

INTEGRAL INVARIANTS OF REDUCED PLUMBING GRAPHS

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ABSTRACT. An algebraic plumbing graph determines an integral trilinear form, a parity class, and a Pontryagin linear form. We prove rank-two rigidity and show that uniqueness fails first in rank three. For each fixed rank $r \geq 5$, the numbers of connected reduced spin presentations are unbounded as the invariant system varies, even with nonzero labels and primitive normalized Pontryagin form. Coprime factorizations of an odd integer $N > 1$ give $2^{\omega(N)-1}$ distinct reduced classes in one fibre. For each N , these graphs present a single oriented diffeomorphism class of six-manifolds admitting positive Ricci curvature. At fixed rank, varying N gives pairwise nonisomorphic graded integral cohomology rings and a common rational homotopy type. They are indecomposable as connected sums and are not sphere-bundle total spaces with positive-dimensional base and fibre. In arbitrary rank, the restricted ambient Euclidean form, together with the trilinear form and parity class, recovers the reduced presentation. The Pontryagin form is then redundant. The proof recovers coordinate covectors from the Voronoi cell and the forest from their closed circuits. For reduced spin graphs with nonzero integer labels, the fifth-order coordinate moment alone also recovers the reduced class.

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1. INTRODUCTION

Let G be an algebraic plumbing graph in the sense of Reiser [7, Section 4.2]. For each integer $k \geq 1$, its invariant system is

$$\mathcal{I}_k(G) = (H_G, \mu_G^k, w_G, p_G^k),$$

where H_G is a finitely generated free abelian group, μ_G^k is an integral symmetric trilinear form, $w_G \in H_G/2H_G$, and $p_G^k \in H_G^*$. For a reduced graph, these are the cup-product and characteristic invariants of a $6k$ -dimensional plumbing realization [7, Definition 4.4 and Lemma 4.6]. For $k > 2$, the construction involves the choice described in [7, Remark 4.5]; every such choice realizes the displayed invariant system up to isomorphism. The rank of G is $\text{rank } H_G$. All isomorphisms of invariant systems are integral.

A reduced graph is a decorated bipartite forest satisfying Definition 2.3. Its reduced class allows graph isomorphisms and, independently on each connected component, the reversal

$$(\alpha, k^+, k^-) \mapsto (-\alpha, k^-, k^+).$$

Reiser's Question 4.10 asks whether

$$\mathcal{I}_k(G) \cong \mathcal{I}_k(G') \implies [G]_{\text{red}} = [G']_{\text{red}} \quad (1.1)$$

for reduced graphs. His Proposition 4.11 gives an affirmative answer in rank one, and Proposition 4.12 treats nonspin graphs of rank two. Remark 4.14 conjectures the remaining spin rank-two case [7].

Theorem A (Rank-two rigidity). *Let G, G' be reduced algebraic plumbing graphs with $\text{rank } H_G = \text{rank } H_{G'} = 2$. If $\mathcal{I}_k(G) \cong \mathcal{I}_k(G')$ for some integer $k \geq 1$, then*

$$[G]_{\text{red}} = [G']_{\text{red}}.$$

For a connected reduced spin graph of rank two, the graph has one unlabelled vertex and three labelled leaves. Its cubic and normalized Pontryagin form, in an integral basis, are

$$\begin{aligned} f_\alpha(x, y) &= \alpha_1 x^3 + \alpha_2 y^3 - \alpha_3 (x + y)^3, \\ \ell_\alpha(x, y) &= (\alpha_1 - \alpha_3)x + (\alpha_2 - \alpha_3)y, \end{aligned}$$

with $\alpha_1 \alpha_2 \alpha_3 \neq 0$. Theorem 3.1 proves that every integral equivalence of such pairs permutes $x, y, -x - y$ up to a common sign. The coefficients may belong to any field of characteristic zero. Neither nonsingularity of the cubic nor nonvanishing of the linear form is required.

Theorem B (Nonuniqueness in rank at least three). *For every integer $r \geq 3$, there are sequences of connected reduced spin graphs $(G_N)_{N \geq 1}$ and $(G'_N)_{N \geq 1}$, all of rank r and with all integer labels nonzero, such that*

$$[G_N]_{\text{red}} \neq [G'_N]_{\text{red}}, \quad \mathcal{I}_k(G_N) \cong \mathcal{I}_k(G'_N) \quad (N \geq 1, k \geq 1).$$

The normalized Pontryagin forms $p_{G_N}^k/\lambda_k$ and $p_{G'_N}^k/\lambda_k$, with λ_k as in (2.3), are primitive. For fixed r , suitable integral bases represent the exhibited isomorphisms by a single matrix $T_r \in \text{SL}(r, \mathbb{Z})$ with $T_r^2 = I$. For each fixed $k \geq 1$, the systems $\mathcal{I}_k(G_N)$ are pairwise nonisomorphic as N varies.

The rank-three example has central label 1 and leaf pairs

$$(2, 1), (-6, -2) \quad \text{and} \quad (-2, -1), (6, 2).$$

In integral kernel coordinates, the equivalence is

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The two branch cubics change sign modulo multiples of the cube of the central coordinate; those multiples cancel. The central vertex and its label rule out a common reversal, and the leaf-label

multisets rule out a graph isomorphism. Thus rank three is the least rank in which (1.1) fails (Corollary 5.11). The same substitution applies locally in larger graphs (Proposition 5.12); connected nonspin counterexamples follow in every rank at least four (Corollary 5.13).

For a paired-leaf tree, the cubic is the product of a linear form and a quadratic form. Theorem 6.2 characterizes equivalence by a congruence $D_b = \varepsilon A^\top D_a A$, the condition $A1 \equiv 1 \pmod{2}$ ensuring an integral substitution, and common-sign conditions on the central label and the diagonal sum. This modulo-two condition is distinct from the parity class w_G , which is zero for these spin graphs. If D_a is definite, the congruence matrix is a signed permutation (Proposition 6.3). In the indefinite case, Proposition 6.5 gives primitive counterexample families in every rank at least four. A second family, with varying content of the Pontryagin form, is given in Proposition 6.6.

Let A_G be the ambient free abelian group in the plumbing construction. Declare its distinguished basis orthonormal, and denote by q_G the restriction of this Euclidean form to H_G . The form q_G is attached to the plumbing presentation. No assertion is made that it is canonically determined by the underlying manifold without a chosen presentation.

Theorem C (Quadratic reconstruction). *Let G, G' be reduced algebraic plumbing graphs, and fix an integer $k \geq 1$. Then*

$$[G]_{\text{red}} = [G']_{\text{red}} \iff (H_G, q_G, \mu_G^k, w_G) \cong (H_{G'}, q_{G'}, \mu_{G'}^k, w_{G'}).$$

Every isomorphism on the right also preserves the Pontryagin linear form.

The preservation in Theorem C is simultaneous. The rank-three examples have isometric quadratic lattices and isomorphic original invariant systems, but no isomorphism realizes both identifications (Section 10).

Two lattice-theoretic results enter the reconstruction. Su and Wagner [8, Theorem 1] show that the integral flow lattice of a regular matroid determines that matroid after contraction of its coloops. The projection formula of McCormick, Peis, Scheidweiler, and Vallentin [5, Theorem 2.3] identifies the Voronoi cell of a totally unimodular kernel with the projection of a cube. We use the latter formula to recover, under each prescribed quadratic isometry, the nonzero ambient coordinate covectors with their exact scales up to sign.

The remaining steps concern the forest and its decorations. The bipartite incidence matrix $B_G = (1_{u \sim v})_{v \in V, u \in U}$ is totally unimodular. On ordinary components, the coordinate covectors represent the dual of its column matroid, and

$$\{N(v) : v \in V\} = \{\text{circuits of this dual matroid which are flats}\}.$$

These closed circuits recover the bipartite forest (Theorem 7.2 and Proposition 8.6). The parity class separates main from additional coordinates. Contractions of the trilinear form recover the attachments and signs of the additional coordinates; independence of the main coordinate cubes recovers the integer labels. The relations at unlabelled vertices make the remaining sign constant on each connected component. Formula (9.5) then recovers the Pontryagin form. Section 8.1 gives the precise relation with the Su–Wagner theorem.

The fibres of the original invariant system need not even have a bound in terms of the rank. Write $\omega(N)$ for the number of distinct prime divisors of N .

Theorem D (Unbounded fibres in fixed rank). *Fix an integer $r \geq 5$. For every odd integer $N > 1$, there is a family of $2^{\omega(N)-1}$ connected reduced spin paired-leaf trees of rank r with pairwise distinct reduced classes and all integer labels nonzero. Their invariant systems are mutually isomorphic for every $k \geq 1$, under integral maps independent of k , and their normalized Pontryagin forms are primitive. For each fixed k , the common systems belonging to different N are nonisomorphic.*

Consequently, for every $M \geq 1$ and every fixed k , infinitely many invariant systems of rank r have at least M such reduced presentations.

The family is indexed by the unordered coprime factorizations $N = ab$. Its branch weights are $a, -a, b, -b, 1, \dots, 1$, and its central label is $r - 4$. The arithmetic step is an integral normal form for

$\text{diag}(a, -a, b, -b)$ which preserves the vector $(1, 1, 1, 1)$ (Lemma 6.7). The associated cubic determines an integral contraction with absolute determinant N^2 (Proposition 6.13). The displayed count is a lower bound for the full fibre; no assertion of an infinite fibre for one fixed system, or of minimality of rank five, is made. Connected nonspin fibres are likewise unbounded in each rank at least six (Corollary 6.11).

In dimension six, Jupp's classification [3], in the formulation of [7, Theorem 4.1], and Reiser's realization theorem [7, Theorem A(2)] give a geometric consequence. For every $r \geq 3$, there are infinitely many oriented diffeomorphism classes of closed simply-connected spin six-manifolds with torsion-free homology, $b_2 = r$, $b_3 = 0$, and primitive $p_1/4$, each admitting positive Ricci curvature and two distinct reduced plumbing presentations (Corollary 10.3). For $r \geq 4$, the two presentations may be chosen to be paired-leaf trees. Theorem 11.3 gives a stronger consequence of Theorem D. At every fixed $r \geq 5$, its manifolds have unbounded presentation multiplicity, pairwise different integral cohomology rings, and a common rational homotopy type. They are indecomposable as connected sums and are not sphere-bundle total spaces with positive-dimensional base and fibre.

Appendix B gives another reconstruction: for a reduced spin graph with nonzero integer labels, the polynomial $F_{d,G} = \sum_u \alpha(u) L_u^d$ determines the reduced class for every odd $d \geq 5$. Five is the least odd degree with this property. These are presentation-dependent tensors, not additional characteristic classes. Positive coordinate weights and an obstruction to replacing the forest hypothesis by total unimodularity are treated in Section 8.2 and Example 10.1.

2. ALGEBRAIC PLUMBING GRAPHS AND THEIR INVARIANTS

All graphs are finite and simple. Graph isomorphisms preserve the bipartition and all decorations. For a finitely generated free abelian group H , write $H^* = \text{Hom}(H, \mathbb{Z})$. All linear dependence relations between integral covectors are taken over $\mathbb{Q}$. Matrices act on column vectors. An isomorphism $T : H' \rightarrow H$ acts on covectors by $T^* \lambda = \lambda \circ T$. A polynomial on $H \otimes \mathbb{Q}$ is determined by its values on H : in an integral basis this follows, one variable at a time, from the fact that a nonzero polynomial over $\mathbb{Q}$ has finitely many roots.

Definition 2.1 (Algebraic plumbing graph). A bipartite graph (U, V, E) , with $E \subseteq U \times V$, together with a map

$$(\alpha, k^+, k^-) : U \longrightarrow \mathbb{Z} \times \mathbb{Z}_{\geq 0}^2,$$

is an *algebraic plumbing graph*, denoted by $G = (U, V, E, (\alpha, k^+, k^-))$. The vertices of U are called *labelled*; those of V are *unlabelled*. The integer $\alpha(u)$ is the label of u , and $(\alpha(u), k^+(u), k^-(u))$ is its decoration. The graph is *spin* if $k^+ = k^- = 0$.

For each $u \in U$, introduce basis vectors u^j , where $-k^-(u) \leq j \leq k^+(u)$, and put

$$A_G = \bigoplus_{u \in U} \bigoplus_{j=-k^-(u)}^{k^+(u)} \mathbb{Z} u^j.$$

Write x_u for the coordinate of u^0 , and $y_{u,j}$ for that of u^j when $j \neq 0$. The *main lattice* and the full lattice are

$$H_{0,G} = \left\{ x \in \mathbb{Z}^U : \sum_{u \sim v} x_u = 0 \text{ for all } v \in V \right\}, \quad (2.1)$$

$$H_G = H_{0,G} \oplus \bigoplus_{u \in U, j \neq 0} \mathbb{Z} u^j \subseteq A_G. \quad (2.2)$$

For each integer $k \geq 1$, Reiser's normalization is

$$\lambda_k = v_k \frac{3 - (-1)^k}{2} (2k - 1)!, \quad \rho_k = \frac{\lambda_k}{4}, \quad c_k = \binom{2k + 1}{k}, \quad (2.3)$$

where

$$v_k = \begin{cases} 2, & k = 1, 4, \\ 8, & k = 2, \\ 1, & k = 3 \text{ or } k > 4. \end{cases}$$

For $k = 1, 2$, one has $\lambda_k = 4, 48$, respectively. For $k \geq 3$, 4 divides $(2k - 1)!$. Thus ρ_k is a positive integer and

$$0 < c_k < \lambda_k. \quad (2.4)$$

For $k = 1, 2$ this follows from $(c_1, \lambda_1) = (3, 4)$ and $(c_2, \lambda_2) = (10, 48)$. For $k \geq 3$,

$$\frac{c_k}{(2k - 1)!} = \frac{2k(2k + 1)}{k!(k + 1)!} < \frac{4}{(k - 1)!k!} \leq \frac{1}{3} < 1.$$

Define a symmetric trilinear form on A_G by

$$\mu^k(u^0, u^0, u^0) = \rho_k \alpha(u), \quad \mu^k(u^0, u^j, u^j) = \text{sgn}(j) \quad (j \neq 0),$$

and by setting all other basis values, apart from permutations of these, equal to zero. Its restriction $\mu_G^k : H_G^3 \rightarrow \mathbb{Z}$ is integral. The associated cubic polynomial $f_G^k(z) = \mu_G^k(z, z, z)$ is

$$f_G^k(x, y) = \rho_k \sum_{u \in U} \alpha(u) x_u^3 + 3 \sum_{u \in U} x_u \sum_{j \neq 0} \text{sgn}(j) y_{u,j}^2. \quad (2.5)$$

The parity class and the Pontryagin linear form are

$$w_G = \sum_{u \in U, j \neq 0} u^j \pmod{2H_G}, \quad (2.6)$$

$$p_G^k(x, y) = \sum_{u \in U} (\lambda_k \alpha(u) + c_k(k^+(u) - k^-(u))) x_u. \quad (2.7)$$

These are Reiser's invariants [7, Section 4.2]. In particular, $w_G \in H_G/2H_G$ and $p_G^k \in H_G^*$. The additional coordinates form a direct summand, so

$$w_G = 0 \iff G \text{ is spin}. \quad (2.8)$$

Definition 2.2 (Invariant system and equivalence). The *invariant system* of G at k is

$$\mathcal{I}_k(G) = (H_G, \mu_G^k, w_G, p_G^k),$$

and the *rank* of G is $\text{rank } H_G$. An isomorphism $\mathcal{I}_k(G') \rightarrow \mathcal{I}_k(G)$ is an integral isomorphism $T : H_{G'} \rightarrow H_G$ satisfying

$$f_G^k \circ T = f_{G'}^k, \quad p_G^k \circ T = p_{G'}^k, \quad T(w_{G'}) = w_G \pmod{2}. \quad (2.9)$$

The graphs are *k-equivalent*, written $G \sim_k G'$, if such an isomorphism exists.

For $f(z) = \mu(z, z, z)$, polarization gives

$$\begin{aligned} 6\mu(a, b, c) &= f(a + b + c) - f(a + b) - f(a + c) - f(b + c) \\ &\quad + f(a) + f(b) + f(c). \end{aligned}$$

Thus equality of the cubic polynomials is equivalent to preservation of the symmetric trilinear forms.

For a spin graph, set $f_G = f_G^1$ and $\ell_G = p_G^1/4$. Then

$$f_G = \sum_u \alpha(u) x_u^3, \quad \ell_G = \sum_u \alpha(u) x_u. \quad (2.10)$$

For every $k \geq 1$,

$$\mu_G^k = \rho_k \mu_G^1, \quad p_G^k = \rho_k p_G^1. \quad (2.11)$$

Consequently, spin equivalence is independent of k .

For $\ell \in H^*$, its *content* $\text{cont}(\ell)$ is the nonnegative generator of $\ell(H) \subseteq \mathbb{Z}$. The form is *primitive* if $\text{cont}(\ell) = 1$. In the spin case,

$$\ell_G = p_G^k / \lambda_k \quad (k \geq 1),$$

so the normalized Pontryagin form is integral and independent of k .

2.1. Reduced graphs. Let $\mathcal{E}$ denote the graph consisting of a single edge from a vertex in V to a vertex in U with decoration $(0, 1, 0)$. Its main coordinate is zero. Its invariant system is

$$H_{\mathcal{E}} = \mathbb{Z}e, \quad f_{\mathcal{E}}^k = p_{\mathcal{E}}^k = 0, \quad w_{\mathcal{E}} = e \pmod{2}. \quad (2.12)$$

Write $-\mathcal{E}$ for its reversal, and call $\mathcal{E}$ and $-\mathcal{E}$ *exceptional*.

Definition 2.3 (Reduced graph [7, Definition 4.8]). An algebraic plumbing graph G is *reduced* if:

- (i) every connected component is a tree;
- (ii) every vertex of V has degree at least three, except for the unlabelled vertex of a possible exceptional component; such a component occurs at most once and, when present, is the only nonspin component;
- (iii) every vertex of U with decoration $(0, 0, 0)$ has degree zero or at least three.

A nonexceptional component is called *ordinary*.

Definition 2.4 (Reduced class). A *sign reversal* on a connected component replaces every decoration by

$$(\alpha, k^+, k^-) \mapsto (-\alpha, k^-, k^+). \quad (2.13)$$

The *reduced class* $[G]_{\text{red}}$ is the set of isomorphism classes of reduced graphs obtained from G by sign reversals on connected components.

The reversals on distinct components are independent. On a connected spin graph all labels therefore have one common sign ambiguity; on isolated labelled vertices the sign ambiguities are independent.

