VOLUME ASYMPTOTICS AND EINSTEIN MODULI ON SPHERES

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Abstract. The moduli spaces of positive Einstein metrics of volume one on S^6 and S^8, modulo all diffeomorphisms, each have countably infinitely many connected components. More generally, this holds on S^{p+q+1} and $S^{p+1} \times Q$ whenever $p, q \geq 2$, $d = p + q \leq 8$, and Q is a closed connected positive Einstein q-manifold. For sequences obtained from Böhm's cap construction and normalized by $\mathrm{Ric} = dg$, we determine the leading volume difference from the limiting sine-cone volume. It is a positive definite quadratic form in the data on a minimal hypersurface. For every such pair, suitable spherical sequences and reflection doubles have volumes converging from above to a positive limit, with successive volume excesses in limiting ratio $\exp(-2\pi\sqrt{(d-1)/(9-d)})$.

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1. Introduction

Böhm constructed infinite sequences of inhomogeneous Einstein metrics on $S^5, \dots, S^9$ and on products $S^{p+1} \times Q$ with an isotropy irreducible factor Q [2, Theorems 3.4 and 3.6]. He conjectured that their Einstein moduli spaces have infinitely many connected components [2, pp. 146 and 153]. The normalized metrics converge to sine-cones [2, Theorem 3.7]. We prove that the leading volume difference is a positive definite quadratic form in the data on a minimal hypersurface. It gives infinitely many distinct volumes and separates connected components of the full Einstein moduli space.

For a closed smooth n-manifold M, with $n \geq 3$, and $d > 0$, put

$$\mathcal{E}_d(M) = \{g \in C^\infty(S^2T^*M) \,:\, g > 0,\ \mathrm{Ric}_g = dg\}/\operatorname{Diff}(M), \tag{1.1}$$

with the quotient of the smooth topology. Volume is locally constant on $\mathcal{E}_d(M)$, by rescaling from the corresponding statement for the Einstein–Hilbert functional on unit-volume Einstein moduli [18, Theorem 3.6(2)].

Theorem A. *Each of $\mathcal{E}_5(S^6)$ and $\mathcal{E}_7(S^8)$ has countably infinitely many connected components.*

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Wang–Ziller constructed homogeneous Einstein metrics on $S^2 \times S^{2m+1}$, $m \geq 1$, whose unit-volume Einstein constants tend to zero [23, Theorem 3 and Corollaries 4–5]. Liu–Sano–Tasin constructed infinitely many Sasaki–Einstein families on every odd-dimensional standard sphere of dimension at least five [13, Theorem 1.1 and Corollary 1.2]; their volume formula separates components in full Einstein moduli [13, Section 2.4, Remark 2.13]. Suitable sequences in both constructions have volumes tending to zero at fixed Einstein constant. Foscolo–Haskins constructed inhomogeneous nearly Kähler Einstein metrics on S^6 and $S^3 \times S^3$ [9, Main Theorem]; Chi constructed a non-round $\mathsf{Sp}(2)\mathsf{Sp}(1)$-invariant positive Einstein metric on S^8 [7, arXiv v5, Theorem 1.3].

On a fixed closed connected n-manifold, the volumes of homogeneous Einstein metrics with $\mathrm{Ric}_g = dg$ form a finite set above every positive lower bound [4, author manuscript, Theorem 1.8]. The normalization $\widehat{g} = \mathrm{Vol}(g)^{-2/n} g$ has scalar curvature $nd\ \mathrm{Vol}(g)^{2/n}$. The cited theorem gives finitely many components above each positive scalar-curvature bound; local constancy of scalar curvature gives finitely many values. The sequences below have distinct volumes with positive limits.

1.1. **Volume asymptotics.** Write $\omega_j = \mathrm{Vol}(S^j, \gamma_j)$ for the volume of the unit round j-sphere. Fix integers

$$p, q \geq 2, \qquad d = p + q \leq 8, \qquad \alpha = \frac{d-1}{2}, \qquad \beta = \frac{\sqrt{(d-1)(9-d)}}{2} > 0. \tag{1.2}$$

Let (Q^q, h_Q) be closed and connected, with $\mathrm{Ric}_{h_Q} = (q-1)h_Q$, and set

$$\begin{gathered} \Omega = \omega_p \,\mathrm{Vol}(Q, h_Q), \qquad v_0 = \left(\frac{p-1}{d-1}\right)^{p/2} \left(\frac{q-1}{d-1}\right)^{q/2}, \\ V_\infty = \Omega v_0 \int_0^\pi \sin^d t \, dt. \end{gathered} \tag{1.3}$$

This is the volume of the sine-cone

$$dt^2 + \sin^2 t \left(\frac{p-1}{d-1}\gamma_p + \frac{q-1}{d-1} h_Q\right), \qquad 0 < t < \pi.$$

For the unit round fibre $Q = S^q$, write

$$V_{p,q} = \omega_p \omega_q \left(\frac{p-1}{d-1}\right)^{p/2} \left(\frac{q-1}{d-1}\right)^{q/2} \int_0^\pi \sin^d t \, dt. \tag{1.4}$$

Theorem B. *For every pair in* (1.2) *and every* (Q, h_Q) *as above,* $S^{p+1} \times Q$ *admits smooth metrics* g_j *with pairwise distinct volumes such that*

$$\mathrm{Ric}_{g_j} = d g_j, \qquad \mathrm{Vol}(g_j) > V_\infty, \qquad \mathrm{Vol}(g_j) \longrightarrow V_\infty. \tag{1.5}$$

When $Q = S^q$ *is unit round, there is also such a sequence on* S^{d+1}*, with limiting volume* $V_{p,q}$*. Each of the corresponding spaces* $\mathcal{E}_d$ *has countably infinitely many connected components. In particular, this holds on* S^5, S^6, S^7, S^8, S^9.

In [2, Theorem 3.4], Q is compact, connected and isotropy irreducible, with positive Einstein constant, and $2 \leq q \leq 6$, $3 \leq p+1 \leq 9-q$. After rescaling h_Q, these are special cases of Theorem B. The replacement of the noncollapsing round factor by an arbitrary positive Einstein manifold was observed in [10, arXiv v1, §3.1, paragraph following (42)]. Thus Theorem B proves the component conclusion conjectured in [2, pp. 146 and 153] for all the manifolds in its Theorems 3.4 and 3.6. Theorem A follows by taking $(p, q) = (2, 3)$ and $(2, 5)$.

1.2. **The volume difference.** For the spherical matchings and reflection doubles of Proposition 6.2, let $Z_j = (x_j, y_j)$ be the data on the common minimal hypersurface, with the outward normal of one cap. The coordinates are defined in Sections 4 and 6. There are constants $\mu, \nu > 0$, depending only on d, such that, as $j \to \infty$,

$$\mathrm{Vol}(g_j) - V_\infty = \frac{\Omega p q v_0}{2\alpha d^2}(\nu x_j^2 + \mu y_j^2) + o(x_j^2 + y_j^2).$$

For spheres, $\Omega = \omega_p \omega_q$ and $V_\infty = V_{p,q}$. The constants are given in Lemma 6.1. The boundary variation formula [17, 1] gives a quadratic primitive for each cap volume. Reversal of the normal cancels the mixed terms; the remaining quadratic form is positive definite (Proposition 7.3).

1.3. **Geometric spacing.**

Theorem 1.1. *For every pair in* (1.2)*, there is a spherical sequence in Theorem B and a constant* $C_{p,q} > 0$ *such that*

$$\mathrm{Vol}(g_j) = V_{p,q} + C_{p,q} e^{-2\pi\alpha j/\beta}(1 + o(1)). \tag{1.6}$$

Its volumes are eventually strictly decreasing, and

$$\lim_{j\to\infty} \frac{\mathrm{Vol}(g_{j+1}) - V_{p,q}}{\mathrm{Vol}(g_j) - V_{p,q}} = \exp\left(-2\pi\sqrt{\frac{d-1}{9-d}}\right). \tag{1.7}$$

For the six- and eight-spheres, the limiting volumes are

$$V_{2,3} = \frac{16\pi^3}{15\sqrt{2}}, \qquad V_{2,5} = \frac{256\pi^4}{945}\sqrt{\frac{2}{3}}, \qquad V_{3,4} = \frac{128\pi^4}{315\sqrt{3}}. \tag{1.8}$$

The ratios in (1.7) are $e^{-2\pi}$ on S^6 and $e^{-2\pi\sqrt{3}}$ on S^8. The sequences on S^8 for $(p, q) = (2, 5)$ and $(3, 4)$ have distinct positive limiting volumes. Reflection doubles on $S^{p+1} \times Q$ satisfy the same geometric asymptotic for every pair in (1.2); see Corollary 7.4.

1.4. **Related results.** Gibbons–Hartnoll–Pope computed the limiting sine-cone volumes and tabulated decreasing volumes for selected compact Böhm metrics [10, arXiv v1, Section 5.2]. Equation (1.6) determines the sign and rate of the leading difference along infinite sequences.

The cap matching construction is due to Böhm [2, Section 7, Theorem 7.3]. Nienhaus–Wink describe minimal-section intersections by cap rotation and comparison with the sine-cone linearization [14, arXiv v3, Introduction, Lemma 3.4 and Remark 5.3]. Siffert's three-warp construction contains uniform C^1 matching estimates in logarithmic scale variables [19, Lemmas 7.5 and 7.8, Theorem 7.9]; her $\mathsf{SU}(3)$-invariant sequence on S^7 has geometric singular-orbit scale spacing [20, Theorem 1.1 and Proposition 8.1]. The cone exponents $-\alpha \pm i\beta$ occur in [12, Section 6, (6.11)] and [10, arXiv v1, Section 6, (126)]. The equatorial cross term computed in Lemma 6.3 gives the spacing in (1.7) for every pair in (1.2).

Further constructions include the metrics on S^{10} in [14], on S^{12} in [5], and on S^{11}, S^{12}, S^{13} and $S^7 \times S^3$ in [24, arXiv v3, Theorem 1.1]. Huang independently constructed a non-product metric on $S^3 \times S^7$ [11, arXiv v2, Theorem A]. For Böhm's sequences, the number of negative Lichnerowicz eigenvalues on transverse-traceless tensors tends to infinity [22, Theorem 4.3].

Eschenburg–Wang established local existence for invariant Einstein metrics near a singular orbit [8]. In the doubly warped family, Lemma 3.1 and Corollary 3.2 give analytic dependence on the Einstein constant. Böhm's complete Ricci-flat caps [3] determine the small-cap asymptotic, whose logarithmic parameter derivative is given in Proposition 5.2.

The metric h_Q need not be homogeneous, and Q need not be orientable. All integrals are taken with respect to the Riemannian density.

2. Boundary variation

At the boundary of a Riemannian manifold, let ν be the outward unit normal and put

$$\mathrm{II}(X, Y) = g(\nabla_X \nu, Y), \qquad H = \mathrm{tr}_{g|_{\partial M}} \mathrm{II}.$$

For a symmetric two-tensor h, write $h^\top$ for its restriction to $T\partial M$.

