FINITENESS THRESHOLDS AND SURFACE BOUNDS FOR GRAPH JACOBIANS

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Abstract. For a finite connected graph G, let $e(G)$ be the exponent of its Jacobian and $\Delta(G)$ its maximum degree. For each integer $b \geq 1$, the simple biconnected graphs with $e(G) \leq b$ form finitely many isomorphism classes if and only if $b \leq 7$; exactly eleven occur with $e(G) \leq 7$. The simple biconnected graphs with $e(G) - \Delta(G) \leq 3$ are K_n ($n \geq 3$), C_4, C_5 and $K_{2,3}$. At exponent eight, the number of vertices is unbounded among simple planar bipartite biconnected graphs of maximum degree four. For every integer $r \geq 1$, there is a simple planar biconnected graph whose Jacobian is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^r$, with explicit divisor generators and an orthogonal decomposition of its monodromy pairing. Every simple 3-connected planar graph satisfies $2|V(G)| - 4 \leq (e(G) - \Delta(G) + 1)^2$ and $2|V(G)| - 5 \leq 3e(G)^2/5$. Every simple graph with a cellular torus embedding whose full inverse image in the universal cover of the torus is 3-connected satisfies $|V(G)| \leq e(G)^2$.

1. Introduction

For a finite connected graph G with at least two vertices, let L be its Laplacian and C a reduced Laplacian. Write $e(G)$ for the exponent of its Jacobian. Then

$$\mathrm{Jac}(G) \simeq \mathbb{Z}^{|V(G)|-1}/C\mathbb{Z}^{|V(G)|-1}, \qquad e(G) = \min\{a \in \mathbb{Z}_{>0} \,:\, aC^{-1} \in \mathrm{Mat}_{|V(G)|-1}(\mathbb{Z})\}. \tag{1.1}$$

The order of $\mathrm{Jac}(G)$ is the number of spanning trees; the terms critical group and sandpile group denote the same finite abelian group [2, 5, 3, 18]. Its monodromy pairing is a nondegenerate symmetric bilinear pairing with values in $\mathbb{Q}/\mathbb{Z}$ [22, Theorem 3.4].

Graphs are finite, undirected and simple, unless parallel edges are explicitly permitted. A graph is *biconnected* if it has at least three vertices and remains connected after deletion of any vertex. It is 3-*connected* if it has at least four vertices and remains connected after deletion of at most two vertices. Write

$$n(G) = |V(G)|, \qquad m(G) = |E(G)|, \qquad \Delta(G) = \max_{v \in V(G)} \deg_G(v).$$

Graph isomorphism is denoted by $\simeq$, and $[G]$ is the isomorphism class of G. The vector $\mathbf{e}_v$ has coordinate one at v and zero at every other vertex.

The bounded-exponent finiteness conjecture attributed to Baker and Shokrieh in [16, Conjecture 1.1, arXiv version] asserts that, for each positive integer b, there are finitely many isomorphism classes of biconnected graphs with $e(G) \leq b$, even when parallel edges are permitted. The cases $b \leq 3$ follow from [12, Lemma 5.2 and Section 5.3]; see also [16, Section 3.1, arXiv version]. For each finite abelian group A, [12, Proposition 5.7] gives an integer N_A such that $n(G) < N_A$ for every simple biconnected graph with $\mathrm{Jac}(G) \simeq (\mathbb{Z}/2\mathbb{Z})^k \oplus A$, $k \geq 0$.

Let S_7 be the graph obtained from K_4 by subdividing once the three edges incident with one chosen vertex. Equivalently,

$$\begin{aligned} V(S_7) &= \{o, x_1, x_2, x_3, y_1, y_2, y_3\}, \\ E(S_7) &= \{ox_i, x_iy_i \,:\, 1 \leq i \leq 3\} \cup \{y_1y_2, y_2y_3, y_3y_1\}. \end{aligned} \tag{1.2}$$

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Theorem 1.1. *For an integer $b \geq 1$, the set*

$$\{[G] \,:\, G \text{ is simple and biconnected, } e(G) \leq b\}$$

is finite if and only if $b \leq 7$. The simple biconnected graphs with $e(G) \leq 7$ are precisely

$$\{C_n \,:\, 3 \leq n \leq 7\} \,\cup\, \{K_n \,:\, 4 \leq n \leq 7\} \,\cup\, \{K_{2,3}, S_7\}. \tag{1.3}$$

Eight is also the least exponent attained by infinitely many such graphs. The same finiteness threshold holds if planarity is required, and if planarity, bipartiteness and maximum degree at most four are all required.

For $t \geq 2$, let G_t have vertex set $\{a_1, b_1, \dots, a_t, b_t\}$ and edges

$$E(G_t) = \{a_i a_{i+1}, a_i b_{i+1}, b_i a_{i+1}, b_i b_{i+1} \,:\, 1 \leq i < t\}. \tag{1.4}$$

Put

$$H_t = G_t + \{a_1 b_1\}, \qquad D_i = \mathbf{e}_{a_i} - \mathbf{e}_{a_{i+1}} \quad (1 \leq i < t). \tag{1.5}$$

For $r \geq 1$, let B_r be the symmetric tridiagonal matrix with diagonal $4, \dots, 4, 5$ and adjacent entries -1; thus $B_1 = (5)$.

Theorem 1.2. *For every $t \geq 2$, the graph H_t is planar and biconnected, with $2t$ vertices, $4t - 3$ edges and maximum degree at most four. The map*

$$(\mathbb{Z}/8\mathbb{Z})^{t-1} \longrightarrow \mathrm{Jac}(H_t), \qquad (c_i) \longmapsto \sum_{i=1}^{t-1} c_i [D_i] \tag{1.6}$$

is an isomorphism. Its monodromy pairing satisfies

$$\langle [D_i], [D_j] \rangle = \frac{(B_{t-1})_{ij}}{8} \pmod{\mathbb{Z}}. \tag{1.7}$$

Writing $\langle u/8 \rangle$ for the pairing $(x, y) \mapsto uxy/8 \pmod{\mathbb{Z}}$ on $\mathbb{Z}/8\mathbb{Z}$, one has

$$\mathrm{Jac}(H_t) \simeq \langle 5/8 \rangle^{\perp \lceil (t-1)/2 \rceil} \perp \langle 7/8 \rangle^{\perp \lfloor (t-1)/2 \rfloor} \tag{1.8}$$

as groups with pairing.

By Lemmas 3.2 and 3.3, the graphs G_t, $t \geq 3$, are planar, bipartite and biconnected, have maximum degree four, and have exponent eight. The graph H_t contains the triangle $a_1 b_1 a_2 a_1$. The multiplicities $t - 1$ in (1.6) are unbounded, contrary to [12, Conjecture 5.9] at modulus eight.

Brandfonbrener et al. [6, Conjecture 6.3] conjecture that, for each vertex x of a simple biconnected graph G, there is a vertex y such that $[\mathbf{e}_x - \mathbf{e}_y]$ has order at least $n(G)$. For G_t with $t \geq 5$, every class is annihilated by eight, so

$$\mathrm{ord}[\mathbf{e}_x - \mathbf{e}_y] \leq 8 < 2t = n(G_t) \qquad (x, y \in V(G_t)). \tag{1.9}$$

Every simple biconnected graph satisfies $e(G) \geq \Delta(G) + 1$ by [12, Lemma 5.2]. For adjacent vertices x, y, [6, Proposition 5.2] gives the sharper estimate

$$\mathrm{ord}[\mathbf{e}_x - \mathbf{e}_y] \geq \deg(x) + \frac{\deg(x) - 1}{\deg(y) - 1}.$$

Theorem 1.3. *Let G be biconnected and put $q = \Delta(G)$. Then $e(G) \geq q + 1$, and*

$e(G) - q$	G
1	K_{q+1}
2	C_4
3	C_5 *or* $K_{2,3}$.

(1.10)

In particular, every noncomplete biconnected graph other than $C_4, C_5, K_{2,3}$ satisfies $e(G) \geq \Delta(G) + 4$.

Theorem 1.4. *Let G be 3-connected and planar, and put $n = n(G)$, $\Delta = \Delta(G)$ and $e = e(G)$. Then*

$$2n - 4 \le (e - \Delta + 1)^2, \qquad 2n - 5 \le \frac{3}{5}e^2. \tag{1.11}$$

Consequently,

$$n \le \min\left\{ \left\lfloor \frac{(e-\Delta+1)^2 + 4}{2} \right\rfloor, \left\lfloor \frac{3e^2}{10} + \frac{5}{2} \right\rfloor \right\}. \tag{1.12}$$

The first inequality in (1.11) *is an equality for K_4. For each fixed b, only finitely many isomorphism classes of such graphs satisfy $e(G) \le b$.*

A torus embedding is *cellular* if every complementary component is an open disk. It is *essentially* 3*-connected* if the full inverse image of the embedded graph in the universal cover of the torus is connected and remains connected after deletion of at most two vertices.

Theorem 1.5. *Let a connected simple graph G admit a cellular, essentially* 3*-connected embedding in the torus. Then*

$$n(G) \le e(G)^2. \tag{1.13}$$

For each fixed b, there are only finitely many isomorphism classes of such simple graphs with $e(G) \le b$.

Let P_t denote the path on t vertices. Without a surface restriction, the graphs $P_t[\overline{K_3}]$, $t \ge 3$, are 3-connected, have exponent eighteen, and satisfy $n(P_t[\overline{K_3}]) = 3t$ (Proposition 7.4).

Integral potentials. Let G be biconnected, put $e = e(G)$, and choose a vertex o of maximum degree q. Write $X = N(o) = \{x_1, \dots, x_q\}$ and $U = V(G) \setminus (\{o\} \cup X)$, and let C be the Laplacian reduced at o. For $1 \le i \le q$, let $Lp_i = e(\mathbf{e}_{x_i} - \mathbf{e}_o)$ and $p_i(o) = 0$. By Lemma 4.1,

$$p_i(v) \in \mathbb{Z}_{>0} \quad (v \ne o), \qquad p_i(v) < p_i(x_i) \quad (v \ne x_i), \qquad \sum_i p_i(v) = e \quad (v \ne o).$$

The matrix $B = (p_i(x_j))$ equals $e(C^{-1})_{XX}$. Assume $s = e - q \in \{1, 2, 3\}$. The proof of Theorem 1.3 gives $B = sI_q + J_q$, where I_r is the identity matrix and J_r is the all-one matrix of order r. Suppose $U \ne \emptyset$. Then $s \ge 2$ by Theorem 1.3. For $u \in U$, let w_u be row u of $-(C_{UU})^{-1}C_{UX}$. These rows give harmonic extension from X to $G - o$ and satisfy $w_u \ge 0$ and $\sum_{i=1}^{q}(w_u)_i = 1$ [9, Lemma II.1(3)]. For u, v in distinct components of $G[U]$, Lemma 4.7 gives

$$e(C^{-1})_{uv} = 1 + s\, w_u \cdot w_v, \qquad s\, w_u \cdot w_v \in \mathbb{Z}.$$

For a 3-connected plane graph, choose consecutive facial vertices a, o, c and potentials $Lf = \mathbf{e}_a - \mathbf{e}_o$, $Lg = \mathbf{e}_c - \mathbf{e}_o$, with $f(o) = g(o) = 0$. Add ac within the face if necessary (Lemma 6.1) and take aoc as the outer face. By the proof of Lemma 6.2, $p = e(G)(f, g)$ is the integral harmonic embedding in the nondegenerate triangle $T = [p(a), p(o), p(c)]$ given by Tutte's theorem [26, (2.8), (7.2), and (9.2)]. Its other $n(G) - 3$ vertex images are distinct and strictly interior. Pick's theorem [4, Theorem 2.8] gives $n(G) - 3 \le |\operatorname{int}(T) \cap \mathbb{Z}^2| \le \operatorname{area}(T) - 1/2$.