2.2. The ambient quadratic form. The distinguished basis of A_G is orthonormal for the standard integral positive-definite quadratic form. Its restriction to H_G is

$$q_G(x, y) = \sum_{u \in U} x_u^2 + \sum_{u \in U, j \neq 0} y_{u,j}^2, \quad (x, y) \in H_G. \quad (2.14)$$

The associated bilinear form is

$$b_G(z, z') = \frac{q_G(z + z') - q_G(z) - q_G(z')}{2}.$$

Definition 2.5 (Enhanced invariant system). The *enhanced invariant system* at k is (H_G, q_G, μ_G^k, w_G) . An isomorphism from the system of G' to that of G is an integral isomorphism $T : H_{G'} \rightarrow H_G$ such that

$$q_G \circ T = q_{G'}, \quad f_G^k \circ T = f_{G'}^k, \quad T(w_{G'}) = w_G \pmod{2}.$$

Lemma 2.6. *Let G be a connected ordinary reduced graph. Then*

$$\text{rank } H_G = |U| - |V| + \sum_u (k^+(u) + k^-(u)) \geq 1 + |V| + \sum_u (k^+(u) + k^-(u)). \quad (2.15)$$

Proof. Every vertex of V has degree at least three. The nonempty connected graph therefore has a vertex in U ; choose one as root. Order the vertices of V so that each is preceded by its parent. An unlabelled vertex v has one old neighbour p and $s = \deg(v) - 1 \geq 2$ new neighbours $q_1, \dots, q_s$. If H_{old} is the kernel for the preceding subtree, its new kernel is the image of

$$H_{\text{old}} \oplus \mathbb{Z}^{s-1} \longrightarrow \mathbb{Z}^{U_{\text{old}} \sqcup \{q_1, \dots, q_s\}}, \quad (x, t) \longmapsto \left(x, t_1, \dots, t_{s-1}, -x_p - \sum_{i < s} t_i \right).$$

This map is injective. Every integral solution of the new equation $x_p + \sum_i x_{q_i} = 0$ has exactly this form; projection onto the old coordinates and the first $s - 1$ new coordinates is its inverse. Starting with the root lattice $\mathbb{Z}$ gives

$$\text{rank } H_{0,G} = 1 + \sum_{v \in V} (\deg(v) - 2).$$

Since G is a tree,

$$\sum_{v \in V} \deg(v) = |E| = |U| + |V| - 1 \geq 3|V|.$$

Thus $\text{rank } H_{0,G} = |U| - |V| \geq 1 + |V|$. Adding the free summand in (2.2) proves (2.15). $\square$

3. INTEGRAL RIGIDITY OF BINARY FORMS

Let $\mathbb{F}$ be a field of characteristic zero. For $\alpha \in \mathbb{F}^3$, put

$$f_\alpha(x, y) = \alpha_1 x^3 + \alpha_2 y^3 - \alpha_3 (x + y)^3, \quad (3.1)$$

$$\ell_\alpha(x, y) = (\alpha_1 - \alpha_3)x + (\alpha_2 - \alpha_3)y. \quad (3.2)$$

Let μ_α be the symmetric $\mathbb{F}$ -trilinear form on $\mathbb{F}^2$ with $\mu_\alpha(z, z, z) = f_\alpha(z)$.

Theorem 3.1. *Let $\mathbb{F}$ be a field of characteristic zero, let $\alpha, \beta \in \mathbb{F}^3$ with $\alpha_1 \alpha_2 \alpha_3 \neq 0$, and let $T \in \text{GL}(2, \mathbb{Z})$. If*

$$f_\alpha \circ T = f_\beta, \quad \ell_\alpha \circ T = \ell_\beta, \quad (3.3)$$

then T permutes the covectors $x, y, -x - y$ up to a common sign, and

$$\beta = \varepsilon(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \alpha_{\pi(3)})$$

for some $\varepsilon \in \{1, -1\}$ and $\pi \in S_3$.

3.1. The hexagonal group. Set

$$L_1 = x, \quad L_2 = y, \quad L_3 = -x - y,$$

so that $f_\alpha = \sum_i \alpha_i L_i^3$ and $\ell_\alpha = \sum_i \alpha_i L_i$. Let

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \langle R, S \rangle. \quad (3.4)$$

Then

$$R^3 = -I, \quad R^6 = S^2 = I, \quad SRS = R^{-1},$$

and

$$\begin{aligned} (L_1, L_2, L_3) \circ S &= (L_2, L_1, L_3), \\ (L_1, L_2, L_3) \circ R &= (-L_2, -L_3, -L_1). \end{aligned}$$

The relation $SRS = R^{-1}$ reduces every word to $R^i S^j$, with $0 \leq i < 6$ and $j \in \{0, 1\}$. The order of R is six, and $\det R = 1$, $\det S = -1$; these twelve matrices are distinct. Their action on the L_i is faithful, since L_1, L_2 form a basis of $(\mathbb{Q}^2)^*$. They therefore realize all twelve permutations with a common sign.

The same group is the full group of linear automorphisms of

$$\mathcal{R} = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}. \quad (3.5)$$

The displayed generators preserve this set and act transitively on it. An automorphism fixing e_1 must send e_2 to a member z of $\mathcal{R}$ such that $e_1 - z \in \mathcal{R}$ and e_1, z are independent. The only possibilities are $z = e_2$ and $z = e_1 - e_2$. Its automorphism group therefore has at most twelve elements, and equals W .

Multiplication of T on either side by an element of W preserves the form of (3.3). More precisely, if $\mathcal{P}_\alpha = (f_\alpha, \ell_\alpha)$ and $N = gTh$, then

$$(\mathcal{P}_\alpha \circ g^{-1}) \circ N = \mathcal{P}_\beta \circ h.$$

The new source and target triples differ from the old ones by signed permutations.

Lemma 3.2. *For every $T \in \text{GL}(2, \mathbb{Z})$, the double coset WTW contains a matrix of one of the following forms:*

$$I, \quad \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \ (h \geq 1), \quad \begin{pmatrix} b+u & b \\ d+v & d \end{pmatrix} \ (b, d, u, v \geq 1, \ ud - bv = 1), \quad (3.6)$$

where all parameters are integers.

Proof. The three lines through $\mathcal{R}$ are $x = 0$, $y = 0$, and $x + y = 0$. If $Te_1, Te_2, T(e_1 - e_2)$ lie on these lines, each is a primitive integral vector on its line, hence belongs to $\mathcal{R}$. Consequently $T(\mathcal{R}) \subseteq \mathcal{R}$. Invertibility gives equality, so $T \in W$ and $WTW = W$.

Otherwise, choose $r \in \mathcal{R}$ with Tr on none of these lines. Choose $h \in W$ with $he_1 = r$ and replace T by Th . The first column $(a, c)^\top$ then satisfies $ac(a + c) \neq 0$. Among $a, c, -a - c$, two have the same sign. The signed-permutation action on these three values permits left multiplication making $a, c > 0$.

Write $(b, d)^\top$ for the second column. If $bd < 0$, then

$$|ad - bc| = a|d| + c|b| \geq a + c \geq 2,$$

contrary to $|\det T| = 1$. Thus the second column is coordinatewise nonnegative or nonpositive. In the latter case, multiplication on the right by

$$U = SR = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

replaces the columns (z, w) by $(z, z - w)$; both new columns are nonnegative. The resulting representative is nonnegative.

If this representative has a zero entry, row and column interchanges, induced by S , place it at $(2, 1)$. Its determinant is then the product of its nonnegative diagonal entries. This product has absolute value one, so both entries are one. The representative is a shear with parameter $h \geq 0$; $h = 0$ gives I .

Suppose all four entries are positive. If $a > b$ and $c < d$, then

$$ad - bc \geq (b + 1)(c + 1) - bc = b + c + 1 \geq 3.$$

If $a < b$ and $c > d$, the same argument gives $bc - ad \geq 3$. Thus the columns are comparable. They cannot be equal, since T is invertible. If exactly one coordinate is equal, use $(z, z - w)$ when $z \geq w$, and $(w, w - z)$ when $w \geq z$. These are right multiplications by U and R , respectively; the result is nonnegative with a zero entry, hence is covered by the preceding case.

The remaining comparison is strict in both coordinates. If $w > z$, right multiplication by R gives $(w, w - z)$, whose first column is strictly larger than its second. A row interchange makes its determinant one if necessary and preserves the strict comparison. Write the columns as $(b + u, d + v)^\top$ and $(b, d)^\top$. All four parameters are positive integers and $1 = (b + u)d - b(d + v) = ud - bv$. $\square$

3.2. **The constraint matrix.** Write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \varphi(t) = t^3 - t.$$

Define

$$\mathcal{A}(T) = \begin{pmatrix} \varphi(a) & \varphi(c) & -\varphi(a+c) \\ \varphi(b) & \varphi(d) & -\varphi(b+d) \\ ab(a-b) & cd(c-d) & -(a+c)(b+d)(a+c-b-d) \end{pmatrix}. \quad (3.7)$$

Lemma 3.3. *Let $\alpha \in \mathbb{F}^3$ and $T \in \text{GL}(2, \mathbb{Z})$. There exists $\beta \in \mathbb{F}^3$ such that $(f_\alpha \circ T, \ell_\alpha \circ T) = (f_\beta, \ell_\beta)$ if and only if*

$$\mathcal{A}(T)\alpha = 0. \quad (3.8)$$

When it exists, β is unique. Equivalently, with $F_\alpha = f_\alpha - \ell_\alpha$, the condition is

$$F_\alpha(Te_1) = F_\alpha(Te_2) = F_\alpha(T(e_1 - e_2)) = 0.$$

Proof. Write

$$f_\alpha \circ T = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3, \quad \ell_\alpha \circ T = Ex + Fy.$$

Comparison with (3.1)–(3.2) gives

$$A = E, \quad D = F, \quad B = C, \quad \beta = (A - B, D - B, -B).$$

Thus the three equalities are necessary and sufficient, and determine β uniquely. Put $z = Te_1$, $w = Te_2$. The first two equalities are $F_\alpha(z) = F_\alpha(w) = 0$ and have the first two rows of $\mathcal{A}(T)$ as their coefficient vectors. Moreover,

$$B - C = \mu_\alpha(z, z, w) - \mu_\alpha(z, w, w) = \sum_i \alpha_i L_i(z) L_i(w) (L_i(z) - L_i(w)),$$

whose coefficient vector is the third row. Hence the three equalities are equivalent to (3.8). Finally,

$$F_\alpha(z - w) = F_\alpha(z) - F_\alpha(w) - 3(B - C).$$

Since 3 is invertible in $\mathbb{F}$, the three stated vanishing conditions are equivalent to the same equalities. $\square$

Lemma 3.4. *Let b, d, u, v be positive integers with $ud - bv = 1$, and put*

$$T = \begin{pmatrix} b+u & b \\ d+v & d \end{pmatrix}.$$

Then

$$\begin{aligned} \det \mathcal{A}(T) &= bv(b+v)(b^2 + 3bv + v^2 - 3) \\ &\quad + 2(b+v)((bv+1)(d^2 + u^2) - 2) \\ &\quad + (bv+1)(d^3 + u^3 - d - u) \\ &\quad + 2bv(b+v)^2(d+u) \geq 72. \end{aligned} \quad (3.9)$$

Equality holds if and only if $b = v = 1$ and $\{d, u\} = \{1, 2\}$.

Proof. Appendix A gives the identity in (3.9). Since $b, v \geq 1$,

$$bv \geq 1, \quad b+v \geq 2, \quad b^2 + 3bv + v^2 - 3 \geq 2.$$

Also $ud = bv + 1 \geq 2$. At least one of the positive integers u, d is at least two, so

$$u+d \geq 3, \quad u^2 + d^2 \geq 5, \quad (u^3 - u) + (d^3 - d) \geq 6.$$

The four summands are bounded below, respectively, by

$$1 \cdot 2 \cdot 2 = 4, \quad 2 \cdot 2(2 \cdot 5 - 2) = 32, \quad 2 \cdot 6 = 12, \quad 2 \cdot 1 \cdot 2^2 \cdot 3 = 24.$$

Their sum is 72. Equality forces equality in the first bound, hence $bv = 1$ and $b = v = 1$. The relation gives $ud = 2$, so $\{d, u\} = \{1, 2\}$. Conversely, these values attain all four bounds. $\square$

Proof of Theorem 3.1. By Lemma 3.2, replace T by a representative of one of the forms (3.6). The corresponding changes to α, β are permutations with common signs, so the three source entries remain nonzero.

For $T = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, Lemma 3.3 gives

$$\begin{aligned} h(h-1)(h+1)\alpha_1 - h(h+1)(h+2)\alpha_3 &= 0, \\ -h(h-1)\alpha_1 + h(h+1)\alpha_3 &= 0. \end{aligned}$$

When $h = 1$, the first equation is $-6\alpha_3 = 0$. When $h \geq 2$, the determinant of this system in (α_1, α_3) is $-h^2(h-1)(h+1) \neq 0$, so $\alpha_1 = \alpha_3 = 0$. Characteristic zero and the nonzero source entries exclude both cases.

For the positive representative, Lemma 3.4 gives a nonzero integer determinant for $\mathcal{A}(T)$. Its image in $\mathbb{F}$ is nonzero. Equation (3.8) would therefore force $\alpha = 0$, again a contradiction.

The representative is I , so the original T belongs to W . Write $L_i \circ T = \varepsilon L_{\sigma(i)}$ with one sign ε and a permutation σ . Then

$$f_\alpha \circ T = \sum_j \varepsilon \alpha_{\sigma^{-1}(j)} L_j^3.$$

The polynomials L_1^3, L_2^3, L_3^3 are independent: the coefficient of x^2y first determines the coefficient of L_3^3 , and the pure terms determine the other two. Thus $\beta_j = \varepsilon \alpha_{\sigma^{-1}(j)}$ for every j . $\square$

4. REDUCED GRAPHS OF RANK TWO

Proposition 4.1. *A reduced spin graph of rank two is either the disjoint union of two isolated labelled vertices or a tree with one unlabelled vertex and three labelled leaves, all with nonzero labels. Up to reduced class, a reduced nonspin graph of rank two is of one of the forms*

$$\mathcal{E} \sqcup \{(\alpha, 0, 0)\}, \quad \{(\alpha, 1, 0)\}, \quad \alpha \in \mathbb{Z},$$

where braces denote an isolated labelled vertex.

Proof. Ranks add over connected components. By Lemma 2.6, every ordinary component has positive rank; its rank is one only when it is an isolated spin vertex. The exceptional component also has rank one, by (2.12).

For a spin component with $V \neq \emptyset$, the rank bound is at least two. If its rank is two, then $|V| = 1$ and $|U| - |V| = 2$, so $|U| = 3$. Connectedness makes these three labelled vertices the leaves of its unique unlabelled vertex. Reducedness makes all three labels nonzero. Otherwise the spin rank-two graph is the disjoint union of two rank-one isolated vertices.

If an exceptional component occurs in a nonspin rank-two graph, reducedness permits no other nonspin component. The remaining rank one is an isolated spin vertex. Reversal makes the exceptional component $\mathcal{E}$. If none occurs, a nonspin ordinary component has at least one additional coordinate, hence rank at least two. It is the whole graph. Equality in (2.15) forces $|V| = 0$ and exactly one additional coordinate. Reversal makes its decoration $(\alpha, 1, 0)$. $\square$

Proposition 4.2. *Let $k \geq 1$. Two reduced spin graphs of rank two are k -equivalent if and only if they have the same reduced class.*

Proof. By (2.11), take $k = 1$. For a connected graph, $u_1^0 - u_3^0, u_2^0 - u_3^0$ form an integral basis of the kernel and give (3.1)–(3.2) with $\alpha_1\alpha_2\alpha_3 \neq 0$. Theorem 3.1 gives permutation and common sign reversal. Conversely, each such operation is induced by the corresponding element of W . A disconnected graph has the binary representation with its third label zero. An equivalence with a connected graph, directed toward the connected graph by inversion if necessary, would contradict Theorem 3.1.

For two isolated vertices write $f = Ax^3 + By^3$, $\ell = Ax + By$. The chain rule gives

$$\text{Hess}(f \circ T) = T^\top (\text{Hess } f \circ T) T, \quad \det \text{Hess } f = 36ABxy.$$

Hence the alternatives $AB = 0$ and $AB \neq 0$ are invariant. In the latter case, writing $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the Hessian identity makes $(ax + by)(cx + dy)$ a nonzero multiple of xy . Thus $ac = bd = 0$. If $a \neq 0$, then $c = 0$, $d \neq 0$, and $b = 0$; if $a = 0$, invertibility gives $b, c \neq 0$ and $d = 0$. Integral unimodularity makes the two nonzero entries ± 1 . The labels therefore differ by a permutation and independent signs.

If $AB = 0$ and $f \neq 0$, exactly one label is nonzero. Its absolute value is $\text{cont}(\ell)$, which is preserved by an integral equivalence. The case $A = B = 0$ is distinguished by $f = 0$. Conversely, a permutation and independent sign changes of the two labels are realized by a signed permutation matrix and preserve both forms. $\square$

Proposition 4.3. *Let $k \geq 1$. Two reduced nonspin graphs of rank two are k -equivalent if and only if they have the same reduced class.*

Proof. By Proposition 4.1, the two families, in coordinates with $w = (0, 1) \pmod{2}$, have invariants

$$\begin{aligned} f_\alpha^I &= \rho_k \alpha x^3, & p_\alpha^I &= \lambda_k \alpha x, \\ f_\alpha^{II} &= \rho_k \alpha x^3 + 3x y^2, & p_\alpha^{II} &= (\lambda_k \alpha + c_k)x. \end{aligned}$$

Their Hessian determinants are zero and $36(\rho_k \alpha x^2 - y^2)$, respectively. The second polynomial is nonzero for every α . The Hessian transformation identity from Proposition 4.2 therefore excludes equivalence between the two families. In the first family, the content of p_α^I is $\lambda_k |\alpha|$, where the content of the zero form is zero. This determines the reduced class. Conversely, if $\beta = \varepsilon \alpha$ with $\varepsilon = \pm 1$, the map $(x, y) \mapsto (\varepsilon x, y)$ identifies these two systems and fixes w .

For the second family set $c_\alpha = \lambda_k \alpha + c_k$. The inequality $0 < c_k < \lambda_k$ gives $c_\alpha \neq 0$ for every $\alpha \in \mathbb{Z}$. Let an equivalence from parameter β to parameter α have first row (a, b) . Equality of the linear forms gives

$$c_\alpha a = c_\beta, \quad c_\alpha b = 0.$$

Thus $b = 0$, and unimodularity gives $a = \varepsilon \in \{1, -1\}$. The second coordinate has the form $v x + \eta y$, with $v \in \mathbb{Z}$ and $\eta = \pm 1$. The coefficients of $x^2 y$ and $x y^2$ in $f_\alpha^{II} \circ T$ are $6\varepsilon v \eta$ and 3ε , respectively. The coefficient of $x y^2$ gives $\varepsilon = 1$; the coefficient of $x^2 y$ then gives $v = 0$, since $\eta = \pm 1$. The coefficient of x^3 then gives $\alpha = \beta$. Conversely, equality of these normalized parameters is realized by the identity, which preserves w . $\square$

Proof of Theorem A. The class w_G vanishes exactly for spin graphs, by (2.8). The assertion follows from Propositions 4.2 and 4.3. $\square$

4.1. Rational equivalence and the hypotheses of rigidity.

Example 4.4. The triples

$$\alpha = (4, 15, 30), \quad \beta = (-6, -5, 60)$$

appear in [7, Remark 4.14]. Put

$$T_Q = \frac{1}{7} \begin{pmatrix} 27 & 25 \\ -16 & -13 \end{pmatrix}.$$

Then $\det T_Q = 1$, and substitution gives

$$\begin{aligned} f_\alpha \circ T_Q &= -66x^3 - 180x^2 y - 180x y^2 - 65y^3 = f_\beta, \\ \ell_\alpha \circ T_Q &= -66x - 65y = \ell_\beta. \end{aligned}$$

The triples are not related by a signed permutation. Theorem 3.1 therefore excludes integral equivalence.

Example 4.5. The nonvanishing condition in Theorem 3.1 is necessary. The shear $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ carries the pair for $(1, 2, 0)$ to the pair for $(0, 2, -1)$, although it does not belong to W .

