LOCAL REALIZATION OF SPECTRAL CLUSTERS

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Abstract. Let λ be an isolated eigenvalue of multiplicity m of a real Schrödinger operator on a compact connected manifold of dimension at least two, with a fixed closed, Dirichlet, or Neumann realization. Every symmetric matrix sufficiently close to λI is the corresponding spectral-cluster matrix of a small scalar-potential perturbation supported in any prescribed nonempty interior open set. The perturbation depends $1/2$-Hölder continuously on the matrix, vanishes at λI, and is real analytic away from that value. Sections with these properties exist for positive Laplace eigenvalues in the full conformal class of a closed manifold and, in dimension at least three, for supported metric perturbations preserving the volume density pointwise.

For potentials and conformal factors, every nonzero scalar contraction of the intrinsic cokernel Hessian has infinite positive and negative index. The first nonzero transverse diagonal term of the contracted eigenfunction kernel is homogeneous and harmonic. On a closed manifold, the multiplicity locus in the full metric space or a full conformal class is a C^1 embedded submanifold exactly where the cluster derivative is onto. Unrestricted metric stability holds through multiplicity seven and can fail at multiplicity eight. For potentials and conformal metrics, the projection connection has arbitrary prescribed curvature at the reference along a compatible matrix germ. In particular, its local holonomy on the exact fixed-eigenvalue level is $\mathrm{SO}(m)$.

For a finite effective symmetry group, the matrix target is the self-adjoint commutant. In corank one, the same index condition gives continuous relative lifts over compact spaces of finite covering dimension. On each fixed harmonic space $\bigoplus_{j=2}^{N} \mathcal{H}_j(S^2)$, the maximum width of the first $N+1$ positive clusters is bounded below by a positive multiple of the squared L^2 norm of every sufficiently small conformal perturbation.

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1. Introduction

Let $P_0 = \Delta_g + V$ be a real Schrödinger operator on a compact connected smooth manifold, and let λ be an isolated eigenvalue of multiplicity m. Perturbation of the operator induces a map to $\mathrm{Sym}_m(\mathbb{R})$ whose eigenvalues are the nearby eigenvalues of P_0, counted with multiplicity. Surjectivity of its derivative gives a smooth local right inverse. A regular zero of the cokernel Hessian gives a $1/2$-Hölder local right inverse through a singular reference.

Choose a contour separating λ from the remaining spectrum. For nearby operators $P(v)$, let $\Pi(v)$ be the resulting rank-m orthogonal projection. An orthonormal basis of $E_\lambda = \ker(P_0 - \lambda)$ extends by the Riesz–Gram construction to a frame $U(v)$, with

$$\mathsf{L}(v) = U(v)^* P(v) U(v), \qquad \mathcal{E}_v = \operatorname{im} \Pi(v), \qquad \nabla = \Pi\, d. \tag{1.1}$$

Then $\mathsf{L}(0) = \lambda I_m$, and $\mathsf{L}^{-1}(\lambda I_m)$ consists exactly of the perturbations for which the selected eigenvalue remains equal to λ with multiplicity m. The bundle $\mathcal{E}$ and its projection connection are defined on this level and on the full parameter neighborhood.

1.1. **Local right inverses.** Throughout, $\Delta_g = -\operatorname{div}_g \nabla_g$, the manifold dimension is at least two, and the coefficients are smooth and real. A boundary, when present, is smooth. The Dirichlet or Neumann realization is fixed. Metric perturbations in boundary problems have compact interior support.

Let $O \subset M^\circ$ be a nonempty open set. Consider the following three parameter spaces, where $\mathrm{Sym}_0(TM, g)$ denotes the g-self-adjoint trace-free endomorphisms of TM.

	Operator and eigenvalue	Perturbations near zero
P	$\Delta_g + V$, $\lambda \in \mathbb{R}$; fixed closed, Dirichlet or Neumann realization	$V + v$, $v \in C_c^\infty(O;\mathbb{R})$
C	Δ_g, $\lambda > 0$; M closed	$(1+v)g$ if $n = 2$; $(1+v)^{4/(n-2)}g$ if $n \geq 3$; $v \in C^\infty(M;\mathbb{R})$
V	Δ_g, $\lambda > 0$; $n \geq 3$; fixed realization	$g(e^H \cdot, \cdot)$, $H \in C_c^\infty(O; \mathrm{Sym}_0(TM, g))$

All matrix norms below are Frobenius norms, with $\langle A, B\rangle_{\mathrm{F}} = \operatorname{tr}(A^T B)$. Spatial norms for tensor parameters are computed relative to the reference metric. Put $D = m(m+1)/2$.

Theorem A (Local sections). *In each of the classes* P, C, *and* V, *there are a finite-dimensional space W of allowed smooth parameters, $\delta > 0$, and*

$$S : B_\delta(0) \subset \mathrm{Sym}_m(\mathbb{R}) \longrightarrow W$$

such that

$$S(0) = 0, \qquad \mathsf{L}(S(Y)) = \lambda I_m + Y, \qquad \|S(Y) - S(Y')\|_{C^k} \leq C_k \|Y - Y'\|^{1/2}. \tag{1.2}$$

Here k is any fixed nonnegative integer. The target ball is independent of k, whereas C_k may depend on k. The section is real analytic on the punctured ball. In P *and* V, *every parameter in W is supported in one compact $K \Subset O$.*

If the allowed derivative is onto, $\dim W = D$ *suffices and S may be analytic at zero with a linear estimate. Otherwise* $\dim W = D + 1$ *suffices, and there is an analytic curve $v(s) \in W$, with $v(0) = 0$, such that*

$$\mathsf{L}(v(s)) = \lambda I_m, \qquad D\mathsf{L}(v(s))|_W : W \longrightarrow \mathrm{Sym}_m(\mathbb{R}) \text{ is onto} \quad (0 < |s| \ll 1). \tag{1.3}$$

At a reference where the derivative is not onto, no exponent greater than $1/2$ can hold uniformly in all target directions in a norm controlling the second derivative of L.

In class P, the cluster map is analytic in an L^∞ neighborhood; in classes C and V it is analytic in a sufficiently small C^k neighborhood, $k \geq 4$. The maps in the theorem take values in a fixed finite-dimensional space of smooth perturbations. Consequently, the same target neighborhood serves every spatial norm. For V, the identity $\det e^H = e^{\operatorname{tr} H} = 1$ gives $dv_{g_H} = dv_g$ pointwise.

1.2. **The cokernel Hessian.** For a fixed parameter space, write

$$F = \mathsf{L} - \lambda I_m, \quad A = DF(0), \quad R = \operatorname{im} A, \quad C = R^{\perp}, \quad K = \ker A, \quad a = \dim R, \quad d = \dim C. \tag{1.4}$$

The intrinsic cokernel quadratic map is

$$Q(v) = \tfrac{1}{2} P_C D^2 F(0)[v, v], \qquad v \in K.$$

For the potential and conformal parameter spaces,

$$\operatorname{ind}_+\langle c, Q\rangle = \operatorname{ind}_-\langle c, Q\rangle = \infty \qquad (c \in C \setminus \{0\}).$$

In the potential case its kernel is determined by

$$b_c(x, y) = \sum_{i,j} c_{ij} u_i(x) u_j(y), \qquad R_\lambda = (P_0 - \lambda)^{-1}|_{E_\lambda^\perp}.$$

The annihilator equation gives $b_c(x, x) = 0$ on the allowed support region. Unique continuation implies that the first nonzero transverse diagonal term has finite degree. This degree is even, and the eigenfunction equation makes the homogeneous term harmonic. Its restriction to the cotangent unit sphere has both signs. The principal symbol of the contracted Hessian therefore has both signs. Disjoint localized oscillations give arbitrarily large definite subspaces; imposing finitely many linear equations preserves both infinite indices. The corresponding conformal identity and the estimates are in Section 3.

A regular zero of Q on a finite-dimensional subspace then gives the section by the weighted implicit equation of Arutyunov–Zhukovskiy [11, Theorem 3.3 and its proof]. The weights are two on a complement of K and one on the quadratic correction directions. A Euclidean radial parameter gives the punctured analyticity and the pairwise $1/2$-Hölder estimate; the reduction is written in Theorem 4.5. For pointwise fixed volume density, the tensor symbol and the compactly supported divergence equation are in Section 5.

1.3. **Strong stability and multiplicity loci.** The strong and weak Arnold hypotheses are those of Colin de Verdière [18]. For the fixed cluster matrix and a specified parameter space, the strong hypothesis is surjectivity of $D\mathsf{L}(0)$. The weak hypothesis states that, in each sufficiently small parameter neighborhood, λI_m remains in the image of every sufficiently small continuous perturbation of L. A right inverse in Theorem A implies the weak hypothesis by Brouwer degree. Appendix B gives a direct Fourier proof for flat two-tori, answering affirmatively the weak-stability question in [18, §3, p. 188].

