and the chart give

$$H_3(Y, Y \setminus \{p\}; \mathbb{Z}) \cong H_3(U, U \setminus \{p\}; \mathbb{Z}) \cong H_3\big(B^{\mathbb{R}^d}_{2\rho}(z_0), B^{\mathbb{R}^d}_{2\rho}(z_0) \setminus \{z_0\}; \mathbb{Z}\big).$$

If $d = 0$, the last pair is $(\{z_0\}, \emptyset)$. If $d \geq 1$, the ball is contractible and its punctured ball deformation retracts onto S^{d-1}. The homology sequence of the pair therefore gives

$$H_3(Y, Y \setminus \{p\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & d = 3, \\ 0, & d \neq 3. \end{cases}$$

This contradicts (11.8). Hence p is not a manifold point, and Y is not a topological manifold. Together with (11.7), this contradicts Cheeger–Colding Conjecture 0.7. □

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