The linear form is also necessary. The shear $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ carries the cubic for $(3, 2, 1)$ to the cubic for $(-1, -4, -3)$, but the transformed linear form is $2x + 5y$, while the target linear form is $2x - y$.

Singular cubics are included: the triple $(4, 4, 1)$ gives

$$f = 3(x + y)(x - y)^2.$$

The triple $(1, 1, 1)$ has $\ell = 0$. Both satisfy Theorem 3.1.

5. NONUNIQUENESS IN RANK AT LEAST THREE

For $a, b \in \mathbb{Z}$, put

$$B_{a,b}(t, x) = at^3 + b(-t - x)^3, \quad b_{a,b}(t, x) = (a - b)x - bt.$$

Lemma 5.1. *Let $q \in \mathbb{Z}$ and suppose $q(a - b) = 2b$. For $\sigma_q(t, x) = (t, qt - x)$ one has*

$$B_{a,b} \circ \sigma_q = -B_{a,b} - b(q + 1)(q + 2)t^3, \quad (5.1)$$

$$b_{a,b} \circ \sigma_q = -b_{a,b}. \quad (5.2)$$

Proof. Set $D = q(a - b) - 2b$. Expansion gives

$$\begin{aligned} B_{a,b}(t, qt - x) + B_{a,b}(t, x) + b(q + 1)(q + 2)t^3 &= D(q^2 t^3 - 3qt^2 x + 3tx^2), \\ b_{a,b}(t, qt - x) + b_{a,b}(t, x) &= Dt. \end{aligned}$$

Both right-hand sides vanish. □

Definition 5.2. Let $m \geq 1$, $h \in \mathbb{Z}$, and $(a_i, b_i) \in \mathbb{Z}^2$ for $1 \leq i \leq m$. The spin graph $\Gamma(h; (a_i, b_i)_{i=1}^m)$ has

$$U = \{u_0\} \sqcup \{u_i^+, u_i^- : 1 \leq i \leq m\}, \quad V = \{v_1, \dots, v_m\},$$

with edges (u_0, v_i) , (u_i^+, v_i) , and (u_i^-, v_i) for each i . The labels of u_0, u_i^+, u_i^- are h, a_i, b_i , respectively. All additional decorations vanish. The three edges incident to v_i form its i th branch.

Proposition 5.3. *Let $\Gamma = \Gamma(h; (a_i, b_i)_{i=1}^m)$ be as in Definition 5.2. Let $I \subseteq \{1, \dots, m\}$, and choose integers $q_i, i \in I$, such that*

$$q_i(a_i - b_i) = 2b_i, \quad \sum_{i \in I} b_i(q_i + 1)(q_i + 2) = 0. \quad (5.3)$$

Changing (a_i, b_i) to $(-a_i, -b_i)$ for $i \in I$, and leaving all other labels unchanged, gives a k -equivalent graph for every $k \geq 1$.

Proof. Let Γ' denote the graph with the indicated label changes. The equations of either graph have the unique integral solution

$$x_{u_0} = t, \quad x_{u_i^+} = x_i, \quad x_{u_i^-} = -t - x_i.$$

Thus their full lattices are identified with $\mathbb{Z}^{m+1}$. In these coordinates the source forms are

$$f = ht^3 + \sum_i B_{a_i, b_i}(t, x_i), \quad \ell = ht + \sum_i b_{a_i, b_i}(t, x_i).$$

Define T by fixing t and replacing x_i by $q_i t - x_i$ for $i \in I$, while fixing the other x_i . Then $T^2 = I$ and $\det T = (-1)^{|I|}$. Lemma 5.1 gives

$$\begin{aligned} f \circ T &= ht^3 + \sum_{i \notin I} B_{a_i, b_i} - \sum_{i \in I} B_{a_i, b_i} - t^3 \sum_{i \in I} b_i(q_i + 1)(q_i + 2) = f_{\Gamma'}, \\ \ell \circ T &= ht + \sum_{i \notin I} b_{a_i, b_i} - \sum_{i \in I} b_{a_i, b_i} = \ell_{\Gamma'}. \end{aligned}$$

The last cubic sum vanishes by (5.3). Both parity classes are zero. Polarization and (2.11) prove the assertion for every $k \geq 1$. □

Remark 5.4. Suppose $ab \neq 0$ and $q(a - b) = 2b$. If $b(q + 1)(q + 2) = 0$, then $q = -1$ or $q = -2$. The value $q = -2$ gives $a = 0$, which is excluded. Thus $q = -1$, $a = -b$, and $(-a, -b) = (b, a)$: a single such branch with zero cubic defect gives only an exchange of its two leaves.

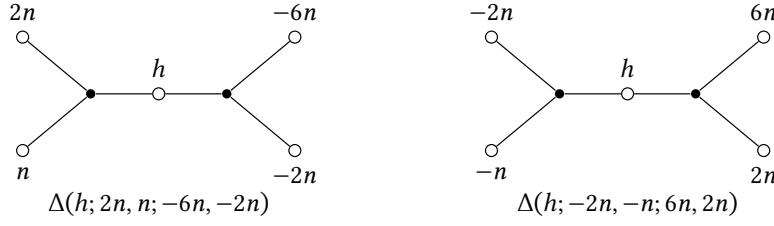


FIGURE 1. The rank-three pair of Theorem 5.5, with $h \neq 0$ and $n \geq 1$. Open circles are labelled vertices, with their integer labels displayed; solid vertices are unlabelled. All additional decorations vanish. The central label is unchanged while the four leaf labels change sign.

5.1. **Rank three.** Let $\Delta(h; a, b; c, d) = \Gamma(h; (a, b), (c, d))$. Thus

$$U = \{u_0, u_1^+, u_1^-, u_2^+, u_2^-\}, \quad V = \{v_1, v_2\},$$

and the edge set is

$$E = \{(u_0, v_i), (u_i^+, v_i), (u_i^-, v_i) : i = 1, 2\}.$$

Its integral coordinates are

$$\iota(t, x, y) = (t, x, -t - x, y, -t - y). \quad (5.4)$$

Figure 1 displays the two graphs of Theorem 5.5.

Theorem 5.5. Let $h \neq 0$ and $n \geq 1$ be integers. The graphs

$$\Delta(h; 2n, n; -6n, -2n), \quad \Delta(h; -2n, -n; 6n, 2n) \quad (5.5)$$

are connected reduced spin graphs of rank three. Their invariant systems are k -equivalent for every $k \geq 1$, whereas their reduced classes differ.

Proof. Each graph is connected and has seven vertices and six edges, hence is a tree. Every unlabelled vertex has degree three. Since $h \neq 0$ and $n \geq 1$, all five labels are nonzero, so both graphs are reduced. The map (5.4) solves the two kernel equations; its inverse reads the central coordinate and the first leaf coordinate of each branch. It is therefore an integral lattice isomorphism.

In Proposition 5.3, take

$$(q_1, a_1, b_1) = (2, 2n, n), \quad (q_2, a_2, b_2) = (1, -6n, -2n).$$

Then $q_i(a_i - b_i) = 2b_i$, and

$$b_1(q_1 + 1)(q_1 + 2) + b_2(q_2 + 1)(q_2 + 2) = 12n - 12n = 0.$$

The corresponding integral map is

$$\tau(t, x, y) = (t, 2t - x, t - y), \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \quad (5.6)$$

It satisfies $T_3^2 = I$ and $\det T_3 = 1$.

Set

$$\begin{aligned} C(t, x, y) &= 2x^3 - (t + x)^3 - 6y^3 + 2(t + y)^3, \\ L(t, x, y) &= t + x - 4y. \end{aligned} \quad (5.7)$$

The two pairs of forms in (5.5) are

$$f_{\pm} = ht^3 \pm nC, \quad \ell_{\pm} = ht \pm nL.$$

Equations (5.1)–(5.2) give

$$C \circ \tau = -C, \quad L \circ \tau = -L. \quad (5.8)$$

Thus $f_+ \circ \tau = f_-$ and $\ell_+ \circ \tau = \ell_-$. The map on the original lattices is $\iota T_3 \iota^{-1}$. It preserves the full invariant systems by polarization and (2.11).

The vertex u_0 is the unique labelled vertex of degree two. A graph isomorphism must therefore preserve it. The leaf-label multisets

$$\{2n, n, -6n, -2n\}, \quad \{-2n, -n, 6n, 2n\}$$

are different, since $-6n$ occurs only in the first. Common sign reversal changes the label of u_0 from h to $-h$, which is different from h . Hence the reduced classes are distinct. $\square$

Example 5.6. For $h = n = 1$, the labels are

$$(1; 2, 1; -6, -2), \quad (1; -2, -1; 6, 2),$$

and

$$\begin{aligned} f_+ &= 2t^3 + x^3 - 4y^3 - 3tx^2 + 6ty^2 - 3t^2x + 6t^2y, \\ f_- &= -x^3 + 4y^3 + 3tx^2 - 6ty^2 + 3t^2x - 6t^2y, \\ \ell_+ &= 2t + x - 4y, \quad \ell_- = -x + 4y. \end{aligned}$$

Both linear forms are primitive and nonzero.

5.2. Rank four and higher ranks. For $r \geq 3$ and $n \geq 1$, define

$$\begin{aligned} G_r^+(n) &= \Gamma(r-2; (2n, n), (-6n, -2n), \underbrace{(1, 1), \dots, (1, 1)}_{r-3}), \\ G_r^-(n) &= \Gamma(r-2; (-2n, -n), (6n, 2n), \underbrace{(1, 1), \dots, (1, 1)}_{r-3}). \end{aligned} \quad (5.9)$$

In particular, $G_3^\pm(n)$ are the graphs of Theorem 5.5 with $h = 1$.

Proposition 5.7. *For $r \geq 3$ and $n \geq 1$, the graphs (5.9) are connected reduced spin graphs of rank r with all labels nonzero. Their reduced classes differ, but they are k -equivalent for every $k \geq 1$ under $T_r = T_3 \oplus I_{r-3}$. Their normalized Pontryagin forms are primitive.*

Proof. Put $m = r - 1$. The graph has $2m + 1$ labelled vertices, m unlabelled vertices, and $3m$ edges. It is connected and has one fewer edge than vertices, hence is a tree. Every unlabelled vertex has degree three. Every integer label is nonzero, so the graph is reduced.

The kernel equations give the integral coordinates $(t, x, y, z_1, \dots, z_{r-3})$. Thus its rank is r . The identity

$$t^3 + z^3 + (-t - z)^3 = -3tz(t + z)$$

gives

$$\begin{aligned} f_\pm &= t^3 \pm nC(t, x, y) - 3t \sum_{j=1}^{r-3} z_j(t + z_j), \\ \ell_\pm &= t \pm nL(t, x, y). \end{aligned} \quad (5.10)$$

Extend τ by fixing every z_j . The result is an integral involution of determinant one. Equation (5.8) proves that it preserves the two invariant systems, in the direction from the minus graph to the plus graph, for every k .

The central vertex is the unique labelled vertex of degree at least two, and its label $r - 2$ is positive. Common sign reversal cannot identify the graphs. Without sign reversal, the leaf label $-6n$ distinguishes them.

Finally,

$$\ell_+ = (n+1)t + nx - 4ny, \quad \ell_- = (1-n)t - nx + 4ny. \quad (5.11)$$

For ℓ_+ the first two coefficients differ by one; for ℓ_- their difference is one as well. Thus each coefficient set has greatest common divisor one. By (2.11), these are precisely the normalized Pontryagin forms. $\square$

Example 5.8. In rank four, one may take

$$\Gamma(2; (2, 1), (-6, -2), (1, 1)), \quad \Gamma(2; (-2, -1), (6, 2), (1, 1)).$$

Each graph has ten vertices and nine edges. In the coordinates (t, x, y, z) , the equivalence is

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The cubic forms are those of Example 5.6, with the common term $-3tz(t+z)$ added. The normalized Pontryagin forms are unchanged.

5.3. Primitive normalized Pontryagin forms. For an integral trilinear form ν on a free abelian group K , let

$$\text{div}_K(\nu) = \gcd\{\nu(a, b, c) : a, b, c \in K\},$$

with nonnegative greatest common divisor. This is the positive generator of the ideal generated by these values when that ideal is nonzero. Multilinearity shows that it is the greatest common divisor of the ordered basis entries of ν , in any integral basis of K .

For an invariant system $\mathcal{I} = (H, \mu, w, p)$ with $p \neq 0$, set

$$d(\mathcal{I}) = \text{div}_{\ker p}(\mu|_{(\ker p)^3}). \quad (5.12)$$

An isomorphism of invariant systems restricts to an isomorphism of the kernels of their linear forms. Hence (5.12) is an invariant.

Proposition 5.9. For $r \geq 3$, $n \geq 1$, and $k \geq 1$,

$$d(\mathcal{I}_k(G_r^\pm(n))) = \rho_k \begin{cases} 2n, & r = 3 \text{ and } n \text{ is odd,} \\ n, & \text{otherwise.} \end{cases} \quad (5.13)$$

Proof. It suffices to consider $G_r^+(n)$ and $k = 1$. Write $\mu = \mu_{G_r^+(n)}^1$. By (5.11), the kernel of the linear form consists exactly of

$$(t, x, y, z_1, \dots, z_{r-3}) = (nu, -(n+1)u + 4v, v, z_1, \dots, z_{r-3}), \quad u, v, z_j \in \mathbb{Z}.$$

Reduction of $(n+1)t + nx - 4ny = 0$ modulo n gives $n \mid t$. Writing $t = nu$ and dividing the equation by $n > 0$ gives $x = -(n+1)u + 4y$. The inverse parametrization reads $u = t/n$, $v = y$, and the unchanged z_j . Thus a full integral basis is

$$v_0 = (n, -n-1, 0, 0, \dots, 0), \quad v_1 = (0, 4, 1, 0, \dots, 0), \quad e_{z_1}, \dots, e_{z_{r-3}}.$$

For $e = (1, -1, 0, \dots, 0)$, one has $\ell_+(e) = 1$. Every $z \in H_G$ decomposes uniquely as $z = (z - \ell_+(z)e) + \ell_+(z)e$, with its first summand in the kernel. Hence adjoining e gives a basis of H_G .

Substituting in (5.10), with all $z_j = 0$, gives

$$\begin{aligned} f_+(uv_0 + vv_1) = & -n(n+1)(5n+1)u^3 + 6n(5n^2 + 8n + 2)u^2v \\ & - 6n(15n+8)uv^2 + 60nv^3. \end{aligned}$$

Consequently the four trilinear entries on $\mathbb{Z}v_0 + \mathbb{Z}v_1$ are

entry	value
$\mu(v_0, v_0, v_0)$	$-n(n+1)(5n+1)$
$\mu(v_0, v_0, v_1)$	$2n(5n^2 + 8n + 2)$
$\mu(v_0, v_1, v_1)$	$-2n(15n + 8)$
$\mu(v_1, v_1, v_1)$	$60n$

(5.14)

After division by n , a common divisor divides the last entry, 60. The only possible odd prime divisors are 3, 5, and $-2(15n + 8) \equiv -1$ modulo each. Thus the common divisor is a power of two. For even n , the first entry $-(n+1)(5n+1)$ is odd, so the greatest common divisor is one. For odd n , every entry is even, whereas $15n + 8$ is odd; the third entry has exactly one factor of two. The greatest common divisor is then two.

For $r \geq 4$, the additional nonzero entries involving one of the basis vectors e_{z_j} , up to symmetry, are

$$\mu(v_0, e_{z_j}, e_{z_j}) = -n, \quad \mu(v_0, v_0, e_{z_j}) = -n^2.$$

All remaining entries involving any e_{z_j} vanish. Every entry is a multiple of n , and $-n$ occurs. The greatest common divisor is therefore n .

Finally, $\ker p_G^k = \ker \ell_G$ and $\mu_G^k = \rho_k \mu_G^1$. Multiplication by ρ_k proves (5.13); equivalence gives the same formula for the minus graph. $\square$

Proof of Theorem B. Fix $r \geq 3$ and set $G_N = G_r^+(2N)$, $G'_N = G_r^-(2N)$. Proposition 5.7 gives distinct reduced classes, primitive normalized Pontryagin forms, and the common involution $T_r = T_3 \oplus I_{r-3}$. Proposition 5.9 gives

$$d(\mathcal{I}_k(G_r^+(2N))) = 2\rho_k N.$$

For fixed k , these positive integers are distinct as N varies. Hence the systems are pairwise nonisomorphic. $\square$

Remark 5.10. The invariant in (5.12) uses the trilinear entries, not the coefficients of the associated cubic polynomial. In (5.14), the two mixed polynomial coefficients must be divided by three.

Corollary 5.11. *For every $k \geq 1$, the least rank of two reduced graphs G, G' with $\mathcal{I}_k(G) \cong \mathcal{I}_k(G')$ and $[G]_{\text{red}} \neq [G']_{\text{red}}$ is three.*

Proof. Theorem 5.5 gives the upper bound. Every nonempty reduced component has rank at least one, by Lemma 2.6 and (2.12); hence rank zero determines the empty graph. A rank-one ordinary component is an isolated spin vertex of label α , and the only other rank-one reduced class is represented by $\mathcal{E}$. The parity class distinguishes these types. For the isolated vertex, $\text{cont}(p_G^k) = \lambda_k |\alpha|$ determines its reduced class. Theorem A proves uniqueness in rank two. $\square$

5.4. Local substitutions and nonspin graphs.

Proposition 5.12. *Let $n \geq 1$ be an integer. Suppose an algebraic plumbing graph contains two distinct unlabelled vertices v_1, v_2 of degree three, both adjacent to a labelled vertex u_0 . Suppose the other neighbours of v_i are distinct labelled leaves u_i^+, u_i^- with no additional decorations. If their labels are*

$$(\alpha(u_1^+), \alpha(u_1^-), \alpha(u_2^+), \alpha(u_2^-)) = (2n, n, -6n, -2n),$$

then changing the signs of these four labels preserves the k -equivalence class for every $k \geq 1$. All other vertices, edges and decorations are unchanged.

Proof. Denote the original graph by G and the graph with the changed labels by G' . Their lattice equations coincide. Write $t = x_{u_0}$, $x = x_{u_1^+}$, and $y = x_{u_2^+}$. The two branch equations give $x_{u_1^-} = -t - x$ and $x_{u_2^-} = -t - y$. Define $T : H_{G'} \rightarrow H_G$ by

$$(x_{u_1^+}, x_{u_1^-}, x_{u_2^+}, x_{u_2^-}) \mapsto (2t - x, -3t + x, t - y, -2t + y)$$

and fix every other coordinate. Both new leaf pairs sum to $-t$. The four vertices are leaves, so no other incidence equation contains one of their coordinates. Therefore T maps the integral kernel to itself. Applying it twice is the identity.

The four original leaf contributions are $\rho_k nC$ and $\lambda_k nL$. Equation (5.8) negates both under T . Every other term in (2.5) and (2.7) is fixed: the only altered coordinates are at leaves with no additional decorations. In particular, t and all additional coordinates are fixed. Thus

$$f_G^k \circ T = f_{G'}^k, \quad p_G^k \circ T = p_{G'}^k, \quad T(w_{G'}) = w_G \pmod{2}.$$

This holds for every $k \geq 1$. $\square$

Corollary 5.13. *For every integer $r \geq 4$ there are connected reduced nonspin graphs of rank r with different reduced classes and k -equivalent invariant systems for every $k \geq 1$.*

Proof. Start with $G_{r-1}^\pm(1)$ and add one positive additional coordinate at the central vertex, leaving its integer label unchanged. This increases the rank by one and preserves reducedness. Proposition 5.12 gives equivalence for every k . The central vertex remains the unique labelled vertex of degree at least two. Its positive label excludes common sign reversal, and the four leaf labels distinguish the graphs without reversal. $\square$

Example 5.14. In rank four the nonspin pair has the underlying rank-three graphs of Example 5.6, with decoration $(\alpha, k^+, k^-) = (1, 1, 0)$ at the central vertex. In coordinates (t, x, y, z) , put $c_k = \binom{2k+1}{k}$. Its forms are

$$f_\pm^k = \rho_k(t^3 \pm C) + 3tz^2, \quad p_\pm^k = \lambda_k(t \pm L) + c_k t, \quad w = e_z \pmod{2H}.$$

The matrix T_4 of Example 5.8 preserves these systems by (2.5)–(2.7).