For a connected simple graph with a cellular, essentially 3-connected torus embedding, [11, Theorem 2.5] gives a unit-weight geodesic equilibrium embedding in $\mathbb{R}^2/\mathbb{Z}^2$. Translate a vertex o to zero and choose lifts $x_v \in \mathbb{R}^2$ with $x_o = 0$. The integral edge translations give $D \in (\mathbb{Z}^2)^{V(G)}$, $\sum_v D_v = 0$ and $Lx = D$ by (6.8)–(6.9). Thus $e(G)x_v \in \mathbb{Z}^2$ by Lemma 2.2, and the embedding induces an injection $V(G) \hookrightarrow e(G)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$ into a set of $e(G)^2$ elements.

Graph substitutions. For a graph F and $m \ge 2$, the graph $F[\overline{K_m}]$ replaces each vertex by an independent part of size m and each edge by all edges between its two parts. The graphs $G_t = P_t[\overline{K_2}]$ are double graphs [21, 20]. Their resistance distances are special cases of [13]. For connected F with at least two vertices, [1, Theorem 3.5] gives

$$L^+_{F[\overline{K_m}]} = \frac{1}{m^2}L_F^+ \otimes J_m + \frac{1}{m}\Delta_F^{-1} \otimes \left(I_m - \frac{1}{m}J_m\right), \tag{1.14}$$

where L^+ is the Moore–Penrose inverse and Δ_F is the degree matrix. Up to an additive constant, the potential in Lemma 3.2 is the specialization of (1.14) to a path.

Jacobson, Niedermaier and Reiner [14, Proposition 9(v)] give an integral similarity reduction for repeated matrix blocks. Dong, Guo and Jiang [10, Proposition 2.1] give an integral reduction for the Laplacians of graphs with successive independent parts of size at least two, including $P_t[\overline{K_m}]$. Resistance formulas are known for substitutions by equal-sized complete or independent parts [23], by disjoint unions of edges and isolated vertices [28], and by arbitrary graphs [27].

The graph H_t is the substitution of K_2 at the first vertex of P_t and of $\overline{K_2}$ at every other vertex. Write $\mathcal{R}_G(u,v)$ for the effective resistance when every edge has resistance one. Set $\alpha_i = 4$ for $i < t$ and $\alpha_t = 2$. For this substitution, [15, Theorem 4.1, arXiv version] gives

$$\begin{aligned} \mathcal{R}_{H_t}(a_i, b_i) &= \frac{2}{\alpha_i}, \\ \mathcal{R}_{H_t}(x_i, y_j) &= \frac{|i-j|}{4} + \frac{1}{2\alpha_i} + \frac{1}{2\alpha_j} \quad \big(i \neq j,\ x_i \in \{a_i, b_i\},\ y_j \in \{a_j, b_j\}\big). \end{aligned} \tag{1.15}$$

In that specialization the weighted base path has conductance four on every edge, hence resistance $|i-j|/4$ between vertices i and j.

For a connected graph G with at least two vertices, fix $o \in V(G)$ and let C be the Laplacian reduced at o. The indexed resistance matrix determines C^{-1} by Lemma 2.4:

$$(C^{-1})_{uv} = \frac{\mathcal{R}_G(u,o) + \mathcal{R}_G(v,o) - \mathcal{R}_G(u,v)}{2} \qquad (u, v \neq o). \tag{1.16}$$

Consequently it also determines the Jacobian and its pairing. In particular, polarization of (1.15) on the divisors D_i gives (1.7).

2. Divisors, potentials and finite groups

Let $G = (V, E)$ be connected, with $n = |V| \geq 2$. For $v \in V$, let $\mathbf{e}_v$ be the corresponding coordinate vector. Write $\mathbf{1}$ for the all-one vector, $\mathbf{1}_A$ for the indicator of $A \subseteq V$, and δ_{ij} for the Kronecker delta. All inner products are Euclidean. Define

$$(Lf)_v = \deg_G(v) f_v - \sum_{w \sim v} f_w, \qquad \mathrm{Div}^0(G) = \left\{ D \in \mathbb{Z}^V \,:\, \sum_{v \in V} D_v = 0 \right\}. \tag{2.1}$$

The principal divisors form the subgroup $L\mathbb{Z}^V \subseteq \mathrm{Div}^0(G)$, and

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/L\mathbb{Z}^V.$$

A *potential* of $D \in \mathrm{Div}^0(G)$ is a function $f \in \mathbb{R}^V$ with $Lf = D$. A function is *harmonic* on $U \subseteq V$ if $(Lf)_v = 0$ for all $v \in U$. Write

$$N_G(v) = \{w \in V \,:\, vw \in E\}, \qquad \partial_G W = \{v \in V \setminus W \,:\, N_G(v) \cap W \neq \emptyset\} \quad (W \subseteq V).$$

Thus $\partial_G W$ is the external vertex boundary of W.

The Laplacian is symmetric, $L\mathbf{1} = 0$, and

$$f^{\mathsf{T}} L f = \sum_{vw \in E} (f_v - f_w)^2 \qquad (f \in \mathbb{R}^V), \tag{2.2}$$

where each edge occurs once. Since G is connected,

$$\ker_{\mathbb{R}} L = \mathbb{R}\mathbf{1}, \qquad L\mathbb{Q}^V = \left\{ D \in \mathbb{Q}^V \,:\, \sum_{v} D_v = 0 \right\}; \tag{2.3}$$

see [3, Sections 2 and 3.3].

Lemma 2.1 ([6, Section 2.1]). *Fix $o \in V$, put $V_o = V \setminus \{o\}$, and let $C = L_{V_o V_o}$. Then C is positive definite, and restriction to V_o induces*

$$\mathrm{Jac}(G) \simeq \mathbb{Z}^{V_o}/C\mathbb{Z}^{V_o}, \qquad |\mathrm{Jac}(G)| = \det C. \tag{2.4}$$

Keep o, V_o and C fixed. For $D \in \mathrm{Div}^0(G)$, write $d = D|_{V_o}$. The potential of D with value zero at o is rational and has restriction $C^{-1}d$. We write $e(G) = \exp \mathrm{Jac}(G)$, with exponent one for the trivial group.

Lemma 2.2. *Let $f \in \mathbb{R}^V$, $Lf = D \in \mathrm{Div}^0(G)$ and $a \in \mathbb{Z}_{>0}$. Then*

$$a[D] = 0 \quad \Longleftrightarrow \quad a(f_v - f_w) \in \mathbb{Z} \quad (v, w \in V). \tag{2.5}$$

In particular,

$$e(G) = \min\{a \in \mathbb{Z}_{>0} \,:\, aC^{-1} \in \mathrm{Mat}_{n-1}(\mathbb{Z})\}. \tag{2.6}$$

Proof. By [6, Lemma 3.1], $a[D] = 0$ if and only if $aC^{-1}d \in \mathbb{Z}^{V_o}$. Since $C^{-1}d = (f_v - f_o)_{v \in V_o}$, this is (2.5). The classes $[\mathbf{e}_v - \mathbf{e}_o]$, $v \in V_o$, generate $\mathrm{Jac}(G)$; their normalized potentials are the columns of C^{-1}. Thus their common annihilators are precisely the integers $a > 0$ for which aC^{-1} is integral. □

Lemma 2.3 ([22, Theorem 3.4 and Proposition 3.7]). *For $D, E \in \mathrm{Div}^0(G)$ and $Lf_E = E$, the formula*

$$\langle [D], [E] \rangle = D^{\mathsf{T}} f_E = d^{\mathsf{T}} C^{-1} d' \pmod{\mathbb{Z}}, \qquad d = D|_{V_o}, \quad d' = E|_{V_o}, \tag{2.7}$$

defines a nondegenerate symmetric bilinear pairing $\mathrm{Jac}(G) \times \mathrm{Jac}(G) \to \mathbb{Q}/\mathbb{Z}$.

This is the *monodromy pairing*. In the formula of [22, Proposition 3.7], the generalized inverse is C^{-1} extended by a zero row and column at o; see [22, Example 3.2].

Lemma 2.4. *For $u, v \in V$, choose $Lf = \mathbf{e}_u - \mathbf{e}_v$ and set $\mathcal{R}_G(u, v) = f_u - f_v$. This is independent of the choice of f, and $\mathcal{R}_G(u, u) = 0$. For $u, v \in V_o$, (1.16) holds. Hence the resistance matrix, with its vertex indices specified, determines C, $\mathrm{Jac}(G)$ and its monodromy pairing.*

Proof. The kernel identity (2.3) gives independence of f. For $Q = C^{-1}$, the resistance and energy formulas in [3, Sections 3.2–3.3] give

$$\mathcal{R}_G(u, o) = Q_{uu}, \qquad \mathcal{R}_G(u, v) = Q_{uu} + Q_{vv} - 2Q_{uv} \qquad (u, v \in V_o).$$

Solving for Q_{uv} gives (1.16). Since $C = Q^{-1}$, the remaining assertions follow from Lemmas 2.1 and 2.3. □

Here $\mathcal{R}_G$ is the effective resistance when every edge has unit resistance [3, Section 3.2].

Lemma 2.5 (Matrix-tree theorem). *Let $\lambda_1, \dots, \lambda_{n-1}$ be the nonzero eigenvalues of L, counted with multiplicity. If $\tau(G)$ is the number of spanning trees, then*

$$\det C = \tau(G) = \frac{1}{n} \prod_{j=1}^{n-1} \lambda_j. \tag{2.8}$$

The determinant formula is [3, Theorem 6.1]; the spectral formula is [18, Section 2, equation (1)].