Lemma 2.1. *Let $g(s)$ be a smooth family of metrics on a fixed compact n-manifold with boundary, $n \geq 3$, such that $\mathrm{Ric}_{g(s)} = \Lambda g(s)$ for a fixed $\Lambda \neq 0$. Suppose that ∂M is minimal for every s. If $h = \partial_s g(s)|_{s=0}$, then*

$$2\Lambda\, \partial_s \mathrm{Vol}(g(s))\big|_{s=0} = -\int_{\partial M} \langle \mathrm{II}, h^\top \rangle \, dA. \tag{2.1}$$

Proof. Put $c = (n-2)\Lambda/2$ and

$$I(g) = -\int_M (\mathrm{scal}_g - 2c)\, dV_g - 2\int_{\partial M} H_g \, dA_g.$$

The first variation formula for this functional [1, arXiv v6, (4.15)–(4.17)] is

$$DI_g(h) = \int_M \left\langle \mathrm{Ric}_g - \tfrac{1}{2}\,\mathrm{scal}_g\, g + cg, h \right\rangle dV_g + \int_{\partial M} \langle \mathrm{II} - Hg^\top, h^\top \rangle \, dA_g.$$

Here $\mathrm{II} = \frac{1}{2}(\mathcal{L}_\nu g)^\top$, with ν outward. Along the given family the interior integrand vanishes and $I(g(s)) = -2\Lambda\,\mathrm{Vol}(g(s))$. Since $H = 0$, differentiation gives (2.1). For a nonorientable M, apply the same formula to its orientation double cover and divide both sides by two. □

3. The singular initial value problem

Fix integers $p, q \geq 2$, put $d = p+q$, and let (Q^q, h_Q) be closed and connected, with $\mathrm{Ric}_{h_Q} = (q-1)h_Q$. Write γ_p for the unit round metric on S^p. On an interval where $a, b > 0$, consider

$$g = dt^2 + a(t)^2\gamma_p + b(t)^2 h_Q, \qquad \mathrm{Ric}_g = \lambda g. \tag{3.1}$$

Put $A = a'/a$, $B = b'/b$, and $H = pA + qB$. The mixed components of the Ricci tensor vanish, and its diagonal components are

$$\mathrm{Ric}_g(\partial_t, \partial_t) = -H' - pA^2 - qB^2,$$

$$\mathrm{Ric}_g\,|_{TS^p} = \big((p-1)a^{-2} - A' - HA\big)g|_{TS^p}, \qquad \mathrm{Ric}_g\,|_{TQ} = \big((q-1)b^{-2} - B' - HB\big)g|_{TQ}.$$

Thus the tangential equations are

$$A' = (p-1)a^{-2} - HA - \lambda, \tag{3.2}$$

$$B' = (q-1)b^{-2} - HB - \lambda. \tag{3.3}$$

Their trace, together with the radial equation, gives

$$H^2 - pA^2 - qB^2 = p(p-1)a^{-2} + q(q-1)b^{-2} - (d-1)\lambda. \tag{3.4}$$

These equations depend on h_Q only through its Einstein constant. For the general cohomogeneity-one singular initial-value problem, see [8].

Lemma 3.1. *Let $I \subset \mathbb{R}$ be a compact interval. There is $t_0 > 0$ such that, for each $\lambda \in I$, a unique pair of even analytic functions $f, g = O(t^2)$ gives a solution*

$$a(t) = te^{f(t)}, \qquad b(t) = e^{g(t)} \tag{3.5}$$

of (3.2)–(3.4) on $(0, t_0]$. The functions depend analytically on λ. The metric extends smoothly over the zero section of $D^{p+1} \times Q$.

Proof. With $\xi = a$ as independent variable, put $v = a'$ and $w = b'$. The tangential equations become

$$v_\xi = -\frac{(p-1)(v^2-1)}{\xi v} - \frac{qw}{b} - \frac{\lambda\xi}{v},$$
$$b_\xi = w/v, \qquad w_\xi = -\frac{pw}{\xi} + \frac{(q-1)(1-w^2)/b - \lambda b}{v}.$$

The initial values are $v(0) = b(0) = 1$, $w(0) = 0$. Set

$$C(b,\lambda) = \frac{(q-1)/b - \lambda b}{p+1}, \qquad D(b,\lambda) = -\frac{qC(b,\lambda)/b + \lambda}{2p}, \qquad \eta(b) = \frac{q}{(p-1)b},$$

and

$$J_* = \frac{v-1}{\xi^2}, \quad W_* = \frac{w}{\xi}, \quad Z = W_* - C, \quad X = J_* - D + \eta Z, \quad E = 1 + \xi^2 J_*.$$

Thus $J_* = D + X - \eta Z$ and $W_* = C + Z$. Put

$$F_0 = \frac{W_*}{E}, \qquad F_1 = \frac{(p-1)J_*^2 + \lambda J_*}{E} - D_b F_0, \qquad F_2 = -\frac{(p+1)CJ_* + (q-1)W_*^2/b}{E} - C_b F_0.$$

The equations are

$$\begin{aligned} \xi b_\xi &= \xi^2 F_0, & \xi\lambda_\xi &= 0, \\ \xi Z_\xi &= -(p+1)Z + \xi^2 F_2, & \xi X_\xi &= -2pX + \xi^2(F_1 + \eta F_2 + \eta_b F_0 Z). \end{aligned} \tag{3.6}$$

The linear term in Z in the last equation vanishes because $(p-1)\eta = q/b$. The F_i are analytic near $(\xi, b, \lambda, X, Z) = (0, 1, \lambda_0, 0, 0)$, where $b = E = 1$.

In [16, arXiv version, Theorem 3], take (b, λ) as regular variables and (X, Z) as the remaining variables. The positive coefficients are $2p$ and $p+1$, and all powers in the regular terms are 2. The theorem gives a unique solution, analytic in (ξ, λ_0), with

$$b(0) = 1, \quad \lambda(0) = \lambda_0, \quad X(0) = Z(0) = 0, \qquad b - 1, X, Z = O(\xi^2).$$

Since the right sides depend on ξ through ξ^2, uniqueness makes b, X, Z, and hence J_*, W_*, even in ξ.

The geodesic coordinate

$$t(\xi, \lambda_0) = \int_0^\xi \frac{d\zeta}{1 + \zeta^2 J_*(\zeta, \lambda_0)}$$

is odd and analytic, with $t_\xi(0, \lambda_0) = 1$. Its analytic inverse $a(t, \lambda_0)$ is odd and equals $t + O(t^3)$; the composite $b(a(t, \lambda_0), \lambda_0)$ is even and equals $1 + O(t^2)$. Thus $f = \log(a/t)$ and $g = \log b$ are even analytic and $O(t^2)$. Conversely, any such pair gives even analytic J_*, W_*; the constant terms of their equations force $W_*(0) = C(1, \lambda)$ and $J_*(0) = D(1, \lambda)$. Hence $X, Z = O(\xi^2)$, and uniqueness in (3.6) gives uniqueness of f, g. A finite cover of the compact interval I by these parameter neighborhoods, with uniqueness on overlaps, gives a common neighborhood of $[-t_0, t_0] \times I$. Every fixed mixed t- and λ-derivative is bounded on this compact set.

For $x \in \mathbb{R}^{p+1}$, put $t = |x|$ and $A_*(t^2) = e^{2f(t)}$. Then

$$dt^2 + a^2 g_p = A_*(|x|^2) \sum_i dx_i^2 + \frac{1 - A_*(|x|^2)}{|x|^2} \left(\sum_i x_i \, dx_i \right)^2.$$

Since $A_*(0) = 1$, the quotient is analytic in $|x|^2$, as is $b^2 = e^{2g(t)}$. The metric therefore extends smoothly, with value equal to the Euclidean normal metric plus h_Q at the zero section.

The tangential equations and the vanishing mixed Ricci components give

$$\mathrm{scal}_g = -H' - pA^2 - qB^2 + d\lambda.$$

This function is constant on each level. For $L = A\, I_{TS^p} \oplus B\, I_{TQ}$, the tangential equations are the metric and shape-operator equations of [8, author manuscript, Lemma 2.2]. That lemma gives

$$\big(\mathrm{Ric}_g(\partial_t, \partial_t) - \lambda\big) a^{2p} b^{2q} \quad \text{constant in } t.$$

Smoothness at the zero section bounds the Ricci tensor, while $a^{2p} b^{2q} = O(t^{2p})$. The constant is zero. Thus $H' = -\lambda - pA^2 - qB^2$, and the trace of the tangential equations gives (3.4). The Einstein equation extends across the zero section by continuity. □

Corollary 3.2. *For $s > 0$, let g_s be s^2 times the normalized metric of Lemma 3.1 with $\lambda = ds^2$. In its geodesic coordinate it has Einstein constant d and initial values $a_s(0) = 0$, $a_s'(0) = 1$, $b_s(0) = s$, $b_s'(0) = 0$. Set*

$$\widetilde{a}_s(t) = s^{-1} a_s(st), \qquad \widetilde{b}_s(t) = s^{-1} b_s(st), \qquad \widetilde{Y}_s = (\widetilde{a}_s, \widetilde{b}_s, \widetilde{a}_s', \widetilde{b}_s').$$

These are the normalized functions of Lemma 3.1 with $\lambda = ds^2$. Let Y_0 denote the normalized Ricci-flat solution. For every fixed T on which Y_0 is regular for $t > 0$, there is $s_T > 0$ such that, for each integer $j \geq 0$,

$$\|\widetilde{Y}_s - Y_0\|_{C^j([0,T])} + \|s\partial_s \widetilde{Y}_s\|_{C^j([0,T])} \leq C_{T,j} s^2 \qquad (0 < s < s_T).$$

The parameter derivative is taken at fixed rescaled geodesic coordinate t.

Proof. Let $Y(t, \lambda)$ be the normalized family of Lemma 3.1. Scaling and uniqueness give $\widetilde{Y}_s(t) = Y(t, ds^2)$. Choose $t_0 \in (0, T)$ in the common initial interval. For $t \geq t_0$, the regular equation is $Y' = \mathcal{F}(Y, \lambda)$, where $Y = (a, b, v, w)$ and

$$\mathcal{F}(Y, \lambda) = \left(v,\, w,\, \frac{(p-1)(1-v^2)}{a} - \frac{qvw}{b} - \lambda a,\, \frac{(q-1)(1-w^2)}{b} - \frac{pvw}{a} - \lambda b \right).$$

The Ricci-flat trajectory on $[t_0, T]$ is compact in $\{a, b > 0\}$. The smooth-flow theorem [21, Theorem 6.1], applied to $(\mathcal{F}, 0)$, extends Y smoothly to a neighborhood of $[0, T] \times [-\lambda_T, \lambda_T]$ for some $\lambda_T > 0$. Every fixed $C^j([0, T])$ norm of $\partial_\lambda Y$ is bounded there. For $ds^2 < \lambda_T$,

$$Y(t, ds^2) - Y(t, 0) = \int_0^{ds^2} \partial_\lambda Y(t, \lambda)\, d\lambda, \qquad s\partial_s Y(t, ds^2) = 2ds^2 \partial_\lambda Y(t, ds^2).$$

These identities prove both estimates. □

4. Ricci-flat cap asymptotics

Fix $4 \leq d \leq 8$. Take $\lambda = d$ for the caps and $\lambda = 0$ for the Ricci-flat cap. Define

$$a = a_0 r e^{qx/d}, \quad b = b_0 r e^{-px/d}, \quad a_0^2 = \frac{p-1}{d-1}, \quad b_0^2 = \frac{q-1}{d-1}, \quad y = x', \quad z = ry, \quad \sigma = \log r. \tag{4.1}$$

Thus $r = (a^p b^q / (a_0^p b_0^q))^{1/d}$ and $x = \log(a/a_0) - \log(b/b_0)$. Put

$$\kappa = \frac{pq}{d^2(d-1)}, \qquad K(x) = \frac{pe^{-2qx/d} + qe^{2px/d}}{d}, \qquad v_0 = a_0^p b_0^q. \tag{4.2}$$

The function K is strictly convex and tends to infinity at both ends of $\mathbb{R}$. Since $K(0) = 1$ and $K'(0) = 0$, it satisfies $K \geq 1$. Also $K''(0) = 4pq/d^2$.