6. PAIRED-LEAF TREES AND QUADRATIC CONGRUENCE

Definition 6.1 (Paired-leaf tree). Let $m \geq 3$, $h \in \mathbb{Z}$, and $a = (a_1, \dots, a_m) \in (\mathbb{Z} \setminus \{0\})^m$. The *paired-leaf tree* $G_m(h; a)$ is the spin graph with

$$\begin{aligned} U &= \{u_0\} \sqcup \{u_i^+, u_i^- : 1 \leq i \leq m\}, \\ V &= \{v_1, \dots, v_m\}, \\ E &= \{(u_0, v_i), (u_i^+, v_i), (u_i^-, v_i) : 1 \leq i \leq m\}. \end{aligned}$$

The vertex u_0 has label h , and both u_i^+ and u_i^- have label a_i . The superscripts on the leaf names distinguish the two leaves.

Every v_i has degree three, the central vertex has degree m , and the leaf labels are nonzero. Thus $G_m(h; a)$ is reduced, including when $h = 0$.

The homomorphism $\mathbb{Z}^{m+1} \rightarrow H_{G_m(h; a)}$ given by

$$x_{u_0} = t, \quad x_{u_i^+} = x_i, \quad x_{u_i^-} = -t - x_i \tag{6.1}$$

is an isomorphism: its inverse reads the coordinates of $u_0, u_1^+, \dots, u_m^+$. Thus the rank is $m + 1$. Write

$$S_a = \sum_i a_i, \quad D_a = \text{diag}(a_1, \dots, a_m), \quad \mathbf{1} = (1, \dots, 1)^\top.$$

The spin pair is

$$f_{h,a}(t, x) = (h - S_a)t^3 - 3t x^\top D_a x - 3t^2 \mathbf{1}^\top D_a x, \tag{6.2}$$

$$\ell_{h,a}(t, x) = (h - S_a)t. \tag{6.3}$$

In the rational coordinates $y = 2x + t\mathbf{1}$,

$$f_{h,a}(t, x) = \frac{t}{4}((4h - S_a)t^2 - 3y^\top D_a y). \tag{6.4}$$

The integral lattice in these coordinates is

$$\{(t, y) \in \mathbb{Z} \oplus \mathbb{Z}^m : y \equiv t\mathbf{1} \pmod{2}\}. \tag{6.5}$$

Theorem 6.2. *Let $m \geq 3$ and $k \geq 1$, let $h, h' \in \mathbb{Z}$, and let $a, b \in (\mathbb{Z} \setminus \{0\})^m$. Then $G_m(h; a) \sim_k G_m(h'; b)$ if and only if there are $\varepsilon \in \{1, -1\}$ and $A \in \text{GL}(m, \mathbb{Z})$ satisfying*

$$h' = \varepsilon h, \quad S_b = \varepsilon S_a, \quad D_b = \varepsilon A^\top D_a A, \quad A\mathbf{1} \equiv \mathbf{1} \pmod{2}. \quad (6.6)$$

In the coordinates (6.1), every equivalence from the spin pair of $G_m(h'; b)$ to that of $G_m(h; a)$ is of the form

$$T(t, x) = \left(\varepsilon t, Ax + \frac{A\mathbf{1} - \varepsilon\mathbf{1}}{2} t \right) \quad (6.7)$$

for such a pair (ε, A) , and every such pair defines an equivalence.

Proof. Spin equivalence is independent of k , by (2.11). Suppose first that (6.6) holds. Set

$$c = \frac{A\mathbf{1} - \varepsilon\mathbf{1}}{2} \in \mathbb{Z}^m.$$

The map $T(t, x) = (\varepsilon t, Ax + ct)$ has determinant $\varepsilon \det A = \pm 1$ and inverse

$$T^{-1}(t, x) = (\varepsilon t, A^{-1}x - \varepsilon A^{-1}ct).$$

In the completed-square coordinates,

$$2(Ax + ct) + \varepsilon t\mathbf{1} = A(2x + t\mathbf{1}) = Ay.$$

Therefore

$$\begin{aligned} f_{h,a} \circ T &= \frac{\varepsilon t}{4} ((4h - S_a)t^2 - 3y^\top A^\top D_a Ay) = f_{h',b}, \\ \ell_{h,a} \circ T &= \varepsilon(h - S_a)t = (h' - S_b)t. \end{aligned}$$

Conversely, let T identify the spin pairs. The factorization $f_{h,a} = tQ_{h,a}$ has a quadratic factor whose restriction to $t = 0$ is $-3x^\top D_a x$. Its matrix rank is therefore at least $m \geq 3$. A product of two linear forms has quadratic rank at most two. Hence $Q_{h,a}$ is irreducible over $\mathbb{Q}$, and t is the unique linear factor of $f_{h,a}$ up to scalar. The same holds for $f_{h',b}$.

Unique factorization in $\mathbb{Q}[t, x_1, \dots, x_m]$ gives $t \circ T = \gamma t$ for some $\gamma \in \mathbb{Q}^\times$. Since T is an integral isomorphism, the image of $t \circ T$ is $\mathbb{Z}$. The covector γt has image $\gamma\mathbb{Z}$; hence $\gamma = \varepsilon \in \{1, -1\}$. Thus

$$T(t, x) = (\varepsilon t, Ax + ct), \quad A \in \text{GL}(m, \mathbb{Z}), \quad c \in \mathbb{Z}^m.$$

Put $d = 2c + \varepsilon\mathbf{1} - A\mathbf{1}$. In the rational coordinates (t, y) , equality of the cubics is

$$\frac{\varepsilon t}{4} ((4h - S_a)t^2 - 3(Ay + dt)^\top D_a (Ay + dt)) = \frac{t}{4} ((4h' - S_b)t^2 - 3y^\top D_b y).$$

The coefficient vector of $t^2 y$ is $-3\varepsilon A^\top D_a d/2$ on the left and zero on the right. Since $A^\top D_a$ is invertible, $d = 0$. Comparing the quadratic coefficients in y gives $D_b = \varepsilon A^\top D_a A$. The remaining equalities are

$$4h' - S_b = \varepsilon(4h - S_a), \quad h' - S_b = \varepsilon(h - S_a).$$

Subtracting gives $h' = \varepsilon h$, and then $S_b = \varepsilon S_a$. Finally $d = 0$ gives the stated shift and the congruence $A\mathbf{1} \equiv \mathbf{1} \pmod{2}$. $\square$

Proposition 6.3 (Definite case). *Under the hypotheses of Theorem 6.2, suppose that all entries of a have the same sign. Then*

$$G_m(h; a) \sim_k G_m(h'; b) \iff [G_m(h; a)]_{\text{red}} = [G_m(h'; b)]_{\text{red}}.$$

Proof. By Theorem 6.2,

$$h' = \varepsilon h, \quad D_b = \varepsilon A^\top D_a A, \quad S_b = \varepsilon S_a.$$

Let σ be the common sign of the a_i . Taking traces and multiplying by σ gives

$$0 = \sigma(\operatorname{tr}(A^\top D_a A) - \operatorname{tr} D_a) = \sum_i |a_i| \left(\sum_j A_{ij}^2 - 1 \right).$$

Every row of A is a nonzero integral vector, so each summand is nonnegative. Equality implies $\sum_j A_{ij}^2 = 1$ for every i . Each row has one nonzero entry, equal to ± 1 , and invertibility places these entries in distinct columns. Thus A is a signed permutation matrix. It follows that $b_i = \varepsilon a_{\pi(i)}$ for a permutation π , and $h' = \varepsilon h$. The centre is the unique labelled vertex of degree greater than one. Consequently these label relations are exactly a branch permutation and a common reversal. Conversely, for any such relations choose the permutation matrix A with $Ae_i = e_{\pi(i)}$. Then $A^\top D_a A = \operatorname{diag}(a_{\pi(1)}, \dots, a_{\pi(m)})$ and $A\mathbf{1} = \mathbf{1}$, so (6.6) holds. $\square$

6.1. A rank-four pair.

Example 6.4. Let

$$a_0 = (1, 5, -6), \quad b_0 = (2, 3, -5), \quad A_0 = \begin{pmatrix} -6 & -3 & 10 \\ -2 & 0 & 3 \\ -3 & -1 & 5 \end{pmatrix}.$$

Multiplication gives

$$\det A_0 = -1, \quad A_0 \mathbf{1} = \mathbf{1}, \quad A_0^\top \operatorname{diag}(1, 5, -6) A_0 = \operatorname{diag}(2, 3, -5).$$

Since $S_{a_0} = S_{b_0} = 0$, Theorem 6.2 gives

$$G_3(1; a_0) \sim_k G_3(1; b_0) \quad (k \geq 1)$$

under the substitution $(t, x) \mapsto (t, A_0 x)$. Both graphs have rank four, and their normalized Pontryagin forms are t . Their maximum absolute labels are 6 and 5, so their reduced classes are distinct.

6.2. Primitive paired-leaf families.

Proposition 6.5. *For every integer $r \geq 4$, there are sequences $(P_{r,N})_{N \geq 1}$ and $(P'_{r,N})_{N \geq 1}$ of connected reduced spin paired-leaf trees of rank r , with all labels nonzero, such that*

$$[P_{r,N}]_{\text{red}} \neq [P'_{r,N}]_{\text{red}}, \quad \mathcal{I}_k(P_{r,N}) \cong \mathcal{I}_k(P'_{r,N}) \quad (N \geq 1, k \geq 1).$$

Their normalized Pontryagin forms are primitive. For each fixed k , the systems $\mathcal{I}_k(P_{r,N})$ are pairwise nonisomorphic as N varies.

Proof. Put $m = r - 1$, $h_r = r - 3$, and

$$\begin{aligned} \tilde{a}^{(r,N)} &= (N, 5N, -6N, \underbrace{1, \dots, 1}_{r-4}), \\ \tilde{b}^{(r,N)} &= (2N, 3N, -5N, \underbrace{1, \dots, 1}_{r-4}). \end{aligned} \tag{6.8}$$

Set $P_{r,N} = G_m(h_r; \tilde{a}^{(r,N)})$ and $P'_{r,N} = G_m(h_r; \tilde{b}^{(r,N)})$. These are reduced trees of rank r with nonzero labels. With $A_r = A_0 \oplus I_{r-4}$, the identities of Example 6.4 give

$$A_r \mathbf{1} = \mathbf{1}, \quad A_r^\top D_{\tilde{a}^{(r,N)}} A_r = D_{\tilde{b}^{(r,N)}}, \quad S_{\tilde{a}^{(r,N)}} = S_{\tilde{b}^{(r,N)}} = r - 4.$$

Theorem 6.2 applies with $\varepsilon = 1$ and $T(t, x) = (t, A_r x)$. Both normalized linear forms are $\ell = (h_r - (r - 4))t = t$.

The central vertex is the unique labelled vertex of degree greater than one. Its positive label excludes common reversal. Without reversal, the leaf label $-6N$, which does not occur in $P'_{r,N}$, excludes an isomorphism. The reduced classes are therefore distinct.

Fix k and set $\mathcal{I} = \mathcal{I}_k(P_{r,N})$, $v = \mu^k/\rho_k$, and $\ell = p^k/\lambda_k$. Both normalized forms are integral. Put $K = \ker \ell$. Since $f_{h,a}$ is divisible by t , polarization gives $v|_{K^3} = 0$. For any z with $\ell(z) = 1$, define

$$B_{\mathcal{I}}(u, v) = -v(z, u, v) \quad (u, v \in K). \quad (6.9)$$

Such z exists because ℓ is primitive. Two choices differ by an element of K ; hence $B_{\mathcal{I}}$ is independent of the choice. If $T : H' \rightarrow H$ is a system isomorphism and $\ell'(z') = 1$, then $\ell(Tz') = 1$ and

$$B_{\mathcal{I}}(Tu, Tv) = -v(Tz', Tu, Tv) = -v'(z', u, v) = B_{\mathcal{I}'}(u, v) \quad (u, v \in \ker \ell').$$

Thus the integral bilinear form is preserved. For $z = (1, 0, \dots, 0)$, equation (6.2) gives, on the coordinate basis $e_1, \dots, e_m$ of K ,

$$(B_{\mathcal{I}}(e_i, e_j))_{1 \leq i, j \leq m} = D_{\tilde{a}^{(r,N)}}.$$

It is nonsingular. Its absolute Gram determinant is independent of the integral basis, since a change of basis multiplies it by the square of a unimodular determinant. Thus

$$|\det B_{\mathcal{I}}| = 30N^3.$$

These integers distinguish the systems as N varies. □

6.3. The rank-five pair. Set

$$a = (-15, 3, 18, -1), \quad b = (-3, 15, 2, -9), \quad h = h' = 1, \quad (6.10)$$

and

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 9 \end{pmatrix}. \quad (6.11)$$

Both diagonal blocks have determinant one. The identities

$$-15(p+2q)^2 + 3(2p+5q)^2 = -3p^2 + 15q^2, \quad (6.12)$$

$$18(r+2s)^2 - (4r+9s)^2 = 2r^2 - 9s^2 \quad (6.13)$$

give $A^T D_a A = D_b$. Moreover,

$$S_a = S_b = 5, \quad A\mathbf{1} = (3, 7, 3, 13)^T \equiv \mathbf{1} \pmod{2}.$$

The corresponding integral equivalence is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 3 & 2 & 5 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 6 & 0 & 0 & 4 & 9 \end{pmatrix} \in \mathrm{SL}(5, \mathbb{Z}). \quad (6.14)$$

It satisfies

$$f_{1,a} \circ T = f_{1,b}, \quad \ell_{1,a} \circ T = \ell_{1,b} = -4t.$$

The graphs are connected and reduced, and all nine labels in each graph are nonzero. The largest absolute labels are 18 and 15. They are invariant under graph isomorphisms and common sign reversal. The reduced classes are therefore different.

Proposition 6.6. *For every $r \geq 5$ there are infinitely many pairwise nonisomorphic invariant systems of rank r , each admitting two connected reduced spin paired-leaf presentations with different reduced classes and all labels nonzero. The two presentations are equivalent for every $k \geq 1$.*

Proof. Fix $r \geq 5$, put $m = r - 1$, and let $N \geq 1$. Take central label 1 and branch weights

$$\begin{aligned} a^{(r,N)} &= (-15N, 3N, 18N, -N, \underbrace{1, \dots, 1}_{r-5}), \\ b^{(r,N)} &= (-3N, 15N, 2N, -9N, \underbrace{1, \dots, 1}_{r-5}). \end{aligned} \quad (6.15)$$

Extend (6.11) by the identity matrix of size $r-5$. The two block identities give the diagonal congruence, and the condition $A1 \equiv 1 \pmod{2}$ is unchanged. Both diagonal sums are $5N + r - 5$. Theorem 6.2 gives equivalence for every k .

The maximum absolute labels are $18N$ and $15N$, so the reduced classes differ. Their common normalized linear form is

$$\ell_{r,N} = (6 - r - 5N)t.$$

Its content is $5N + r - 6$. For fixed r , these positive integers are distinct as N varies. Content is invariant under integral equivalence; therefore the resulting invariant systems are pairwise nonisomorphic. $\square$

6.4. A pointed integral normal form. Put

$$D(a, b) = \text{diag}(a, -a, b, -b), \quad e = (1, 1, 1, 1)^\top, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 6.7. *Let a, b be positive odd coprime integers and put $N = ab$. There is $P_{a,b} \in \text{GL}(4, \mathbb{Z})$ with first column e such that*

$$P_{a,b}^\top D(a, b) P_{a,b} = J \oplus NH. \quad (6.16)$$

If a', b' satisfy the same hypotheses and $a'b' = N$, there is $A \in \text{GL}(4, \mathbb{Z})$ such that

$$A^\top D(a, b) A = D(a', b'), \quad Ae = e. \quad (6.17)$$

Proof. Choose $s, t \in \mathbb{Z}$ with $as + bt = 1$. In $\mathbb{Z}^4$ set

$$u = (1, 1, 0, 0)^\top, \quad v = (0, 0, 1, 1)^\top, \quad u' = (1, 0, 0, 0)^\top, \quad v' = (0, 0, 1, 0)^\top.$$

These vectors form an integral basis. Denote the bilinear form with Gram matrix $D(a, b)$ by $(\ , \)$. Its only nonzero pairings on this basis, apart from symmetry, are

$$(u, u') = (u', u') = a, \quad (v, v') = (v', v') = b.$$

Define

$$\begin{aligned} e &= u + v, & f_0 &= su' + tv', \\ z &= btu - asv, & w_0 &= bu' - av', \\ w &= w_0 - N(s - t)e, & \eta &= w - \frac{a+b}{2}z, \\ f &= f_0 + \frac{1 - as^2 - bt^2}{2}e. \end{aligned}$$

Both halves are integers, since $a + b$ is even and $as^2 + bt^2 \equiv as + bt \equiv 1 \pmod{2}$. Relative to (u, v, u', v') , the change to (e, z, f_0, w_0) has diagonal blocks

$$\begin{pmatrix} 1 & bt \\ 1 & -as \end{pmatrix}, \quad \begin{pmatrix} s & b \\ t & -a \end{pmatrix},$$

each of determinant -1 . Integral column additions and a permutation then give (e, f, z, η) , which is therefore an integral basis.

The defining pairings give

$$\begin{aligned} (e, e) &= 0, & (e, f_0) &= 1, & (f_0, f_0) &= as^2 + bt^2, \\ (z, e) &= (z, f_0) = (z, z) = 0, & (z, w_0) &= ab(as + bt) = N, \\ (e, w_0) &= 0, & (f_0, w_0) &= N(s - t), & (w_0, w_0) &= N(a + b). \end{aligned}$$

Consequently $(e, f) = 1$, $(f, f) = 1$, and both z and w are orthogonal to e, f : the displayed pairings give $(z, f) = (z, e) = 0$ and

$$(w, e) = 0, \quad (w, f) = (w_0, f_0) - N(s - t)(e, f_0) = 0.$$

Also $w^2 = w_0^2 = N(a + b)$ and $(z, w) = N$. Therefore

$$(z, \eta) = N, \quad (z, z) = 0, \quad (\eta, \eta) = N(a + b) - (a + b)N = 0.$$

The matrix with columns (e, f, z, η) satisfies (6.16). Finally, set $A = P_{a,b}P_{a',b'}^{-1}$. Writing $P = P_{a,b}$ and $P' = P_{a',b'}$ gives

$$A^\top D(a, b)A = P'^{-\top}(J \oplus NH)P'^{-1} = D(a', b'), \quad Ae = PP'^{-1}P'e_1 = Pe_1 = e.$$

Both P -matrices are unimodular, so $A \in \text{GL}(4, \mathbb{Z})$. □

6.5. Fibres in fixed rank. Fix $r \geq 5$ and an odd integer $N > 1$. For an unordered factorization $N = ab$ with $a, b > 0$ and $\gcd(a, b) = 1$, choose $a \leq b$ and set

$$\mathcal{G}_{r,N}(a, b) = G_{r-1}(r-4; a, -a, b, -b, \underbrace{1, \dots, 1}_{r-5}). \quad (6.18)$$

Interchanging a, b merely permutes branches. Denote the vector of branch weights by $\mathbf{a}_r(a, b)$, and put

$$D_r(a, b) = D(a, b) \oplus I_{r-5}.$$

Proof of Theorem D. The graphs (6.18) are trees. Their $r - 1$ unlabelled vertices are trivalent; the central labelled vertex has degree $r - 1$ and label $r - 4 > 0$; all leaf labels are nonzero. They are reduced and spin, and (6.1) gives rank r . The sum of their branch weights is $r - 5$, so (6.3) gives

$$\ell_{\mathcal{G}_{r,N}(a,b)} = t. \quad (6.19)$$

This covector is primitive.