On a closed manifold, a nonzero annihilator c for unrestricted metric variations at a positive Laplace eigenvalue satisfies

$$\sum c_{ij} u_i u_j = 0, \qquad \sum c_{ij}\, du_i \otimes du_j = 0.$$

Theorem 6.1 gives $n_+(c), n_-(c) \geq 4$. Thus every positive eigenvalue of multiplicity at most seven satisfies the Strong Arnold Hypothesis; the flat-torus example of multiplicity eight is sharp. This answers Question 4.7 of Greilhuber–Kepplinger [28].

For a closed manifold, in the full metric space or full conformal class, put

$$\Gamma = \mathsf{L} - \frac{\operatorname{tr} \mathsf{L}}{m} I_m, \qquad \mathcal{Z} = \Gamma^{-1}(0).$$

Theorem 6.6 states that $\mathcal{Z}$ is a C^1 embedded submanifold near the reference if and only if $D\mathsf{L}(0)$ is onto. In that case its codimension is $m(m+1)/2 - 1$. At a singular reference, the intrinsic quadratic equation has exact-level velocities whose difference is not a quadratic zero; hence the velocities cannot form a linear tangent space.

For the exact value level, the contingent tangent cone is

$$T_0 F^{-1}(0) = \{v \in K \,:\, Q(v) = 0\}. \tag{1.5}$$

Every vector on the right is the velocity of a two-sided $C^{1,1/2}$ curve in that level. More generally, a smooth target germ f admits an exact C^1 lift with prescribed derivative L precisely when

$$AL = Df(0), \qquad P_C D^2 f(0)[u,v] = P_C D^2 F(0)[Lu, Lv]. \tag{1.6}$$

These statements hold for the potential and conformal parameter spaces, and for the Banach-space maps of Section 6 satisfying its infinite-index hypothesis.

1.4. **Curvature on spectral levels.** Let Ω denote the curvature of the connection in (1.1). For a varying metric, the Hilbert spaces are identified with $L^2(M, dv_g)$ by multiplication by $(dv_{g_v}/dv_g)^{1/2}$. This identification is part of the definition of the connection.

If $F \circ h = f$ and $h(0) = 0$, the chain rule requires $Df(0)(\mathbb{R}^r) \subset R$. In classes P and C, this is the only condition on a smooth matrix germ when the derivative of h is free.

Theorem B (Matrix germs and curvature). *Suppose $m \geq 2$ and the variation class is* P *or* C. *Let $r \geq 1$, let $f : (\mathbb{R}^r, 0) \to (\mathrm{Sym}_m(\mathbb{R}), 0)$ be smooth with $Df(0)(\mathbb{R}^r) \subset R$, and prescribe matrices*

$$\mathscr{R}_{ij} = -\mathscr{R}_{ji} \in \mathfrak{so}(m) \qquad (1 \leq i, j \leq r).$$

There are a finite-dimensional space V of admissible perturbations and an embedded smooth germ $h : (\mathbb{R}^r, 0) \to (V, 0)$ such that

$$F(h(t)) = f(t), \qquad (h^*\Omega)_0(e_i, e_j) = \mathscr{R}_{ij}, \tag{1.7}$$

where Ω is the curvature of the projection connection. If $d > 0$, one may take $\dim V = a + r(d+1)$, and the cluster derivative on V is onto at every sufficiently small $t \neq 0$. If $d = 0$, $\dim V = D + r$ suffices. Analytic f gives analytic h.

For $r = m$, $f = 0$, and $\mathscr{R}_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the elementary matrix, every element of $SO(m)$ is the holonomy of a smooth based loop in the exact level $\mathsf{L} = \lambda I_m$, in every sufficiently small parameter neighborhood. These loops may be chosen in one embedded m-parameter family and are contractible within that family.

The space V may depend on f and $\mathscr{R}$. The curvature equation is the equality of alternating bilinear maps $(h^*\Omega)_0 = \mathscr{R}$ on $T_0\mathbb{R}^r$. The conformal factors range over $C^\infty(M)$. For the equation $F \circ h = f$ alone, Theorem 4.9 supplies one finite-dimensional space for all germs with a fixed parameter dimension.

For a fixed $T \in \mathrm{SO}(m) \setminus \{I_m\}$, the infimum of the total variation of such loops with amplitude at most ε is bounded above and below by positive multiples of ε^{-1}. The parameter norm and the constants are specified in Section 7.

For example, take the eigenvalue 2 of the round two-sphere, with $m = 3$. Scalar potentials supported in any given nonempty open set admit a nine-dimensional perturbation space with the following property: every sufficiently small parameter neighborhood contains a loop on which the eigenvalue remains 2 with multiplicity three and whose holonomy is any prescribed element of SO(3). The loop is contractible in the same exact spectral level; see Section 7.

1.5. **Relative families and simultaneous spectral conditions.** In corank one, continuous lifts prescribed on a closed subset of a compact parameter space of finite covering dimension extend into a larger prescribed neighborhood. The smaller source and target neighborhoods depend on the covering dimension of the parameter space; the precise statement is Theorem 8.5. For a separated part of a cluster, the map to a real Grassmannian determines the spectral subbundle. In these nested neighborhoods, extension of a spherical family through the gap space is equivalent to nullhomotopy of its Grassmannian map.

For a finite effective symmetry group G, invariant perturbations give local sections with target

$$\mathrm{Sym}_G(E_\lambda) = \{Y = Y^* : Y\rho(\gamma) = \rho(\gamma)Y \quad (\gamma \in G)\}.$$

Theorem 9.1 gives the corresponding residual irreducibility statement. For positive-dimensional groups, larger common symmetries may restrict the image: on the quaternionic shells of Section 9,

left Sp(1)-invariant Euclidean domains and scalar potentials are $O(4)$-invariant. Their degree-two angular eigenspace gives the stated obstruction to generic irreducibility.

On the unit-round S^2, let $\mathcal{H}_j$ denote the real degree-j spherical harmonics, put $d\mu = dv_\gamma/(4\pi)$, and set $V_N = \bigoplus_{j=2}^{N} \mathcal{H}_j$. If $w_\ell(v)$ is the width of the cluster issuing from $\ell(\ell+1)$ for $(1+v)\gamma$, Theorem 11.1 gives

$$\max_{1\le \ell \le N+1} w_\ell(v) \ge c_N \|v\|^2_{L^2(d\mu)} \qquad (v \in V_N,\ \|v\|_{L^2} < \epsilon_N). \tag{1.8}$$

Both constants depend on the fixed integer N. The normalization gives area 4π and removes the degree-one tangent directions of conformal automorphisms. Each common cluster value is free to vary. The incompatibility of the joint second-order equations is established in Sections 10–11.

1.6. **Related results.**

1.6.1. *Multiplicity and transversality.* Arnol'd's comparison of eigenvalue multiplicities with the strata of symmetric matrices [6] led to the strong and weak hypotheses formulated by Colin de Verdière [18]. Uhlenbeck proved generic simplicity and eigenfunction transversality for variable metrics [58]; Bleecker–Wilson and Bando–Urakawa obtained splitting and generic simplicity by perturbation theory [16, 13]. For Neumann realizations, Gomes–Marrocos proved generic simplicity and studied analytic metric variation [27]. Theorems of Teytel and Micheletti–Pistoia describe noncrossing and multiplicity loci under transversality hypotheses [57, 43].

Colin de Verdière and Besson [18, 15] identified the annihilator of the metric derivative at a positive Laplace eigenvalue with the symmetric matrices C satisfying

$$F_C = \sum C_{ij} u_i u_j = 0, \qquad T_C = \sum C_{ij}\, du_i \otimes du_j = 0.$$

For a full conformal class, the annihilator is defined by $F_C = 0$. Proposition 2.1 gives the characterization for metric variations. Theorem 6.1 proves that $F_C = T_C = 0$ and $C \neq 0$ imply that C has at least four eigenvalues of each sign.

Greilhuber–Kepplinger prove that conformally unstable metrics form a set of infinite transversality codimension, and deduce maximal noncrossing rules [28, Theorems 1.3–1.4]. Their low-multiplicity theorem gives unrestricted stability through multiplicity six. Their Lemma 4.5 proves weak conformal stability on a surface when the conformal cokernel is one-dimensional and a reversing isometric involution is present. Theorem A concerns every given reference, without these two additional assumptions. The contracted resolvent kernel in [28, Proposition 3.1 and Lemma 4.5] also occurs in Section 3. Its first nonzero transverse diagonal term is harmonic; the associated scalar cokernel Hessian has infinite positive and negative indices. Theorems 6.1 and 6.6 address Questions 4.7 and 4.8 of [28], respectively. The numbering of [28] is that of version 4.