Lemma 4.1. *For a cap with $\lambda = d$, before the first zero of H one has $r' > 0$ and*

$$r'^2 + r^2 = K(x) + \kappa z^2, \tag{4.3}$$

$$x_\sigma = z/f, \qquad z_\sigma = -(d-1)z + \frac{d-1}{f}(e^{-2qx/d} - e^{2px/d}), \tag{4.4}$$

$$f = \sqrt{K(x) + \kappa z^2 - r^2} = r'. \tag{4.5}$$

For

$$\mathcal{A} = \frac{1}{2}z^2 + \frac{K(x)-1}{2\kappa}, \tag{4.6}$$

the geodesic derivatives satisfy

$$\mathcal{A}' = -(d-1)\frac{r'}{r}z^2, \qquad r'^2 + r^2 = 1 + 2\kappa\mathcal{A}, \qquad r'' = -r - \frac{pq}{d^2}\frac{z^2}{r}. \tag{4.7}$$

Proof. The definitions give

$$A = \frac{r'}{r} + \frac{q}{d}y, \quad B = \frac{r'}{r} - \frac{p}{d}y, \quad H = d\frac{r'}{r}, \qquad pA^2 + qB^2 = d\left(\frac{r'}{r}\right)^2 + \frac{pq}{d}y^2.$$

As $t \downarrow 0$, $H = p/t + O(t) > 0$; hence $r' > 0$ until its first zero. Moreover,

$$p(p-1)a^{-2} + q(q-1)b^{-2} = d(d-1)r^{-2}K(x).$$

Substitution in (3.4) gives (4.3). Subtracting the tangential equations and observing that $A - B = y$ gives

$$y' = -d\frac{r'}{r}y + (d-1)r^{-2}(e^{-2qx/d} - e^{2px/d}). \tag{4.8}$$

Consequently

$$z' = -(d-1)\frac{r'}{r}z + \frac{d-1}{r}(e^{-2qx/d} - e^{2px/d}).$$

Division by $\sigma' = r'/r$ gives (4.4). Finally,

$$K'(x) = -2\kappa(d-1)(e^{-2qx/d} - e^{2px/d}), \qquad x' = z/r,$$

so

$$\mathcal{A}' = -(d-1)\frac{r'}{r}z^2 + \frac{d-1}{r}z(e^{-2qx/d} - e^{2px/d}) + \frac{K'(x)z}{2\kappa r} = -(d-1)\frac{r'}{r}z^2.$$

The constraint is $r'^2 + r^2 = 1 + 2\kappa\mathcal{A}$. The radial Einstein equation $H' = -d - pA^2 - qB^2$ gives

$$d\left(\frac{r''}{r} - \frac{r'^2}{r^2}\right) = -d - d\frac{r'^2}{r^2} - \frac{pq}{d}y^2.$$

This proves the last identity in (4.7) wherever $a, b > 0$, including at a regular minimal hypersurface. □

For $\lambda = 0$, the same calculation gives $r'^2 = K(x) + \kappa z^2$ and (4.4) with $f = \sqrt{K(x) + \kappa z^2}$. The energy identity in σ is $\mathcal{A}_\sigma = -(d-1)z^2$.

Lemma 4.2. *The normalized Ricci-flat cap exists for all $t \geq 0$, is complete, and satisfies $U = (x, z) \to 0$ as $r \to \infty$. Moreover,*

$$a(t)/t \to a_0, \quad b(t)/t \to b_0, \quad a'(t) \to a_0, \quad b'(t) \to b_0.$$

Proof. For $Q = S^q$, the product action of $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)$ on $\mathbb{R}^{p+1}\times S^q$ has principal orbit $S^p \times S^q$, singular orbit S^q, and invariant principal-orbit metrics $a^2\gamma_p + b^2\gamma_q$. The product submersion onto S^q has zero O'Neill tensor. Thus the parameters in [3, §11] are

$$k = p \geq 2, \qquad d_Q = q, \qquad \mathrm{Ric}^Q = q - 1, \qquad \|A\| = 0, \qquad \widehat{D} = (q-1)^2 > 0.$$

Theorem 6.1 of that paper, whose case $r = 0$ is treated in §11, gives a complete Ricci-flat cap. The uniqueness theorem [2, Theorem 2.3], recalled in [3, p. 174], identifies its profiles with those of Lemma 3.1 after normalization. If (a_h, b_h) denotes the profiles with initial data $(0, 1, h, 0)$, homothety gives

$$a_h(t) = ha_1(t/h), \qquad b_h(t) = hb_1(t/h).$$

Hence every $h > 0$ gives a complete cap. For linear Ricci-flat profiles $a = \eta t$, $b = \xi t$, the tangential equations give $(p-1)\eta^{-2} = (q-1)\xi^{-2} = d-1$. Convergence Theorem 11.1 and the geodesic-time convergence in [3, §10, pp. 171–172] therefore give

$$(a_h, b_h) \longrightarrow (a_0 t, b_0 t) \quad \text{in } C^\infty_{\mathrm{loc}}((0,\infty)) \quad (h \downarrow 0).$$

Evaluate this convergence and its first derivative at $t = 1$, with $h = 1/T$. The scaling identities give the four stated limits.

For general Q, the scalar equations and normalized initial values are unchanged. Their analytic solution and regular continuation therefore have the same positive profiles. The metric is smooth at the zero section by Lemma 3.1. Its radial sublevel sets are compact and $|dt|_g = 1$ for $t > 0$; every curve leaving all compact sets has infinite length.

The profile limits imply $r/t \to 1$, $x \to 0$, and

$$z = \frac{r}{t}\left(\frac{a'}{a/t} - \frac{b'}{b/t}\right) \longrightarrow 0.$$

Near zero, $r = Ct^{p/d}(1 + O(t^2))$, with $C > 0$ and $r' > 0$. The constraint $r'^2 = K + \kappa z^2 \geq 1$ preserves this sign. Thus r increases from zero to infinity. □

Lemma 4.3. *Suppose $4 \leq d \leq 8$. For the normalized Ricci-flat cap, there is $c \in \mathbb{R}^2 \setminus \{0\}$ such that*

$$U_{\mathrm{RF}}(r) = r^{A_0} c + O(r^{-2\alpha}), \qquad A_0 = \begin{pmatrix} 0 & 1 \\ -2(d-1) & -(d-1) \end{pmatrix}. \tag{4.9}$$

Here α, β are given by (1.2)*. For every fixed $j \geq 0$, the remainder after j applications of $r\partial_r$ is also $O(r^{-2\alpha})$.*

Proof. The Ricci-flat vector field F is holomorphic near zero: the analytic function $K(x) + \kappa z^2$ has value one there, so its square root has a holomorphic branch with value one. Moreover, $F(0) = 0$ and $DF(0) = A_0$. The eigenvalues $\lambda_\pm = -\alpha \pm i\beta$ have $\alpha, \beta > 0$, their convex hull does not contain zero, and

$$\mathrm{Re}(k\lambda_+ + \ell\lambda_- - \lambda_\varepsilon) = -(k + \ell - 1)\alpha < 0 \quad (k, \ell \in \mathbb{Z}_{\geq 0},\ k + \ell \geq 2,\ \varepsilon \in \{+, -\}).$$

There are no resonances. The Poincaré linearization theorem [6, arXiv version, Theorem 1.2] gives a holomorphic local diffeomorphism L, with $L(0) = 0$ and $DL(U)F(U) = A_0 L(U)$. Its linear part commutes with A_0; multiplication by its inverse therefore makes $DL(0) = I$. On real U, the real part $L_{\mathbb{R}} = \mathrm{Re}\, L$ still satisfies the conjugacy equation and has derivative I. Its real analytic inverse has the form $h(V) = V + O(|V|^2)$.

For all sufficiently large $\sigma = \log r$, Lemma 4.2 places the cap in this neighborhood. Then

$$V(\sigma) = L_{\mathbb{R}}(U_{\mathrm{RF}}(e^\sigma)) = e^{A_0\sigma} c, \qquad c \in \mathbb{R}^2.$$

If $c = 0$, regular uniqueness would give $x = 0$ throughout the cap, hence $a/a_0 = b/b_0$, contradicting $a(0) = 0$, $b(0) = 1$. Thus $c \neq 0$.

The distinct eigenvalues of A_0 give $|V(\sigma)| = O(e^{-\alpha\sigma})$. Put $R(V) = h(V) - V$ and $\mathscr{L} = (A_0 V) \cdot \nabla_V$. This derivation preserves the order of vanishing at zero, so every $\mathscr{L}^j R$ vanishes to order at least two. Hence, for each fixed $j \geq 0$,

$$\partial_\sigma^j R(V(\sigma)) = (\mathscr{L}^j R)(V(\sigma)) = O(e^{-2\alpha\sigma}).$$

Since $U_{\mathrm{RF}} = V + R(V)$, this proves the assertions. □

5. Dependence on the cap parameter

Fix $r_0 \in (0, 1)$ and write $U = (x, z)^{\mathsf{T}}$. Constants in this section may depend on p, q, r_0; dependence on an additional parameter R is indicated by a subscript. On $0 < r \leq r_0$ the shape equation for $\lambda = d$ has the form

$$U_\sigma = A(r)U + N(r, U), \qquad A(r) = \begin{pmatrix} 0 & (1-r^2)^{-1/2} \\ -2(d-1)(1-r^2)^{-1/2} & -(d-1) \end{pmatrix}. \tag{5.1}$$

Since $f \geq \sqrt{1-r_0^2} > 0$, its vector field is smooth also at $r = 0$ for U in a fixed ball. Its value at $U = 0$ is zero and its derivative there is $A(r)$. Taylor's formula therefore gives

$$|N(r,U)| \leq C|U|^2, \qquad |D_U N(r,U)| \leq C|U|, \tag{5.2}$$

uniformly for $0 \leq r \leq r_0$ in that ball. Also $A(r) - A_0 = O(r^2)$.

Lemma 5.1. *The linear equation in* (5.1) *has a unique fundamental matrix with* $r^{-A_0}\Phi(r) \to I$ *as* $r \downarrow 0$. *It satisfies*

$$\Phi(r) = r^{A_0}(I + O(r^2)), \qquad \|\Phi(r)\Phi(\rho)^{-1}\| \leq C(r/\rho)^{-\alpha}, \qquad \|\Phi(r)^{-1}\| \leq Cr^{\alpha} \tag{5.3}$$

for $0 < \rho \leq r \leq r_0$.

Proof. Put $s_\pm = -\alpha \pm i\beta$, $e_\pm = (1, s_\pm)^{\mathsf{T}}$, and $E = (e_+, e_-)$. Then $A_0 E = E\,\mathrm{diag}(s_+, s_-)$ and

$$\|r^{A_0}\| \leq Cr^{-\alpha}, \qquad \|r^{-A_0}\| \leq Cr^{\alpha}.$$

Elimination of $z = \sqrt{1-r^2}\, rh'(r)$ gives

$$r^2 h'' + \frac{d - (d+1)r^2}{1-r^2}\, rh' + \frac{2(d-1)}{1-r^2} h = 0. \tag{5.4}$$

The coefficients are even and holomorphic at zero. The indicial roots $s_\pm$ differ by $2i\beta \notin \mathbb{Z}$. Frobenius theory [15, §2.7(i), (2.7.3)–(2.7.5)] gives conjugate solutions $h_\pm = r^{s_\pm}(1 + O(r^2))$, with differentiated remainders; uniqueness of the normalized analytic factors gives their evenness. Set

$$\psi_\pm = \begin{pmatrix} h_\pm \\ \sqrt{1-r^2}\, rh'_\pm \end{pmatrix}, \qquad \Phi = (\psi_+, \psi_-)E^{-1}.$$

The conjugate columns make Φ real, and

$$(\psi_+, \psi_-) = (E + O(r^2))\,\mathrm{diag}(r^{s_+}, r^{s_-}).$$

Since both exponents have real part $-\alpha$, $Y = r^{-A_0}\Phi = I + O(r^2)$. The matrix Φ extends as an invertible fundamental matrix to $(0, r_0]$, so Y, Y^{-1} are bounded there. The identities

$$\Phi(r)\Phi(\rho)^{-1} = r^{A_0} Y(r) Y(\rho)^{-1} \rho^{-A_0}, \qquad \Phi(r)^{-1} = Y(r)^{-1} r^{-A_0}$$

give (5.3) and $\|\Phi(r)\|\,\|\Phi(r)^{-1}\| \leq C$. If $\widetilde{\Phi}$ has the same normalization, the constant matrix $\Phi^{-1}\widetilde{\Phi}$ tends to I at zero; hence it equals I. □

Proposition 5.2. *Let* U_s *be the cap with* $\lambda = d$ *and noncollapsing radius* s. *For all sufficiently small* s, *it reaches* $r = r_0$ *before its first minimal hypersurface, and*

$$U_s(r_0) = \Phi(r_0) s^{-A_0} c + o(s^\alpha), \tag{5.5}$$

$$s\partial_s U_s(r_0) = -\Phi(r_0) s^{-A_0} A_0 c + o(s^\alpha), \tag{5.6}$$

where $c \neq 0$ *is the vector in* (4.9). *The parameter derivative is taken at fixed* r.