Lemma 2.6 (Dirichlet equations). *Let $\emptyset \neq X \subseteq V$ and $U = V \setminus X$. Prescribed real values on X have a unique extension f harmonic on U. If $U \neq \emptyset$, then*

$$f_U = -L_{UU}^{-1} L_{UX} f_X.$$

For each component K of $G[U]$, its external vertex boundary $\partial_G V(K)$ is a nonempty subset of X, and

$$\min_{\partial_G V(K)} f \le f(v) \le \max_{\partial_G V(K)} f \qquad (v \in V(K)).$$

If either bound is attained at a vertex of K, then f is constant on $V(K) \cup \partial_G V(K)$.

Proof. If $U = \emptyset$, there is nothing to extend. Otherwise, the existence and uniqueness principles in [19, Section 2.1] apply to simple random walk on G: the boundary data are bounded, and every vertex of U has a path to X. Uniqueness for zero boundary data makes L_{UU} invertible, and the harmonicity equations give the displayed formula.

For a component K, put $B = \partial_G V(K)$. Connectedness of G and maximality of the component give $\emptyset \neq B \subseteq X$. The induced graph $G[V(K) \cup B]$ is connected and has the same harmonicity equations on $V(K)$. Its maximum principle [19, Section 2.1], applied to f and $-f$ with boundary B, gives both bounds and the equality assertion. □

3. Exponent eight and homocyclic pairings

For integers $m, t \geq 2$, define

$$V(G_{m,t}) = \{1, \dots, t\} \times \{1, \dots, m\}, \qquad (v, a) \sim (w, b) \iff |v - w| = 1.$$

Thus $G_t = G_{2,t}$, with $a_v = (v, 1)$ and $b_v = (v, 2)$. The vertex connectivity of a graph is the least cardinality of a vertex set whose deletion disconnects it or leaves at most one vertex. Put $d_v = \deg_{P_t}(v)$. Each vertex (v, a) has degree md_v; the endpoint values of d_v are one and the interior values are two.

Lemma 3.1. *If $s \in \mathbb{Z}^t$ and $\sum_v s_v = 0$, the equations $L_{P_t} z = s$ have an integral solution. One is specified by*

$$z_t = 0, \qquad z_v - z_{v+1} = \sum_{j=1}^{v} s_j \quad (1 \leq v < t). \tag{3.1}$$

Proof. By Lemmas 2.1 and 2.5, $|\operatorname{Jac}(P_t)| = \tau(P_t) = 1$. Thus $s = L_{P_t} z$ for some $z \in \mathbb{Z}^t$. Subtracting $z_t \mathbf{1}$ gives $z_t = 0$. Summing the equations over $\{1, \dots, v\}$ gives (3.1). □

Lemma 3.2. *For $m, t \geq 2$, the graph $G_{m,t}$ is bipartite and has vertex connectivity m. Its exponent is*

$$e(G_{m,t}) = \begin{cases} m^2, & t = 2, \\ 2m^2, & t \geq 3. \end{cases}$$

Proof. For $D \in \operatorname{Div}^0(G_{m,t})$, put $s_v = \sum_a D_{v,a}$. Then $\sum_v s_v = 0$. Let z be the integral solution in Lemma 3.1, and put $\bar{z} = t^{-1} \sum_v z_v$. Since P_t is connected and $d_v > 0$, the formula of [1, Theorem 3.5], with base graph P_t and independent parts of size m, applies. As $L^+_{P_t} s = z - \bar{z}\mathbf{1}$, it gives

$$f_{v,a} = \frac{z_v}{m^2} + \frac{mD_{v,a} - s_v}{m^2 d_v}, \qquad f = L^+_{G_{m,t}} D + \frac{\bar{z}}{m^2}\mathbf{1}. \tag{3.2}$$

In particular $L_{G_{m,t}} f = D$. Suppose $t \geq 3$. Then $2m^2 f \in \mathbb{Z}^{V(G_{m,t})}$, since $d_v \in \{1, 2\}$. Hence $e(G_{m,t}) \mid 2m^2$. For $D = \mathbf{e}_{(1,1)} - \mathbf{e}_{(2,1)}$, one has $z = (1, 0, \dots, 0)$ and

$$f_{2,2} = \frac{1}{2m^2}, \qquad f_{3,1} = 0.$$

By Lemma 2.2, $a[D] = 0$ implies $2m^2 \mid a$. Thus $e(G_{m,t}) = 2m^2$. For $t = 2$, one has $G_{m,2} = K_{m,m}$, whose exponent is m^2 by [14, Corollary 5] with $k = 2$ and equal part size $m \geq 2$.

The unions of the odd and even parts form a bipartition. If $|S| < m$, every part contains a vertex outside S. Vertices in different parts are joined by a path through surviving vertices in the intervening parts; two vertices in the same part have a common surviving neighbour in an adjacent part. Thus $G_{m,t} - S$ is connected and has at least $mt - (m - 1) = m(t - 1) + 1 > 1$ vertices. Deleting the second part leaves the $m \geq 2$ first-part vertices isolated. The vertex connectivity is therefore m. □

Lemma 3.3. *For every $t \geq 2$, the graphs G_t and H_t are planar and biconnected. They have $2t$ vertices and respectively $4t - 4$ and $4t - 3$ edges. Their maximum degrees are at most four; that of G_t equals four for $t \geq 3$.*

Proof. The $t-1$ pairs of consecutive parts contribute four edges each. The additional edge a_1b_1 belongs to H_t alone. The degrees in G_t are two at the end parts and four at interior parts; in H_t the two first-part degrees increase from two to three. Both graphs have $2t \geq 4$ vertices. By Lemma 3.2, $G_t - v$ is connected for every vertex v, and it is a spanning subgraph of $H_t - v$. Thus both graphs are biconnected.

Embed G_2 as the cycle a_1, a_2, b_1, b_2, a_1 in the sphere. For H_2, draw a_1b_1 in one face. In either case there is a face F with disk closure and with a_2, b_2 on its boundary. Suppose an embedding has such a face with a_t, b_t on its boundary. Draw two simple arcs from a_t to b_t whose interiors are disjoint and contained in F, and subdivide them at a_{t+1} and b_{t+1}, respectively. The four new edges are precisely those between parts t and $t+1$. The arcs bound a disk whose interior is a face of the enlarged embedding and whose boundary is $a_t, a_{t+1}, b_t, b_{t+1}, a_t$. This face contains a_{t+1}, b_{t+1} on its boundary, so induction gives the required embeddings. □

Lemma 3.4. *For $t \geq 2$, set*

$$\alpha_v = \begin{cases} 4, & v < t, \\ 2, & v = t. \end{cases}$$

Let $D \in \mathrm{Div}^0(H_t)$, put $s_v = D_{a_v} + D_{b_v}$, $q_v = D_{a_v} - D_{b_v}$, and choose an integral solution of $L_{P_t} z = s$. Then a potential of D on H_t is

$$f_{a_v} = \frac{z_v}{4} + \frac{q_v}{2\alpha_v}, \qquad f_{b_v} = \frac{z_v}{4} - \frac{q_v}{2\alpha_v}. \tag{3.3}$$

In particular $8\,\mathrm{Jac}(H_t) = 0$, *and*

$$|\,\mathrm{Jac}(H_t)| = 8^{t-1}. \tag{3.4}$$

Proof. Put $h_v = \mathbf{e}_{a_v} - \mathbf{e}_{b_v}$, and define $J : \mathbb{R}^t \to \mathbb{R}^{V(H_t)}$ by $(Ju)_{a_v} = (Ju)_{b_v} = u_v$. The orthogonal decomposition

$$\mathbb{R}^{V(H_t)} = J\mathbb{R}^t \oplus \bigoplus_{v=1}^{t} \mathbb{R}h_v$$

is preserved by L_{H_t}, with

$$L_{H_t} Ju = J(2L_{P_t} u), \qquad L_{H_t} h_v = (2d_v + 2\delta_{v1})h_v = \alpha_v h_v.$$

Since $D = J(s/2) + \sum_v (q_v/2)h_v$ and $L_{P_t} z = s$,

$$L_{H_t}\left(J(z/4) + \sum_v \frac{q_v}{2\alpha_v} h_v \right) = D.$$

This gives (3.3). Moreover,

$$8f_{a_v} = 2z_v + \frac{4}{\alpha_v} q_v \in \mathbb{Z}, \qquad 8f_{b_v} = 2z_v - \frac{4}{\alpha_v} q_v \in \mathbb{Z},$$

so $8\,\mathrm{Jac}(H_t) = 0$.

The $t-1$ nonzero Laplacian eigenvalues of P_t have product t, by Lemma 2.5. The same lemma and the invariant decomposition above give

$$|\,\mathrm{Jac}(H_t)| = \frac{2^{t-1} t \prod_{v=1}^{t} \alpha_v}{2t} = \frac{2^{t-1} t\, 4^{t-1} \cdot 2}{2t} = 8^{t-1}.$$

□

Lemma 3.5. *The divisors in* (1.5) *satisfy* (1.7).

Proof. Put $c_v = (2\alpha_v)^{-1}$ and $R_{ij} = \mathcal{R}_{H_t}(a_i, a_j)$. By (1.15),

$$R_{ii} = 0, \qquad R_{ij} = \frac{|i-j|}{4} + c_i + c_j \quad (i \neq j).$$

By Lemmas 2.3 and 2.4,

$$\begin{aligned} \langle [D_i], [D_j] \rangle &= \frac{R_{i,j+1} + R_{i+1,j} - R_{ij} - R_{i+1,j+1}}{2} \\ &= \begin{cases} \frac{1}{4} + c_i + c_{i+1}, & i = j, \\ -c_{\max(i,j)}, & |i-j| = 1, \\ 0, & |i-j| \geq 2 \end{cases} \pmod{\mathbb{Z}}. \end{aligned} \tag{3.5}$$

Here $c_v = 1/8$ for $v < t$ and $c_t = 1/4$. The diagonal entries are therefore $4/8$ except for the last, which is $5/8$; each adjacent entry is $-1/8$. For $t = 2$ the matrix is $(5/8)$. □

Proof of Theorem 1.2. By Lemma 3.3, H_t is planar and biconnected, with $2t$ vertices, $4t-3$ edges and maximum degree at most four. Put $r = t-1$ and $R = \mathbb{Z}/8\mathbb{Z}$. Since $8[D_i] = 0$ by Lemma 3.4, (1.6) defines a homomorphism $\phi : R^r \to \mathrm{Jac}(H_t)$.