1.6.2. *Quadratic maps and singular inverse problems.* Regular zeros and finite-dimensional restrictions of quadratic maps were studied by Agrachev and Agrachev–Sarychev [3, 4]; rescaled inverse equations appear in [4, Lemmas 9.2–9.3 and Proposition 9.4]. If the intrinsic quadratic map has a regular zero, Arutyunov–Karamzin [8, Theorem 1] obtain a continuous local inverse with the estimate

$$\|S(y) - S(0)\| \le C\big(|y| + |P_C y|^{1/2}\big).$$

Their later work treats square-root metric regularity and tangent cones in Hilbert spaces [9]. For regular weighted polynomial truncations, Arutyunov–Zhukovskiy [11, Theorems 3.2–3.3] give coordinatewise inverse estimates and an implicit equation after weighted rescaling. The finite-dimensional section in Theorem 4.5 follows from that equation with weight two on a complement of the derivative kernel and weight one on a regular quadratic slice. Integration of the derivative estimate gives the $1/2$-Hölder bound between two target points; a Euclidean radial parameter gives smoothness off zero.

The corollary in Arutyunov–Zhukovskiy [12, §3] also applies to the finite-dimensional equation $F(x+z) = Y$, with the singleton constraint and weights two on the derivative complement and one on the quadratic slice. It gives $|x| = O(|Y|)$ and $|z| = O(|Y|^{1/2})$ for a solution at each target Y.

Tangent-cone formulas under 2-regularity are due to Izmailov–Solodov [36]. Ramos [52, Theorem 1] obtains inclusions for higher-order tangent sets under directional metric pseudo-subregularity. Arutyunov [10, Theorems 2.2, 2.4, and 2.6] constructs smooth exact curves with prescribed regular quadratic-null direction and compatible higher jets. Under the infinite-index hypothesis below, regular zeros are dense in the quadratic null cone, so its equality with the contingent tangent cone also follows from the 2-regular tangent-cone theorem. In Theorem 6.7, finitely many mixed Taylor coefficients vanish after restriction to a finite-codimensional subspace. The augmented equation, with weights 2 on the parameter coordinates and $2k+1$ on the correction coordinates, then has a regular weighted truncation. The implicit equation in [11, proof of Theorem 3.3, equations (4.19)–(4.22)], together with polar derivative estimates, gives the prescribed $C^{k,1/2}$ lift. For $k=1$, every quadratic-null direction, including a singular one, is realized by a two-sided $C^{1,1/2}$ curve.

The quadratic maps in Section 4 remain surjective after restriction to every finite-codimensional subspace and admit prescribed finite symmetric and alternating Gram arrays. In corank one, a finite-dimensional nondegenerate scalar restriction gives the relative lifting theorem of Section 8; its smaller neighborhoods depend on the covering dimension of the parameter space. Related relative extension results for endpoint maps are due to Agrachev–Boarotto–Lerario [1]. High-connectivity results for intersections of real quadrics are due to Larsen–Lindenstrauss [39].

The smooth graph completion in Appendix D follows from symmetric Taylor factorization and the parameter-dependent implicit function theorem. Smooth cutoffs preserve a given graph near a closed subset of the external parameter space; a flat correction preserves an entire formal jet.

1.6.3. *Prescription and geometric constraints.* Taking $Y = \pm sI_m$, $s > 0$, gives metrics with the same volume density for which every eigenvalue in the selected cluster equals $\lambda \pm s$. Thus, in dimension at least three, every positive ordered eigenvalue can be increased and decreased by arbitrarily small smooth metric perturbations supported in O and preserving the volume density. For the first positive eigenvalue, the absence of local minima under a fixed-total-volume constraint is due to El Soufi–Ilias [25, Corollary 1.1(ii)]. On the round S^2, Hersch's inequality [33] excludes an increase of the first positive eigenvalue at fixed area. Section 5 constructs, in every dimension at least three, a metric and an open set on which all sufficiently small nonzero conformal perturbations decrease a specified simple eigenvalue.

Colin de Verdière constructed metrics with prescribed finite spectra, including multiplicities [19]. Lohkamp imposed additional geometric constraints [40]; He–Wang prescribed volume and finite Dirichlet spectra [32]. Theorem A gives a local section through every reference in the stated parameter spaces. In the potential and fixed-density cases, the perturbations have support in one compact subset of the prescribed open set.

Pseudodifferential representations of metric Hessians also occur in Okikiolu's work on spectral zeta functions [48]. The fixed-density calculation here concerns one isolated cluster. Its trace-free tensor Hessian has the transverse quadratic symbol

$$t_C\big(\operatorname{tr} D^2 - (\operatorname{tr} D)^2\big).$$

Both signs occur for $n \geq 3$. When the stress tensor vanishes, Section 5 instead solves a compactly supported divergence equation. First variations of ordered eigenvalues and extremal metrics are studied in [26, 37].

1.6.4. *Bundles, connections, and holonomy.* A section of the matrix map pulls back the tautological bundle from a Grassmannian. Grassmannians also occur in Agrachev's multiplicity filtrations and in the Morse theory of ordered eigenvalues developed by Berkolaiko–Zelenko [2, 14]. For Dirac families, nontrivial spectral bundles obstruct extension across a ball [47]. The families below have the prescribed bundles as spectral subbundles of scalar operators. On the round S^2, the multiplication isomorphism of Colin de Verdière [18, §2] gives a local potential section smooth at zero.

The universal-connection theorem of Narasimhan–Ramanan [46] realizes connections as pullbacks of a Grassmannian connection. Ramadas determines the homotopy type of the corresponding space

of classifying maps [51]. The spectral interpretation of the projection connection and its holonomy is developed by Simon, Wilczek–Zee, and Wu [54, 59, 60]. The generation of holonomy by curvature is the content of the Ambrose–Singer theorem [5]; curvature-spanning constructions for families of Hamiltonians appear in [61]. Isoholonomic length and optimization problems are treated in [56, 55].

In Theorem B, the projections are spectral projections of scalar Schrödinger or Laplace operators. The equations $F \circ h = f$ and $(h^*\Omega)_0 = \mathscr{R}$ prescribe the matrix germ and the curvature at zero. For $f = 0$, the eigenvalue and its multiplicity remain constant along every curve in the parameter family.

1.6.5. *Symmetry and multiplicity rigidity.* For a finite normal cover, Zelditch proved generic irreducibility when the manifold dimension is at least the dimension of every irreducible real representation of the deck group [64]. Theorem 9.1 gives commutant-valued local sections, and its genericity consequence has no representation-degree restriction. Shelukhin has announced generic irreducibility under finite symmetry in joint work with Morozov [53]. For positive-dimensional symmetry, Cianci–Judge–Lin–Sutton prove generic real irreducibility for free torus actions, with dimension larger than the orbit dimension [17]. These statements concern invariant Riemannian metrics.

For invariant Euclidean domains, Pereira [49] and Marrocos–Pereira [41, 42] study the corresponding irreducibility problem. The quaternionic example in Section 9 disproves the compact-group conjecture in [41, Conjecture 1, p. 2]. Its Euclidean domains and scalar potentials share a larger orthogonal symmetry. Larger commutants also occur in the homogeneous-space analysis of [20].

Zelditch formulated the multiplicity-rigidity problem for the round two-sphere in [63, 62]. Odd infinitesimal conformal deformations also occur in Guillemin's theory of Zoll surfaces [31]. Theorem 11.1 gives local uniqueness in each finite harmonic space V_N, together with a quadratic lower bound for cluster splitting. The scalar value of each cluster may vary; the constants depend on N.

2. Spectral projections and perturbations

All adjoints are taken in a specified real Hilbert space, after complexification only for contour integration. Matrix pairings are $\langle A, B\rangle_{\mathrm{F}} = \operatorname{tr}(A^T B)$. In particular, on symmetric matrices this is $\operatorname{tr}(AB)$, and on skew matrices it is $-\operatorname{tr}(AB)$. If the parameter changes a density, the identification with the reference Hilbert space is multiplication by the positive square root of the density ratio. The sign and indexing conventions are

$$\Delta_g = -\operatorname{div}_g \nabla_g, \qquad 0 = \lambda_0(g) < \lambda_1(g) \le \lambda_2(g) \le \cdots \tag{2.1}$$

for a closed connected manifold. Let $\mathcal{D}$ denote the operator domain with its graph norm. Contour projections are chosen before any finite-dimensional family is constructed.