Proof. Let R be large and fixed. In rescaled geodesic time $\tau = t/s$, the volume radius satisfies $r_s(s\tau) = s\widetilde{r}_s(\tau)$, while x and z are unchanged by the rescaling. The equation $\widetilde{r}_0(\tau) = R$ has a unique solution and derivative $\widetilde{r}_0' \geq 1$. On a compact interval containing this solution, Corollary 3.2 and the implicit function theorem give a smooth solution $\tau_s(R)$. Its dependence is through ds^2; the derivative with respect to this variable is bounded for fixed R. Thus

$$\tau_s(R) - \tau_0(R) = O_R(s^2), \qquad s\partial_s \tau_s(R) = O_R(s^2).$$

The rescaled radial derivative is positive up to this section. For small τ, Lemma 3.1 gives

$$\widetilde{r}_s'(\tau) = \frac{p}{d} v_0^{-1/d} \tau^{p/d-1}(1 + O(\tau^2)),$$

with a remainder uniform for small s. This derivative is therefore positive on a common initial interval. On the remaining compact interval it converges uniformly to $\widetilde{r}_0' \geq 1$. Composition with $\tau_s(R)$ gives

$$U_s(sR) = U_{\mathrm{RF}}(R) + \varepsilon_R(s), \qquad |\varepsilon_R(s)| + |s\varepsilon_R'(s)| \leq C_R s^2. \tag{5.7}$$

By Lemma 4.3, there are $R_1, K > 0$ such that

$$|U_{\mathrm{RF}}(R)| + |(U_{\mathrm{RF}})_\sigma(R)| \leq KR^{-\alpha} \qquad (R \geq R_1).$$

For each fixed $R \geq R_1$, choose $s_R > 0$ so that $s_R R < r_0$ and $C_R s_R^2 \leq R^{-\alpha}$. Constants without a subscript R below are independent of $R \geq R_1$ and $0 < s < s_R$. The number s_R may be decreased separately for each fixed R. As long as the solution stays in the ball of (5.2), variation of constants gives

$$\begin{aligned} |U_s(r)| \leq C(r/(sR))^{-\alpha}|U_s(sR)| \\ + C\int_{sR}^{r} (r/\rho)^{-\alpha}|U_s(\rho)|^2 \frac{d\rho}{\rho}. \end{aligned} \tag{5.8}$$

Put $M_s(r) = \sup_{sR \leq \rho \leq r} (\rho/s)^\alpha |U_s(\rho)|$. The bound $R^\alpha |U_s(sR)| \leq K + 1$ is independent of R, s. Multiplication by $(r/s)^\alpha$ and integration of $(s/\rho)^\alpha \, d\rho/\rho$ give

$$M_s(r) \leq C_0 + C_1 R^{-\alpha} M_s(r)^2.$$

Increase R_1 so that $4C_0C_1R_1^{-\alpha} < 1/2$ and the ball of radius $2C_0R_1^{-\alpha}$ lies strictly inside the fixed shape ball. The continuous function M_s starts below $2C_0$. At a first point where it equalled $2C_0$, the preceding inequality would give $2C_0 \leq C_0 + 4C_1R^{-\alpha}C_0^2 < 3C_0/2$, a contradiction. Thus

$$|U_s(r)| \leq C(s/r)^\alpha \qquad (sR \leq r \leq r_0). \tag{5.9}$$

For each fixed s, this bound and $sR \leq r \leq r_0$ keep a, b bounded above and away from zero. The formulas

$$a' = \frac{a}{r}\left(f + \frac{q}{d}z\right), \qquad b' = \frac{b}{r}\left(f - \frac{p}{d}z\right)$$

keep a', b' bounded as well. Hence the original regular first-order system continues until $r = r_0$. The geodesic time is $t(sR) + \int_{sR}^{r} f^{-1}\, d\rho$; it is finite because $f \geq \sqrt{1 - r_0^2} > 0$. Thus $r' = f > 0$ throughout this interval.

Smooth cap dependence and transversality of every radius section give a smooth function $s \mapsto U_s(r)$ for fixed r. Put $W_s = s\partial_s U_s$ at fixed r. Differentiation gives

$$(W_s)_\sigma = (A(r) + D_U N(r, U_s))W_s. \tag{5.10}$$

The chain rule on the moving section is $s\frac{d}{ds}U_s(sR) = W_s(sR) + (U_s)_\sigma(sR)$, so

$$W_s(sR) = -(U_s)_\sigma(sR) + O_R(s^2). \tag{5.11}$$

The vector field for $\lambda = d$ at $(sR, U_s(sR))$ differs from the Ricci-flat vector field at $U_{\mathrm{RF}}(R)$ by $O_R(s^2)$. After decreasing s_R, the initial bound is therefore $|W_s(sR)| \leq CR^{-\alpha}$. For $Y_s = \Phi^{-1}W_s$, equation (5.10) becomes

$$(Y_s)_\sigma = \Phi^{-1}D_U N(r, U_s)\Phi Y_s, \qquad \|\Phi^{-1}D_U N(r, U_s)\Phi\| \leq C|U_s|.$$

One has $|Y_s(sR)| \leq C(sR)^\alpha R^{-\alpha} = Cs^\alpha$, and

$$\int_{sR}^{r_0} |U_s(r)| \frac{dr}{r} \leq CR^{-\alpha}.$$

Gronwall's inequality [21, Lemma 2.7, (2.36)] gives $|Y_s(r)| \leq Cs^\alpha e^{CR^{-\alpha}}$, and hence $|W_s(r)| \leq C(s/r)^\alpha$.

The two nonlinear integrals satisfy

$$\begin{aligned} \int_{sR}^{r_0} \|\Phi(r)^{-1}\|\big(|N(r, U_s)| + |D_U N(r, U_s)W_s|\big)\frac{dr}{r} \\ \leq Cs^{2\alpha}\int_{sR}^{r_0} r^{-\alpha}\frac{dr}{r} \leq Cs^\alpha R^{-\alpha}. \end{aligned} \tag{5.12}$$

Also $\Phi(sR)^{-1} = (I + O_R(s^2))(sR)^{-A_0}$ and $(sR)^{-A_0}R^{A_0} = s^{-A_0}$. Thus (4.9) and (5.7) give

$$\Phi(sR)^{-1}U_s(sR) = s^{-A_0}c + O(s^\alpha R^{-\alpha}) + O_R(s^{\alpha+2}). \tag{5.13}$$

The differentiated Ricci-flat asymptotic is $(U_{\mathrm{RF}})_\sigma(R) = R^{A_0}A_0c + O(R^{-2\alpha})$. Multiplication of (5.11) by $\Phi(sR)^{-1}$ therefore gives

$$\Phi(sR)^{-1}W_s(sR) = -s^{-A_0}A_0c + O(s^\alpha R^{-\alpha}) + O_R(s^{\alpha+2}).$$

Set

$$E_{0,s} = U_s(r_0) - \Phi(r_0)s^{-A_0}c, \qquad E_{1,s} = W_s(r_0) + \Phi(r_0)s^{-A_0}A_0c.$$

Integrating the equations for $\Phi^{-1}U_s$ and $\Phi^{-1}W_s$ and applying (5.12) yields, for every fixed $R \geq R_1$,

$$\limsup_{s\downarrow 0} s^{-\alpha}(|E_{0,s}| + |E_{1,s}|) \leq CR^{-\alpha}. \tag{5.14}$$

The left side is independent of R, and the constant C is uniform for $R \geq R_1$. Given $\epsilon > 0$, choose R with $CR^{-\alpha} < \epsilon$ and then let $s \downarrow 0$ with this R fixed. This proves (5.5) and (5.6). □

Lemma 5.3. *Every cap with $\lambda = d$ reaches a unique hypersurface $\{t\} \times S^p \times Q$ with $H = 0$, at a time $T_s < \pi/2$. Its data on this hypersurface depend smoothly on the initial cap radius $s > 0$. Near the sine-cone, the transition from $r = r_0$ to the section $H = 0$ is a smooth local diffeomorphism taking zero shape data to zero shape data.*

Proof. The radial equation gives

$$H' = -d - pA^2 - qB^2 = -d - \frac{H^2}{d} - \frac{pq}{d}(A - B)^2, \tag{5.15}$$

and therefore

$$\frac{d}{dt}\arctan(H/d) = -1 - \frac{pq(A-B)^2}{d^2 + H^2} \leq -1. \tag{5.16}$$

For $Q = S^q$, existence of a zero is [2, §4(e), pp. 156–157], with $d_S = p$, $d_Q = q$, $d_P = k = d$, $\|A\| = 0$, $\mathrm{Ric}^Q = q - 1 > 0$ and $D = (q-1)^2 > 0$. His mean curvature is $\ell = pw'/w + dh'/h = H$, since $w = a/b$ and $h = b$. For general Q, the normalized profiles coincide with the round profiles by Lemma 3.1 and regular uniqueness. Equation (5.15) makes the zero unique.

As $t \downarrow 0$, $H = p/t + O(t)$ and $A - B = t^{-1} + O(t)$. The last integrand in (5.16) tends to $q/p > 0$. Integration gives

$$T_s = \frac{\pi}{2} - \int_0^{T_s} \frac{pq(A-B)^2}{d^2 + H^2}\, dt < \frac{\pi}{2}.$$

The cap depends smoothly on s on regular compact intervals. Since $H' \leq -d$ at the zero, the implicit function theorem gives smooth dependence of T_s and its induced data.

Put $\omega = r'$ and

$$\mathscr{C} = \{F = 0\}, \qquad F(r, \omega, x, y) = \omega^2 + r^2 - K(x) - \kappa r^2 y^2. \tag{5.17}$$

The geodesic vector field is

$$r' = \omega, \quad \omega' = -r - \frac{pq}{d^2}ry^2, \quad x' = y, \quad y' = -d\frac{\omega}{r}y + (d-1)r^{-2}(e^{-2qx/d} - e^{2px/d}).$$

It is smooth for $r > 0$. The identities $\kappa(d-1) = pq/d^2$ and $K' = -2\kappa(d-1)(e^{-2qx/d} - e^{2px/d})$ give $F' = 0$. Along the sine-cone segment from r_0 to its equator,

$$(r, \omega, x, y) = (\sin t, \cos t, 0, 0), \qquad dF = (2r, 2\omega, 0, 0) \neq 0.$$

Thus $\mathscr{C}$ is a smooth three-manifold near that segment.