For two factorizations, extend A from Lemma 6.7 by I_{r-5} and denote the resulting matrix by $\hat{A}$. Then

$$\hat{A}\mathbf{1} = \mathbf{1}, \quad \hat{A}^\top D_r(a, b)\hat{A} = D_r(a', b').$$

The equality

$$\mathbf{1}^\top D_r(a, b)\hat{A} = \mathbf{1}^\top \hat{A}^\top D_r(a, b)\hat{A} = \mathbf{1}^\top D_r(a', b')$$

and (6.2) show that $T(t, x) = (t, \hat{A}x)$ identifies the two cubics. It also preserves (6.19). Equation (2.11) gives the same equivalence for every k .

The central vertex is the unique labelled vertex of degree greater than one. Its positive label excludes a common reversal. An isomorphism without reversal preserves the multiset of negative branch weights $\{-a, -b\}$, and hence the unordered factorization. Conversely, an interchange of the factors is a branch permutation. If $N = \prod_{i=1}^s p_i^{e_i}$, coprimality assigns each $p_i^{e_i}$ wholly to one factor. There are 2^s ordered assignments. An assignment fixed by interchange would have $a = b$; coprimality would then give $a = b = 1$, contrary to $N > 1$. Thus every interchange orbit has size two, and the number of reduced classes in the displayed family is $2^{s-1} = 2^{\omega(N)-1}$.

Fix k and normalize $v = \mu^k/\rho_k$ and $\ell = p^k/\lambda_k = t$. These tensors are integral by (2.11). Put $K = \ker \ell$. The cubic vanishes on K ; polarization gives $v|_{K^3} = 0$. For any z with $\ell(z) = 1$, set

$$B(u, v) = -v(z, u, v) \quad (u, v \in K).$$

Such z exists by primitivity, and two choices differ by an element of K , so B is independent of the choice. An isomorphism of invariant systems preserves ℓ , ν , and therefore B . In the coordinate basis of K , with $z = (1, 0, \dots, 0)$, its Gram matrix is $D_r(a, b)$. Hence

$$|\det B| = a^2 b^2 = N^2. \quad (6.20)$$

The absolute determinant is unchanged by an integral basis change. Different N therefore give different systems.

Given $M \geq 1$, choose $s \geq 1$ with $2^{s-1} \geq M$. Fix $s - 1$ distinct odd primes and let a further odd prime vary outside that finite set. Their products give infinitely many N with $\omega(N) = s$. $\square$

Example 6.8. For $r = 5$ and $N = 105$, the factorizations

$$\{1, 105\}, \quad \{3, 35\}, \quad \{5, 21\}, \quad \{7, 15\}$$

give four distinct reduced classes of the form $G_4(1; a, -a, b, -b)$ and one common invariant system. A product of seven distinct odd primes gives 64 displayed reduced classes in rank five.

Proposition 6.9. *For fixed rank, an invariant system has only finitely many reduced presentations within the paired-leaf class of Definition 6.1.*

Proof. Fix one presentation $G_m(h; a)$. By Theorem 6.2, any equivalent $G_m(h'; b)$ has $h' = \pm h$ and

$$\left| \prod_{i=1}^m b_i \right| = \left| \prod_{i=1}^m a_i \right| =: D > 0.$$

Every $|b_i|$ is a positive divisor of D . There are finitely many ordered signed m -tuples of these divisors and at most two central labels. Passing to reduced classes cannot increase their number. $\square$

Remark 6.10. The count in Theorem D is a lower bound for the full fibre, not a classification of all its presentations. Proposition 6.9 concerns only paired-leaf trees. Neither statement determines the least rank with unbounded fibre sizes.

Corollary 6.11. *For each fixed $r \geq 6$, the number of connected reduced nonspin presentations in one invariant-system class is unbounded as the class varies. All integer labels may be required to be nonzero, and the displayed equivalences hold for every $k \geq 1$.*

Proof. Apply (6.18) in rank $r - 1$ and add one positive additional coordinate z at the central vertex. Its main label is $r - 5 > 0$. Reducedness is preserved and the rank increases by one. The forms become

$$f^k = \rho_k f + 3tz^2, \quad p^k = (\lambda_k + c_k)t, \quad w = e_z \pmod{2}.$$

The maps used in Theorem D fix t ; extend them by fixing z . They preserve the three displayed data. The centre is the unique labelled nonleaf and has positive label. Reversal is impossible, and the negative branch weights distinguish the unordered factorizations.

For fixed k , $p^k/(\lambda_k + c_k) = t$ is a primitive integral covector. Put $K = \ker t$ and choose v with $t(v) = 1$. The cubic vanishes on K , so $-\mu^k(v, -, -)|_{K^2}$ is independent of v and is preserved by system isomorphisms. In the displayed coordinates its Gram matrix is

$$\rho_k D_{r-1}(a, b) \oplus (-1),$$

with absolute determinant $\rho_k^{r-2} N^2$. Thus distinct N give distinct classes of invariant systems. Their displayed presentation counts $2^{\omega(N)-1}$ are unbounded as in Theorem D. $\square$

6.6. A discriminant of the cubic.

Lemma 6.12. *Let H be a free abelian group of finite rank at least two, and let μ be an integral symmetric trilinear form. Suppose the nonzero cubic $f(x) = \mu(x, x, x)$ has a unique rational linear factor up to*

nonzero scalar. Choose a primitive integral covector τ on that factor line, put $K = \ker \tau$, and choose $z \in H$ with $\tau(z) = 1$. Then

$$Q_\mu(u, v) = \mu(z, u, v) \quad (u, v \in K) \quad (6.21)$$

is independent of z . The number

$$\Delta(\mu) = |\det Q_\mu| \quad (6.22)$$

is independent of the integral basis of K and the sign of τ . It is preserved by integral equivalence of tensors, including equivalence up to multiplication of the tensor by -1 .

Proof. A rational line in $H^* \otimes \mathbb{Q}$ contains exactly two primitive integral generators. Primitivity gives $H = \mathbb{Z}z \oplus K$. Since τ divides f , one has $f|_K = 0$ and hence $\mu|_{K^3} = 0$ by polarization. Replacing z by $z + v$, $v \in K$, leaves (6.21) unchanged. Replacing τ by $-\tau$ permits the lift $-z$ and negates Q_μ . An integral basis change with matrix U replaces a Gram matrix Q by $U^\top Q U$, whose determinant is $(\det U)^2 \det Q = \det Q$.

Let $T : H' \rightarrow H$ satisfy $\mu(Tx, Ty, Tw) = \epsilon \mu'(x, y, w)$, where $\epsilon = \pm 1$. Uniqueness of the factor line and primitivity give $\tau \circ T = s\tau'$ for some $s = \pm 1$. Thus $T(K') = K$. For $\tau'(z') = 1$, the vector sTz' has τ -value one; hence

$$Q_\mu(Tu, Tv) = \mu(sTz', Tu, Tv) = s\epsilon Q_{\mu'}(u, v) \quad (u, v \in K').$$

This sign and the integral change of basis preserve the absolute determinant. $\square$

Proposition 6.13. *For the graphs (6.18), the normalized integral cubic has a unique rational linear-factor line, generated by t , and*

$$\Delta(\mu_{G_{r,N}(a,b)}^1) = N^2.$$

For fixed r , all these normalized cubics are equivalent over $\mathbb{Q}$ as N varies.

Proof. Put $y = 2x + t1$. Equation (6.4) gives

$$f(t, y) = \frac{t}{4} ((3r - 11)t^2 - 3y^\top D_r(a, b)y). \quad (6.23)$$

The quadratic factor has rank at least $r - 1 \geq 4$. A product of two linear forms has rank at most two, so this quadratic factor is irreducible over $\mathbb{Q}$. Unique factorization makes $\mathbb{Q}t$ the unique linear-factor line. In the original integral coordinates, the form (6.21) has Gram matrix $-D_r(a, b)$. Thus its absolute determinant is N^2 .

For any $d \in \mathbb{Q}^\times$,

$$R_d = \begin{pmatrix} 1 & 1/(2d) \\ 1 & -1/(2d) \end{pmatrix}, \quad R_d^\top \text{diag}(d, -d)R_d = \mathbb{H}.$$

Therefore $D_r(a, b)$ is rationally congruent to $\mathbb{H} \oplus \mathbb{H} \oplus I_{r-5}$, independently of a, b . The coefficient $3r - 11$ in (6.23) depends only on r . These congruences, together with the invertible rational change $y = 2x + t1$, identify all the cubics at fixed rank. $\square$

7. COORDINATE CONFIGURATIONS OF FORESTS

Let (U, V, E) be a bipartite forest in which every vertex of V has degree at least three. Set

$$H_0 = \left\{ x \in \mathbb{Z}^U : \sum_{u \sim v} x_u = 0 \text{ for all } v \in V \right\}, \quad L_u(x) = x_u.$$

Linear dependence of the covectors L_u is taken in $H_0^* \otimes \mathbb{Q}$.

Definition 7.1 (Closed circuit). A *circuit* of $(L_u)_{u \in U}$ is a nonempty subset $C \subseteq U$ for which $(L_u)_{u \in C}$ is linearly dependent and every proper subset is linearly independent. It is *closed* if, for every $w \in U$,

$$L_w \in \text{span}_{\mathbb{Q}}\{L_u : u \in C\} \implies w \in C. \quad (7.1)$$

Circuits and closed circuits are unchanged by independent nonzero rescaling of the covectors.

Theorem 7.2. *Let (U, V, E) be a bipartite forest with $\deg(v) \geq 3$ for every $v \in V$. For its coordinate configuration $(L_u)_{u \in U}$:*

- (i) *every L_u is primitive and nonzero, and no two are proportional;*
- (ii) *the closed circuits are precisely the neighbour sets $N(v) = \{u \in U : u \sim v\}$, for $v \in V$;*
- (iii) *for every integer $d \geq 2$, the homogeneous polynomials L_u^d , $u \in U$, are linearly independent over $\mathbb{Q}$.*

Proof. Root each component at a labelled vertex. Such a root exists because an isolated unlabelled vertex is excluded by the degree hypothesis. Initially the root coordinates are independent. Adjoin each unlabelled vertex after its parent: it joins an old labelled vertex p to $s \geq 2$ new labelled vertices $q_1, \dots, q_s$. Set $b_i = L_{q_i}$. The new equation and an integral parametrization are

$$L_p + b_1 + \dots + b_s = 0, \quad (7.2)$$

$$H_{\text{new}} = H_{\text{old}} \oplus \mathbb{Z}^{s-1}, \quad b_i = t_i \ (i < s), \quad b_s = -L_p - \sum_{i < s} t_i. \quad (7.3)$$

The inverse of this parametrization reads the old coordinates and $b_1, \dots, b_{s-1}$.

An old primitive covector remains primitive under pullback by the projection onto H_{old} . Each b_i has a coefficient 1 or -1 on a new variable, so is primitive. In particular all these covectors are nonzero. Put $E_{\text{old}} = H_{\text{old}}^* \otimes \mathbb{Q}$. Modulo E_{old} , the only relation among the images $\bar{b}_i$ is $\sum_i \bar{b}_i = 0$: the coefficients of each t_i force all relation coefficients to agree. Each image is nonzero. Thus no new covector is proportional to an old one. When $s \geq 3$, any two images are independent. When $s = 2$, an equality $-L_p - t_1 = ct_1$ would give $c = -1$ and $L_p = 0$, which is impossible. Induction proves (i).

In any relation involving a new covector, passage to the quotient forces all s new covectors to occur with the same nonzero coefficient. Consequently

$$C_0 = \{p, q_1, \dots, q_s\} \quad (7.4)$$

is a circuit: (7.2) is a dependence, while omitting a new covector forces all coefficients to vanish, and omitting p would leave a nonzero multiple of L_p .

Let C be any circuit containing a new covector. If $p \in C$, the inclusion $C_0 \subseteq C$ and minimality give $C = C_0$. Otherwise put $S = C \cap U_{\text{old}}$. Minimality makes the old covectors indexed by S independent. Their relation with the b_i , followed by (7.2), expresses L_p as a linear combination of these covectors. No proper subset of S spans L_p , since it would give a dependence on a proper subset of C . Hence $D = S \cup \{p\}$ is an old circuit and

$$C = (D \setminus \{p\}) \cup \{q_1, \dots, q_s\}. \quad (7.5)$$

Conversely, let D be an old circuit containing p . Substituting $L_p = -\sum_i b_i$ in its relation gives a dependence on (7.5). A nonzero dependence on a proper subset containing a new covector would have to contain all of them, with equal coefficients; substitution back would give a dependence on a proper subset of D . A dependence containing no new covector would lie in $D \setminus \{p\}$, which is independent. Thus every set (7.5) is a circuit.

The old covectors embed in E_{old} and no new one lies there, so the old closed circuits remain exactly the old closed circuits. The quotient relation also gives

$$\text{span}_{\mathbb{Q}}\{L_u : u \in C_0\} \cap E_{\text{old}} = \mathbb{Q}L_p.$$

By (i), no other old covector lies on this line. Thus C_0 is closed. A circuit (7.5) is not closed: it omits p , although $L_p = -\sum_i b_i$ belongs to its span. Each attachment therefore adds exactly its neighbour set to the closed circuits. Initially there are no circuits. This proves (ii).

For (iii), fix $d \geq 2$ and suppose

$$F_{\text{old}} + \sum_{i < s} c_i t_i^d + c_s \left(-L_p - \sum_{i < s} t_i \right)^d = 0,$$

where F_{old} is a linear combination of old powers. Set $t_2 = \dots = t_{s-1} = 0$. As a polynomial in t_1 with coefficients in the old polynomial ring, its coefficient of t_1^{d-1} is $d(-1)^d c_s L_p$. It vanishes only when $c_s = 0$, since $d \neq 0$ in $\mathbb{Q}$ and $L_p \neq 0$. The coefficient of each t_i^d then gives $c_i = 0$, leaving $F_{\text{old}} = 0$. The powers of the initial independent root coordinates are independent, so induction proves (iii). $\square$

Corollary 7.3. *The configuration of coordinate lines determines the bipartite forest up to isomorphism. Suppose that an isomorphism of two coordinate spaces satisfies $L_u \mapsto s_u L'_{\pi(u)}$, with $s_u \in \{1, -1\}$ and π a bijection of their indexing sets. Then s_u is constant on the labelled vertices of each connected component.*

Proof. The distinct coordinate lines give the set U . By Theorem 7.2(ii), their closed circuits give the sets $N(v)$. Distinct unlabelled vertices have distinct neighbour sets: if two such sets agreed, their cardinality at least three would provide two common neighbours and a four-cycle. Thus take V to be the set of closed circuits and define incidence by membership. A labelled vertex incident to an edge lies in its neighbour-set circuit; an isolated labelled vertex is an independent direct-summand coordinate and lies in no circuit. This reconstructs also the isolated components.

An isomorphism sending L_u to $s_u L'_{\pi(u)}$ preserves dependence and spans, hence carries closed circuits to closed circuits. On $N(v)$ it sends the relation to

$$\sum_{u \in N(v)} s_u L'_{\pi(u)} = 0.$$

The target circuit is $N(v')$. Its relation space is one-dimensional: removal of any one element gives an independent set of size $|N(v')| - 1$. The vector with all coefficients one spans that relation space, so all s_u on $N(v)$ agree. Every path between labelled vertices in a component passes through such neighbour sets. The signs therefore agree on the whole component; for an isolated vertex the assertion is automatic. $\square$

8. QUADRATIC LATTICES AND COORDINATE RECOVERY

A matrix is *totally unimodular* if every square minor belongs to $\{0, 1, -1\}$.

Lemma 8.1. *For a finite bipartite forest (U, V, E) , the incidence matrix $B = (1_{u \sim v})_{v \in V, u \in U}$ is totally unimodular.*

Proof. The support graph of a square submatrix is a forest. If it has an isolated vertex, the determinant is zero. Otherwise it has a vertex of degree one. Expansion along the corresponding row or column reduces the determinant, up to sign, to a smaller minor. Induction, starting with the empty minor of determinant one, proves the assertion. $\square$

Let B now be any totally unimodular matrix with N columns. Put

$$K = \ker_{\mathbb{R}} B, \quad L = K \cap \mathbb{Z}^N,$$

Give K the Euclidean inner product, with quadratic form $q(x) = \|x\|^2$. A rational basis of $\ker_{\mathbb{Q}} B$ can be multiplied by positive integers to give integral vectors spanning K . Since $L \subseteq \mathbb{Z}^N$ is discrete and contains these vectors, it is a full lattice in K . Let $P : \mathbb{R}^N \rightarrow K$ be orthogonal projection.

Lemma 8.2. *Let B be totally unimodular and $K = \ker_{\mathbb{R}} B$. Every nonzero vector of K with inclusion-minimal support is a real multiple of an integral vector with entries in $\{0, 1, -1\}$. Every $h \in K$ is a sum*

$$h = \sum_{v=1}^s t_v c_v, \quad t_v > 0, \tag{8.1}$$

where each c_v is an integral vector of inclusion-minimal nonempty support with entries in $\{0, 1, -1\}$, and is conformal to h : $\text{supp } c_v \subseteq \text{supp } h$ and the two vectors have the same sign on $\text{supp } c_v$. Consequently,

$$\|h\|_1 = \sum_v t_v \|c_v\|_1.$$

For $h = 0$, the sum is empty.

Proof. Let S be a minimal nonempty support of a vector in K , and put $s = |S|$. The submatrix B_S has a one-dimensional kernel. Indeed, any two independent vectors in that kernel admit a nonzero linear combination whose prescribed coordinate is zero, contradicting the minimality of S . Thus $\text{rank } B_S = s - 1$.

If $s > 1$, choose $s - 1$ independent rows of B_S to obtain an $(s - 1) \times s$ matrix C . They span its row space, so $\ker C = \ker B_S$. For $1 \leq i \leq s$, let $C_{\hat{i}}$ omit column i . Laplace expansion gives a kernel vector

$$c_i = (-1)^{i-1} \det C_{\hat{i}}.$$

At least one coordinate is nonzero because C has rank $s - 1$. Minimality of S makes every coordinate nonzero. Total unimodularity then gives $c_i \in \{1, -1\}$. Extending by zero outside S proves the first assertion. If $s = 1$, the corresponding column of B is zero and the unit vector gives the same conclusion.