Let $b(x, y) = x^{\mathsf{T}} B_r y$ on R^r. Set

$$u_1 = 5, \qquad u_{j+1} = 4 - u_j^{-1} \quad (1 \leq j < r).$$

In R, $5^{-1} = 5$ and $7^{-1} = 7$; hence $u_j = 5$ for odd j and $u_j = 7$ for even j. Define $g_r = \mathbf{e}_r$ and, successively for $k = r-1, \dots, 1$,

$$g_k = \mathbf{e}_k + u_{r-k}^{-1} g_{k+1}.$$

Each g_k belongs to $\mathbf{e}_k + \mathrm{span}_R\{\mathbf{e}_{k+1}, \dots, \mathbf{e}_r\}$; thus the g_k form a basis. For $k < r$, one has $b(\mathbf{e}_k, g_{k+1}) = -1$ and $b(\mathbf{e}_k, g_j) = 0$ for $j > k+1$. Descending induction from $b(g_r, g_r) = 5$ gives

$$b(g_k, g_{k+1}) = -1 + u_{r-k}^{-1} u_{r-k} = 0, \qquad b(g_k, g_k) = 4 - u_{r-k}^{-1} = u_{r-k+1},$$

and $b(g_k, g_j) = 0$ for $j > k+1$. Consequently the matrix T with columns $g_1, \dots, g_r$ satisfies

$$T \in \mathrm{GL}_r(R), \qquad T^{\mathsf{T}} B_r T = \mathrm{diag}(u_r, \dots, u_1). \tag{3.6}$$

Every u_j is a unit, so B_r is invertible over R. If $c \in \ker \phi$, Lemma 3.5 gives

$$0 = \langle [D_j], \phi(c) \rangle = \frac{(B_r c)_j}{8} \pmod{\mathbb{Z}} \qquad (1 \leq j \leq r),$$

where the quotient is taken after an integer lift. Thus $B_r c = 0$ in R^r, and $c = 0$. Both R^r and $\mathrm{Jac}(H_t)$ have order 8^r, by (3.4); hence ϕ is an isomorphism.

By Lemma 3.5 and (3.6), the elements $\phi(g_1), \dots, \phi(g_r)$ form an orthogonal basis, with diagonal pairings $u_r/8, \dots, u_1/8$ in $\mathbb{Q}/\mathbb{Z}$. There are $\lceil r/2 \rceil$ entries $5/8$ and $\lfloor r/2 \rfloor$ entries $7/8$, which gives (1.8). □

4. Boundary potentials and the first three degree differences

Let G be biconnected, and fix a vertex o of degree $q \geq 2$. Put

$$X = N(o) = \{x_1, \dots, x_q\}, \qquad U = V(G) \setminus (\{o\} \cup X), \qquad e = e(G).$$

Let C be the Laplacian reduced at o. For each i let p_i be the unique grounded potential

$$L p_i = e(\mathbf{e}_{x_i} - \mathbf{e}_o), \qquad p_i(o) = 0. \tag{4.1}$$

Write $p(v) = (p_1(v), \dots, p_q(v))$ and

$$B_{ij} = p_i(x_j), \qquad M_i = B_{ii}. \tag{4.2}$$

Lemma 4.1. *The potentials p_i are integral and satisfy*

$$\begin{gathered} p_i(v) \geq 1 \quad (v \neq o), \qquad p_i(v) < M_i \quad (v \neq x_i), \\ \sum_{i=1}^{q} p_i(v) = e \quad (v \neq o). \end{gathered} \tag{4.3}$$

The matrix B is symmetric positive definite, has positive integer entries, and has every row sum equal to e. Moreover,

$$2 \leq M_i \leq e - q + 1, \qquad e \geq q + 1. \tag{4.4}$$

Proof. By Lemma 2.2, $p_i \in \mathbb{Z}^V$. Since C is positive definite,

$$M_i = e(C^{-1})_{x_i x_i} > 0.$$

The graphs $G - o$ and $G - x_i$ are connected. Hence every component of $G - \{o, x_i\}$ has external boundary $\{o, x_i\}$. The function p_i is harmonic there and has boundary values $0, M_i$. By Lemma 2.6 (see also [6, Remark 5.3]),

$$0 < p_i(v) < M_i \qquad (v \notin \{o, x_i\}).$$

Together with $p_i(o) = 0$, $p_i(x_i) = M_i$ and integrality, this gives the inequalities in (4.3).

The functions $\sum_i p_i$ and $e\mathbf{1}_{V\setminus\{o\}}$ both vanish at o and both have Laplacian $e(\sum_i \mathbf{e}_{x_i} - q\mathbf{e}_o)$. Their difference is zero by (2.3). Moreover,

$$B = e(C^{-1})_{XX}.$$

Thus B is symmetric positive definite. The sum identity gives its column sums e, and symmetry gives its row sums. All its entries are positive integers. For $j \neq i$, $1 \leq B_{ij} < M_i$, so $M_i \geq 2$. Finally $e = M_i + \sum_{j\neq i} B_{ij} \geq M_i + q - 1$, giving both inequalities in (4.4). □

Lemma 4.2. *If $M_i \geq 3$ and $j \neq i$, then $B_{ij} \neq M_i - 1$.*

Proof. Suppose $p_i(x_j) = M_i - 1$. In $\sum_{w\sim x_j}(p_i(x_j) - p_i(w))$, the neighbour o contributes $M_i - 1$. By Lemma 4.1, the only possible negative term is -1, from x_i; it occurs at most once since G is simple. Thus

$$0 = (Lp_i)_{x_j} = \sum_{w\sim x_j} (p_i(x_j) - p_i(w)) \geq M_i - 2 > 0,$$

a contradiction. □

Lemma 4.3. *Let $Z \subseteq V(G)$ be nonempty, let $u \notin Z$, and suppose $V(G) \setminus (Z \cup \{u\}) \neq \emptyset$. If $\partial_G Z \subseteq \{u\}$, then G is not biconnected.*

Proof. After deletion of u there is no edge from Z to the nonempty set $V(G) \setminus (Z \cup \{u\})$. Thus $G - u$ is disconnected. □

4.1. **The matrices $sI_q + J_q$.** Assume for this subsection that

$$B = sI_q + J_q, \qquad e = q + s, \qquad s \in \mathbb{Z}_{\geq 2}. \tag{4.5}$$

For $v \neq o$, put $z_i(v) = p_i(v) - 1$. Then $z_i(x_a) = s\delta_{ia}$. For $v \in U$ define $T(v) = \{i : z_i(v) > 0\}$.

Lemma 4.4. *The set X is independent. For $v \in U$,*

$$0 \leq z_i(v) \leq s - 1, \qquad \sum_i z_i(v) = s, \qquad 2 \leq |T(v)| \leq \min\{q, s\}. \tag{4.6}$$

For every component K of $G[U]$,

$$T(v) = \{i : x_i \in \partial_G V(K)\} \qquad (v \in V(K)).$$

For $a \neq j$,

$$\sum_{v\in U:\ v\sim x_a} z_j(v) = 1. \tag{4.7}$$

Proof. Suppose $x_a \sim x_b$. At x_a, the terms of Lp_b corresponding to o and x_b are 1 and $-s$; every other term is nonpositive by Lemma 4.1. Thus

$$0 = (Lp_b)_{x_a} \leq 1 - s < 0,$$

a contradiction. The bounds in (4.6) follow from (4.3), since $M_i = s + 1$. Their sum is $e - q = s$. A single positive coordinate would have value s, contrary to its upper bound. Each positive coordinate is at least one, so there are at most s of them.

Let K be a component of $G[U]$. Since $G - o$ is connected, $\emptyset \neq \partial_G V(K) \subseteq X$. No vertex of U is adjacent to o, so each z_i is harmonic on U in $G - o$. Its boundary values are zero except for the value s at x_i. By Lemma 2.6, z_i vanishes on K if $x_i \notin \partial_G V(K)$, and is strictly positive on K otherwise. This gives the formula for $T(v)$.

For $a \neq j$, the equation $Lp_j(x_a) = 0$ reads

$$\deg(x_a) = \sum_{v \in U\,:\; v \sim x_a} (1 + z_j(v)),$$

because o is its only neighbour outside U and has value zero. There are $\deg(x_a) - 1$ terms, giving (4.7). □

For $T \subseteq \{1, \dots, q\}$ put $U_T = \{v \in U \,:\, T(v) = T\}$. Edges from U_T can end only in U_T or in the boundary vertices indexed by T.

Lemma 4.5. *Assume* (4.5) *and* $T = \{a, b\}$. *If* $U_T \neq \emptyset$, *its vertices are exactly the internal vertices of one path of length* s *from* x_b *to* x_a. *Along that path the values of* z_a *are* $0, 1, \dots, s$. *Each internal vertex has degree two in* G.

Proof. Put $Y = U_T$, $K = G[Y \cup \{x_a, x_b\}]$ and $t = z_a|_{V(K)}$. Then $t(x_a) = s$, $t(x_b) = 0$, and $1 \leq t \leq s - 1$ on Y. Every neighbour in G of a vertex in Y belongs to K, so $(L_K t)|_Y = 0$. For $v \in Y$, one has $z_b(v) = s - t(v)$. Equation (4.7) therefore permits at most one edge from x_a to Y, ending at a vertex of value $s - 1$, and at most one edge from x_b to Y, ending at a vertex of value one.

By Lemma 4.4, every component of $G[Y]$ has external boundary $\{x_a, x_b\}$. The bounds on the number of attachments imply that $G[Y]$ is connected and

$$(L_K t)(x_a) = 1, \qquad (L_K t)(x_b) = -1.$$

For $1 \leq j \leq s$, let $Y_j = \{v \in V(K) \,:\, t(v) \geq j\}$. Then

$$1 = \sum_{v \in Y_j} (L_K t)_v = \sum_{\substack{vw \in E(K) \\ v \in Y_j,\; w \notin Y_j}} (t(v) - t(w)). \tag{4.8}$$

Each summand is a positive integer. Thus exactly one edge crosses this cut, and its endpoint values are $j, j - 1$. No edge can join levels differing by more than one.

Set $Z_0 = \{x_b\}$, $Z_s = \{x_a\}$ and $Z_j = \{v \in Y \,:\, t(v) = j\}$ for $1 \leq j < s$. The cut equations imply $Z_j \neq \emptyset$ for every j. For $1 \leq j < s$ and $v \in Z_j$, adjacent values differ by at most one, so harmonicity gives

$$0 = (L_K t)_v = |N_K(v) \cap Z_{j-1}| - |N_K(v) \cap Z_{j+1}|. \tag{4.9}$$

There is exactly one edge to each adjacent level, by (4.8). Their endpoints in Z_j therefore coincide; call this endpoint v_j. Every other vertex of Z_j has all its neighbours in Z_j, whence

$$\partial_G(Z_j \setminus \{v_j\}) \subseteq \{v_j\}.$$

If $Z_j \setminus \{v_j\}$ were nonempty, deleting v_j would separate it from o, by Lemma 4.3. Hence $Z_j = \{v_j\}$. The only edges of K join successive vertices of $x_b, v_1, \dots, v_{s-1}, x_a$: the cut equations exclude other interlevel edges, and simplicity excludes edges within singleton levels. Each internal vertex has degree two in G. □

Lemma 4.6. *Assume* (4.5).