At $r = r_0$, $\omega = \sqrt{1 - r_0^2} > 0$: the section is transverse, and solving $F = 0$ for ω gives coordinates (x, y). At $\omega = 0$, the sine-cone has $r = 1$, $\omega' = -1$ and $\partial_r F = 2$: this section is also transverse, with coordinates (x, y) obtained by solving for r. Smooth flow on the intervening compact segment and transverse hitting give a smooth transition $\mathcal{P}_{xy}$. Backward flow gives its smooth local inverse. At the initial section $z = r_0 y$, so

$$J_{r_0}(x, z) = (x, z/r_0), \qquad \mathcal{P} = \mathcal{P}_{xy} \circ J_{r_0}, \qquad D\mathcal{P}(0) = D\mathcal{P}_{xy}(0) \begin{pmatrix} 1 & 0 \\ 0 & r_0^{-1} \end{pmatrix}. \tag{5.18}$$

Both factors are invertible. □

When $Q = S^q$ is round, interchange (p, a) and (q, b). This gives the cap in which the S^q factor collapses. The volume radius r is unchanged, while the exchanged shape coordinates are $(-x, -z)$. The matrices A_0 and $A(r)$ depend only on d, so the same fundamental matrix Φ occurs. If $\widehat{c} \neq 0$ is the Ricci-flat coefficient for the exchanged dimensions, the second cap therefore satisfies, in the original coordinates,

$$\begin{aligned} U_{+,s}(r_0) &= \Phi(r_0) s^{-A_0} c_+ + o(s^\alpha), \\ s\partial_s U_{+,s}(r_0) &= -\Phi(r_0) s^{-A_0} A_0 c_+ + o(s^\alpha), \end{aligned} \qquad c_+ = -\widehat{c} \neq 0.$$

If $\widehat{\mathcal{P}}$ is the hitting map of Lemma 5.3 for the exchanged dimensions, then the hitting map in the original coordinates is $Y \mapsto -\widehat{\mathcal{P}}(-Y)$. Writing $K_{p,q}$ for the function in (4.2), the identity $K_{q,p}(-x) = K_{p,q}(x)$ and uniqueness of the geodesic flow identify this map with $\mathcal{P}$.

6. The equatorial equation and gluing

Let $\Gamma_-(s) = (x, y)$ be the data on the first minimal hypersurface of an a-collapsing cap. For a round fibre, let $\Gamma_+(s)$ be the data of a b-collapsing cap, in the same (x, y) coordinates. In both cases the radial normal points out of the cap. The sine-cone has data on the minimal hypersurface $(0, 0)$.

6.1. **The equatorial Wronskian.** The shape equation, linearized at $r = \sin t$, $x = y = 0$, is

$$u'' + d \cot t \, u' + 2(d - 1) \csc^2 t \, u = 0. \tag{6.1}$$

Here the primes denote differentiation in the geodesic variable t.

Lemma 6.1. *For $4 \leq d \leq 8$, there are constants $\mu, \nu > 0$ and $\delta_d \in \mathbb{R}$, depending only on d, such that*

$$\mu\nu - \delta_d^2 = \beta^2. \tag{6.2}$$

There is a pair $u_e, v_e \in \mathbb{C}$, depending only on d, and for each cap type a constant $C \in \mathbb{C} \setminus \{0\}$ such that

$$\Gamma(s) = \operatorname{Re}\left[C s^{\alpha - i\beta} \begin{pmatrix} u_e \\ v_e \end{pmatrix} \right] + o(s^\alpha), \tag{6.3}$$

$$s\Gamma'(s) = \operatorname{Re}\left[(\alpha - i\beta) C s^{\alpha - i\beta} \begin{pmatrix} u_e \\ v_e \end{pmatrix} \right] + o(s^\alpha). \tag{6.4}$$

Consequently, for positive constants c_1, c_2,

$$\begin{gathered} s\Gamma'(s) = \mathcal{B}_d \Gamma(s) + o(s^\alpha), \qquad \mathcal{B}_d = \begin{pmatrix} \alpha + \delta_d & -\mu \\ \nu & \alpha - \delta_d \end{pmatrix}, \\ c_1 s^\alpha \leq |\Gamma(s)| \leq c_2 s^\alpha. \end{gathered} \tag{6.5}$$

Proof. Let h_+ be the normalized Frobenius solution in the proof of Lemma 5.1. The function $u(t) = h_+(\sin t)$ satisfies (6.1) for $0 < t \leq \arcsin r_0$. The normalization of h_+ gives

$$u(t) = t^{-\alpha + i\beta} \sum_{j \geq 0} a_j t^{2j}, \qquad a_0 = 1,$$

with convergent series near zero. Thus $u = t^{-\alpha+i\beta}(1 + O(t^2))$ and $tu' = t^{-\alpha+i\beta}(-\alpha + i\beta + O(t^2))$. The solution extends to $\pi/2$ by [21, Theorem 3.9], whose continuous-coefficient hypothesis holds on $(0, \pi)$. This normalized solution depends only on d. Equation (6.1) gives

$$\frac{d}{dt}\left(\sin^d t\, \overline{u}u'\right) = \sin^d t\, |u'|^2 - 2(d-1)\sin^{d-2} t\, |u|^2 \in \mathbb{R}.$$

Hence $\sin^d t\ \mathrm{Im}(\overline{u}u')$ is constant. Since $\overline{u}u' = t^{-2\alpha-1}(-\alpha + i\beta + O(t^2))$, this constant equals β, since $d - 2\alpha - 1 = 0$. Put

$$u_e = u(\pi/2), \quad v_e = u'(\pi/2), \quad \mu = |u_e|^2, \quad \nu = |v_e|^2, \quad \delta_d = \mathrm{Re}(\overline{u_e}v_e). \tag{6.6}$$

Then

$$\mathrm{Im}(\overline{u_e}v_e) = \beta > 0. \tag{6.7}$$

This gives $u_e, v_e \neq 0$, (6.2), and a real fundamental pair $\mathrm{Re}\, u, \mathrm{Im}\, u$.

Let $c = (c_x, c_z)^\mathsf{T} \neq 0$ be the coefficient of Proposition 5.2, and put

$$s_+ = -\alpha + i\beta, \qquad e_+ = (1, s_+)^\mathsf{T}, \qquad C = c_x - i\frac{c_z + \alpha c_x}{\beta}. \tag{6.8}$$

Thus $A_0 e_+ = s_+ e_+$, $c = \mathrm{Re}(Ce_+)$, $C \neq 0$, and $s^{-A_0}c = \mathrm{Re}(Cs^{\alpha-i\beta}e_+)$. For $0 < r < 1$, define

$$\psi(r) = \left(u(\arcsin r),\ ru'(\arcsin r)\right)^\mathsf{T}. \tag{6.9}$$

By the construction of Φ in Lemma 5.1, $\psi(r) = \Phi(r)e_+$ for $0 < r \leq r_0$.

Let $\mathcal{P}$ be the hitting map in (5.18). At the initial section $r = r_0$, a pure shape variation has $\delta r = \delta\omega = 0$: $K'(0) = 0$, and the y terms in the constraint are quadratic. The linearized radial equations are $(\delta r)' = \delta\omega$ and $(\delta\omega)' = -\delta r$. Thus both variations vanish up to the final section. The variation of the hitting time is $-\delta\omega/\omega' = 0$, since $\omega' = -1$ at the equator. The remaining linearized equation is (6.1). Extending the real differential $D\mathcal{P}(0)$ complex linearly gives

$$D\mathcal{P}(0)\,\Phi(r_0)e_+ = (u_e, v_e)^\mathsf{T}. \tag{6.10}$$

Write $\mathcal{P}(Y) = D\mathcal{P}(0)Y + \mathcal{R}(Y)$, where $|\mathcal{R}(Y)| \leq C_1|Y|^2$ and $|D\mathcal{R}(Y)| \leq C_1|Y|$. By Proposition 5.2, both $U_s(r_0)$ and $s\partial_s U_s(r_0)$ are $O(s^\alpha)$. Consequently

$$|\mathcal{R}(U_s(r_0))| + |D\mathcal{R}(U_s(r_0))\, s\partial_s U_s(r_0)| = O(s^{2\alpha}).$$

The second term is the logarithmic derivative of the composed remainder by the chain rule. Equations (6.8) and (6.10) now give (6.3) and (6.4).

For the leading term, set $w = Cs^{\alpha-i\beta}u_e$. Then

$$x = \mathrm{Re}\, w, \qquad y = \frac{\delta_d\, \mathrm{Re}\, w - \beta\, \mathrm{Im}\, w}{\mu}.$$

Since $s\partial_s w = (\alpha - i\beta)w$,

$$s\partial_s x = (\alpha + \delta_d)x - \mu y, \qquad s\partial_s y = \nu x + (\alpha - \delta_d)y.$$

Here $(\delta_d^2 + \beta^2)/\mu = \nu$ by (6.2). This gives $\mathcal{B}_d$. This real linear map from w to (x, y) is invertible, and $|w| = |Cu_e|s^\alpha$. The two norm bounds follow, including the $o(s^\alpha)$ remainder. □

A continuous angle lift of a sufficiently short cap curve satisfies

$$\frac{d\theta}{d\log s} = \frac{\nu x^2 - 2\delta_d xy + \mu y^2}{x^2 + y^2} + o(1). \tag{6.11}$$

The error in this formula is $o(1)$ because $|\Gamma(s)| \geq c_1 s^\alpha$. The numerator is positive definite by (6.2). Thus the angular velocity is bounded above and below by positive constants, and $\theta(s) \to -\infty$ as $s \downarrow 0$. Normal reversal is

$$\mathsf{S}(x, y) = (x, -y). \tag{6.12}$$

It reverses the orientation of the angle.

Proposition 6.2. *For a round fibre there are $s_j, t_j \downarrow 0$ such that*

$$\Gamma_-(s_j) = \mathsf{S}\Gamma_+(t_j) = Z_j \neq 0, \qquad |Z_j| \asymp s_j^\alpha \asymp t_j^\alpha. \tag{6.13}$$

They give smooth Einstein metrics on S^{d+1}. For every (Q, h_Q) in Theorem B, there are $s_j \downarrow 0$ with $\Gamma_-(s_j) = (x_j, 0)$, $x_j \neq 0$. Even reflection gives a smooth Einstein metric on $S^{p+1} \times Q$.

Proof. For $Q = S^q$, apply Böhm's construction [2, Theorem 7.3, pp. 172–174] to the product action in Lemma 4.2, with

$$d_S = p, \qquad d_Q = q, \qquad d_P = d, \qquad k = d, \qquad \|A\| = 0, \qquad \mathrm{Ric}^Q = q - 1 > 0.$$

The dimension hypotheses are $p, q \geq 2$ and $d \leq 8$. For every sufficiently small $\varepsilon > 0$, the two curves on [2, p. 173], restricted to initial parameters in $(0, \varepsilon)$, converge to the cone point and have unbounded opposite angle lifts [2, Lemmas 7.1–7.2]. The intersection construction in the proof of the cited theorem therefore gives a spherical gluing with both initial parameters in $(0, \varepsilon)$. Taking $\varepsilon \downarrow 0$ gives $s_j, t_j \to 0$; a subsequence makes both sequences strictly decreasing. Their unique minimal hypersurfaces coincide (Lemma 5.3). Thus $\Gamma_-(s_j) = \mathsf{S}\Gamma_+(t_j) = Z_j$, and (6.5) gives $0 < |Z_j| \asymp s_j^\alpha \asymp t_j^\alpha$.