Let $h \neq 0$ and set

$$\mathcal{C}_h = \{z \in K : z_i = 0 \text{ if } h_i = 0, h_i z_i \geq 0 \text{ otherwise}\}.$$

This set contains h . Choose a nonzero $z \in \mathcal{C}_h$ with minimal support among its nonzero elements. Suppose $v \in K \setminus \{0\}$ has $\text{supp } v \subsetneq \text{supp } z$. Let

$$r = \min_{v_i \neq 0} \frac{|z_i|}{|v_i|} > 0, \quad \sigma = \text{sgn}(z_{i_0} v_{i_0}),$$

where i_0 attains the minimum. Since $r|v_i| \leq |z_i|$ on $\text{supp } v$, every coordinate of $z - \sigma r v$ has the sign of z or is zero. Its i_0 -coordinate vanishes, and any coordinate of z outside $\text{supp } v$ is unchanged. Hence it is a nonzero element of $\mathcal{C}_h$ with smaller support, a contradiction. The support of z is therefore a minimal nonempty kernel support. The first assertion gives a primitive $c \in \{0, \pm 1\}^N$ spanning $\mathbb{R}z$; choose the sign so that $c \in \mathcal{C}_h$.

Put $t = \min_{c_i \neq 0} |h_i| > 0$. Then $h - tc \in \mathcal{C}_h$ and its support is strictly smaller than that of h . Induction on the finite support produces (8.1). At each step every surviving coordinate has the sign of the original h , so the summands are conformal to h . Summing their coordinatewise absolute values gives $\|h\|_1 = \sum_v t_v \|c_v\|_1$. For $h = 0$ the empty sum applies. $\square$

The Voronoi cell of (L, q) is

$$\text{Vor}(L) = \{z \in K : \|z\| \leq \|z - h\| \text{ for all } h \in L\}.$$

Proposition 8.3 (Projection formula [5, Theorem 2.3]). *Let B be a totally unimodular matrix with N columns, let $K = \ker_{\mathbb{R}} B$ and $L = K \cap \mathbb{Z}^N$, and let $P : \mathbb{R}^N \rightarrow K$ be orthogonal projection for the standard Euclidean inner product. Then*

$$\text{Vor}(L) = P([-1/2, 1/2]^N). \quad (8.2)$$

Proof. For $z \in K$ and $h \in L$, expansion of $\|z - h\|^2$ shows that the Voronoi condition is

$$\langle z, h \rangle \leq \frac{1}{2} \|h\|^2 \quad (h \in L). \quad (8.3)$$

If $a \in [-1/2, 1/2]^N$, then

$$\langle Pa, h \rangle = \langle a, h \rangle \leq \frac{1}{2} \sum_i |h_i| \leq \frac{1}{2} \sum_i h_i^2$$

for integral h . Thus $P([-1/2, 1/2]^N) \subseteq \text{Vor}(L)$.

Conversely, take $z \in \text{Vor}(L)$ and $h \in K$. For the conformal expansion $h = \sum_v t_v c_v$ of Lemma 8.2, $\|c_v\|^2 = \|c_v\|_1$ because its coordinates are 0, ± 1 . Applying (8.3) to each $c_v \in L$ gives

$$\langle z, h \rangle \leq \frac{1}{2} \sum_v t_v \|c_v\|_1 = \frac{1}{2} \|h\|_1.$$

For the compact convex set $Z = P([-1/2, 1/2]^N)$, its support function is

$$\max_{a \in [-1/2, 1/2]^N} \langle Pa, h \rangle = \max_{|a_i| \leq 1/2} \sum_i a_i h_i = \frac{1}{2} \|h\|_1 \quad (h \in K).$$

If $z \notin Z$, strict separation from Z would give some $h \in K$ violating this upper bound. Hence $z \in Z$. $\square$

Proposition 8.4. *Let K, L, P be as in Proposition 8.3. Suppose the nonzero restrictions to K of the coordinate covectors of $\mathbb{R}^N$ are pairwise nonproportional. The quadratic lattice (L, q) determines each of these covectors up to sign. Every integral isometry between two such quadratic lattices carries their coordinate configurations to one another. In rank zero the recovered configuration is empty.*

Proof. Put $g_i = Pe_i$ and omit the zero vectors. Projection and quadratic duality give $\langle g_i, x \rangle = x_i$ for $x \in K$; consequently the g_i are pairwise nonparallel. They span K because P is surjective. Proposition 8.3 gives

$$\text{Vor}(L) = \sum_i [-g_i/2, g_i/2]. \quad (8.4)$$

An edge vector is the difference of its endpoints, considered up to sign.

Suppose $\dim K \geq 2$ and fix i . In the hyperplane $A_i = \{\phi \in K^* : \phi(g_i) = 0\}$, each subspace $A_i \cap \{\phi : \phi(g_j) = 0\}$, $j \neq i$, is proper: otherwise g_j would be proportional to g_i . A finite union of proper real linear subspaces cannot exhaust A_i . Choose $\phi \in A_i$ outside this union. The face maximizing ϕ in (8.4) is

$$\sum_{j \neq i} \frac{\text{sgn}(\phi(g_j))}{2} g_j + [-g_i/2, g_i/2].$$

It is an edge with edge vector g_i .

Every edge of a polytope is exposed. For an exposing functional ψ , the maximizing face in (8.4) is the sum of a fixed endpoint for each $\psi(g_j) \neq 0$ and the full segment for each $\psi(g_j) = 0$. Its direction space is the span of the latter generators. This space has dimension one for an edge. Nonparallelism therefore permits exactly one such generator. The edge vector is its positive or negative.

If $\dim K = 1$, there is exactly one nonzero generator by the hypothesis, and it is the endpoint difference of the Voronoi interval. If $K = 0$, there are none. Thus the intrinsic Voronoi cell recovers all the pairs $\{g_i, -g_i\}$. Their quadratic duals are the coordinate covectors with their original scales. An integral isometry extends to a real isometry; the defining distance inequalities show that it carries the two Voronoi cells to one another. It therefore transports the edge vectors and their quadratic duals, as asserted. $\square$

Corollary 8.5. *Let (U, V, E) and (U', V', E') be finite bipartite forests whose unlabelled vertices have degree at least three, and let B, B' be their incidence matrices. If*

$$\left(\ker_{\mathbb{Z}} B, x \mapsto \sum_{u \in U} x_u^2 \right) \cong \left(\ker_{\mathbb{Z}} B', x' \mapsto \sum_{u' \in U'} (x'_{u'})^2 \right)$$

as integral quadratic lattices, then the forests are isomorphic as bipartite graphs.

Proof. Lemma 8.1, Theorem 7.2(i), and Proposition 8.4 recover the coordinate covectors up to sign. Their closed circuits recover the forest by Corollary 7.3. $\square$

8.1. Relation with Su–Wagner’s theorem. For a matrix D with columns indexed by a finite set S , let $\mathbf{M}_D$ denote its column matroid over $\mathbb{Q}$: a subset of S is independent precisely when the corresponding columns are linearly independent. A matroid is *regular* if it admits a totally unimodular representation. The bases of the dual matroid $\mathbf{M}^*$ are the complements of the bases of $\mathbf{M}$. A *coloop* belongs to every basis. Write $\mathbf{M}_.$ for the minor obtained by contracting all coloops. A *flat* is a subset containing every element in its span; in particular, the closed circuits of Definition 7.1 are exactly the circuits which are flats.

If D is totally unimodular, its integral flow lattice is

$$\Lambda(M_D) = \left(\ker_{\mathbb{Z}} D, x \mapsto \sum_{s \in S} x_s^2 \right).$$

Its isometry class is independent of the totally unimodular representation [8, Sections 2.2–2.3]. For regular matroids M, N , Su and Wagner prove [8, Theorem 1]

$$\Lambda(M) \cong \Lambda(N) \iff M. \cong N., \quad (8.5)$$

where the isomorphism on the left is an isometry of integral quadratic lattices.

For a bipartite forest, the ground set of M_B is the labelled vertex set U . The matrix B records relations indexed by V ; it is not the signed vertex–edge incidence matrix of the forest.

Proposition 8.6 (The relation matroid of a forest). *Let $F = (U, V, E)$ be a finite bipartite forest with $\deg(v) \geq 3$ for every $v \in V$. Let $B = (1_{u \sim v})_{v \in V, u \in U}$, $H_0 = \ker_{\mathbb{Z}} B$, and $L_u(x) = x_u$. Then:*

- (i) M_B is regular and has no coloops; the matroid represented by $(L_u)_{u \in U}$ is M_B^* ;
- (ii) the map $v \mapsto N(v)$ is a bijection from V to the circuits of M_B^* which are flats;
- (iii) every isomorphism between the relation matroids of two such forests extends uniquely to a bipartite graph isomorphism between the forests.

Proof. Regularity follows from Lemma 8.1. Put $K = H_0 \otimes \mathbb{Q}$, $n = |U|$, and $d = \text{rank } B$. For $J \subseteq U$ with $|J| = n - d$, the covectors $(L_u)_{u \in J}$ form a basis of K^* if and only if the coordinate projection $K \rightarrow \mathbb{Q}^J$ is an isomorphism. Since the dimensions agree, this is equivalent to

$$K \cap \mathbb{Q}^{U \setminus J} = 0,$$

where $\mathbb{Q}^{U \setminus J}$ is the coordinate subspace supported on $U \setminus J$. The latter condition says that the d columns of B indexed by $U \setminus J$ are independent, hence form a basis of M_B . The two matroids are therefore dual. This argument also includes $d = 0$ and $K = 0$.

For any rational matrix, a column indexed by u is a coloop precisely when removing it lowers the rank, equivalently when it is not spanned by the other columns. A kernel vector with nonzero u -coordinate, solved for that column, is precisely an expression in the span of the others. Hence the coloop condition is equivalent to every kernel vector having zero u -coordinate. Thus

$$u \text{ is a coloop of } M_B \iff L_u = 0. \quad (8.6)$$

Theorem 7.2(i) excludes this equality and proves (i). Part (ii) is Theorem 7.2(ii), since closedness is the flat condition. The neighbour sets are distinct: two vertices of V with the same neighbour set would have two common neighbours and give a cycle.

An isomorphism $\varphi : M_B \rightarrow M_{B'}$ also identifies the dual matroids and preserves their circuits and flats. By (ii), for every $v \in V$ there is a unique $v' \in V'$ such that $\varphi(N(v)) = N(v')$. Define $\varphi(v) = v'$. Applying the same construction to φ^{-1} gives its inverse on V , so it is a bijection. Since $u \in N(v)$ exactly when $\varphi(u) \in N(\varphi(v))$, it preserves incidence. Together with the given bijection on U , it is the unique bipartite graph isomorphism extending φ . Conversely, a bipartite graph isomorphism permutes the rows and columns of the relation matrix and hence induces an isomorphism of its column matroid. $\square$

For these forests, $(M_B). = M_B$. Consequently, (8.5) and Proposition 8.6 give

$$(H_0, q) \cong (H'_0, q') \iff M_B \cong M_{B'} \iff F \cong F'. \quad (8.7)$$

Thus Corollary 8.5 is also a consequence of the Su–Wagner theorem and the closed-circuit description of the bipartite incidence relation. The coordinate recovery of Proposition 8.4 gives the signed coordinate map for each prescribed lattice isometry, as required for simultaneous preservation of the trilinear form and the parity class.

8.2. Positive coordinate weights. Let B be totally unimodular, $K = \ker_{\mathbb{R}} B$, and $L = K \cap \mathbb{Z}^n$. For positive real numbers $\eta = (\eta_1, \dots, \eta_n)$, equip $\mathbb{R}^n$ with

$$\langle x, y \rangle_{\eta} = \sum_i \eta_i x_i y_i, \quad q_{\eta}(x) = \sum_i \eta_i x_i^2,$$

and let $P_{\eta} : \mathbb{R}^n \rightarrow K$ be orthogonal projection for this inner product.

Proposition 8.7 (Weighted projection [5, Theorem 2.3]). *The Voronoi cell of (L, q_{η}) is*

$$\text{Vor}(L, q_{\eta}) = P_{\eta}([-1/2, 1/2]^n).$$

Proof. For $a \in [-1/2, 1/2]^n$ and $h \in L$,

$$\langle P_{\eta} a, h \rangle_{\eta} = \langle a, h \rangle_{\eta} \leq \frac{1}{2} \sum_i \eta_i |h_i| \leq \frac{1}{2} q_{\eta}(h).$$

The last inequality follows from $|h_i| \leq h_i^2$ for integral h_i . This proves the inclusion from right to left.

Conversely, let $z \in \text{Vor}(L, q_{\eta})$ and $h \in K$. Apply Lemma 8.2 to write $h = \sum_v t_v c_v$, with $t_v > 0$ and conformal integral $c_v \in \{0, \pm 1\}^n$. The Voronoi inequalities give

$$\langle z, h \rangle_{\eta} \leq \frac{1}{2} \sum_v t_v q_{\eta}(c_v) = \frac{1}{2} \sum_i \eta_i |h_i|.$$

For $h \in K$, the support function is

$$\max_{|a_i| \leq 1/2} \langle P_{\eta} a, h \rangle_{\eta} = \max_{|a_i| \leq 1/2} \sum_i \eta_i a_i h_i = \frac{1}{2} \sum_i \eta_i |h_i|.$$

Strict separation would contradict the preceding inequality if z were outside the projected cube. This proves the other inclusion. $\square$

Corollary 8.8. *Suppose the nonzero restrictions L_i of the coordinate covectors to L are primitive and pairwise nonproportional. The metric lattice (L, q_{η}) determines the pairs $\{L_i, -L_i\}$ and their positive weights η_i for every nonzero restriction. Every integral isometry between two such metric lattices transports these weighted configurations. No weight attached to an identically zero restriction is determined by this assertion.*

Proof. Put $g_i = P_{\eta} e_i$ and omit zero vectors. Their metric duals are

$$\langle g_i, x \rangle_{\eta} = \eta_i L_i(x) \quad (x \in K).$$

Thus the g_i are pairwise nonparallel. By Proposition 8.7, the Voronoi cell is the Minkowski sum of the segments $[-g_i/2, g_i/2]$. The exposed-edge argument in Proposition 8.4 applies to this sum and recovers exactly the edge vectors $\pm g_i$, including in ranks zero and one. Metric duality recovers $\pm \eta_i L_i$. Let λ be either primitive integral generator of its real line. Since L_i is primitive, the two possibilities are $\lambda = \pm L_i$. Choosing $x \in L$ with $\lambda(x) = 1$ recovers the weight as the absolute value of the recovered covector at x , namely η_i . It is independent of this choice because the two covectors are proportional. All these constructions commute with an integral isometry. $\square$

9. RECONSTRUCTION OF THE DECORATIONS

Let G be reduced. Append zero columns to its bipartite incidence matrix for the additional coordinates. The resulting matrix is totally unimodular, and its integral kernel is H_G with the form q_G . On ordinary components, the main coordinate covectors are nonzero and pairwise nonproportional by Theorem 7.2. The additional coordinates are independent orthogonal unit directions, so they introduce no proportionalities. Only a possible exceptional component has a zero main coordinate.

Proposition 8.4 thus recovers every nonzero coordinate covector up to sign. Regard each pair $\{\lambda, -\lambda\}$ as one index. When forming circuits, choose one representative of each pair; the resulting circuits and their closures do not depend on that choice.

Lemma 9.1. *For a reduced graph G , the recovered nonzero coordinate covectors split into two intrinsic sets: the ordinary main covectors, which vanish on w_G modulo 2, and the additional covectors, which take the value 1 on w_G modulo 2.*

Proof. Let $\tilde{w} = \sum_{u,j \neq 0} u^j \in H_G$ represent w_G . Every main covector has $L_u(\tilde{w}) = 0$, whereas every additional coordinate covector has $y_{u,j}(\tilde{w}) = 1$. Evaluation modulo two is independent of the representative of w_G , since the covectors are integral. Negating a covector does not change its value modulo two. These two values therefore distinguish exactly the stated sets. $\square$

Apply the closed-circuit construction to the recovered main covectors. It gives all the ordinary components of the underlying bipartite forest. For an additional coordinate indexed by a , let e_a be its q_G -dual vector. In the original presentation this is its integral unit vector, and the reconstructed data determine it up to sign. Define

$$C_a = \mu_G^k(e_a, e_a, -) \in H_G^*. \quad (9.1)$$

This definition is independent of the sign of e_a .

Lemma 9.2. *Let G be reduced. For an additional coordinate a attached to an ordinary vertex u with sign $\sigma_a \in \{1, -1\}$, one has*

$$C_a = \sigma_a L_u. \quad (9.2)$$

Thus C_a determines the parent u and the sign relative to a chosen representative of $\{L_u, -L_u\}$. The equality $C_a = 0$ holds exactly when a belongs to an exceptional component.

Proof. In the original splitting (2.2), the quadratic dual e_a of the additional covector is its integral unit vector. The ambient basis definition of μ^k gives

$$\mu_G^k(e_a, e_a, z) = \sigma_a L_u(z) \quad (z \in H_G).$$

Negating e_a has no effect because it occurs twice. Thus $C_a = \sigma_a L_u$. On ordinary vertices the main covectors are nonzero and pairwise nonproportional by Theorem 7.2(i). Consequently a nonzero C_a determines a unique parent line; its coefficient relative to a chosen primitive representative gives the sign in that orientation.

On an exceptional component, the main coordinate vanishes on the kernel, so $C_a = 0$. No ordinary parent can give zero. Definition 2.3 permits at most one exceptional component, and it has exactly one additional coordinate. This proves also the assertion about zero contractions. $\square$

Remark 9.3 (The full relation matroid). Let $\hat{B}_G$ be the matrix obtained from B_G by adjoining the zero columns of the additional coordinates. These columns represent loops of $M_{\hat{B}_G}$, that is, elements whose singleton sets are circuits. By (8.6), its coloops are exactly the coordinates which vanish on H_G . For a reduced graph, the only possible coloop is the zero main coordinate of an exceptional component; its relation matrix is $\begin{pmatrix} 1 & 0 \end{pmatrix}$. Contracting that coloop leaves a single loop, the same matroid as an isolated main coordinate. The values of w_G distinguish the two cases. On ordinary components, the contractions in (9.2) recover the attachments and signs of the additional coordinates, and the coefficients in (9.4) recover the integer labels. These decorations are not part of the relation matroid.

Proof of Theorem C. Let $T : H_{G'} \rightarrow H_G$ preserve q, μ^k , and w . The extended relation matrix, including the zero columns for additional coordinates, is totally unimodular. Its nonzero coordinate restrictions are pairwise nonproportional, as established at the start of this section. Proposition 8.4 and Lemma 9.1 therefore give a bijection π of the ordinary main coordinates and signs s_u with

$$L_u \circ T = s_u L'_{\pi(u)}. \quad (9.3)$$

Closed circuits and Corollary 7.3 extend π to a bipartite graph isomorphism of the ordinary forests and make $s_u = s_C$ constant on each component C .

For an additional coordinate write $y_a \circ T = \tau_a y'_{a'}$, where $\tau_a = \pm 1$. For every $z \in H_{G'}$, metric duality gives

$$b_G(e_a, Tz) = \tau_a b_{G'}(e'_{a'}, z) = b_G(\tau_a T e'_{a'}, Tz).$$

Surjectivity of T and nondegeneracy of b_G imply $e_a = \tau_a T e'_{a'}$. Hence

$$C_a \circ T = \mu_G^k(e_a, e_a, T-) = \mu_{G'}^k(e'_{a'}, e'_{a'}, -) = C'_{a'}.$$

If a has ordinary parent $u \in C$, then $C_a \neq 0$. Since T is invertible, $C'_{a'} = C_a \circ T \neq 0$; hence Lemma 9.2 gives an ordinary parent u' for a' . Thus

$$\sigma_a s_C L'_{\pi(u)} = \sigma'_{a'} L'_{u'}.$$

Nonproportionality of main covectors forces $u' = \pi(u)$ and $\sigma'_{a'} = s_C \sigma_a$. Thus the numbers of positive and negative additional coordinates at each parent are preserved if $s_C = 1$ and interchanged if $s_C = -1$. The zero contractions correspond bijectively as well. They identify the possible exceptional component, including its forced zero label and single unlabelled neighbour, up to reversal.