(i) *If $s = 3$ and $|T| = 3$, then U_T is empty or consists of one vertex adjacent exactly to the three vertices indexed by T.*

(ii) *If $s = 4$, $q = 3$ and $T = \{1, 2, 3\}$, then U_T is empty or consists of three vertices forming a triangle. Their shifted vectors are the three permutations of $(2, 1, 1)$, and the vertex with $z_a = 2$ is adjacent to x_a and to no other boundary vertex.*

Proof. Assume $U_T \neq \emptyset$. For (i), $z(v)$ equals one on T and zero elsewhere. Write $\varepsilon_a(v) = \mathbf{1}_{v\sim x_a}$ for $a \in T$. Harmonicity gives

$$3\varepsilon_a(v) = \sum_{b\in T} \varepsilon_b(v) \qquad (a \in T).$$

Thus all three incidences are equal. By (4.7), at most one vertex v_T has all three incidences one. Some component has an external neighbour, so v_T exists. All remaining vertices have no boundary neighbour; hence $\partial_G(U_T \setminus \{v_T\}) \subseteq \{v_T\}$. If this difference were nonempty, Lemma 4.3, with o outside, would contradict biconnectivity. Hence $U_T = \{v_T\}$. Its neighbour set is exactly $\{x_a : a \in T\}$.

For (ii), define $Z_a = \{v \in U_T : z_a(v) = 2\}$. The three sets Z_a partition U_T, since every positive integer triple of sum four is a permutation of $(2, 1, 1)$. A vertex of Z_a cannot be adjacent to x_b for $b \neq a$: its contribution two would contradict (4.7) for (b, a). That identity also permits at most one $v_a \in Z_a$ adjacent to x_a. At $v \in Z_a$, the ath harmonic equation is

$$2\mathbf{1}_{v\sim x_a} = |N_G(v) \cap (U_T \setminus Z_a)|. \tag{4.10}$$

Every vertex of Z_a not adjacent to x_a therefore has neighbours only in Z_a. Suppose $Z_a \neq \emptyset$. If no v_a exists, then $\partial_G Z_a = \emptyset$, contrary to connectedness. For this unique vertex v_a one has $\partial_G(Z_a \setminus \{v_a\}) \subseteq \{v_a\}$. If this difference were nonempty, Lemma 4.3, with o outside, would contradict biconnectivity. Thus every nonempty Z_a is the singleton $\{v_a\}$. Since $U_T \neq \emptyset$, choose one such v_a. Equation (4.10) requires two distinct neighbours in the other two singleton sets. All three v_a exist and are pairwise adjacent, and $N_G(v_a) = \{x_a\} \cup \{v_b : b \neq a\}$. □

Lemma 4.7. *Assume $U \neq \emptyset$. Order the reduced Laplacian at o by X, U:*

$$C = \begin{pmatrix} A_0 & -H \\ -H^{\mathsf{T}} & T_0 \end{pmatrix}, \qquad W = T_0^{-1} H^{\mathsf{T}}.$$

The matrix T_0 is positive definite and block diagonal on the components of $G[U]$. The entries of W are nonnegative and its row sums are one. Moreover,

$$(C^{-1})_{UU} = T_0^{-1} + W \frac{B}{e} W^{\mathsf{T}}. \tag{4.11}$$

Under (4.5)*, for vertices u, v in different components,*

$$e(C^{-1})_{uv} = 1 + s\, w_u \cdot w_v, \tag{4.12}$$

where w_u, w_v are their rows of W. If an interior component consists of the internal vertices of a path of length s whose endpoints lie in X, with no other edges incident to the component, its nonzero row entries are $j/s, (s-j)/s$, $1 \leq j < s$. For a component consisting of one vertex with exactly three neighbours, all in X, these entries are $1/3$.

Proof. The matrix T_0 is a positive definite principal submatrix of C, and its entries between distinct components of $G[U]$ are zero. Put $S = A_0 - HT_0^{-1}H^{\mathsf{T}}$. Since C and T_0 are nonsingular, the block inverse formula [24, equation (1.4), p. 406], applied with the order U, X, gives

$$S^{-1} = (C^{-1})_{XX} = \frac{B}{e},$$

and (4.11) follows.

No vertex of U is adjacent to o, so the Laplacian row sums on U give $T_0\mathbf{1}_U = H^\mathsf{T}\mathbf{1}_X$. Hence $W\mathbf{1}_X = \mathbf{1}_U$. For boundary values g on X, harmonicity on U in $G - o$ is equivalent to $T_0 f_U = H^\mathsf{T} g$, whence $f_U = Wg$. Apply Lemma 2.6 with g equal to each coordinate vector of $\mathbb{R}^X$; this gives $W \geq 0$.

For vertices in distinct components, $(T_0^{-1})_{uv} = 0$, so

$$e(C^{-1})_{uv} = w_u(sI_q + J_q)w_v^\mathsf{T} = s\, w_u \cdot w_v + (w_u\mathbf{1}_X)(w_v\mathbf{1}_X) = 1 + s\, w_u \cdot w_v.$$

On a path, $2f_j = f_{j-1} + f_{j+1}$ implies $f_j = (1 - j/s)f_0 + (j/s)f_s$. At the centre of $K_{1,3}$, $f = (f_1 + f_2 + f_3)/3$. □

4.2. **Degree differences.**

Proof of Theorem 1.3. Choose o of maximum degree q. By Lemma 4.1, $e \geq q + 1$. If $e = q + 1$, then $M_i = 2$ for all i, and positivity and the row sums give $B = I_q + J_q$. An interior vertex would have all its coordinates $p_i \leq 1$, giving $\sum_i p_i \leq q < e$. Thus $U = \emptyset$ and

$$C^{-1} = \frac{I_q + J_q}{q+1}, \qquad C = (q+1)I_q - J_q.$$

These are the matrices of K_{q+1} grounded at one vertex, so $G = K_{q+1}$.

Suppose $e = q + 2$. A diagonal entry two has every off-diagonal entry at most one, so its row sum is at most $q + 1$. Hence every $M_i = 3$. By Lemma 4.2 its off-diagonal entries are all one, so $B = 2I_q + J_q$. Every interior support has two indices, and Lemma 4.5 makes the corresponding interior vertices the internal vertices of one path of length two. Equation (4.7) requires one such path for each unordered pair of boundary vertices. If $q \geq 3$, choose two pairs meeting at one index. The two internal vertices lie in distinct components, and Lemma 4.7 gives

$$e(C^{-1})_{uv} = 1 + 2\left(\frac{1}{2}\right)^2 = \frac{3}{2} \notin \mathbb{Z}. \tag{4.13}$$

This contradicts Lemma 2.2. Thus $q = 2$ and $G = C_4$.

Suppose $e = q + 3$. Diagonal entries two are excluded by row sums. If a diagonal entry were three, Lemma 4.2 would make its other entries one, giving row sum $q + 2$. Thus every $M_i = 4$; the other $q - 1$ positive entries must sum to $q - 1$, so $B = 3I_q + J_q$. By Lemmas 4.5 and 4.6, each component Q of $G[U]$ has a support T_Q of size two or three. The induced graph $G[V(Q) \cup \{x_a : a \in T_Q\}]$ is respectively a path of length three or a copy of $K_{1,3}$. Its contribution to the boundary sum for distinct indices a, b is one if $\{a, b\} \subseteq T_Q$, and zero otherwise. Hence (4.7) gives

$$\sum_Q \mathbf{1}_{\{a,b\}\subseteq T_Q} = 1 \qquad (a \neq b).$$

Thus every unordered pair belongs to exactly one component support, and distinct supports intersect in at most one index. Since $q \geq 2$, there is at least one interior component.

Suppose that there are at least two interior components. If their supports were pairwise disjoint, a pair chosen from two of them could not be covered. Hence two supports intersect, necessarily at one index. Choose an internal vertex in each component. Their harmonic weights at that index are $c/3, d/3$ with $c, d \in \{1, 2\}$. Formula (4.12) gives

$$e(C^{-1})_{uv} = 1 + \frac{cd}{3} \notin \mathbb{Z}, \tag{4.14}$$

since $cd \in \{1, 2, 4\}$. There is therefore only one interior component. Its support must contain all boundary indices, because it covers every pair. The induced graph on the component and its boundary vertices is either a path of length three, with $q = 2$, or a copy of $K_{1,3}$, with $q = 3$. Adding the edges from o gives respectively C_5 and $K_{2,3}$. Conversely, Lemma 5.2 gives

$$e(K_{q+1}) = q + 1, \qquad e(C_4) = 4, \qquad e(C_5) = 5, \qquad e(K_{2,3}) = 6.$$

Their maximum degrees are respectively $q, 2, 2, 3$. □

5. Classification below exponent eight

5.1. Maximum degree three and exponent seven.

Proposition 5.1. *If $\Delta(G) = 3$ and $e(G) = 7$ for a biconnected graph G, then $G \simeq S_7$.*

Proof. Take o of degree three. By Lemmas 4.1 and 4.2, the diagonal entries of its boundary matrix are four or five: a diagonal two gives row sum at most four, and a diagonal three gives row sum at most five. A diagonal five has off-diagonal entries one and one. A diagonal four has off-diagonal entries one and two. Since B is symmetric and each row contains at most one entry two, the pairs $\{i, j\}$ with $B_{ij} = 2$ are disjoint. On three indices, at most one pair can occur. Hence, up to simultaneous permutation, the two possibilities are:

$$B = 4I_3 + J_3, \qquad B = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}. \tag{5.1}$$

Case $B = 4I_3 + J_3$. The positive supports in (4.6) have size two or three. By Lemmas 4.5 and 4.6, the vertices of a nonempty U_T with $|T| = 2$ are the internal vertices of a path of length four, while $|T| = 3$ gives a triangle with the three specified boundary attachments. If that triangle occurs, its attachments already contribute one to every boundary sum (4.7). Any vertex v of another component adjacent to x_a would have some $j \in T(v) \setminus \{a\}$, and $z_j(v) \geq 1$ would make the sum in (4.7) at least two. Thus no other interior component can have a neighbour in X. Every interior component has a neighbour in X. Hence there is no other component, and the graph is S_7. If no three-element support occurs, then every nonzero term $z_j(v)$ in (4.7) has $T(v) = \{a, j\}$, by Lemma 4.4. Thus $U_{\{a,j\}} \neq \emptyset$ for every $a \neq j$, and Lemma 4.5 gives one path of length four for each boundary pair. Choose u on the x_1x_2-path and v on the x_1x_3-path, each at distance three from x_1. They lie in distinct components of $G[U]$, by Lemma 4.4. In the notation of Lemma 4.7, their rows are

$$w_u = (1/4, 3/4, 0), \qquad w_v = (1/4, 0, 3/4).$$

Thus (4.12) gives

$$7(C^{-1})_{uv} = 1 + 4\left(\frac{1}{4}\right)^2 = \frac{5}{4} \notin \mathbb{Z}, \tag{5.2}$$

contradicting Lemma 2.2.