At a regular minimal hypersurface, the constraint and $H = 0$ give

$$r^2 = \frac{K(x)}{1 - \kappa y^2}, \qquad A = \frac{q}{d} y, \qquad B = -\frac{p}{d} y. \tag{6.14}$$

Here $r^2(1 - \kappa y^2) = K(x) > 0$. Hence any sufficiently small matching $\Gamma_-(s) = \mathsf{S}\Gamma_+(t)$ gives equal r, a, b and opposite outward derivatives a', b'. In a signed normal coordinate, the two caps have the same Cauchy data (a, b, a', b'). Since $a, b > 0$, uniqueness for the regular Einstein ODE [21, Theorem 2.2] identifies their smooth extensions across the common hypersurface. The identity product-boundary gluing of $D^{p+1} \times S^q$ and $S^p \times D^{q+1}$ gives the standard S^{p+q+1} [2, §7, p. 172].

For reflection doubles, take the round-fibre solutions in [2, Theorem 3.3 and §6, pp. 170–172]. Their initial radii decrease to zero, and their profiles satisfy $a' = b' = 0$ at the middle hypersurface. Replacing the noncollapsing round factor by (Q, h_Q) preserves the Einstein equations and the endpoint conditions [10, arXiv v1, §3.1, paragraph following (42)]. Thus $\Gamma_-(s_j) = (x_j, 0)$, and (6.5) gives $x_j \neq 0$ for all sufficiently large j. At every sufficiently small zero of $y(s)$, (6.14) gives $a' = b' = 0$. The even extensions of a, b satisfy the same regular ODE and the same Cauchy data; uniqueness makes the reflected metric smooth. The identity boundary gluing gives $S^{p+1} \times Q$. Smoothness at the two zero sections follows from Lemma 3.1. □

6.2. **The equatorial constants and geometric spacing.**

Lemma 6.3. *For $4 \leq d \leq 8$, the constants of Lemma 6.1 satisfy*

$$\delta_d = -\frac{\beta \sin(\pi\alpha)}{\sinh(\pi\beta)}, \qquad |\delta_d| < \frac{1}{\pi} < \alpha. \tag{6.15}$$

For the two spherical cap types, there are nonzero vectors $v_-, v_+ \in \mathbb{R}^2$ such that

$$\Gamma_\pm(s) = e^{\mathcal{B}_d \log s} v_\pm + o(s^\alpha), \tag{6.16}$$

$$s\Gamma_\pm'(s) = \mathcal{B}_d \Gamma_\pm(s) + o(s^\alpha). \tag{6.17}$$

In particular,

$$\mathcal{B}_5 = \begin{pmatrix} 2 & -5/8 \\ 32/5 & 2 \end{pmatrix}, \qquad \mathcal{B}_7 = \begin{pmatrix} 3 & -7/16 \\ 48/7 & 3 \end{pmatrix}. \tag{6.18}$$

Proof. Let u be the normalized solution in Lemma 6.1. Put $\tau = \log\tan(t/2)$ and $u = (\cosh\tau)^\alpha f$. Since $2\alpha = d - 1$ and $\alpha^2 + \beta^2 = 2(d-1)$, equation (6.1) becomes

$$f_{\tau\tau} + \left(\beta^2 + \alpha(\alpha+1)\operatorname{sech}^2\tau\right) f = 0.$$

With $\xi = -\tanh\tau = \cos t$, this is the associated Legendre equation of degree α and order $-i\beta$ [15, (14.2.2)]. The Ferrers representation [15, (14.3.1), (14.3.3)] gives

$$\begin{aligned} u(t) &= 2^{i\beta}(\sin t)^{-\alpha}\Gamma(1+i\beta)\mathsf{P}_\alpha^{-i\beta}(\cos t) \\ &= (\sin t)^{-\alpha}\big(2\tan(t/2)\big)^{i\beta}\,{}_2F_1\left(-\alpha,\alpha+1;1+i\beta;\frac{1-\cos t}{2}\right). \end{aligned}$$

All powers of positive numbers are defined by the real logarithm. The hypergeometric argument belongs to $[0,1/2]$ for $0 \le t \le \pi/2$, and its denominator parameter $1+i\beta$ is not a nonpositive integer. Its convergent series at zero gives $u(t) = t^{-\alpha+i\beta}(1+O(t^2))$, with an even analytic quotient of value one at zero. Thus the normalization agrees with that of Lemma 6.1.

The Ferrers values at zero [15, (14.5.1)–(14.5.2)] give

$$\begin{aligned} u_e &= \frac{\sqrt{\pi}\,\Gamma(1+i\beta)}{\Gamma(1+(\alpha+i\beta)/2)\Gamma((1-\alpha+i\beta)/2)}, \\ v_e &= \frac{2\sqrt{\pi}\,\Gamma(1+i\beta)}{\Gamma((1+\alpha+i\beta)/2)\Gamma((-\alpha+i\beta)/2)}. \end{aligned} \tag{6.19}$$

The derivative formula includes $d(\cos t)/dt = -1$ at $t = \pi/2$. The identities hold for the complex order $-i\beta$ by holomorphy of the regularized hypergeometric function in its parameters [15, §15.2(ii)]. Every Gamma argument in the denominators has imaginary part $\beta/2 \neq 0$; the factors are finite and nonzero [15, §5.2(i)].

Set $k = i\beta$, $z = (-\alpha+k)/2$, and $w = (1-\alpha-k)/2$. The reflection and recurrence formulas [15, (5.5.1), (5.5.3)] give

$$\begin{aligned} \overline{u_e}v_e &= \frac{2\pi\,\Gamma(1-k)\Gamma(1+k)}{\Gamma(z)\Gamma(1-z)\Gamma(w)\Gamma(1-w)} \\ &= \frac{2\Gamma(1-k)\Gamma(1+k)}{\pi}\sin(\pi z)\sin(\pi w), \\ \Gamma(1-k)\Gamma(1+k) &= \frac{\pi k}{\sin(\pi k)} = \frac{\pi\beta}{\sinh(\pi\beta)}. \end{aligned}$$

Moreover,

$$2\sin(\pi z)\sin(\pi w) = 2\sin\left(\frac{-\pi\alpha+\pi k}{2}\right)\cos\left(\frac{\pi\alpha+\pi k}{2}\right) = \sin(\pi k) - \sin(\pi\alpha).$$

Consequently,

$$\overline{u_e}v_e = i\beta - \frac{\beta\sin(\pi\alpha)}{\sinh(\pi\beta)}.$$

Taking real parts proves the identity in (6.15). The inequalities follow from $\sinh(\pi\beta) > \pi\beta$ and $\alpha \ge 3/2$.

For $\alpha = 2, 3$, recurrence and duplication [15, (5.5.1), (5.5.5)], applied to (6.19), give

$$\mu = \begin{cases} (\beta^2+1)/(\beta^2+4), & \alpha = 2, \\ \beta^2(\beta^2+4)/((\beta^2+1)(\beta^2+9)), & \alpha = 3. \end{cases}$$

In these two cases $\delta_d = 0$ and $\nu = \beta^2/\mu$. The values $(\alpha,\beta^2) = (2,4)$ and $(3,3)$ give (6.18).

The leading term in (6.3) satisfies the constant-coefficient equation with matrix $\mathcal{B}_d$. Its value at $\log s = 0$ is $v_\pm = \mathrm{Re}(C_\pm(u_e,v_e)^{\mathsf{T}})$. The map $C \mapsto \mathrm{Re}(C(u_e,v_e)^{\mathsf{T}})$ is invertible by (6.7), so $v_\pm \neq 0$. Equations (6.3) and (6.4) give (6.16) and (6.17). □

Proposition 6.4. *For every pair in* (1.2)*, there are constants $S_-, S_+ > 0$ and a sequence of smooth spherical gluings with strictly decreasing cap radii $s_j, t_j \downarrow 0$ such that*

$$s_j = S_- e^{-\pi j/\beta}(1+o(1)), \qquad t_j = S_+ e^{-\pi j/\beta}(1+o(1)). \tag{6.20}$$

Their common data $Z_j = \Gamma_-(s_j) = \mathrm{S}\Gamma_+(t_j)$ *satisfy*

$$\begin{aligned} \mathcal{R}(Z_j) = R_* e^{-\pi\alpha j/\beta}(1+o(1)), \qquad R_* > 0, \\ \mathcal{R}(x,y)^2 = \nu x^2 + \mu y^2. \end{aligned} \tag{6.21}$$

Proof. Put $\delta = \delta_d$, $\ell = \log s$, $P = \pi/\beta$, and

$$D = \begin{pmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}, \qquad \gamma = \sqrt{\mu\nu} = \sqrt{\beta^2+\delta^2}, \qquad N = \begin{pmatrix} \delta & -\gamma \\ \gamma & -\delta \end{pmatrix}.$$

Then

$$\begin{aligned} D\mathcal{B}_d D^{-1} = \alpha I + N, \qquad N^2 = -\beta^2 I, \\ e^{N\ell} = I\cos(\beta\ell) + \beta^{-1} N \sin(\beta\ell). \end{aligned} \tag{6.22}$$

Write

$$\widetilde{Z}_\pm(\ell) = D\Gamma_\pm(e^\ell), \qquad W_\pm(\ell) = e^{(\alpha I+N)\ell} D v_\pm.$$

Equations (6.16) and (6.17) give

$$|\widetilde{Z}_\pm - W_\pm| + |\widetilde{Z}'_\pm - W'_\pm| = o(e^{\alpha\ell}).$$

The matrices $e^{N\ell}$ and $e^{-N\ell}$ are bounded on $\mathbb{R}$. Since $v_\pm \neq 0$, the quantities $e^{-\alpha\ell}|W_\pm(\ell)|$ have positive upper and lower bounds. The same holds for $e^{-\alpha\ell}|\widetilde{Z}_\pm(\ell)|$ on a sufficiently negative half-line.

Let

$$L_\pm = \log|\widetilde{Z}_\pm|, \qquad L^0_\pm = \log|W_\pm|,$$

and choose angle lifts $\theta_\pm$, $\theta^0_\pm$ of these curves. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, the quotient $\widetilde{Z}_\pm/W_\pm$ tends to one. Its argument near one fixes lifts for which

$$L_\pm - L^0_\pm = o(1), \qquad \theta_\pm - \theta^0_\pm = o(1) \quad (\ell \to -\infty). \tag{6.23}$$

Polar differentiation gives

$$L'_\pm = \alpha + \delta\cos(2\theta_\pm) + o(1), \qquad (L^0_\pm)' = \alpha + \delta\cos(2\theta^0_\pm), \tag{6.24}$$

$$\theta'_\pm = \gamma - \delta\sin(2\theta_\pm) + o(1), \qquad (\theta^0_\pm)' = \gamma - \delta\sin(2\theta^0_\pm). \tag{6.25}$$

Set

$$m = \alpha - |\delta| > 0, \quad M = \alpha + |\delta|, \qquad c = \gamma - |\delta| > 0, \quad C = \gamma + |\delta|.$$

The leading derivatives satisfy

$$m \le (L^0_\pm)' \le M, \qquad c \le (\theta^0_\pm)' \le C.$$

For all sufficiently negative ℓ,

$$L'_\pm(\ell) \ge m/2, \qquad \theta'_\pm(\ell) \ge c/2. \tag{6.26}$$

Equation (6.22) implies $W_\pm(\ell - P) = -e^{-\alpha P} W_\pm(\ell)$, and hence

$$L^0_\pm(\ell - P) = L^0_\pm(\ell) - \alpha P, \qquad \theta^0_\pm(\ell - P) = \theta^0_\pm(\ell) - \pi. \tag{6.27}$$

For $z \in \mathbb{R}^2 \setminus \{0\}$ and $0 < h < P$,

$$\det(z, e^{Nh}z) = \frac{\sin(\beta h)}{\beta}\big(\gamma|z|^2 - 2\delta z_1 z_2\big) > 0,$$

since $\gamma > |\delta|$ and $0 < \beta h < \pi$. Applying this to $z = W_\pm(\ell)$ and using $W_\pm(\ell + h) = e^{\alpha h} e^{Nh} W_\pm(\ell)$ gives, by continuity of the angle lifts,

$$0 < \theta^0_\pm(\ell + h) - \theta^0_\pm(\ell) < \pi \qquad (0 < h < P).$$

At $h = P$, $e^{NP} = -I$; the angle increment is therefore π. This proves the angular identity in (6.27).