Set

$$X_G = \bigcap_{a \text{ additional}} \ker y_a \subseteq H_G.$$

The coordinate identities give $T(X_{G'}) = X_G$. This is the ordinary main kernel; the exceptional main coordinate, if present, is zero. The restricted cubic is

$$f_{G,0}^k = \rho_k \sum_{u \text{ ordinary}} \alpha(u) L_u^3. \quad (9.4)$$

Theorem 7.2(iii) makes these cubes independent on X_G . Comparing the restricted cubics under T and using $s_C^3 = s_C$ gives

$$\alpha'(\pi(u)) = s_C \alpha(u) \quad (u \in C),$$

since $\rho_k \neq 0$. These are exactly graph isomorphism and the componentwise reversal (2.13).

Put $d(u) = k^+(u) - k^-(u)$. The additional-coordinate count gives $d'(\pi(u)) = s_C d(u)$. Together with (9.3),

$$(\lambda_k \alpha(u) + c_k d(u)) L_u \circ T = (\lambda_k \alpha'(\pi(u)) + c_k d'(\pi(u))) L'_{\pi(u)}.$$

Summing over ordinary vertices proves $p_G^k \circ T = p_{G'}^k$. An exceptional component contributes zero.

Conversely, a decorated graph isomorphism permutes ambient coordinates and preserves all defining expressions. It remains to check one component reversal. Negate each main coordinate and match an additional coordinate of sign σ with one of sign $-\sigma$ in the reversed graph, without changing its value. The homogeneous kernel equations remain satisfied; this signed coordinate map is an integral isometry. It preserves w modulo two. Each original main contribution becomes $\rho_k \alpha(-x)^3 = \rho_k (-\alpha)x^3$, and each mixed contribution becomes $3\sigma(-x)y^2 = 3(-\sigma)xy^2$, exactly the reversed formulas. Thus it gives an enhanced isomorphism. Applying these maps independently on the components proves the converse.

By Lemma 2.6 and (2.12), rank zero forces the empty graph. In rank one the only possibilities are an isolated spin vertex and the exceptional component; their parity classes separate them. Thus no zero-rank or rank-one case is omitted. $\square$

Corollary 9.4. *Let G be reduced and fix $k \geq 1$. Choose signs for the recovered ordinary main covectors and write the unique expansion*

$$f_{G,0}^k = \sum_u a_u L_u^3.$$

Then

$$p_G^k = 4 \sum_u a_u L_u + c_k \sum_{a \text{ additional}} C_a. \quad (9.5)$$

This expression is independent of the chosen signs. In particular, every isomorphism of enhanced invariant systems preserves the Pontryagin form.

Proof. In the original coordinate orientations, $a_u = \rho_k \alpha(u)$ by (9.4), and Lemma 9.2 gives

$$\sum_{a \text{ additional}} C_a = \sum_{u \text{ ordinary}} (k^+(u) - k^-(u)) L_u.$$

Exceptional contractions are zero. Since $4\rho_k = \lambda_k$, (9.5) is exactly (2.7). Replacing L_u by $-L_u$ replaces its unique cubic coefficient a_u by $-a_u$ and leaves $a_u L_u$ unchanged. Each C_a is independent of the sign of e_a . The coordinate recovery and contraction formulas commute with a prescribed enhanced isomorphism, so the displayed expression, and hence p_G^k , is preserved by that isomorphism. $\square$

Corollary 9.5. *For reduced graphs G, G' , the following are equivalent:*

- (i) *their enhanced invariant systems are isomorphic for some $k \geq 1$;*
- (ii) *their enhanced invariant systems are isomorphic for every $k \geq 1$.*

More precisely, every isomorphism of enhanced invariant systems at one value of k is an isomorphism at every value of $k \geq 1$. On the class of reduced spin graphs, the triple (H_G, q_G, μ_G^k) determines the reduced class for every fixed $k \geq 1$.

Proof. Let T be an enhanced isomorphism at $k_0 \geq 1$. By the proof of Theorem C, it satisfies

$$L_u \circ T = s_C L'_{\pi(u)}, \quad \alpha'(\pi(u)) = s_C \alpha(u), \quad y_a \circ T = \tau_a y'_{a'}, \quad \sigma'_{a'} = s_C \sigma_a,$$

where $u \in C$, the coordinate a is attached to u , and $s_C, \tau_a \in \{1, -1\}$. For any $k \geq 1$ these identities give

$$\rho_k \alpha(u) (L_u \circ T)^3 = \rho_k \alpha'(\pi(u)) (L'_{\pi(u)})^3,$$

$$3\sigma_a (L_u \circ T) (y_a \circ T)^2 = 3\sigma'_{a'} L'_{\pi(u)} (y'_{a'})^2.$$

Zero exceptional terms remain zero. Thus T preserves μ^k for all k by polarization. The same label and sign identities preserve (2.7) for all k . The metric and parity data are independent of k and were already preserved by T . This proves (i) $\Rightarrow$ (ii) with the same map; (ii) $\Rightarrow$ (i) follows by taking $k = 1$. For spin graphs $w_G = 0$, so Theorem C requires only the stated triple. $\square$

10. CONSEQUENCES AND HYPOTHESES

10.1. Simultaneous preservation. For the graphs $G_r^\pm(n)$ of (5.9), the ambient-coordinate quadratic form is

$$q_G = r t^2 + 2t \left(x + y + \sum_j z_j \right) + 2 \left(x^2 + y^2 + \sum_j z_j^2 \right). \quad (10.1)$$

With $e_0 = (1, 0, \dots, 0)$ and $T_r = T_3 \oplus I_{r-3}$, one has

$$q_G(e_0) = r, \quad q_G(T_r e_0) = r + 16.$$

Thus T_r is not an isometry. The two quadratic lattices are nevertheless identical in these coordinates. By Theorem C, no isomorphism of the original invariant systems preserves this quadratic form.

For the pair in (6.10), the quadratic form of both graphs, in coordinates (6.1), is

$$q = 5t^2 + 2t(x_1 + x_2 + x_3 + x_4) + 2 \sum_{i=1}^4 x_i^2. \quad (10.2)$$

For $e = (0, 1, 0, 0, 0)$ and the matrix (6.14),

$$q(e) = 2, \quad q(Te) = q(0, 1, 2, 0, 0) = 10.$$

Thus T does not preserve q . Since the reduced classes are different, Theorem C excludes every simultaneous isomorphism, not just this one. Hence separate isometry of the quadratic lattices and separate equivalence of the original systems do not suffice.

10.2. Reduction and parity. A vertex in V of degree two has two main coordinate covectors which may be proportional. For a graph consisting of one such vertex and leaves of labels A, B ,

$$(x_1, x_2) = (z, -z), \quad q = 2z^2, \quad f^k = \rho_k(A - B)z^3, \quad p^k = \lambda_k(A - B)z.$$

The pairs $(A, B) = (1, 2)$ and $(2, 3)$ give identical enhanced systems but are not related by a permutation and common sign. Thus the degree restriction cannot simply be omitted from the reconstruction theorem.

The parity class is necessary in the nonspin theorem. An exceptional component and an isolated spin vertex of label zero both have

$$H = \mathbb{Z}, \quad q(z) = z^2, \quad \mu^k = p^k = 0.$$

Their parity classes are nonzero and zero, respectively, and their reduced classes differ.

Example 10.1. The forest hypothesis cannot be replaced by total unimodularity of a chosen relation matrix. Consider

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

All entries and all two-by-two minors are 0 or ± 1 , so both matrices are totally unimodular. Subtracting the first row of B_1 from its second row gives B_2 ; hence their integral kernels in $\mathbb{Z}^6$ coincide. The bipartite graph of B_1 is connected and has a four-cycle, whereas that of B_2 is the disjoint union of two three-leaf stars. Give all six labelled vertices the decoration $(1, 0, 0)$ in both graphs. Their kernel lattices, restricted Euclidean forms, cubic and Pontryagin forms, and parity classes are identical. The same is true of $\sum_i L_i^d$ in every degree. These graphs are not isomorphic; the first is outside the reduced class.

10.3. A finite equivalence criterion.

Proposition 10.2. Fix $k \geq 1$. Let two enhanced invariant systems of reduced algebraic plumbing graphs be given in integral bases. There is a finite procedure deciding whether their reduced classes are equal. If the common rank is positive and Q, Q' are the quadratic Gram matrices, every isomorphism $T : H_{G'} \rightarrow H_G$ satisfies

$$|T_{ji}|^2 \leq (Q^{-1})_{jj} Q'_{ii} \quad \text{for all } i, j.$$

Proof. Unequal ranks exclude an isomorphism. Rank zero identifies two empty graphs by Lemma 2.6 and (2.12). Suppose the common rank is $r > 0$. Let Q, Q' be the given Gram matrices and write t_i for column i of an isometry $T : H_{G'} \rightarrow H_G$. Then $t_i^\top Q t_i = Q'_{ii}$. For the Q -inner product,

$$|T_{ji}|^2 = |\langle Q^{-1} e_j, t_i \rangle_Q|^2 \leq (Q^{-1})_{jj} Q'_{ii}. \quad (10.3)$$

The quantities on the right are positive rational numbers, so they give finite computable bounds on every integral entry of T . Enumerate the resulting finite set of matrices. For each, test

$$T^\top Q T = Q', \quad \det T \in \{1, -1\}, \quad T w' = w \pmod{2},$$

and

$$\mu_G^k(T e_i, T e_j, T e_l) = \mu_{G'}^k(e_i, e_j, e_l) \quad (1 \leq i, j, l \leq r).$$

Multilinearity makes the last tests equivalent to preservation on all triples. A matrix passes exactly when it is an enhanced isomorphism. Theorem C identifies its existence with equality of reduced classes. $\square$

10.4. Six-dimensional manifolds. Let G be a reduced spin tree, and let M_G be the oriented six-manifold associated to G in [7, Definition 4.4], with dimension parameter $k = 1$. It is closed and simply connected, has torsion-free homology, and has invariant system $\mathcal{I}_1(G)$ [7, Lemma 4.6]. Every vertex of V has degree at least three, so [7, Corollary 3.15(2)] gives $b_3(M_G) = 0$.

For oriented closed simply-connected smooth six-manifolds with torsion-free homology, Jupp's classification identifies the oriented diffeomorphism class by b_3 and the integral system (H^2, μ, w_2, p_1) [3]; see [7, Theorem 4.1].

Corollary 10.3. *For every integer $r \geq 3$, there exist infinitely many pairwise distinct oriented diffeomorphism classes of closed simply-connected spin six-manifolds with torsion-free homology,*

$$b_2 = r, \quad b_3 = 0, \quad p_1/4 \text{ primitive},$$

each admitting a metric of positive Ricci curvature and at least two reduced spin plumbing presentations with distinct reduced classes and all integer labels nonzero. For $r \geq 4$, the two presentations may be chosen to be paired-leaf trees.

Proof. For $N \geq 1$, take the two trees $G_r^+(2N), G_r^-(2N)$ of (5.9). Their associated six-manifolds have $b_3 = 0$. Theorem B identifies their integral invariant systems, so Jupp's classification [3, Theorem 1], in the formulation of [7, Theorem 4.1], gives an orientation-preserving diffeomorphism

$$M_{G_r^+(2N)} \cong M_{G_r^-(2N)}.$$

The reduced classes differ by Proposition 5.7. The same proposition proves that $p_1/4$ is primitive. Each graph is a tree; hence [7, Theorem A(2)] supplies a core metric, and therefore a metric of positive Ricci curvature, on its associated manifold. Since $\rho_1 = 1$, Proposition 5.9 gives

$$\text{div}_{\ker p_1}(\mu|_{(\ker p_1)^3}) = 2N.$$

An oriented diffeomorphism preserves the cup product and p_1 , so it preserves this integer. Distinct values of N therefore give distinct oriented diffeomorphism classes.

For $r \geq 4$, use $P_{r,N}, P'_{r,N}$ from Proposition 6.5. The same realization and classification results give two paired-leaf presentations of one oriented diffeomorphism class, with primitive $p_1/4$ and $b_3 = 0$. The associated manifold admits a core metric. The bilinear form (6.9), defined by $v = \mu$ and $\ell = p_1/4$, is preserved by oriented diffeomorphisms. Its absolute determinant is $30N^3$, so these classes are pairwise distinct. $\square$

Corollary 10.4. *For every integer $r \geq 5$, there exist infinitely many pairwise distinct oriented diffeomorphism classes of closed simply-connected spin six-manifolds with torsion-free homology and*

$$b_2 = r, \quad b_3 = 0,$$

each admitting a metric of positive Ricci curvature and at least two reduced spin plumbing presentations with distinct reduced classes and all integer labels nonzero. They may be chosen with $\text{cont}(p_1) = 4(5N + r - 6)$, $N \geq 1$.

Proof. Let $G_{r,N}$ and $G'_{r,N}$ be the two trees of (6.15). Their associated six-manifolds have $b_3 = 0$, and Proposition 6.6 gives equality of their integral invariant systems. Jupp's classification [3, Theorem 1], in the formulation of [7, Theorem 4.1], gives an orientation-preserving diffeomorphism

$$M_{G_{r,N}} \cong M_{G'_{r,N}}.$$

The reduced classes of the graphs differ. Since both graphs are trees, [7, Theorem A(2)] gives a core metric, and hence a metric of positive Ricci curvature, on each plumbing manifold. Finally, the content of its Pontryagin form is

$$\text{cont}(p_1) = 4(5N + r - 6),$$

so different N give different oriented diffeomorphism classes. $\square$

Remark 10.5. For $k > 1$, the equality furnished by Theorem B is an isomorphism of the systems

$$(H^{2k}, \mu, w_2^k, p_k).$$

The oriented diffeomorphism assertion in Corollary 10.3 has the additional hypothesis $k = 1$. In Theorem C, the classified objects are the reduced plumbing presentations, for every $k \geq 1$ and every rank.

11. SIX-MANIFOLDS WITH MANY REDUCED PRESENTATIONS

All manifolds in this section are smooth, closed, connected, and oriented. A core metric is a positive-Ricci-curvature metric together with an embedded disc whose boundary has the unit round induced metric and positive semidefinite second fundamental form with respect to the inward normal, in the convention of [7, Definition 2.1].

11.1. An algebraic indecomposability criterion.

Lemma 11.1. *Let $V = \mathbb{Q}e_0 \oplus W$, with $\dim W \geq 2$, and let μ be a symmetric trilinear form such that*

$$\mu|_{W^3} = 0, \quad \mu(e_0, e_0, W) = 0.$$

Suppose $b(u, v) = \mu(e_0, u, v)$ is nonsingular on W . Every $P \in \text{End}_{\mathbb{Q}}(V)$ satisfying

$$\mu(Pu, v, w) = \mu(u, Pv, w) \quad (u, v, w \in V) \quad (11.1)$$

is scalar. Consequently, there is no decomposition $V = V_1 \oplus V_2$ with both $V_i \neq 0$ and all mixed values of μ equal to zero.

Proof. Write

$$P(e_0) = \alpha e_0 + c, \quad P(x) = \beta(x)e_0 + E(x) \quad (x \in W),$$

where $c \in W$, $\beta \in W^*$, and $E \in \text{End}_{\mathbb{Q}}(W)$. On $x, y, z \in W$, equation (11.1) becomes

$$\beta(x)b(y, z) = \beta(y)b(x, z).$$

If $\beta \neq 0$, choose x with $\beta(x) \neq 0$. Since $\dim W \geq 2$, there is a nonzero $y \in \ker \beta$. The displayed equality then gives $b(y, W) = 0$, a contradiction. Thus $\beta = 0$. Evaluating (11.1) on (e_0, x, y) gives $\alpha b(x, y) = b(E(x), y)$, whence $E = \alpha I$. Evaluation on (e_0, x, e_0) gives $b(c, x) = 0$ for all x , so $c = 0$.

For an orthogonal direct sum $V = V_1 \oplus V_2$, let P be the projection onto V_1 . Write $u = u_1 + u_2$, $v = v_1 + v_2$, $w = w_1 + w_2$. Vanishing of all mixed values gives $\mu(Pu, v, w) = \mu(u_1, v_1, w_1) = \mu(u, Pv, w)$. Thus P is scalar. Since $P^2 = P$, its scalar is zero or one, contrary to both summands being nonzero. $\square$

11.2. An integral obstruction to a sphere-bundle structure.

Lemma 11.2. *Let M be a simply-connected six-manifold with torsion-free homology. Suppose its cup-product cubic has a unique rational linear-factor line and $\Delta(\mu_M) \neq 1$. Then M is not the total space of a smooth S^2 -bundle over a closed four-manifold.*

Proof. Suppose $\pi : M \rightarrow B$ is such a bundle. The homotopy exact sequence gives $\pi_1(B) = 0$. Thus B is orientable and $H^3(B; \mathbb{Z}) \cong H_1(B; \mathbb{Z}) = 0$. The orientation local system of the fibre is trivial. Choose compatible base and fibre orientations. The integral Gysin sequence has the segment

$$H^{-1}(B; \mathbb{Z}) \longrightarrow H^2(B; \mathbb{Z}) \xrightarrow{\pi^*} H^2(M; \mathbb{Z}) \xrightarrow{\pi_!} H^0(B; \mathbb{Z}) \longrightarrow H^3(B; \mathbb{Z}).$$

Its two end groups vanish and $H^0(B; \mathbb{Z}) = \mathbb{Z}$. Hence

$$0 \longrightarrow H^2(B; \mathbb{Z}) \xrightarrow{\pi^*} H^2(M; \mathbb{Z}) \xrightarrow{\pi_!} \mathbb{Z} \longrightarrow 0. \quad (11.2)$$

In particular, $\pi_!$ is a primitive integral covector and $\ker \pi_! = \pi^* H^2(B; \mathbb{Z})$. The cubic vanishes on this kernel, because $H^6(B; \mathbb{Z}) = 0$. A polynomial vanishing on the hyperplane of a nonzero linear form is

divisible by that form over $\mathbb{Q}$: choose coordinates in which the form is the first coordinate and set it equal to zero. Hence uniqueness identifies π_1 with the primitive covector τ of Lemma 6.12, up to sign.

Choose $z \in H^2(M; \mathbb{Z})$ with $\pi_1(z) = 1$. For $u, v \in H^2(B; \mathbb{Z})$, the projection formula for integration along the fibre gives

$$\mu_M(z, \pi^* u, \pi^* v) = \langle z \pi^*(uv), [M] \rangle = \langle uv, [B] \rangle.$$

The group $H^2(B; \mathbb{Z})$ is free, by the universal coefficient theorem and $H_1(B; \mathbb{Z}) = 0$. Its intersection pairing is unimodular by integral Poincaré duality. Equation (11.2) identifies this pairing with (6.21), up to sign. Therefore $\Delta(\mu_M) = 1$, a contradiction. $\square$

11.3. Fixed rank and fixed rational homotopy type.

Theorem 11.3. *Fix $r \geq 5$. For every odd integer $N > 1$, there is a simply-connected spin six-manifold $M_{r,N}$ with torsion-free homology and*

$$b_2(M_{r,N}) = r, \quad b_3(M_{r,N}) = 0, \quad p_1/4 \text{ primitive},$$

with the following properties.