Case of the second matrix. Name the boundary vertices a, b, c. Their potential vectors are

$$p(a) = (5, 1, 1), \qquad p(b) = (1, 4, 2), \qquad p(c) = (1, 2, 4).$$

For $i \in \{2, 3\}$, the function p_i is harmonic on U in the connected graph $G - o$, since no vertex of U is adjacent to o. Suppose $p_i(z) = 1$ at a vertex of a component K of $G[U]$. All values of p_i on $G - o$ are at least one. By Lemma 2.6, $p_i = 1$ on $V(K) \cup \partial_G V(K)$. The only vertex of X with this coordinate one is a, so $\partial_G V(K) = \{a\}$. Since $o \notin V(K) \cup \{a\}$, this contradicts Lemma 4.3.

At any interior vertex the second and third coordinates are therefore in $\{2, 3\}$, by the strict maximum. The first coordinate is positive, and the coordinate sum is seven. Hence its vector belongs to

$$\kappa = (1, 3, 3), \qquad u = (3, 2, 2), \qquad v = (2, 2, 3), \qquad w = (2, 3, 2). \tag{5.3}$$

For $\xi \in \{\kappa, u, v, w\}$ put $U_\xi = \{z \in U : p(z) = \xi\}$.

The edge ab is impossible: the second coordinate is harmonic at a, where its value is one, its degree is at most three, and b would contribute four. The third coordinate similarly excludes ac. Every neighbour of a other than o therefore lies in U and has second and third coordinates at least two. If $d = \deg(a)$, harmonicity in either coordinate gives $2(d - 1) \leq d$. Since $d \geq 2$, one has $d = 2$. Its one neighbour outside $\{o\}$ has second and third coordinates two, hence is some $u_* \in U_u$.

At b, the vector equation is

$$\sum_{y\sim b} p(y) = \deg(b)(1,4,2) - (0,7,0).$$

If $\deg(b) = 2$, the single neighbour different from o would have vector $(2,1,4)$, which is neither a boundary vector nor one of (5.3). Hence $\deg(b) = 3$, and the other two neighbour vectors sum to $(3,5,6)$. The first-coordinate sum is $1+2$, so the possible vector sums are

pair	$\{p(c), v\}$	$\{p(c), w\}$	$\{\kappa, v\}$	$\{\kappa, w\}$
sum	$(3,4,7)$	$(3,5,6)$	$(3,5,6)$	$(3,6,5)$.

Thus $N_G(b) \setminus \{o\}$ is one of

$$\{c, w_*\} \quad (w_* \in U_w), \qquad \{k_*, v_*\} \quad (k_* \in U_\kappa,\ v_* \in U_v). \tag{5.4}$$

Interchanging the second and third coordinates gives at c the alternatives $\{b, v_*\}$ or $\{k_*, w_*\}$.

At a vertex in U_v, the second coordinate equals two. It is adjacent neither to o nor to a, whose two neighbours have been determined. Every possible remaining neighbour has second coordinate at least two. Harmonicity therefore excludes b, whose coordinate is four. Thus the second alternative in (5.4) is impossible. At a vertex in U_w, the third-coordinate equation likewise excludes c. Consequently

$$b\,c, \quad b\,w_*, \quad c v_* \tag{5.5}$$

are edges; hence $N(b) = \{o, c, w_*\}$ and $N(c) = \{o, b, v_*\}$.

A vertex in U_κ has first coordinate one. Its neighbours must also have that coordinate one, so they can only lie in U_κ or be b, c. The neighbour sets of b, c have already been determined. Hence U_κ, if nonempty, has no edge to its complement, contradicting connectedness. Thus $U_\kappa = \emptyset$.

For $z \in U_v$, the second-coordinate equation permits neighbours only in $\{c\} \cup U_u \cup U_v$. Since $p_1(z) = 2$, $p_1(c) = 1$ and $p_1 = 3$ on U_u, its first-coordinate equation is

$$|N_G(z) \cap U_u| = \mathbf{1}_{z\sim c}. \tag{5.6}$$

There is precisely one vertex v_* adjacent to c, because of (5.5), and it has one neighbour in U_u. For $Z = U_v \setminus \{v_*\}$, equation (5.6) gives $N_G(z) \subseteq U_v$ for every $z \in Z$; hence $\partial_G Z \subseteq \{v_*\}$. Since $o \notin Z \cup \{v_*\}$, Lemma 4.3 rules out $Z \neq \emptyset$. Interchanging the second and third coordinates gives $|N_G(z) \cap U_u| = \mathbf{1}_{z\sim b}$ for $z \in U_w$ and $\partial_G(U_w \setminus \{w_*\}) \subseteq \{w_*\}$. The same separation argument gives $U_w = \{w_*\}$, and both v_*, w_* have one neighbour in U_u.

For $z \in U_u$, all possible neighbours lie in $\{a, v_*, w_*\} \cup U_u$. The terms $p_1(z) - p_1(y)$ in its first-coordinate Laplacian equation are respectively $-2, 1, 1, 0$. Hence

$$2\mathbf{1}_{z\sim a} = |N_G(z) \cap \{v_*, w_*\}|. \tag{5.7}$$

Equation (5.7) gives $u_* v_*, u_* w_* \in E$. For $Z = U_u \setminus \{u_*\}$ it gives $N_G(z) \subseteq U_u$ for each $z \in Z$, because a is adjacent to no other vertex of U_u. Thus $\partial_G Z \subseteq \{u_*\}$, and Lemma 4.3 rules out $Z \neq \emptyset$. Consequently

$$V = \{o, a, b, c, u_*, v_*, w_*\}, \qquad E = \{o\,a, o\,b, o\,c, b\,c, b\,w_*, c v_*, a u_*, u_* v_*, u_* w_*\}.$$

The vertices o, b, c form a triangle and are joined to u_* by the paths $u_* a o$, $u_* w_* b$, $u_* v_* c$, each of length two. This is S_7. □

5.2. **Jacobians of the eleven graphs.**

Lemma 5.2. *For $n \geq 3$,*

$$\mathrm{Jac}(C_n) \simeq \mathbb{Z}/n\mathbb{Z}, \qquad \mathrm{Jac}(K_n) \simeq (\mathbb{Z}/n\mathbb{Z})^{n-2}.$$

Moreover,

$$\mathrm{Jac}(K_{2,3}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}, \qquad \mathrm{Jac}(S_7) \simeq (\mathbb{Z}/7\mathbb{Z})^2.$$

All these graphs are biconnected.

Proof. The cycle formula is [17, p. 280, Remark], and the complete-graph formula is [17, p. 278, Lemma]. For $K_{2,3}$, take $p = 2$ and $q = 3$ in [17, p. 280, Example].

For S_7, the reduced Laplacian at o, with the x_i before the y_i, is

$$C = \begin{pmatrix} 2I_3 & -I_3 \\ -I_3 & 4I_3 - J_3 \end{pmatrix}.$$

By Lemma 2.1, its Jacobian has generators ξ_i, η_i corresponding to x_i, y_i and relations

$$2\xi_i - \eta_i = 0, \qquad -\xi_i + 4\eta_i - \sum_{j=1}^{3} \eta_j = 0 \qquad (1 \le i \le 3).$$

Eliminating $\eta_i = 2\xi_i$ gives

$$7\xi_i - 2s = 0 \qquad (1 \le i \le 3), \qquad s = \xi_1 + \xi_2 + \xi_3.$$

Since $\sum_i (7\xi_i - 2s) = s$, these relations are equivalent to $s = 0$ and $7\xi_i = 0$ for $1 \le i \le 3$. Eliminating $\xi_3 = -\xi_1 - \xi_2$ leaves

$$\mathrm{Jac}(S_7) \simeq \mathbb{Z}^2/7\mathbb{Z}^2 \simeq (\mathbb{Z}/7\mathbb{Z})^2.$$

Deleting a vertex from a cycle leaves a path, and deleting a vertex from a complete graph leaves a complete graph. Deleting a vertex from $K_{2,3}$ leaves either $K_{1,3}$ or $K_{2,2}$, both connected. For S_7, deletion of o leaves a triangle with one pendent edge at each vertex. Deletion of x_i leaves the triangle connected to o through the other two paths. After deletion of y_i, the vertex x_i has the sole neighbour o, while the two remaining triangle vertices are adjacent and joined to o through their two paths. Thus $S_7 - v$ is connected for every $v \in V(S_7)$. □

Proof of Theorem 1.1. First suppose $e(G) \le 7$, and put $q = \Delta(G) \ge 2$. If $q = 2$, then G is connected and every vertex has degree two. Thus $G \simeq C_n$ for some $n \ge 3$ [17, p. 280, Remark], and $n = e(G) \le 7$ by Lemma 5.2. If $q \ge 4$, then $1 \le e - q \le 3$. The noncomplete cases in Theorem 1.3 have maximum degree at most three, so only K_{q+1} remains. If $q = 3$ and $e \le 6$, the same theorem leaves K_4 and $K_{2,3}$, the listed cases of maximum degree three. The last case, $q = 3, e = 7$, is Proposition 5.1. The inequality $e \ge q + 1 \ge 3$ excludes exponents one and two. Hence G belongs to (1.3). By Lemma 5.2,

e	G	$\mathrm{Jac}(G)$
3	C_3	$\mathbb{Z}/3\mathbb{Z}$
4	$C_4,\ K_4$	$\mathbb{Z}/4\mathbb{Z},\ (\mathbb{Z}/4\mathbb{Z})^2$
5	$C_5,\ K_5$	$\mathbb{Z}/5\mathbb{Z},\ (\mathbb{Z}/5\mathbb{Z})^3$
6	$C_6,\ K_{2,3},\ K_6$	$\mathbb{Z}/6\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z},\ (\mathbb{Z}/6\mathbb{Z})^4$
7	$C_7,\ S_7,\ K_7$	$\mathbb{Z}/7\mathbb{Z},\ (\mathbb{Z}/7\mathbb{Z})^2,\ (\mathbb{Z}/7\mathbb{Z})^5$.

(5.8)

All listed graphs are simple and biconnected. Their degree sequences, with multiplicities indicated by exponents, are

$$C_n : (2^n), \qquad K_n : ((n-1)^n), \qquad K_{2,3} : (3^2, 2^3), \qquad S_7 : (3^4, 2^3).$$

These sequences and the vertex counts distinguish the eleven graphs.