2.1. **Fixed Hilbert spaces and spectral projections.** Fix a reference density dv_g. For g' near g, put $\rho_{g'} = dv_{g'}/dv_g$ and

$$J_{g'} u = \rho_{g'}^{1/2} u, \qquad T_{g'} = J_{g'} \Delta_{g'} J_{g'}^{-1} \quad \text{on } L^2(M, dv_g). \tag{2.2}$$

The map $J_{g'}$ is unitary from $L^2(M, dv_{g'})$ to $L^2(M, dv_g)$, since $\int |J_{g'}u|^2 dv_g = \int |u|^2 dv_{g'}$. For closed M, the operator domain is $H^2(M)$. For a smooth boundary, write

$$\mathcal{D}_D = H^2(M) \cap H_0^1(M), \qquad \mathcal{D}_N = \{u \in H^2(M) \,:\, \partial_\nu u = 0\}.$$

In the fixed-manifold metric perturbations considered here, $g' = g$ on a collar of ∂M. Thus $J_{g'} = I$ on that collar and the conormal derivative is independent of g'. The domains of $T_{g'}$ are consequently the fixed spaces $\mathcal{D}_D$ or $\mathcal{D}_N$. The Euclidean domain families in Section 9 are treated by their separate radial and meridional realizations.

The form $q_g(u) = \int_M |du|_g^2 \, dv_g$, with domain $H_0^1(M)$ for Dirichlet data and $H^1(M)$ otherwise, is closed and nonnegative. Indeed, $q_g(u) + \|u\|_2^2$ is an equivalent squared norm on the indicated form domain. Its inclusion into L^2 is compact. The associated self-adjoint operator therefore has compact resolvent. For closed M, the domain and regularity follow from [24, Theorems 15.1 and 16.1]. For

the boundary domains, see [30, (2.4) and Theorem 2.2, p. 833]; the elliptic regularity implication on compact manifolds is [29, Theorem 6.8, 3°, and Corollary 6.10, pp. 40–41].

The boundary principal symbols satisfy the following complementing condition. In inward boundary normal coordinates, the principal part of Δ_g at a boundary point is

$$-\partial_t^2 - h^{\alpha\beta}(y,0)\partial_{y_\alpha}\partial_{y_\beta}.$$

For a nonzero tangential covector ξ', set $\kappa = (h^{\alpha\beta}\xi'_\alpha\xi'_\beta)^{1/2} > 0$. The solutions of $-w'' + \kappa^2 w = 0$ decaying as $t \to +\infty$ are $w(t) = ce^{-\kappa t}$. Their Dirichlet data are c, and their outward normal derivatives are κc. Either homogeneous boundary condition therefore forces $c = 0$. Both boundary operators are normal. Applying the cited regularity theorem to $1 + \Delta_g$ meets its invertible Dirichlet-realization hypothesis: the form $q_g + \|\cdot\|_2^2$ is coercive. Adding the constant 1 does not change the principal or boundary symbols.

For the closed, Dirichlet, and Neumann realizations, and every integer $s \geq 0$, the resulting estimate is

$$\|u\|_{H^{s+2}} \leq C_s\big(\|\Delta_g u\|_{H^s} + \|u\|_{L^2}\big) \qquad (u \in H^{s+2}(M) \cap \mathcal{D}). \tag{2.3}$$

Here $\mathcal{D}$ denotes the fixed H^2 domain. Put $R_+ = (1 + \Delta_g)^{-1}$. Regularity gives $R_+(H^s) \subset H^{s+2} \cap \mathcal{D}$. This map has closed graph: convergence in H^{s+2} implies convergence in L^2, where R_+ is bounded. Thus $R_+ : H^s \to H^{s+2}$ is bounded. For $s > 0$, the Sobolev interpolation inequality

$$\|u\|_{H^s} \leq \epsilon\|u\|_{H^{s+2}} + C_{s,\epsilon}\|u\|_{L^2}$$

absorbs the additional $\|u\|_{H^s}$ term in the bound for $R_+(1 + \Delta_g)u$; for $s = 0$ the L^2 resolvent bound gives the estimate directly. Repeated application to $\Delta_g u = \lambda u$ makes every eigenfunction smooth up to the boundary. Write $m' = \sqrt{\det g'}$ in a coordinate chart. The differential expression on the fixed Hilbert space is

$$T_{g'}f = -\rho_{g'}^{1/2}(m')^{-1}\partial_i\Big(m' g'^{ij}\partial_j(\rho_{g'}^{-1/2}f)\Big). \tag{2.4}$$

For each $k \geq 4$, the maps $g' \mapsto g'^{-1}$, m', and $\rho_{g'}^{\pm 1/2}$ are real analytic between the corresponding C^k spaces on a uniformly positive neighborhood. Indeed, matrix inversion follows from a locally convergent Neumann series; the determinant is polynomial; and the positive scalar powers have locally convergent binomial series in the Banach algebra C^k. After expanding the two derivatives in (2.4), each coefficient is a product of these functions and at most two metric derivatives. Thus $g' \mapsto T_{g'}$ is real analytic into $\mathcal{L}(\mathcal{D}, L^2)$, where $\mathcal{D}$ is the fixed H^2 domain stated above. Multiplication by a bounded coefficient has L^2 norm at most its supremum norm, and each coordinate derivative of order at most two maps H^2 continuously to L^2. A finite partition of unity sums these estimates. In particular, on a smaller neighborhood,

$$\big\|T_{g'} - T_g\big\|_{\mathcal{D}\to L^2} \leq C\,\big\|g' - g\big\|_{C^2}. \tag{2.5}$$

The same expansion gives all parameter derivatives on finite-dimensional smooth families. For a fixed high Sobolev exponent, replace $H^2 \to L^2$ by $H^{s+2} \to H^s$ and require the corresponding finite number of bounded coefficient derivatives. Choose a positively oriented rectangular contour γ, invariant as a set under complex conjugation, whose real intersections are the endpoints of I and whose interior contains no reference eigenvalue other than λ. Let

$$d_\gamma = \min_{z\in\gamma} \operatorname{dist}(z, \operatorname{spec} T_g) > 0, \qquad R_0(z) = (z - T_g)^{-1}.$$

For every $\nu \in \operatorname{spec} T_g$ and $z \in \gamma$,

$$\frac{1+\nu}{|z-\nu|} \leq 1 + \frac{1+|z|}{d_\gamma}.$$

Expansion in the reference eigenbasis and (2.3) therefore give

$$\sup_{z\in\gamma}\|R_0(z)\|_{L^2\to\boldsymbol{D}}<\infty. \tag{2.6}$$

The perturbation bound (2.5) now applies uniformly on γ. If $\sup_\gamma \|(T_{g'}-T_g)R_0(z)\|_{L^2\to L^2}<1/2$, then

$$(z-T_{g'})^{-1}=R_0(z)\sum_{j=0}^{\infty}\big((T_{g'}-T_g)R_0(z)\big)^j. \tag{2.7}$$

The identity

$$z-T_{g'}=\big[I-(T_{g'}-T_g)R_0(z)\big](z-T_g)$$

proves that (2.7) is a two-sided inverse with domain $\boldsymbol{D}$. The geometric-series bound and (2.6) imply uniform convergence in $\mathcal{L}(L^2,\boldsymbol{D})$. The resolvent identity at two parameters is

$$R_{g''}(z)-R_{g'}(z)=R_{g''}(z)(T_{g''}-T_{g'})R_{g'}(z).$$

Division by the parameter increment, followed by the operator-norm limit, gives $DR[h]=R\,DT[h]\,R$. In particular,

$$\begin{aligned}D^2R[h_1,h_2]=R\,D^2T[h_1,h_2]R+R\,DT[h_1]R\,DT[h_2]R\\+R\,DT[h_2]R\,DT[h_1]R.\end{aligned}$$

Induction by the product rule gives each higher derivative as a finite sum of products of these bounded operators and derivatives of T. The estimates are uniform on the compact contour and justify differentiation under its integral. The locally normally convergent series (2.7) also proves real analyticity in the metric parameter. In particular, no eigenvalue crosses either endpoint of I in this neighborhood. Define

$$P_{g'}=\frac{1}{2\pi i}\int_\gamma (z-T_{g'})^{-1}\,dz. \tag{2.8}$$

The Riesz projection theorem [38, Chapter III, Section 6.4], applied to the self-adjoint realization above, identifies $P_{g'}$ with the orthogonal projection onto the indicated cluster. It commutes with complex conjugation and therefore preserves real functions. Its rank is m. Indeed, if P' and P are orthogonal projections with $\|P'-P\|<1$ and $v\in\operatorname{im}P$ satisfies $P'v=0$, then

$$\|v\|=\|(P-P')v\|\le\|P-P'\|\,\|v\|,$$

so $v=0$. Interchanging the projections proves injectivity in the other direction. Since $\operatorname{im}P$ has dimension m, both inequalities on the dimensions follow. Let $U:\mathbb{R}^m\to L^2(M,dv_g)$ have columns an orthonormal real eigenbasis $u_1,\dots,u_m$. Put

$$U_{g'}=P_{g'}U\,(U^*P_{g'}U)^{-1/2},\qquad \mathsf{L}(g')=U_{g'}^*T_{g'}U_{g'}. \tag{2.9}$$