- (i) $M_{r,N}$ admits a core metric, and therefore a metric of positive Ricci curvature.
- (ii) It admits at least $2^{\omega(N)-1}$ connected reduced spin paired-leaf presentations with pairwise distinct reduced classes and all integer labels nonzero.
- (iii) For fixed r , different N give nonisomorphic graded integral cohomology rings, even allowing orientation reversal. In particular, the manifolds are pairwise homotopy inequivalent.
- (iv) For fixed r , all $M_{r,N}$ have the same rational homotopy type.
- (v) Every connected-sum decomposition of $M_{r,N}$ has a summand diffeomorphic to S^6 . The manifold is not the total space of a smooth sphere bundle with positive-dimensional base and fibre.

Proof. Let $M = M_{r,N}$ be the manifold associated to $\mathcal{G}_{r,N}(1, N)$ in (6.18), at $k = 1$. The graph is a reduced spin tree, so [7, Lemma 4.6 and Corollary 3.15(2)] give simple connectivity, torsion-free homology, $b_2 = r$, $b_3 = 0$, and the invariant system of that graph. Equation (6.19) gives $p_1 = 4t$, with t primitive. Theorem A(2) of [7] gives a core metric.

For every coprime factorization $N = ab$, Theorem D identifies the integral invariant system of $\mathcal{G}_{r,N}(a, b)$ with that of $\mathcal{G}_{r,N}(1, N)$. Both associated manifolds have $b_3 = 0$. Jupp's classification [3, Theorem 1], as stated in [7, Theorem 4.1], therefore gives an orientation-preserving diffeomorphism. The distinct reduced classes counted in Theorem D prove (ii).

A graded integral cohomology-ring isomorphism sends the generator of $H^6 \cong \mathbb{Z}$ to one of its two signs. Its restriction to H^2 hence identifies the integral cup-product cubics up to sign. Lemma 6.12 and Proposition 6.13 give

$$\Delta(\mu_{M_{r,N}}) = N^2.$$

This proves (iii).

For (iv), put $V = H^2(M_{r,N}; \mathbb{Q})$ and fix the orientation generator $\Omega \in H^6(M_{r,N}; \mathbb{Q})$. Poincaré duality identifies $H^4(M_{r,N}; \mathbb{Q})$ with V^* by $\xi \mapsto (x \mapsto \langle x\xi, [M_{r,N}] \rangle)$. There is no odd-degree rational cohomology. Apart from multiplication by the unit, the algebra structure is given by

$$(xy)(z) = \mu_{M_{r,N}}(x, y, z), \quad x\xi = \xi(x)\Omega \quad (x, y, z \in V, \xi \in V^*).$$

All other products of positive-degree elements are determined by these identities or vanish for dimensional reasons. For a second such cubic on V' , let $T : V' \rightarrow V$ satisfy $\mu(Tx, Ty, Tz) = \mu'(x, y, z)$. Use T in degree two, $(T^{-1})^* : (V')^* \rightarrow V^*$ in degree four, and $\Omega' \mapsto \Omega$ in degree six. For $x, y \in V'$ and $w \in V$,

$$\mu(Tx, Ty, w) = \mu'(x, y, T^{-1}w), \quad ((T^{-1})^*\xi)(Tx) = \xi(x).$$

These identities verify both nontrivial products, so the resulting map is a graded rational algebra isomorphism. Proposition 6.13 supplies the cubic equivalences for fixed r .

Miller's theorem [6] applies to a closed orientable $(\kappa - 1)$ -connected manifold of dimension at most $4\kappa - 2$; here $\kappa = 2$ and the dimension is six. Thus $\Omega^*(M; \mathbb{R})$ is formal; see also [1, Introduction]. Let $\mathcal{M}_M$ be a Sullivan minimal model over $\mathbb{Q}$ of the algebra $A_{PL}(M; \mathbb{Q})$ of polynomial forms. The manifold M is simply connected and has finite CW type, so $\mathcal{M}_M$ is a nilpotent minimal algebra of finite type. The comparison of polynomial and smooth forms [9, Section 7] identifies $\mathcal{M}_M \otimes_{\mathbb{Q}} \mathbb{R}$ with a minimal model of $\Omega^*(M; \mathbb{R})$, up to isomorphism. It is therefore formal. By [9, Theorem 12.1], $\mathcal{M}_M$ is formal over $\mathbb{Q}$; see also [2, Theorem 6.8]. It is consequently a minimal model of $(H^*(M; \mathbb{Q}), 0)$. Uniqueness of minimal models [9, Theorem 5.1] and the graded algebra isomorphisms above give isomorphic rational minimal models for all $M_{r,N}$ at fixed r . Their rational homotopy types are therefore equal by [9, Theorem 10.1]. This proves (iv).

For (v), use the rational coordinates of (6.23). With $W = \{t = 0\}$ and $e_0 = (1, 0, \dots, 0)$, the tensor satisfies

$$\mu|_{W^3} = 0, \quad \mu(e_0, e_0, W) = 0, \quad (\mu(e_0, e_i, e_j))_{i,j \geq 1} = -\frac{1}{4}D_r(a, b).$$

The last matrix is nonsingular and $\dim W = r - 1 \geq 4$. Lemma 11.1 excludes a nontrivial orthogonal decomposition of the rational cubic.

Suppose $M_{r,N} = X \# Y$. Van Kampen's theorem gives $\pi_1(M_{r,N}) = \pi_1(X) * \pi_1(Y)$, so both summands are simply connected. In degrees 1 through 5, Mayer–Vietoris identifies the homology of $M_{r,N}$ with the direct sum of those of X and Y . The summands therefore have torsion-free homology and $b_3 = 0$. In degree two the two summands identify with their pullbacks under the collapse maps from $M_{r,N}$. Classes belonging to different summands have zero product: equivalently, they come from different factors of $X \vee Y$ after collapsing the separating sphere. Thus their rational cubics give an orthogonal direct sum. By Lemma 11.1, one degree-two space vanishes; assume it is $H^2(X; \mathbb{Q})$. The free integral group $H_2(X; \mathbb{Z})$ then vanishes. Simple connectivity gives $H_1(X; \mathbb{Z}) = 0$, and torsion-freeness with $b_3(X) = 0$ gives $H_3(X; \mathbb{Z}) = 0$. Poincaré duality yields $H_4(X; \mathbb{Z}) = H_5(X; \mathbb{Z}) = 0$. Thus X is an integral homology six-sphere. Its degree-two group, parity class, trilinear form, and Pontryagin form are all zero, and $b_3(X) = 0$. These are the invariants of S^6 . Both manifolds are smooth, closed, oriented, and simply connected with torsion-free homology. Jupp's classification therefore gives an orientation-preserving diffeomorphism $X \cong S^6$.

Since the only nonzero Betti numbers are $b_0 = b_6 = 1$ and $b_2 = b_4 = r$, one has $\chi(M_{r,N}) = 2 + 2r > 0$. Suppose it were a smooth S^d -bundle with both dimensions positive. Then $1 \leq d \leq 5$. The base is compact as a continuous image of M ; it has no boundary, since local trivializations would otherwise give boundary points of M . For odd d , multiplicativity of Euler characteristic gives $\chi(M) = \chi(B)\chi(S^d) = 0$, a contradiction. For $d = 4$, the homotopy exact sequence makes the closed surface B simply connected, hence $B = S^2$. In total degree two, the rational Serre spectral sequence has only $E_2^{2,0} = \mathbb{Q}$, with no possible incoming or outgoing differential; hence $b_2(M) = 1$, a contradiction. For $d = 2$, Lemma 11.2 applies because the cubic has its unique factor line and $\Delta(\mu_M) = N^2 > 1$. All possible fibre dimensions are excluded. $\square$

Remark 11.4. The multiplicity in Theorem 11.3 concerns reduced plumbing presentations, not components of a space of Ricci-positive metrics. Its diffeomorphism conclusions use $k = 1$. No higher-dimensional classification follows here from equivalence of the four entries of $\mathcal{I}_k$.

APPENDIX A. THE DETERMINANT IDENTITY

Let $T = \begin{pmatrix} b+u & b \\ d+v & d \end{pmatrix}$, and set

$$\delta = ud - bv, \quad \eta = bv, \quad \theta = du, \quad s = b + v, \quad t = d + u.$$

Define

$$H = (\eta + 2\theta)s + (\theta + 2\eta)t, \tag{A.1}$$

$$K = \eta(s^2 - 2\eta)(s + 2t) + \theta(t^2 - 2\theta)(t + 2s) + 3\eta\theta(s + t). \tag{A.2}$$

Lemma A.1. *For indeterminates b, d, u, v , the following identity holds in $\mathbb{Z}[b, d, u, v]$:*

$$\det \mathcal{A}(T) = \delta(K - (\delta^2 + 1)H). \quad (\text{A.3})$$

Proof. Work first in $\mathbb{Q}[b, d, u, v]$. The identities

$$\varphi(x - y) = \varphi(x) - \varphi(y) - 3xy(x - y), \quad \varphi(x + y) = \varphi(x) + \varphi(y) + 3xy(x + y)$$

give the following row and column operations on (3.7):

$$R_3 \leftarrow R_1 - R_2 - 3R_3, \quad C_3 \leftarrow -\frac{1}{3}(C_1 + C_2 + C_3).$$

The column operation uses the columns after the row operation. The determinant is multiplied successively by -3 and $-1/3$, so is unchanged. Thus

$$\det \mathcal{A}(T) = \det \begin{pmatrix} \varphi(b+u) & \varphi(d+v) & (b+u)(d+v)(b+d+u+v) \\ \varphi(b) & \varphi(d) & bd(b+d) \\ \varphi(u) & \varphi(v) & uv(u+v) \end{pmatrix}. \quad (\text{A.4})$$

For polynomials P_1, P_2, P_3 in X, Y , write $[P_1, P_2, P_3]$ for their evaluation determinant on

$$(b+u, d+v), \quad (b, d), \quad (u, v).$$

Set $Z = XY(X+Y)$ and $D_{ij} = [X^i, Y^j, Z]$. Multilinearity gives

$$\det \mathcal{A}(T) = D_{33} - D_{31} - D_{13} + D_{11}. \quad (\text{A.5})$$

Put

$$P_0 = b^2 + v^2, \quad Q_0 = d^2 + u^2, \quad \Sigma = s + t.$$

Then

$$H = Z(b+u, d+v) - Z(b, d) - Z(u, v), \quad (\text{A.6})$$

$$H = \eta(s+2t) + \theta(t+2s),$$

$$K = \eta P_0(s+2t) + \theta Q_0(t+2s) + 3\eta\theta\Sigma. \quad (\text{A.7})$$

In each D_{ij} , subtract the second and third rows from the first. For D_{11} the first two entries become zero, and the third is H . Hence

$$D_{11} = H(bv - du) = -\delta H.$$

For D_{33} the first row becomes $(3bu(b+u), 3dv(d+v), H)$. Its cofactor expansion is

$$\begin{aligned} D_{33} &= 3\eta\theta E - \delta(\eta^2 + \eta\theta + \theta^2)H, \\ E &= (b+u)(d^2u(u+v) - v^2b(b+d)) \\ &\quad - (d+v)(b^2v(u+v) - u^2d(b+d)). \end{aligned}$$

The two identities

$$d(b+u)(u+v) + u(d+v)(b+d) = H - \eta\Sigma,$$

$$v(b+u)(b+d) + b(d+v)(u+v) = H - \theta\Sigma$$

give

$$E = \theta(H - \eta\Sigma) - \eta(H - \theta\Sigma) = \delta H.$$

It follows that

$$D_{33} = -\delta(\eta^2 - 2\eta\theta + \theta^2)H = -\delta^3 H.$$

For $D_{31} + D_{13}$ the same cofactor expansion gives

$$\begin{aligned} D_{31} + D_{13} &= 3\eta\theta F + H(\eta P_0 - \theta Q_0), \\ F &= (b + u)(u(u + v) - b(b + d)) \\ &\quad - (d + v)(v(u + v) - d(b + d)) \\ &= \Sigma(Q_0 - P_0 - \delta). \end{aligned}$$

Since $\delta = \theta - \eta$, (A.7) implies

$$H(\eta P_0 - \theta Q_0) + \delta K = 3\eta\theta\Sigma(P_0 - Q_0 + \delta).$$

Consequently $D_{31} + D_{13} = -\delta K$. Substitution in (A.5) proves (A.3). Both sides belong to $\mathbb{Z}[b, d, u, v]$, so the identity holds in that ring. $\square$

If $ud - bv = 1$, then $\delta = 1$ and $\theta = \eta + 1$. Formula (A.3) becomes $K - 2H$. Grouping its terms first by s and t gives

$$\begin{aligned} K - 2H &= s(\eta(b^2 + v^2) + 2\theta(d^2 + u^2) + 3\eta\theta - 2\eta - 4\theta) \\ &\quad + t(\theta(d^2 + u^2) + 2\eta(b^2 + v^2) + 3\eta\theta - 2\theta - 4\eta) \\ &= \eta s(b^2 + 3\eta + v^2 - 3) + 2s(\theta(d^2 + u^2) - 2) \\ &\quad + \theta(d^3 + u^3 - d - u) + 2\eta s^2 t. \end{aligned}$$

Indeed, the coefficient of s in both expressions is

$$\eta(b^2 + v^2) + 2\theta(d^2 + u^2) + 3\eta^2 - 3\eta - 4,$$

and the coefficient of t is

$$\theta(d^2 + u^2) + 2\eta(b^2 + v^2) + 3\eta^2 - 3\eta - 2,$$

using $d^3 + u^3 = (d + u)(d^2 - du + u^2)$. Replacing η, s, t, θ by $bv, b + v, d + u, bv + 1$ yields (3.9).

APPENDIX B. RECONSTRUCTION FROM HIGHER ODD MOMENTS

Let G be a reduced spin graph whose integer labels are all nonzero. For $d \geq 1$, define the homogeneous polynomial

$$F_{d,G} = \sum_{u \in U} \alpha(u) L_u^d \quad \text{on } H_G, \quad L_u(x) = x_u. \quad (\text{B.1})$$

Thus $F_{1,G} = \ell_G$ and $F_{3,G} = f_G$. These higher moments are attached to the plumbing presentation. They are not asserted to be ordinary characteristic classes or higher cup products of its associated six-manifold.

Lemma B.1. *Let V be a finite-dimensional vector space over a field of characteristic zero, and let $L_1, \dots, L_n \in V^*$ have linearly independent squares. If $L \in V^* \setminus \{0\}$ satisfies*

$$L^3 \in \text{span}\{L_1^3, \dots, L_n^3\},$$

then L is proportional to one of the L_i .

Proof. Write $L^3 = \sum_i c_i L_i^3$. For every $v \in \ker L$, directional differentiation gives

$$0 = 3 \sum_i c_i L_i(v) L_i^2.$$

Independence of the squares implies $c_i L_i(v) = 0$ for every i . Choose an index with $c_i \neq 0$, which exists since $L^3 \neq 0$. Then $\ker L \subseteq \ker L_i$. Both covectors are nonzero, so their kernels have codimension one and coincide. They are proportional. $\square$

Theorem B.2 (Odd-moment reconstruction). *Fix an odd integer $d \geq 5$. On the class of reduced spin graphs with all integer labels nonzero, the integral pair $(H_G, F_{d,G})$ determines the reduced class. More precisely, every integral isomorphism $T : H_{G'} \rightarrow H_G$ with $F_{d,G} \circ T = F_{d,G'}$ induces a bipartite graph isomorphism $\pi : U \sqcup V \rightarrow U' \sqcup V'$ and signs $s_C \in \{1, -1\}$, one for each connected component C of G , such that*

$$L_u \circ T = s_C L'_{\pi(u)}, \quad \alpha'(\pi(u)) = s_C \alpha(u) \quad (u \in C).$$

Conversely, equality of reduced classes gives such an integral isomorphism preserving the degree- d moment.

Proof. Put $V_G = H_G \otimes \mathbb{Q}$. The $(d-3)$ rd directional derivative of $F_{d,G}$ defines an intrinsic linear map

$$C_{F_{d,G}} : \text{Sym}^{d-3} V_G \longrightarrow \text{Sym}^3 V_G^*.$$

On decomposable symmetric products it is

$$C_{F_{d,G}}(v_1 \cdots v_{d-3}) = \frac{d!}{3!} \sum_u \alpha(u) \left(\prod_{j=1}^{d-3} L_u(v_j) \right) L_u^3. \quad (\text{B.2})$$

Since $d-3 \geq 2$, Theorem 7.2(iii) makes $\mathbb{Q}^U \rightarrow \text{Sym}^{d-3} V_G^*$, $e_u \mapsto L_u^{d-3}$, injective. Its dual is the surjective evaluation map

$$\text{Sym}^{d-3} V_G \longrightarrow \mathbb{Q}^U, \quad v_1 \cdots v_{d-3} \longmapsto \left(\prod_j L_u(v_j) \right)_{u \in U}.$$

All $\alpha(u)$ are nonzero. Equation (B.2) therefore gives

$$\text{im } C_{F_{d,G}} = \text{span}_{\mathbb{Q}}\{L_u^3 : u \in U\}.$$

By Theorem 7.2(iii), the squares L_u^2 are independent. Lemma B.1 identifies the coordinate lines as exactly the lines $\mathbb{Q}L$ with $L \neq 0$ and $L^3 \in \text{im } C_{F_{d,G}}$. For $v_1, \dots, v_{d-3} \in V_{G'}$, the chain rule gives

$$C_{F_{d,G'}}(v_1 \cdots v_{d-3}) = T^*(C_{F_{d,G}}(Tv_1 \cdots Tv_{d-3})).$$

Since T is invertible, it transports the two derivative images and hence their pure-cube lines.

Each coordinate covector is primitive by Theorem 7.2(i). On a recovered rational line, the intersection with the dual integral lattice is an infinite cyclic group with exactly two primitive generators. The integral isomorphism T^* preserves this group and its primitive generators. Thus it transports the covectors with signs, not arbitrary rational factors. Corollary 7.3 gives π and makes those signs constant on each connected component. The independence of the degree- d powers and the equality of moments give

$$\alpha'(\pi(u)) = s_C^d \alpha(u) = s_C \alpha(u).$$

These are precisely the reduced-class identifications. When the rank is zero, both graphs are empty by Lemma 2.6; the argument is otherwise valid also for isolated vertices and rank one.

Conversely, a graph isomorphism permutes coordinates, and a component reversal negates its main coordinates and labels. Since d is odd, $(-\alpha)(-L)^d = \alpha L^d$. The resulting integral coordinate map preserves (B.1). $\square$

Corollary B.3. *Five is the least positive odd degree for which the degree- d moment alone determines every reduced class in the class of Theorem B.2.*

Proof. Theorem B.2 applies at $d = 5$. The two graphs of Theorem 5.5 have nonzero labels and different reduced classes, but an integral isomorphism identifies both $F_3 = f$ and $F_1 = \ell$. Thus neither $d = 1$ nor $d = 3$ has the stated property. $\square$

For $d = 5$, formula (B.2) is the second derivative and its factor is 20. Grouping a fifth-order tensor into factors of degrees $(2, 2, 1)$ relates this argument to the uniqueness criterion of Leurgans, Ross, and Abel [4]: two factor families are linearly independent and the third has no proportional pair. The proof above uses the forest power-independence and circuit statements directly; no general tensor-decomposition theorem is required as a premise.

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