Each $L^0_\pm$ is a strictly increasing bijection of $\mathbb{R}$. Define $T_0 : \mathbb{R} \to \mathbb{R}$ by

$$L^0_-(\ell) = L^0_+(T_0(\ell)).$$

The derivative bounds and (6.27) give

$$\frac{m}{M} \le T_0'(\ell) \le \frac{M}{m}, \qquad T_0(\ell - P) = T_0(\ell) - P. \tag{6.28}$$

Thus $L^0_\pm(\ell) - \alpha\ell$, $\theta^0_\pm(\ell) - \beta\ell$, and $T_0(\ell) - \ell$ are P-periodic. Define

$$F_0(\ell) = \theta^0_-(\ell) + \theta^0_+(T_0(\ell)).$$

Then

$$c(1 + m/M) \le F_0'(\ell) \le C(1 + M/m), \qquad F_0(\ell - P) = F_0(\ell) - 2\pi. \tag{6.29}$$

There is a unique $\ell_* \in \mathbb{R}$ with $F_0(\ell_*) = 0$. Put $m_* = T_0(\ell_*)$. The solutions of $F_0(\ell) = -2\pi j$ are

$$\ell^0_j = \ell_* - jP, \qquad T_0(\ell^0_j) = m_* - jP. \tag{6.30}$$

By (6.26), for every sufficiently negative ℓ there is a unique $T(\ell)$ such that

$$L_-(\ell) = L_+(T(\ell)).$$

The function T is smooth, $T' > 0$, and $T(\ell) \to -\infty$ as $\ell \to -\infty$. The radius bounds give

$$L_\pm(\ell) = \alpha\ell + O(1), \qquad L^0_\pm(\ell) = \alpha\ell + O(1),$$

and therefore $T(\ell) - \ell = O(1)$ and $T_0(\ell) - \ell = O(1)$. Write $\eta_\pm = L_\pm - L^0_\pm = o(1)$. Subtracting the two radius equalities yields

$$L^0_+(T(\ell)) - L^0_+(T_0(\ell)) = \eta_-(\ell) - \eta_+(T(\ell)) = o(1).$$

Since $(L^0_+)' \ge m$, the mean-value theorem gives

$$T(\ell) - T_0(\ell) = o(1). \tag{6.31}$$

Put $F(\ell) = \theta_-(\ell) + \theta_+(T(\ell))$ and $\varepsilon_\pm = \theta_\pm - \theta^0_\pm = o(1)$. The bound $(\theta^0_+)' \le C$ gives

$$|F(\ell) - F_0(\ell)| \le |\varepsilon_-(\ell)| + |\varepsilon_+(T(\ell))| + C|T(\ell) - T_0(\ell)| = o(1).$$

Moreover,

$$F'(\ell) = \theta_-'(\ell) + \theta_+'(T(\ell))T'(\ell) > 0$$

on a sufficiently negative half-line, and $F(\ell) \to -\infty$ as $\ell \to -\infty$.

For every sufficiently large integer j, there is therefore a unique solution ℓ_j of $F(\ell_j) = -2\pi j$ on this half-line. The solutions strictly decrease to $-\infty$. Set $m_j = T(\ell_j)$, $s_j = e^{\ell_j}$, and $t_j = e^{m_j}$. The matrix D commutes with S. Thus the radius equality and the phase equation give

$$\Gamma_-(s_j) = \mathsf{S}\Gamma_+(t_j) = Z_j \ne 0.$$

These are the smooth spherical gluings of Proposition 6.2. Since $T' > 0$, both cap sequences are strictly decreasing.

At these roots,

$$F_0(\ell_j) - F_0(\ell^0_j) = F_0(\ell_j) - F(\ell_j) = o(1).$$

The lower derivative bound in (6.29) implies $\ell_j - \ell^0_j = o(1)$. Equations (6.28) and (6.31) then give

$$m_j - (m_* - jP) = T(\ell_j) - T_0(\ell_j) + T_0(\ell_j) - T_0(\ell^0_j) = o(1).$$

Hence (6.20) holds with $S_- = e^{\ell_*}$ and $S_+ = e^{m_*}$.

Finally, set $Z_* = e^{\mathcal{B}_d \ell_*} v_- \ne 0$. Since $(\mathcal{B}_d - \alpha I)^2 = -\beta^2 I$, equation (6.16) and the root asymptotic give

$$Z_j = (-1)^j e^{-\alpha P j}\big(Z_* + o(1)\big).$$

Thus (6.21) holds with

$$R_* = \mathcal{R}(Z_*) = \mathcal{R}(e^{\mathcal{B}_d \ell_*} v_-) > 0.$$

□

7. The volume asymptotic

Let $\Omega = \omega_p \operatorname{Vol}(Q, h_Q)$ and let V_∞ be as in (1.3). Since $a^p b^q = v_0 r^d$, a level hypersurface has volume $\Omega v_0 r^d$. Let $V_-(s)$ be the volume of an a-collapsing cap up to its first minimal hypersurface. For a round fibre, let $V_+(s)$ be the volume of a b-collapsing cap up to its first minimal hypersurface; in this case $V_\infty = V_{p,q}$.

Lemma 7.1. *For each cap type,*

$$\lim_{s\downarrow 0} V_\pm(s) = \frac{V_\infty}{2}. \tag{7.1}$$

Proof. Write T_s for the first minimal time and $r_{\max} = r_s(T_s)$. The radius increases on $(0, T_s)$ and vanishes at the zero section. By (4.7),

$$r_{\max}^2 = 1 + 2\kappa \mathcal{A}_s(r_{\max}) \geq 1.$$

Fix a sufficiently small $r_1 > 0$. Equation (5.5), applied at the section $r = r_1$, gives $U_s(r_1) = O(s^\alpha)$. Since $K(0) = 1$ and $K'(0) = 0$,

$$0 \leq \mathcal{A}_s(r_1) = \frac{1}{2} z_s(r_1)^2 + \frac{K(x_s(r_1)) - 1}{2\kappa} = O(s^{2\alpha}).$$

The constant in this estimate may depend on r_1. The energy decreases on the increasing branch, and hence

$$1 \leq r_{\max}^2 \leq 1 + 2\kappa \mathcal{A}_s(r_1).$$

In particular, $r_{\max} \to 1$.

The inequality $r'' \leq -r$ in (4.7) gives

$$r'(t)^2 \geq r_{\max}^2 - r(t)^2 \qquad (0 < t < T_s). \tag{7.2}$$

Since $r'(T_s) = 0$,

$$r'(t)^2 + r(t)^2 - r_{\max}^2 = -\int_t^{T_s} 2r'(u)\big(r''(u) + r(u)\big)\, du \geq 0.$$

Changing variables from t to r therefore gives

$$\begin{aligned} V_\pm(s) &= \Omega v_0 \int_0^{r_{\max}} \frac{r^d}{r'}\, dr \\ &\leq \Omega v_0 r_{\max}^d \int_0^1 \frac{u^d}{\sqrt{1-u^2}}\, du. \end{aligned} \tag{7.3}$$

The last integrand belongs to $L^1(0,1)$.

For $0 < \delta < 1 - r_1$ and $r_1 \leq r \leq 1 - \delta$, the constraint and energy monotonicity give

$$0 < r'^2 = 1 - r^2 + 2\kappa \mathcal{A}_s(r) \leq 1 - r^2 + 2\kappa \mathcal{A}_s(r_1).$$

Consequently

$$V_\pm(s) \geq \Omega v_0 \int_{r_1}^{1-\delta} \frac{r^d}{\sqrt{1 - r^2 + 2\kappa \mathcal{A}_s(r_1)}}\, dr. \tag{7.4}$$

On this fixed interval the denominator is at least $\sqrt{1-(1-\delta)^2} > 0$. Thus the integrand converges uniformly as $s \downarrow 0$. Taking the lower limit in (7.4), then letting $r_1 \downarrow 0$ and $\delta \downarrow 0$, gives

$$\liminf_{s\downarrow 0} V_\pm(s) \geq \Omega v_0 \int_0^1 \frac{r^d}{\sqrt{1-r^2}}\, dr.$$

Equation (7.3) gives the reverse upper bound, since $r_{\max} \to 1$. Finally, $r = \sin t$ on $[0, \pi/2]$ and $\sin(\pi - t) = \sin t$, so this common limit is $V_\infty/2$. □

Lemma 7.2. *For the constants of Lemma 6.1, put* $C_0 = \Omega pq/d^2$ *and*

$$\mathcal{Q}(x,y) = \frac{v_0}{4\alpha}\big(-\nu x^2 + 2(\alpha+\delta_d)xy - \mu y^2\big). \tag{7.5}$$

For each cap type there is a differentiable remainder $E_\pm(s)$ *such that*

$$\begin{aligned} V_\pm(s) - \frac{V_\infty}{2} &= -C_0\mathcal{Q}(\Gamma_\pm(s)) + E_\pm(s), \\ E_\pm(s) = o(s^{2\alpha}), &\qquad sE_\pm'(s) = o(s^{2\alpha}). \end{aligned} \tag{7.6}$$

Proof. For s near a fixed positive value, pull the cap back to $D^{p+1}\times Q$ by $t = T_s\rho$, $0\le\rho\le 1$. The function T_s is smooth by Lemma 5.3. In normal disk coordinates the pullback is the dilation $u\mapsto T_s u$, which is smooth also at the zero section. The cap therefore gives a smooth family on a fixed compact manifold whose boundary is minimal. For the other cap type the fixed manifold is $S^p\times D^{q+1}$.

At its boundary, put $Z=(x,y)$. By (6.14), the density relative to $dA_{\gamma_p}\,dA_{h_Q}$ is

$$v(Z) = a^p b^q = v_0\left(\frac{K(x)}{1-\kappa y^2}\right)^{d/2}. \tag{7.7}$$

On a sufficiently small ball about zero, this is a smooth positive function. Its value is v_0 and its differential is zero at the origin. Taylor's formula gives, on that ball,

$$|v(Z)-v_0|\le C|Z|^2, \qquad |Dv(Z)|\le C|Z|.$$

The outward second fundamental form has eigenvalues A and B on the two factors. The induced variation differentiates the boundary values $a_s(T_s)$ and $b_s(T_s)$, including the dependence of T_s on s. Thus

$$\begin{aligned} \langle \mathrm{II}, h^\top\rangle &= 2pA\,\frac{d}{ds}\log a_s(T_s) + 2qB\,\frac{d}{ds}\log b_s(T_s) \\ &= 2H\,\frac{d}{ds}\log r_s(T_s) + \frac{2pq}{d}(A-B)\frac{dx}{ds} \\ &= \frac{2pq}{d}\,y\,\frac{dx}{ds}. \end{aligned} \tag{7.8}$$

Here $H=0$ and $A-B=x'=y$. Since $\mathrm{Ric}=dg$, Lemma 2.1 now gives

$$\frac{dV_\pm}{ds} = -C_0 v(\Gamma_\pm(s))\,y(s)\frac{dx(s)}{ds}. \tag{7.9}$$

Write $\delta=\delta_d$ in the next calculation. One has

$$D\mathcal{Q}(Z) = \frac{v_0}{2\alpha}\big(-\nu x + (\alpha+\delta)y,\ (\alpha+\delta)x - \mu y\big).$$