The Gram matrix equals I_m at g and remains positive. Its positive inverse square root is real analytic by the convergent power series for $(I+X)^{-1/2}$ when $\|X\|<1$. Direct multiplication gives $U_{g'}^*U_{g'}=I_m$. Its image has dimension m and equals $\operatorname{im}P_{g'}$. The matrix $\mathsf{L}(g')$ is therefore real analytic on the same metric neighborhood. Consequently

$$T_{g'}U_{g'}=U_{g'}\mathsf{L}(g'),\qquad \mathsf{L}(g')^T=\mathsf{L}(g'). \tag{2.10}$$

The corresponding frame for the original metric is

$$V_{g'}=J_{g'}^{-1}U_{g'},\qquad \Delta_{g'}V_{g'}=V_{g'}\mathsf{L}(g'),\qquad V_{g'}^*V_{g'}=I_m, \tag{2.11}$$

where the last adjoint is taken in $L^2(M,dv_{g'})$. In particular,

$$V_{g'}\ker(\mathsf{L}(g')-\theta I_m)=\ker(\Delta_{g'}-\theta)\cap\operatorname{im}V_{g'}.$$

The spectral frame also depends smoothly on the spatial variables. Let t range over a finite-dimensional ball and let g_t have coefficients smooth in (t, x), with the same boundary realization as above. For each nonnegative integer s, put

$$\mathcal{D}_s = \{u \in H^{s+2}(M) \,:\, u \text{ satisfies the fixed boundary condition}\};$$

in the closed case there is no boundary condition. On a smaller parameter ball, the elliptic estimate and the resolvent bound give

$$\sup_{\substack{t\in B_\epsilon(0)\\ z\in\gamma}} \|(z - T_t)^{-1}\|_{H^s\to\mathcal{D}_s} < \infty. \tag{2.12}$$

Choose a compact parameter ball $\mathcal{P}$ contained in the neighborhood where the contour avoids the spectrum. Fix s and $(t_0, z_0) \in \mathcal{P} \times \gamma$. Regularity and the closed graph theorem give

$$(z_0 - T_{t_0})^{-1} \,:\, H^s \longrightarrow \mathcal{D}_s$$

as a bounded map. The coefficients of T_t depend continuously on t in the norms required for $\mathcal{L}(\mathcal{D}_s, H^s)$. Therefore, near (t_0, z_0),

$$\big\|\big((T_t - T_{t_0}) + (z_0 - z)I\big)(z_0 - T_{t_0})^{-1}\big\|_{H^s\to H^s} < \tfrac{1}{2}.$$

The Neumann series gives a uniform $H^s \to \mathcal{D}_s$ bound on this neighborhood. A finite cover of $\mathcal{P} \times \gamma$ proves (2.12). The compact ball $\mathcal{P}$ depends only on the L^2 contour condition and is independent of s; the bounds and finite covers may depend on s.

For fixed smooth f, let $u_t(z) = (z - T_t)^{-1} f$. Differentiating its equation gives, for a nonzero parameter multi-index α,

$$(z - T_t)\partial_t^\alpha u_t = \sum_{0<\eta\le\alpha} \binom{\alpha}{\eta} (\partial_t^\eta T_t)\partial_t^{\alpha-\eta} u_t. \tag{2.13}$$

The resolvent identity first proves this equality in $\mathcal{D}_0$ by difference quotients. Estimate (2.12) and induction on $|\alpha|$ prove it in every $\mathcal{D}_s$. Thus $t \mapsto u_t(z)$ is smooth with values in each Sobolev space, uniformly on the contour. Sobolev embedding, with s chosen arbitrarily large, makes $u_t(z, x)$ jointly smooth in (t, x). Integration over γ, followed by the finite-dimensional Gram normalization, proves joint smoothness of the frames $V_{g_t} = J_{g_t}^{-1} U_{g_t}$ in Sections 8–7.5. Let g_t be a smooth one-parameter family through g. Differentiating $(z - T_t)(z - T_t)^{-1} = I$ gives

$$\frac{d}{dt}(z - T_t)^{-1} = (z - T_t)^{-1} \dot{T}_t (z - T_t)^{-1}. \tag{2.14}$$

The series (2.7) converges uniformly on the contour, and the differentiated series does so on any smaller parameter neighborhood. Hence

$$\dot{P} = \frac{1}{2\pi i} \int_\gamma (z - T_g)^{-1} \dot{T} (z - T_g)^{-1}\, dz. \tag{2.15}$$

For example,

$$\|\dot{P}\|_{L^2\to\mathcal{D}} \le \frac{\text{length}(\gamma)}{2\pi} \sup_{z\in\gamma} \|(z - T_g)^{-1}\|_{L^2\to\mathcal{D}}^2 \, \|\dot{T}\|_{\mathcal{D}\to L^2}.$$

The norm on $\mathcal{D}$ includes the L^2 norm. The constant depends on the fixed contour and its distance from the excluded spectrum. Differentiating $P_t^2 = P_t$ and multiplying on both sides by P gives

$$P\dot{P}P = 0.$$

Since $PU = U$, it follows that $\frac{d}{dt}(U^* P_t U)|_0 = 0$. The derivative of its inverse square root at the identity is therefore zero, and

$$\dot{U} = \dot{P}U, \qquad P\dot{U} = 0, \qquad U^*\dot{U} = 0. \tag{2.16}$$

For a different orthonormal frame $\widetilde{U}_t = U_t O_t$ with $O_0 = I_m$, one has

$$\Pi \dot{\widetilde{U}}_0 = U \dot{O}_0.$$

Thus orthogonality of the frame derivative to the reference eigenspace is the normalization in (2.16). The same construction applies when the reference cluster has eigenvalues $\theta_1, \dots, \theta_m$, not necessarily equal. Choose U so that $T_g U = U\Theta$, $\Theta = \operatorname{diag}(\theta_1, \dots, \theta_m)$. Equations (2.10) and (2.16) give

$$\dot{\mathsf{L}} = U^* \dot{T} U, \qquad (T_g - \theta_a)\dot{u}_a = -(I - P)\dot{T} u_a. \tag{2.17}$$

Indeed, differentiation of $\mathsf{L} = U_t^* T_t U_t$ gives two frame terms. They are $(U^*\dot{U})^*\Theta$ and $\Theta U^*\dot{U}$, both zero. Differentiation of $T_t U_t = U_t \mathsf{L}_t$ and projection to $\ker P$ gives the second equality. Define

$$R_a = (T_g - \theta_a)^{-1}\big|_{\ker P}, \qquad R_a|_{\operatorname{im} P} = 0.$$

The excluded spectrum is disjoint from every θ_a, so R_a exists and

$$\dot{u}_a = -R_a \dot{T} u_a. \tag{2.18}$$

Both formulas remain valid when some of the θ_a coincide. At a smooth reference metric, take a complete orthonormal eigenbasis e_j with eigenvalues $\nu_j \geq 0$, and define the Hilbert scale

$$\|f\|_{\mathcal{H}^s}^2 = \sum_j (1 + \nu_j)^s |\langle f, e_j \rangle|^2.$$

Let $d_a = \inf\{|\nu_j - \theta_a| \,:\, e_j \in \ker P\} > 0$. For every excluded eigenvalue,

$$\frac{1 + \nu_j}{|\nu_j - \theta_a|} \leq 1 + \frac{1 + \theta_a}{d_a}.$$

Squaring this inequality, multiplying by $(1 + \nu_j)^{-1}|f_j|^2$, and summing proves

$$\|R_a f\|_{\mathcal{H}^1} \leq \left(1 + \frac{1 + \theta_a}{d_a}\right) \|f\|_{\mathcal{H}^{-1}}. \tag{2.19}$$

The $\mathcal{H}^1$ norm is the norm of the Dirichlet energy plus L^2 mass, on H^1 in the closed and Neumann cases and on H_0^1 in the Dirichlet case. This follows first on finite eigenfunction sums and then by completion. The extension in (2.19) is self-adjoint in the associated dual pairing, since its real spectral multipliers are symmetric. Another smooth real orthonormal frame is $U_{g'} O(g')$, with $O(g') \in O(m)$. After constant conjugation, assume $O(g) = I_m$. Along a curve through g, set $A = \dot{O}(0)$ and $L(t) = \mathsf{L}(g_t)$. Since $O^T O = I_m$ and $L(0) = \lambda I_m$,

$$\frac{d}{dt}(O^T L O)\bigg|_0 = \dot{L}(0), \qquad \frac{d^2}{dt^2}(O^T L O)\bigg|_0 = \ddot{L}(0) + 2A^T \dot{L}(0) + 2\dot{L}(0)A.$$

Thus the second derivative is unchanged when $\dot{g}_0 \in \ker D\mathsf{L}(g)$. For a parameter chart θ with $\theta(0) = g$, the chain rule gives

$$D^2(\mathsf{L} \circ \theta)(0)[a, a] = D^2\mathsf{L}(g)[D\theta(0)a, D\theta(0)a] + D\mathsf{L}(g) D^2\theta(0)[a, a].$$

Pairing with a cokernel matrix removes the last term. The contracted Hessian on the derivative kernel is therefore independent of the frame and source chart.