For $b \le 7$ the class in the theorem is a subset of this finite list. For $b \ge 8$, take $G_t = G_{2,t}$ with $t \ge 3$. By Lemmas 3.2 and 3.3, these graphs are planar, bipartite and biconnected, have maximum degree four, and have exponent eight. Their vertex counts $2t$ are distinct. Thus each of the three sets in the theorem is infinite for $b \ge 8$. For every $1 \le a \le 7$, the graphs of exact exponent a form a finite subset of (1.3), whereas $e(G_t) = 8$ for every $t \ge 3$. □

6. Quantitative finiteness on surfaces

6.1. Integral embeddings in the plane.

Tutte's embedding theorem ([26, (2.8), (7.2), and (9.2)]). *Let G be a simple 3-connected plane graph with a triangular outer face. Assign its boundary vertices to three noncollinear points of $\mathbb{R}^2$. The map obtained by extending each coordinate harmonically to the other vertices is a straight-line embedding with that outer triangle. Distinct vertices have distinct images, all other vertices lie strictly inside the triangle, and each edge is the segment joining its endpoint images.*

Lemma 6.1. *In a simple 3-connected plane graph, each face has an induced cycle as boundary and a closed disk as its closure on the sphere. If a, o, c are consecutive vertices on a nontriangular face boundary, one may add the edge ac within that face. The resulting graph remains simple and 3-connected and has a triangular face aoc.*

Proof. By [8, Proposition 4.2.7], each face boundary is an induced cycle. The closure of each face on the sphere is a closed disk by [8, Theorem 4.1.4]. Let F be a nontriangular face with consecutive boundary vertices a, o, c. Since a, c are nonconsecutive on this induced cycle, $ac \notin E(G)$. An arc from a to c with interior in F cuts off a disk bounded by aoc. Adding this arc gives a simple plane graph $G + ac$ with the same vertices and with face aoc. For every $S \subseteq V(G)$ with $|S| \leq 2$, the graph $(G + ac) - S$ contains the connected spanning subgraph $G - S$. Thus $G + ac$ is 3-connected. □

Lemma 6.2. *Let G be 3-connected and plane, and let a, o, c be consecutive vertices on one facial cycle. Let f, g be grounded potentials on G with*

$$Lf = \mathbf{e}_a - \mathbf{e}_o, \qquad Lg = \mathbf{e}_c - \mathbf{e}_o, \qquad f(o) = g(o) = 0.$$

Put $e = e(G)$ and $n = n(G)$, and set

$$R = \begin{pmatrix} f(a) & g(a) \\ f(c) & g(c) \end{pmatrix}. \tag{6.1}$$

Then R is symmetric positive definite, eR has positive integer entries, and

$$2n(G) - 5 \leq \det(eR). \tag{6.2}$$

Proof. Let C be the reduced Laplacian grounded at o. The vectors $f|_{V\setminus\{o\}}$ and $g|_{V\setminus\{o\}}$ are the ath and cth columns of C^{-1}. Thus $R = (C^{-1})_{\{a,c\},\{a,c\}}$ is symmetric positive definite. Since G is biconnected and $a, c \in N(o)$, the functions ef, eg satisfy (4.1); by Lemma 4.1 their values outside o are positive integers.

If the face is triangular, let $G' = G$. Otherwise let $G' = G + ac$, with ac inside the face as in Lemma 6.1. In either case G' is simple and 3-connected, with face aoc. Choose this face as the outer face, and set

$$p(v) = e(f(v), g(v)) \in \mathbb{Z}^2. \tag{6.3}$$

The boundary images $p(o), p(a), p(c)$ are the origin and the two rows of eR, and hence are noncollinear. At every vertex outside $\{o, a, c\}$, both coordinates of p are harmonic on G. The only possible added edge joins a to c, so the same equations hold on G'. By Lemma 2.6, p is the unique harmonic extension of these boundary positions. Tutte's theorem gives a straight-line embedding of G' in the triangle $T = [p(o), p(a), p(c)]$. The $n(G) - 3$ other vertex images are distinct integer points strictly inside T. Put

$$I_T = |\operatorname{int}(T) \cap \mathbb{Z}^2|, \qquad B_T = |\partial T \cap \mathbb{Z}^2|.$$

Then $I_T \geq n(G) - 3$ and $B_T \geq 3$. Pick's theorem [4, Theorem 2.8] gives

$$\det(eR) = 2 \operatorname{area} T = 2I_T + B_T - 2 \geq 2n(G) - 5.$$

□

Proposition 6.3. *If a simple* 3*-connected plane graph has a face of length* k*, then*

$$2n(G) - 5 \le \frac{k-2}{k} e(G)^2. \tag{6.4}$$

Proof. Write the facial cycle as $C_k = (o, a, \dots, c, o)$, with $k \ge 3$. Let f, g and R be as in Lemma 6.2, and let R_k be the analogous matrix for the unit-resistance cycle C_k. The series and parallel laws [19, Section 2.3] give

$$\mathcal{R}_{C_k}(o,a) = \mathcal{R}_{C_k}(o,c) = \frac{k-1}{k}, \qquad \mathcal{R}_{C_k}(a,c) = \frac{2(k-2)}{k}.$$

Hence Lemma 2.4 gives

$$R_k = \begin{pmatrix} (k-1)/k & 1/k \\ 1/k & (k-1)/k \end{pmatrix}.$$

For $t = (t_1, t_2)^{\mathsf{T}} \in \mathbb{R}^2$, put $D = t_1(\mathbf{e}_a - \mathbf{e}_o) + t_2(\mathbf{e}_c - \mathbf{e}_o)$. The current on G with divergence D has potential $t_1 f + t_2 g$ and energy $D^{\mathsf{T}}(t_1 f + t_2 g) = t^{\mathsf{T}} R t$. The corresponding current on C_k has energy $t^{\mathsf{T}} R_k t$. Extend the latter current by zero on $E(G) \setminus E(C_k)$. Its divergence on G is still D, and its energy is unchanged. Thomson's principle [19, Section 2.4, p. 35] therefore gives

$$t^{\mathsf{T}} R t \le t^{\mathsf{T}} R_k t \qquad (t \in \mathbb{R}^2).$$

Thus $R_k - R$ is positive semidefinite. Both R and R_k are positive definite, so Weyl's monotonicity principle [25, Corollary 9.9] gives

$$\det R \le \det R_k = \frac{k-2}{k}.$$

Together with (6.2), this yields

$$2n(G) - 5 \le e(G)^2 \det R \le \frac{k-2}{k} e(G)^2.$$

□

Proof of Theorem 1.4. Fix a plane embedding of G. Choose o of degree Δ and consecutive neighbours a, c in its cyclic order. The sector between oa and oc belongs to a face, whose boundary cycle contains the path aoc by Lemma 6.1. Write the matrix in Lemma 6.2 as

$$eR = \begin{pmatrix} u & v \\ v & w \end{pmatrix}, \qquad S = e - \Delta + 2.$$

All entries are positive integers. For $d \in N(o)$, let $Lp_d = e(\mathbf{e}_d - \mathbf{e}_o)$ and $p_d(o) = 0$. Then $u = p_a(a)$, $v = p_c(a) = p_a(c)$, and $w = p_c(c)$. The sum identity in Lemma 4.1 gives

$$e = u + v + \sum_{d \in N(o) \setminus \{a,c\}} p_d(a), \qquad e = v + w + \sum_{d \in N(o) \setminus \{a,c\}} p_d(c).$$

The remaining $\Delta - 2$ terms in each sum are at least one; hence

$$u + v \le S, \qquad v + w \le S. \tag{6.5}$$

By Lemma 4.1, $e \ge \Delta + 1$, so $S \ge 3$. Since $v \ge 1$ and $u, w \le S - v$,

$$\begin{aligned} \det(eR) &= uw - v^2 \le (S-v)^2 - v^2 \\ &= S^2 - 2Sv \le S(S-2) = (e - \Delta + 1)^2 - 1. \end{aligned} \tag{6.6}$$

Together with (6.2), this gives $2n - 5 \le (e - \Delta + 1)^2 - 1$, and hence $2n - 4 \le (e - \Delta + 1)^2$. In particular $e \ge \Delta - 1 + \lceil \sqrt{2n-4} \rceil$, since $e - \Delta + 1 \ge 2$. For K_4, one has $n = 4$, $\Delta = 3$ and $e = 4$ by Lemma 5.2, so equality holds.

Write $m = m(G)$ and let F be the number of faces. Since the minimum degree is at least three, $2m \geq 3n$. Euler's formula [8, Theorem 4.2.9] gives

$$6F - 2m = 12 - 6n + 4m \geq 12.$$

The sum of the face lengths is $2m$, so some face has length $3 \leq k \leq 5$. Proposition 6.3 gives

$$2n - 5 \leq \frac{k-2}{k} e^2 \leq \frac{3}{5} e^2.$$

Since n is an integer, the two inequalities give (1.12). Since $\Delta \geq 3$ and $e \geq \Delta + 1$, one has $0 < e - \Delta + 1 \leq e - 2$ and hence

$$n \leq \min \left\{ \left\lfloor \frac{e^2 - 4e + 8}{2} \right\rfloor, \left\lfloor \frac{3e^2}{10} + \frac{5}{2} \right\rfloor \right\}. \tag{6.7}$$

For $e \leq b$, put $N_b = \lfloor 3b^2/10 + 5/2 \rfloor$. Then $n \leq N_b$, and the number of isomorphism classes is at most

$$\sum_{j=1}^{N_b} 2^{\binom{j}{2}} < \infty.$$

□

6.2. **Essentially 3-connected torus embeddings.** An embedding is *essentially simple* if its universal covering graph has no loops or parallel edges. A geodesic in a flat torus is the projection of a line segment in its Euclidean universal cover.

Toroidal embedding theorem ([11, Theorem 2.5]; for simplicial triangulations, [7, Theorems 1–2]). *Let Γ be a cellular, essentially simple, essentially 3-connected embedding of a finite graph in a flat torus. Assign a positive real number to each edge. Up to translation there is a unique geodesic drawing homotopic to Γ for which, at every vertex, the sum of the outgoing edge displacement vectors multiplied by their assigned numbers is zero. This drawing is an embedding.*

Proof of Theorem 1.5. Identify the torus with $\mathbb{R}^2/\mathbb{Z}^2$ by a homeomorphism. Cellularity and the universal covering graph are preserved. Since G is simple and its universal covering graph is connected, $n(G) \geq 2$; moreover, the covering graph is simple. Apply the toroidal embedding theorem with all edge weights equal to one. Translate the resulting geodesic embedding so that a chosen vertex q lies at zero, and choose lifts $x_v \in \mathbb{R}^2$ with $x_q = 0$.