Multiplication by $\mathcal{B}_d Z = ((\alpha+\delta)x-\mu y,\ \nu x + (\alpha-\delta)y)$ gives

$$D\mathcal{Q}(Z)[\mathcal{B}_d Z] = v_0\big((\alpha+\delta)xy - \mu y^2\big) = v_0 y(\mathcal{B}_d Z)_x. \tag{7.10}$$

Fix one cap type and write $Z(s)=\Gamma_\pm(s)$. Put $\ell=\log s$, $\dot Z = sZ'(s)$, and $e_s = \dot Z - \mathcal{B}_d Z$. Equation (6.5) gives

$$|Z(s)| + |\dot Z(s)| \le Cs^\alpha, \qquad e_s = o(s^\alpha).$$

For sufficiently small $s>0$, define

$$\eta(s) = \sup_{0<u\le s} u^{-\alpha}|e_u|.$$

This function is finite, nondecreasing, and tends to zero at zero. Define $E(s) = V_\pm(s) - V_\infty/2 + C_0\mathcal{Q}(Z(s))$. Equations (7.9) and (7.10) imply

$$sE'(s) = C_0\left(D\mathcal{Q}(Z)[e_s] - v_0 y(e_s)_x - (v(Z)-v_0)y\dot x\right).$$

The first two terms have absolute value at most $C|Z||e_s|$. The last has absolute value at most $C|Z|^3|\dot{Z}|$. Hence

$$|sE'(s)| \le C\eta(s)s^{2\alpha} + Cs^{4\alpha}. \tag{7.11}$$

By Lemma 7.1 and $Z(s) \to 0$, one has $E(s) \to 0$. Since $\alpha > 0$, the right side of (7.11), divided by s, is integrable at zero. Integration gives

$$|E(s)| \le \frac{C}{2\alpha}\eta(s)s^{2\alpha} + \frac{C}{4\alpha}s^{4\alpha}.$$

Thus $E(s) = o(s^{2\alpha})$; equation (7.11) gives $sE'(s) = o(s^{2\alpha})$. □

Proposition 7.3. *For the sphere matchings and reflection doubles of Proposition 6.2, let $Z_j = (x_j, y_j)$ be the data on the common minimal hypersurface, with the outward normal of one cap. Then*

$$\mathrm{Vol}(g_j) - V_\infty = \frac{\Omega p q v_0}{2\alpha d^2}(\nu x_j^2 + \mu y_j^2) + o(|Z_j|^2) > 0 \tag{7.12}$$

for all sufficiently large j. For spheres, $\Omega = \omega_p\omega_q$ and $V_\infty = V_{p,q}$; for reflection doubles, $y_j = 0$.

Proof. The outward data of the two caps are Z_j and $\mathsf{S}Z_j = (x_j, -y_j)$. For spherical gluings, (6.13) gives $s_j^{2\alpha} \asymp |Z_j|^2 \asymp t_j^{2\alpha}$. Thus each remainder in (7.6) is $o(|Z_j|^2)$. Adding the cap volumes and applying

$$\mathcal{Q}(Z) + \mathcal{Q}(\mathsf{S}Z) = -\frac{v_0}{2\alpha}(\nu x^2 + \mu y^2) \tag{7.13}$$

gives the displayed formula. For a reflection double, $\mathsf{S}Z_j = Z_j$, and the same identity applies to twice the cap volume; the norm bounds in (6.5) again give an $o(|Z_j|^2)$ remainder.

All factors in $\Omega p q v_0/(2\alpha d^2)$ are positive, and

$$\nu x_j^2 + \mu y_j^2 \ge \min\{\mu, \nu\}|Z_j|^2.$$

Since $Z_j \neq 0$ and the remainder divided by $|Z_j|^2$ tends to zero, the volume excess is positive for all sufficiently large j. It tends to zero because $Z_j \to 0$. □

Proof of Theorem B. The metrics in Proposition 6.2 are smooth and satisfy $\mathrm{Ric} = dg$. By Proposition 7.3, their volume excesses are eventually positive and tend to zero. After choosing one such term, choose each subsequent term with excess less than half the preceding excess. This gives the required sequence of pairwise distinct volumes.

Let M be one of these closed connected manifolds and put $n = \dim M$. Let $\mathscr{E}(M)$ be the unit-volume Einstein moduli space, modulo $\mathrm{Diff}(M)$, and let $\mathscr{E}_+(M)$ be its positive locus. The inverse normalization maps

$$g \longmapsto \widehat{g} = \mathrm{Vol}(g)^{-2/n} g, \qquad h \longmapsto \frac{\lambda(h)}{d}h, \quad \mathrm{Ric}_h = \lambda(h)h, \quad \lambda(h) > 0, \quad \mathrm{Vol}(h) = 1,$$

are continuous and $\mathrm{Diff}(M)$-equivariant. They induce a homeomorphism $\mathcal{E}_d(M) \cong \mathscr{E}_+(M)$, and

$$\mathcal{S}(\widehat{g}) = nd\ \mathrm{Vol}(g)^{2/n}, \qquad \mathcal{S}(h) = \int_M \mathrm{scal}_h\ dV_h.$$

By [18, Definition 3.1 and Theorems 3.6(2)–(3), 3.9(i)], $\mathcal{S}$ is locally constant on the full quotient $\mathscr{E}(M)$, which has at most countably many connected components. It follows that volume is constant on each connected component of $\mathcal{E}_d(M)$, so the distinct volumes obtained above give infinitely many components. The positive locus is a union of components of $\mathscr{E}(M)$; hence both $\mathcal{E}_d(M)$ and $\mathscr{E}(M)$ have countably infinitely many components. The functional is defined with respect to the Riemannian density [18, Definition 2.1], so orientability is not required. Finally, $p = 2$ and $q = 2, 3, 4, 5, 6$ give S^5, S^6, S^7, S^8, S^9. □

Corollary 7.4. *Order the reflection doubles by successive sufficiently small zeros of $y(s)$, with $s_j \downarrow 0$. There are constants $s_*, C_* > 0$ such that*

$$s_j = s_* e^{-\pi j/\beta}(1 + o(1)), \qquad \mathrm{Vol}(g_j) - V_\infty = C_* e^{-2\pi\alpha j/\beta}(1 + o(1)). \tag{7.14}$$

Their volumes are eventually strictly decreasing.

Proof. Let $C \neq 0$ be the coefficient of the a-collapsing cap in Lemma 6.1, and put $\ell = \log s$. Equations (6.3) and (6.4) give

$$e^{-\alpha\ell} y(e^\ell) = f(\ell) + \varepsilon(\ell), \qquad f(\ell) = \mathrm{Re}(C v_e e^{-i\beta\ell}), \qquad \varepsilon(\ell), \varepsilon'(\ell) \longrightarrow 0 \quad (\ell \to -\infty).$$

Here $v_e \neq 0$, so f is a nonzero sinusoid. Write its successive zeros tending to $-\infty$ as $\ell_j^0 = \ell_* - \pi j/\beta$. Choose h with $0 < h < \pi/(4\beta)$. On each interval $|\ell - \ell_j^0| \leq h$, the derivative of f has constant sign and absolute value at least $\beta|Cv_e|\cos(\beta h) > 0$. At its endpoints, f has opposite signs and absolute value $|Cv_e|\sin(\beta h)$. Outside these intervals, $|f| \geq |Cv_e|\sin(\beta h)$. For all sufficiently large j, the errors ε and ε' are smaller than half the corresponding bounds. The perturbed function therefore has exactly one zero in each interval and none between them. The derivative bound and $f(\ell_j) = -\varepsilon(\ell_j)$ give $\ell_j - \ell_j^0 = o(1)$. Exponentiation proves the first formula in (7.14), with $s_* = e^{\ell_*} > 0$.

At a leading zero, $Cv_e e^{-i\beta\ell_j^0}$ equals $\pm i|Cv_e|$. Since

$$\frac{u_e}{v_e} = \frac{\delta_d - i\beta}{\nu}, \qquad \nu = |v_e|^2,$$

the corresponding leading value of $e^{-\alpha\ell_j^0} x(e^{\ell_j^0})$ is $\pm b_*$, where

$$b_* = \frac{\beta|Cv_e|}{\nu} = \frac{\beta|C|}{|v_e|} > 0.$$

The zero displacements and the value remainder in (6.3) tend to zero. Thus

$$x(s_j)^2 = b_*^2 s_j^{2\alpha}(1 + o(1)).$$

Equation (7.12), with $y(s_j) = 0$, now gives the second formula in (7.14), where

$$C_* = \frac{\Omega p q v_0}{2\alpha d^2} \nu\, b_*^2 s_*^{2\alpha} > 0.$$

The ratio of consecutive positive excesses tends to $e^{-2\pi\alpha/\beta} < 1$. It is therefore less than one for all sufficiently large j, proving strict decrease. □

Proof of Theorem 1.1. For the sequence of Proposition 6.4, put $C_0 = \omega_p \omega_q pq/d^2$. By (7.12),

$$\mathrm{Vol}(g_j) - V_{p,q} = \frac{C_0 v_0}{2\alpha} \mathcal{R}(Z_j)^2 + o(\mathcal{R}(Z_j)^2), \tag{7.15}$$

since $\mathcal{R}(x, y)^2 = \nu x^2 + \mu y^2$ and $\mu, \nu > 0$. Equation (6.21) gives (1.6), with

$$C_{p,q} = \frac{C_0 v_0 R_*^2}{2\alpha} > 0.$$

The consecutive volume-excess ratio tends to $e^{-2\pi\alpha/\beta} < 1$, so the volumes are eventually strictly decreasing. Since $\alpha/\beta = \sqrt{(d-1)/(9-d)}$, this also gives (1.7).

Euler's beta integral [15, (5.12.1)–(5.12.2)] gives

$$I_d := \int_0^\pi \sin^d t\, dt = \mathrm{B}\big((d+1)/2, 1/2\big), \qquad I_5 = \frac{16}{15}, \quad I_7 = \frac{32}{35}.$$

Substitution of $\omega_2 = 4\pi$, $\omega_3 = 2\pi^2$, $\omega_4 = 8\pi^2/3$, $\omega_5 = \pi^3$ in (1.4) yields (1.8). □

Corollary 7.5. *For any sequence in Theorem 1.1 or Corollary 7.4, denote its limiting volume by V_* and put $\delta_j = \mathrm{Vol}(g_j)/V_* - 1$. For fixed sufficiently small $\epsilon_0 > 0$,*

$$\#\{j \,:\, \epsilon \le \delta_j \le \epsilon_0\} = \frac{\beta}{2\pi\alpha}\log(1/\epsilon) + O(1) \qquad (\epsilon \downarrow 0). \tag{7.16}$$

After omission of finitely many terms, the spherical sequences for $(p,q) = (2,5)$ and $(3,4)$ lie in disjoint sets of connected components of $\mathcal{E}_7(S^8)$.

Proof. By (1.6) or (7.14), $\log \delta_j = -b\,j + c + o(1)$ for some $c \in \mathbb{R}$, where $b = 2\pi\alpha/\beta > 0$. The sequence is eventually positive and strictly decreasing. If $N(\epsilon) = \max\{j \,:\, \delta_j \ge \epsilon\}$, then

$$bN(\epsilon) - c + o(1) \le \log(1/\epsilon) < b(N(\epsilon)+1) - c + o(1).$$

Thus $N(\epsilon) = b^{-1}\log(1/\epsilon) + O(1)$; subtracting the fixed upper-threshold count proves (7.16).

Since $V_{2,5}/V_{3,4} = 2\sqrt{2}/3 \ne 1$, the two S^8 sequences eventually have volumes in disjoint intervals about their limits. Volume is constant on each connected component of $\mathcal{E}_7(S^8)$, so no component contains sufficiently late terms of both sequences. □

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