2.1.1. *The metric derivative.* For $g_t = g + th + O(t^2)$,

$$\frac{d}{dt} g_t^{-1}\bigg|_0 = -h^{\sharp\sharp}, \qquad \frac{d}{dt} dv_{g_t}\bigg|_0 = \tfrac{1}{2} \operatorname{tr}_g(h)\, dv_g. \tag{2.20}$$

The first identity follows by differentiating $g_t g_t^{-1} = I$. The second follows from $\frac{d}{dt} \det g_t = (\det g_t) \operatorname{tr}(g_t^{-1} \dot{g}_t)$. Let $v_i(t) = J_{g_t}^{-1} u_i(t)$ be the columns of the frame in $L^2(M, dv_{g_t})$. They are orthonormal there. At $t = 0$, differentiation gives

$$\int_M (\dot{v}_i u_j + u_i \dot{v}_j)\, dv_g = -\tfrac{1}{2} \int_M \operatorname{tr}_g h\, u_i u_j\, dv_g. \tag{2.21}$$

The matrix entry is $\int g_t^{ab}\,\partial_a v_i(t)\partial_b v_j(t)\,dv_{g_t}$. Its derivative has the inverse-metric term, the density term, and the two terms

$$\int_M \left(\left\langle d\dot v_i, du_j\right\rangle + \left\langle du_i, d\dot v_j\right\rangle\right) dv_g = \lambda \int_M (\dot v_i u_j + u_i \dot v_j)\, dv_g.$$

Green's formula proves this equality. For Dirichlet data, $\dot v_i|_{\partial M} = 0$; for Neumann data, $\partial_\nu u_j = 0$. These identities remove the boundary terms. There is no boundary term when M is closed. Substituting (2.21) therefore gives

$$\begin{aligned} D\mathsf{L}(g)[h]_{ij} = \int_M \Big[&- h(\nabla u_i, \nabla u_j) \\ &+ \tfrac{1}{2}\operatorname{tr}_g h \left(\left\langle du_i, du_j\right\rangle_g - \lambda u_i u_j\right)\Big] dv_g. \end{aligned} \tag{2.22}$$

The formulas hold for all closed-manifold variations and for the interior-supported boundary variations considered here. For a family with $dv_{g_t} = dv_g$ and initial derivative h, the coordinate divergence expression gives

$$\dot\Delta f = \operatorname{div}_g(h^{\sharp\sharp} df), \qquad \left\langle \dot\Delta f, z\right\rangle = -\int_M h(\nabla f, \nabla z)\, dv_g. \tag{2.23}$$

The second identity follows by integration by parts. It is valid for $z \in H^1$ when h has interior support, by density of smooth functions in H^1 and continuity of both sides. Here $h^{\sharp\sharp} df$ has components $g^{ia} g^{jb} h_{ab}\,\partial_j f$. For such a family the density conjugation is the identity. Thus (2.17) and (2.18) become

$$\dot{\mathsf{L}}_{ab} = -\int_M h(\nabla u_a, \nabla u_b)\, dv_g, \qquad \dot u_a = -R_a \dot\Delta u_a. \tag{2.24}$$

If $\eta \in \mathcal{H}^{-1}$ annihilates the cluster and $w_a = R_a \eta$, equations (2.19) and (2.23) also give the weak pairing identity

$$\langle \eta, \dot u_a\rangle = -\left\langle R_a\eta, \dot\Delta u_a\right\rangle = \int_M h(\nabla u_a, \nabla w_a)\, dv_g. \tag{2.25}$$

The integral is finite because u_a and h are smooth and $w_a \in \mathcal{H}^1$. Finite spectral sums prove the dual self-adjointness equality first; (2.19) extends it to the stated data. Assume $\lambda > 0$. For $C \in \operatorname{Sym}_m(\mathbb{R})$ set

$$F_C = \sum C_{ij} u_i u_j, \qquad T_C = \sum C_{ij}\, du_i \otimes du_j, \qquad \tau_C = \operatorname{tr}_g T_C. \tag{2.26}$$

The tensor T_C is symmetric. For unrestricted variations, the annihilator equation obtained from (2.22) is

$$T_C = \tfrac{1}{2}(\tau_C - \lambda F_C) g. \tag{2.27}$$

Localization is valid because all smooth symmetric tensors supported in any coordinate ball are allowed as test variations. Taking the trace in (2.27), and then applying the product rule, gives

$$(n-2)\tau_C = n\lambda F_C, \qquad \Delta_g F_C = 2\lambda F_C - 2\tau_C. \tag{2.28}$$

For $n > 2$, these imply $\Delta_g F_C = -4\lambda F_C/(n-2)$. Multiplication by F_C and integration gives

$$\int_M |dF_C|^2\, dv_g = -\frac{4\lambda}{n-2}\int_M F_C^2\, dv_g. \tag{2.29}$$

Since $\lambda > 0$, the identity

$$\int_M |dF_C|^2\, dv_g + \frac{4\lambda}{n-2}\int_M F_C^2\, dv_g = 0$$

gives $F_C = 0$, followed by $\tau_C = 0$ and $T_C = 0$. For $n = 2$, the first equation in (2.28) gives $2\lambda F_C = 0$; its second gives $\tau_C = 0$, and (2.27) gives $T_C = 0$.

Proposition 2.1 (Colin de Verdière–Besson characterization). *Let $\lambda > 0$ be an isolated Laplace eigenvalue on a closed connected manifold. The cokernel of $D\mathsf{L}(g)$ for unrestricted metric variations consists precisely of the matrices C for which $F_C = T_C = 0$. Every such matrix is trace-free.*

Proof. If C annihilates $D\mathsf{L}(g)$, localization gives (2.27), hence $F_C = T_C = 0$ by (2.28)–(2.29). Conversely, these two identities make the contraction of the integrand in (2.22) zero for every h. Moreover,

$$\operatorname{tr} C = \sum_{i,j} C_{ij} \int_M u_i u_j \, dv_g = \int_M F_C \, dv_g = 0.$$

□

See [15, Sections 4A and 7A] and [28, Proposition 2.3]. For $g_t = (1+t)g$, $\Delta_{g_t} = (1+t)^{-1}\Delta_g$. Therefore

$$D\mathsf{L}(g)[g] = -\lambda I_m. \tag{2.30}$$

For the full metric space and the full conformal class, $D\mathsf{L}(g)$ is onto if and only if $D\Gamma(g)$ is onto Sym_m^0. Indeed, if $D\Gamma(g)h = B \in \operatorname{Sym}_m^0$, write $D\mathsf{L}(g)h = B + cI_m$. Equation (2.30) gives

$$D\mathsf{L}(g)\big[h + (c/\lambda)g\big] = B.$$

Adding a further multiple of g gives every scalar component. Conversely, surjectivity of $D\mathsf{L}(g)$ implies surjectivity after projection onto Sym_m^0.

The derivative ranges on smooth variations and on C^k variations coincide. In fact, smooth functions and symmetric tensors are dense in the corresponding C^k spaces. The derivative is continuous, and its image on the smooth subspace is a finite-dimensional, hence closed, subspace of Sym_m. Taking limits therefore gives equality of the two ranges. This observation applies both to the unrestricted space and to the full conformal class. Aronszajn's theorem [7, pp. 235–239] implies that a smooth solution of $(\Delta_g + V - \lambda)u = 0$, with $V \in C^\infty(M;\mathbb{R})$, which vanishes to infinite order at an interior point vanishes on its connected interior component. In an interior chart the principal matrix is smooth and uniformly positive definite on compact subsets, and the lower-order coefficients are locally bounded. The theorem applies to every constant linear combination of eigenfunctions with eigenvalue λ. Vanishing on an open set implies infinite-order vanishing at each of its points.

The interior of a connected smooth manifold with boundary is connected. Indeed, each boundary point has a half-ball neighborhood whose interior part is connected, and hence belongs to one component of M°. Assign the boundary point to that component; nested half-ball neighborhoods show that the assignment is independent of the chart. Each interior component, together with its assigned boundary points, is open in M. These sets form a disjoint open partition of M. Connectedness leaves a single set. Unique continuation on M° and continuity on M therefore imply global vanishing.