For an oriented edge $d : v \to w$, the lift starting at x_v ends at $x_w + \tau_d$ for a unique $\tau_d \in \mathbb{Z}^2$. Reversing this lift and translating it by $-\tau_d$ gives $\tau_{\bar{d}} = -\tau_d$. Put

$$D_v = \sum_{d\colon\, v \to w} \tau_d \in \mathbb{Z}^2. \tag{6.8}$$

The two orientations of each edge give opposite contributions, so $\sum_v D_v = 0$. Equilibrium gives

$$0 = \sum_{d\colon\, v \to w} (x_w + \tau_d - x_v) = -(Lx)_v + D_v. \tag{6.9}$$

Thus $Lx = D$. Each coordinate of D is an integral degree-zero divisor. Since $x_q = 0$, Lemma 2.2 gives, with $e = e(G)$,

$$e x_v \in \mathbb{Z}^2 \qquad (v \in V). \tag{6.10}$$

The embedding therefore induces an injection

$$V(G) \longrightarrow e^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \simeq (\mathbb{Z}/e\mathbb{Z})^2.$$

Hence $n(G) \leq e^2$. If $e(G) \leq b$, then $n(G) \leq b^2$. The finite sum in the proof of Theorem 1.4, with $N_b = \lfloor b^2 \rfloor$, bounds the number of isomorphism classes. □

7. Consequences

Corollary 7.1. *Every simple biconnected graph satisfies*

$$e(G) \geq \min\{n(G), 8\}.$$

The simple 3-connected graphs of exponent at most seven are exactly K_4, K_5, K_6, K_7.

Proof. Put $n = n(G)$ and $e = e(G)$. If $e \leq 7$, Theorem 1.1 and Lemma 5.2 give $n = e$ for the cycles, complete graphs and S_7, and $(n, e) = (5, 6)$ for $K_{2,3}$. Thus $e \geq n$ in this case. If $e \geq 8$, then $e \geq \min\{n, 8\}$ directly.

Among the classified graphs, every cycle, $K_{2,3}$ and S_7 has a vertex of degree two. A 3-connected graph has minimum degree at least three, so none of these is 3-connected. The graphs K_4, K_5, K_6, K_7 remain complete on at least two vertices after deletion of at most two vertices, and their exponents are $4, 5, 6, 7$, respectively. □

Corollary 7.2. *Let* $3 \leq a \leq 7$. *If* G *is simple and biconnected and* $\mathrm{Jac}(G) \simeq (\mathbb{Z}/a\mathbb{Z})^r$ *with* $r \geq 1$, *then* $r \leq a - 2$. *Equality is attained by* K_a. *At* $a = 8$ *there is no bound on* r, *even among simple planar biconnected graphs of maximum degree at most four.*

Proof. The group $(\mathbb{Z}/a\mathbb{Z})^r$ has exponent a and r nontrivial invariant factors, so its minimum number of generators is r. Since $a \leq 7$, Corollary 7.1 gives $n(G) \leq a$. The graph G is connected and simple, with minimum degree at least two, so [17, p. 278, Lemma] gives

$$r \leq n(G) - 2 \leq a - 2.$$

By Lemma 5.2, $\mathrm{Jac}(K_a) \simeq (\mathbb{Z}/a\mathbb{Z})^{a-2}$. For every $r \geq 1$, Theorem 1.2, with $t = r + 1$, gives $\mathrm{Jac}(H_{r+1}) \simeq (\mathbb{Z}/8\mathbb{Z})^r$; these graphs are simple, planar and biconnected and have maximum degree at most four. □

Corollary 7.3. *If* G *is simple, planar and* 3*-connected and* $\mathrm{Jac}(G) \simeq (\mathbb{Z}/a\mathbb{Z})^r$, *with* $a \geq 2$ *and* $r \geq 1$, *then*

$$r \leq n(G) - 2 \leq \min\left\{ \left\lfloor \frac{a^2 - 4a + 4}{2} \right\rfloor, \left\lfloor \frac{3a^2}{10} + \frac{1}{2} \right\rfloor \right\}. \tag{7.1}$$

Proof. Put $n = n(G)$. The r invariant factors a are nontrivial, so the minimum number of generators of $\mathrm{Jac}(G)$ is r. Since G is connected and simple, with minimum degree at least three, [17, p. 278, Lemma] gives $r \leq n - 2$. Since $e(G) = a$, subtracting two in (6.7) gives the second inequality, with $\lfloor x \rfloor - 2 = \lfloor x - 2 \rfloor$. □

Proposition 7.4. *For each integer* $m \geq 2$, *infinitely many pairwise nonisomorphic simple bipartite graphs have vertex connectivity* m *and Jacobian exponent* $2m^2$. *In particular, abstract* 3*-connectivity does not imply bounded-exponent finiteness.*

Proof. For fixed $m \geq 2$ and $t \geq 3$, the graph $G_{m,t}$ is simple by its definition: vertices in consecutive independent parts are joined by exactly one edge. By Lemma 3.2, it is bipartite, has vertex connectivity m and exponent $2m^2$. Since $n(G_{m,t}) = mt$, distinct values of t give nonisomorphic graphs.

For $m = 2$, these are the planar graphs G_t of Lemma 3.3. For $m = 3$, these are 3-connected graphs of exponent eighteen. Two consecutive parts induce $K_{3,3}$, so every $G_{3,t}$ is nonplanar by [8, Corollary 4.2.11]. □

References

[1] A. Azimi, R. B. Bapat and M. Farrokhi D. G., *Resistance distance of blowups of trees*, Discrete Math. **344** (2021), no. 7, Article 112387. 10.1016/j.disc.2021.112387.

[2] R. Bacher, P. de la Harpe and T. Nagnibeda, *The lattice of integral flows and the lattice of integral cuts on a finite graph*, Bull. Soc. Math. France **125** (1997), no. 2, 167–198. 10.24033/bsmf.2303.

[3] M. Baker and F. Shokrieh, *Chip-firing games, potential theory on graphs, and spanning trees*, J. Combin. Theory Ser. A **120** (2013), no. 1, 164–182. 10.1016/j.jcta.2012.07.011.

[4] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, second ed., Undergraduate Texts in Mathematics, Springer, New York, 2015. 10.1007/978-1-4939-2969-6.

[5] N. Biggs, *Algebraic potential theory on graphs*, Bull. London Math. Soc. **29** (1997), no. 6, 641–682. 10.1112/S0024609397003305.
[6] D. Brandfonbrener, P. Devlin, N. Friedenberg, Y. Ke, S. Marcus, H. Reichard and E. Sciamma, *Two-vertex generators of Jacobians of graphs*, Electron. J. Combin. **25** (2018), no. 1, P1.15, 18 pp. 10.37236/7302.
[7] Y. Colin de Verdière, *Comment rendre géodésique une triangulation d'une surface?*, Enseign. Math. (2) **37** (1991), nos. 3–4, 201–212.
[8] R. Diestel, *Graph Theory*, fifth ed., Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2017. 10.1007/978-3-662-53622-3.
[9] F. Dörfler and F. Bullo, *Kron reduction of graphs with applications to electrical networks*, IEEE Trans. Circuits Syst. I **60** (2013), no. 1, 150–163. 10.1109/TCSI.2012.2215780.
[10] X. Dong, W. Guo and G. Jiang, *On the critical group of the k-partite graph*, preprint (2024), arXiv:2409.02654v1.
[11] J. Erickson and P. Lin, *A toroidal Maxwell–Cremona–Delaunay correspondence*, J. Comput. Geom. **12** (2021), no. 2, 55–85. 10.20382/jocg.v12i2a4.
[12] L. Gaudet, D. Jensen, D. Ranganathan, N. Wawrykow and T. Weisman, *Realization of groups with pairing as Jacobians of finite graphs*, Ann. Comb. **22** (2018), no. 4, 781–801. 10.1007/s00026-018-0406-0.
[13] Q. Huang, H. Chen and Q. Deng, *Resistance distances and the Kirchhoff index in double graphs*, J. Appl. Math. Comput. **50** (2016), 1–14. 10.1007/s12190-014-0855-5.
[14] B. Jacobson, A. Niedermaier and V. Reiner, *Critical groups for complete multipartite graphs and Cartesian products of complete graphs*, J. Graph Theory **44** (2003), no. 3, 231–250. 10.1002/jgt.10139.
[15] H. Liu, L. Li, L. You, H. Hua and L. Chen, *Enumeration of spanning trees and resistance distances of generalized blow-up graphs*, Discrete Appl. Math. **381** (2026), 276–291. 10.1016/j.dam.2025.11.014. arXiv:2504.15314v1 (2025).
[16] H. Lheem, D. Li, C. J. Quines and J. Zhang, *Exponents of Jacobians of graphs and regular matroids*, Rose-Hulman Undergrad. Math. J. **21** (2020), no. 2, Article 6. arXiv:1910.06442v2 (2021).
[17] D. Lorenzini, *A finite group attached to the Laplacian of a graph*, Discrete Math. **91** (1991), no. 3, 277–282. 10.1016/0012-365X(90)90236-B.
[18] D. Lorenzini, *Smith normal form and Laplacians*, J. Combin. Theory Ser. B **98** (2008), no. 6, 1271–1300. 10.1016/j.jctb.2008.02.002.
[19] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 42, Cambridge University Press, New York, 2016. 10.1017/9781316672815.
[20] M. C. Marino and N. Zagaglia Salvi, *Generalizing double graphs*, Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur. **85** (2007), Article C1A0702002. 10.1478/C1A0702002.
[21] E. Munarini, C. Perelli Cippo, A. Scagliola and N. Zagaglia Salvi, *Double graphs*, Discrete Math. **308** (2008), nos. 2–3, 242–254. 10.1016/j.disc.2006.11.038.
[22] F. Shokrieh, *The monodromy pairing and discrete logarithm on the Jacobian of finite graphs*, J. Math. Cryptol. **4** (2010), no. 1, 43–56. 10.1515/JMC.2010.002.
[23] W. Sun, Y. Yang and S.-J. Xu, *Resistance distance and Kirchhoff index of unbalanced blowups of graphs*, Discrete Math. **348** (2025), no. 3, Article 114327. 10.1016/j.disc.2024.114327.
[24] Y. Tian and Y. Takane, *Schur complements and Banachiewicz–Schur forms*, Electron. J. Linear Algebra **13** (2005), 405–418. 10.13001/1081-3810.1171.
[25] J. A. Tropp, *ACM 204: Matrix Analysis*, Caltech CMS Lecture Notes 2022-01, California Institute of Technology, Pasadena, 2022. 10.7907/m421-yb89.
[26] W. T. Tutte, *How to draw a graph*, Proc. London Math. Soc. (3) **13** (1963), 743–767. 10.1112/plms/s3-13.1.743.
[27] Y. Wang, D. Li and W. Yan, *Resistance distance in blowup graphs*, Discrete Math. **349** (2026), no. 2, Article 114765. 10.1016/j.disc.2025.114765.
[28] S. Xu and K. Xu, *Resistance distances in generalized join graphs*, Discrete Appl. Math. **362** (2025), 18–33. 10.1016/j.dam.2024.11.013.

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