2.2. **Supported scalar potentials.** Let (M^n, g) be compact, connected, and smooth, with $n \geq 2$. If $\partial M \neq \emptyset$, fix either the Dirichlet or the Neumann realization throughout this section. Let $V \in C^\infty(M;\mathbb{R})$ and

$$P = \Delta_g + V.$$

Fix an eigenvalue $\lambda \in \mathbb{R}$ of multiplicity m, an isolating interval I, and a nonempty open set $O \subset M^\circ$. The metric, measure, and boundary realization are fixed.

Multiplication by the real smooth V is bounded and self-adjoint on $L^2(M, dv_g)$. On the fixed domain of the Laplacian, P is self-adjoint by bounded perturbation; its resolvent is compact. Indeed, for a real $z < -\|V\|_\infty - 1$, the inverse $(z - \Delta_g)^{-1}$ is compact and $\|M_V(z-\Delta_g)^{-1}\| < 1$. The identity

$$(z-P)^{-1} = (z-\Delta_g)^{-1}[I - M_V(z-\Delta_g)^{-1}]^{-1}$$

expresses one resolvent as a compact operator times a bounded operator. The resolvent identity gives compactness at every point of the resolvent set. It is bounded below by $-\|V\|_\infty$. For an eigenfunction, $\Delta_g u = (\lambda - V)u$. Multiplication by the smooth function $\lambda - V$ preserves every H^s. Starting with

$u \in H^2$, equation (2.3) gives $u \in H^{s+2}$ whenever $u \in H^s$; hence $u \in C^\infty(M)$, up to the boundary. In particular, an arbitrary real eigenvalue λ has finite multiplicity and positive distance from the excluded spectrum. For a sufficiently small real potential q, let $\mathsf{L}(q)$ be the symmetric matrix of the I-cluster of $P+q$ in its Riesz–Gram frame, based at a fixed real orthonormal basis of $\ker(P-\lambda)$. Thus $\mathsf{L}(0) = \lambda I_m$.

2.2.1. *The localized annihilator.* Let $E = \ker(P-\lambda)$, let $u_1, \dots, u_m$ be its real orthonormal basis, and let $U : \mathbb{R}^m \to E$ be the associated isometry. Denote the orthogonal projection by Π. For z on a fixed contour enclosing the cluster, the identity

$$(z - P - q)^{-1} = (z-P)^{-1}[I - M_q(z-P)^{-1}]^{-1}$$

and the norm-convergent Neumann series apply whenever $\|q\|_\infty \sup_z \|(z-P)^{-1}\| < 1$. Both the Riesz projection $\Pi(q)$ and the bounded spectral operator

$$A(q) = \frac{1}{2\pi i}\int_\gamma z(z - P - M_q)^{-1}\, dz = (P + M_q)\Pi(q)$$

are analytic in the L^∞ norm of q. If U_q is the Gram-normalized frame, then $\mathsf{L}(q) = U_q^* A(q) U_q$ is analytic as well. The family has the fixed domain $\boldsymbol{\mathcal{D}}$ and reference measure dv_g [38].

Differentiating at $q = 0$ gives

$$D\mathsf{L}(0)[f]_{ij} = \int_M f u_i u_j \, dv_g. \tag{2.31}$$

Hence the annihilator and kernel are, respectively,

$$\mathcal{C}_O = \left\{ C \in \mathrm{Sym}_m \,:\, F_C := \sum_{i,j} C_{ij} u_i u_j = 0 \text{ on } O \right\}, \tag{2.32}$$

$$\boldsymbol{\mathcal{K}}_O = \left\{ f \in C_c^\infty(O) \,:\, \Pi M_f \Pi = 0 \right\}. \tag{2.33}$$

Indeed, the vanishing of $\int f F_C$ for every $f \in C_c^\infty(O)$ is equivalent to $F_C|_O = 0$ by the defining property of distributions, and F_C is smooth. Writing $\mathcal{R}_O = D\mathsf{L}(0)(C_c^\infty(O))$, one has

$$\mathrm{Sym}_m = \mathcal{R}_O \oplus \mathcal{C}_O, \qquad \operatorname{tr} C = \int_M F_C \, dv_g.$$

Every basis of $\mathcal{R}_O$ has preimages in $C_c^\infty(O)$. The cokernel is taken in Sym_m, including its scalar subspace.

Set

$$R = (P-\lambda)^{-1}|_{E^\perp}, \qquad R|_E = 0, \qquad B_C = \sum_{i,j} C_{ij} M_{u_i} R M_{u_j}.$$

The operator B_C is self-adjoint because $R^* = R$ and $C^T = C$.

Moreover,

$$\|B_C\|_{L^2\to L^2} \le \|R\|_{L^2\to L^2} \sum_{i,j} |C_{ij}|\, \|u_i\|_\infty \|u_j\|_\infty < \infty.$$

The norm is global; the relation $F_C = 0$ holds on O.

Lemma 2.2. *For $C \in \mathcal{C}_O$ and $f \in \boldsymbol{\mathcal{K}}_O$,*

$$\langle C, D^2\mathsf{L}(0)[f,f]\rangle = -2\langle f, B_C f\rangle_{L^2(M)}. \tag{2.34}$$

Proof. Write $(P + tM_f)U(t) = U(t)\mathsf{L}(t)$, where $U(t)^* U(t) = I_m$. Equation (2.31) gives $\mathsf{L}'(0) = 0$. The first differentiated equation, projected onto $E^\perp$, gives

$$(I - \Pi)U'(0) = -R M_f U.$$

Apply U^* to the second differentiated equation. The terms containing PU'' and $\lambda U''$ cancel, and the term $2U'\mathsf{L}'$ is zero. Thus

$$\mathsf{L}''(0) = 2U^* M_f U'(0) = -2U^* M_f R M_f U.$$

The E-component of $U'(0)$ contributes zero because $U^* M_f U = 0$. Contracting with C proves (2.34). □

2.3. **Conformal metrics.** The reference operator in this subsection is the Laplacian. The eigenvalue satisfies $\lambda > 0$. In boundary realizations, every conformal factor equals one on a fixed boundary collar.

2.3.1. *The generalized eigenvalue equation.* For $n \geq 3$, put $\beta = 4/(n-2)$ and

$$H = 1 + v > 0, \qquad g_v = H^\beta g. \tag{2.35}$$

For $n = 2$, put $g_v = (1+v)g$ with $1 + v > 0$. All parameters are real and smooth. On a manifold with boundary, v is compactly supported in the interior. For $n \geq 3$, $dv_{g_v} = H^{2n/(n-2)} dv_g$ and $g_v^{-1} = H^{-4/(n-2)} g^{-1}$. Hence

$$\Delta_{g_v} u = H^{-(\beta+2)} \big(H^2 \Delta_g u - 2H \langle dH, du \rangle \big). \tag{2.36}$$

The product rule for the positive Laplacian gives $\Delta_g(Hu) = H\Delta_g u + u\Delta_g H - 2\langle dH, du \rangle$. Substituting $\Phi = Hu$ into (2.36) yields

$$A_v \Phi = \theta B_v \Phi, \qquad A_v = \Delta_g - H^{-1}\Delta_g H, \qquad B_v = H^\beta. \tag{2.37}$$

Here $H^{-1}\Delta_g H$ acts by multiplication, not composition. The mass identity is

$$\int_M u^2 \, dv_{g_v} = \int_M H^\beta \Phi^2 \, dv_g. \tag{2.38}$$

Thus $B_v^{-1/2} A_v B_v^{-1/2}$ is the fixed-L^2 self-adjoint realization of the pencil. It agrees with (2.2), since $B_v^{1/2}\Phi = H^{n/(n-2)} u$. Near a boundary $H = 1$, so the Dirichlet and Neumann domains are unchanged. In dimension two, $A_v = \Delta_g$ and $B_v = 1 + v$. Along $v = ta$, define

$$\mathcal{A} = \begin{cases} \Delta_g + \beta\lambda, & n \geq 3, \\ \lambda I, & n = 2. \end{cases} \tag{2.39}$$

At $t = 0$, $A_0 = \Delta_g$ and $B_0 = I$. Differentiation of (2.37) gives

$$A_0' - \lambda B_0' = -M_{\mathcal{A}a}. \tag{2.40}$$

For $n \geq 3$ its direct second derivative is

$$A_0'' - \lambda B_0'' = M_{2a\Delta_g a - \lambda\beta(\beta-1)a^2}. \tag{2.41}$$

In dimension two this direct second derivative is zero. Let $\Phi(v)$ be the frame corresponding to (2.9), normalized by $\Phi(v)^* B_v \Phi(v) = I_m$. Differentiating $A_v\Phi(v) = B_v\Phi(v)\mathsf{L}(v)$ at zero and taking inner products with E_λ gives

$$D\mathsf{L}(0)[a]_{ij} = -\int_M (\mathcal{A}a) u_i u_j \, dv_g. \tag{2.42}$$

The terms $\Delta_g\Phi'$ and $\lambda\Phi'$ cancel on projection. This derivation includes the mass derivative B_0'.

More explicitly, left multiplication of the differentiated pencil equation by U^* gives

$$U^*(A_0' - \lambda B_0')U + U^*(\Delta_g - \lambda)\Phi' = \mathsf{L}'.$$

The second summand is zero because the columns of U solve $\Delta_g u_i = \lambda u_i$ and Φ' satisfies the fixed boundary realization. Thus $\Pi\Phi'$ contributes zero to the first variation.

2.3.2. *The relative cokernel.* Let $Z \subset M$ be closed, with $Z \cap M^\circ$ having empty interior. Set $O = M^\circ \setminus Z$ and $\mathcal{V} = C_c^\infty(O)$. For closed M this notation also permits $Z = \emptyset$ and $\mathcal{V} = C^\infty(M)$. The allowed region O is dense in M°. If C annihilates (2.42) on $\mathcal{V}$, then $\int a\,\mathcal{A}F_C\,dv_g = 0$ for all $a \in \mathcal{V}$. Therefore $\mathcal{A}F_C = 0$ on O. This function is smooth; density extends the equality to M°. For Dirichlet eigenfunctions, $F_C|_{\partial M} = 0$. For Neumann eigenfunctions, $\partial_\nu F_C = \sum C_{ij}(u_i\partial_\nu u_j + u_j\partial_\nu u_i) = 0$. For $n \geq 3$, integration by parts therefore yields

$$0 = \int_M F_C \mathcal{A}F_C\,dv_g = \int_M |dF_C|^2\,dv_g + \beta\lambda \int_M F_C^2\,dv_g. \tag{2.43}$$

The boundary term vanishes in each realization. Thus $F_C = 0$. For $n = 2$, the equation $\mathcal{A}F_C = \lambda F_C = 0$ gives the same conclusion. Consequently

$$\mathcal{C} = \operatorname{coker} D\mathsf{L}(0) = \{C \in \operatorname{Sym}_m(\mathbb{R}) \,:\, F_C = 0\}. \tag{2.44}$$

The converse follows by substituting $F_C = 0$ into the integral in (2.42). Orthonormality gives $\operatorname{tr} C = \int_M F_C\,dv_g = 0$ for every $C \in \mathcal{C}$. Put

$$\mathcal{K} = \{a \in \mathcal{V} \,:\, PM_{\mathcal{A}a}P = 0\}, \qquad P = P_g, \tag{2.45}$$

and let

$$R = (\Delta_g - \lambda)^{-1}\big|_{E_\lambda^\perp}, \qquad R|_{E_\lambda} = 0. \tag{2.46}$$

The inverse is bounded on L^2 because the excluded spectrum has positive distance from λ; it maps L^2 to the elliptic domain. It is real and self-adjoint.

2.3.3. *The second variation.* For $C \in \mathcal{C}$, define

$$B_C = \sum_{i,j} C_{ij} M_{u_i} R M_{u_j}. \tag{2.47}$$

This contracted operator occurs in [28, Proposition 3.1 and Lemma 4.5]. It is real and self-adjoint: taking its adjoint exchanges i, j, while $C_{ij} = C_{ji}$. Its Schwartz kernel relative to $dv_g(y)$ is $b_C(x,y)R(x,y)$, where

$$b_C(x,y) = \sum_{i,j} C_{ij} u_i(x) u_j(y). \tag{2.48}$$

For $a \in \mathcal{K}$, $\mathsf{L}'(0) = 0$. Differentiating $A_t\Phi_t = B_t\Phi_t\mathsf{L}_t$ once at $t = 0$ gives

$$(\Delta_g - \lambda)\Phi' = -(A_0' - \lambda B_0')U + U\mathsf{L}'.$$

Since $\mathsf{L}' = 0$ and $A_0' - \lambda B_0' = -M_{\mathcal{A}a}$, projection to $E_\lambda^\perp$ and the definition of R give

$$(I - P)\Phi'(0) = RM_{\mathcal{A}a}U. \tag{2.49}$$

Thus $\Phi' = RM_{\mathcal{A}a}U + UZ$ for some matrix Z. The second differentiated pencil equation, before any projection, is

$$\begin{aligned} A_0''U + 2A_0'\Phi' + \Delta_g\Phi'' = {} & \lambda B_0''U + 2\lambda B_0'\Phi' + \lambda\Phi'' \\ & + 2(B_0'U + \Phi')\mathsf{L}' + U\mathsf{L}''. \end{aligned}$$

Since $\mathsf{L}' = 0$ and $U^*(\Delta_g - \lambda)\Phi'' = 0$, multiplication by U^* gives

$$\mathsf{L}''(0) = U^*(A_0'' - \lambda B_0'')U + 2U^*(A_0' - \lambda B_0')\Phi'(0). \tag{2.50}$$

The identity $U^*(A_0' - \lambda B_0')U = 0$ gives $U^*(A_0' - \lambda B_0')UZ = 0$. By (2.41), the direct term in (2.50) is multiplication by a smooth scalar function, or zero if $n = 2$. Its contraction with C is the integral of that scalar function times F_C, hence is zero. Substituting (2.40) and (2.49) gives

$$\big\langle C, D^2\mathsf{L}(0)[a,a]\big\rangle = -2\,\langle \mathcal{A}a, B_C\mathcal{A}a\rangle_{L^2} \qquad (a \in \mathcal{K}). \tag{2.51}$$

The matrix pairing is $\langle C, D\rangle = \operatorname{tr}(CD)$. Polarization determines the contracted mixed Hessian on $\mathcal{K} \times \mathcal{K}$. On a closed manifold, every annihilator for the full metric derivative lies in (2.44). Every $a \in \mathcal{K}$ determines a conformal tangent vector in the kernel of the full derivative. The contracted Hessian (2.51) is therefore available for full-metric annihilators as well. Its value is independent of the acceleration of the conformal chart by the cokernel observation following (2.10).

2.4. **The projection connection.** On the rank-m bundle $\mathcal{E}_v = \operatorname{im}\Pi(v)$, set

$$\nabla s = \Pi\, ds, \qquad \omega = U(v)^* dU(v), \qquad \Omega = d\omega + \omega \wedge \omega. \tag{2.52}$$

The connection preserves the real inner product. Its connection form is skew, since differentiation of $U(v)^*U(v) = I$ gives $\omega^* = -\omega$. Parallel transport along $v(s)$ is the solution of

$$H'(s) = -\omega_{v(s)}(v'(s))H(s), \qquad H(0) = I.$$

Differentiation of H^*H gives zero, so $H(s) \in SO(m)$. The sign of the determinant follows from continuity at the identity.

Lemma 2.3 (First and second response formulas). *Let $P(v)$ be a self-adjoint family on a real Hilbert space $\mathcal{H}$, with common domain $\mathcal{D}$ endowed with the graph norm of $P(0)$. Suppose that $v \mapsto P(v)$ is smooth with values in $\mathcal{L}(\mathcal{D}, \mathcal{H})$, and let λ be an isolated eigenvalue of P_0 of finite multiplicity m. Let $U : \mathbb{R}^m \to E_\lambda$ be an isometry. Write $P_a = DP(0)[a]$, $P_{ab} = D^2P(0)[a, b]$, and let $R_\lambda = (P_0 - \lambda)^{-1}$ on $E_\lambda^\perp$, extended by zero. Then*

$$D\mathsf{L}(0)[a] = U^* P_a U, \tag{2.53}$$

$$(I - \Pi)DU(0)[a] = -R_\lambda P_a U, \tag{2.54}$$

$$\Omega_0(a, b) = U^* P_a R_\lambda^2 P_b U - U^* P_b R_\lambda^2 P_a U. \tag{2.55}$$

If $a, b \in \ker D\mathsf{L}(0)$ and C annihilates the derivative range, then

$$\langle C, D^2\mathsf{L}(0)[a, b]\rangle = \left\langle C, U^* P_{ab} U - U^* P_a R_\lambda P_b U - U^* P_b R_\lambda P_a U \right\rangle. \tag{2.56}$$

Proof. Write $U_a = DU(0)[a]$ and $U_{ab} = D^2U(0)[a, b]$. The normalization of the Riesz–Gram frame gives $U^*U_a = 0$ and $\Pi U_a = 0$. Differentiating $P(v)U(v) = U(v)\mathsf{L}(v)$ yields

$$(P_0 - \lambda)U_a = -P_a U + U\mathsf{L}_a.$$

Since $U^*(P_0 - \lambda) = 0$ and $U^*U = I_m$, left multiplication by U^* gives $\mathsf{L}_a = U^* P_a U$. Applying R_λ instead gives $U_a = -R_\lambda P_a U$. At zero the connection form vanishes. For constant parameter vectors a, b,

$$(d\omega)_0(a, b) = U_a^* U_b - U_b^* U_a + U^*(U_{ab} - U_{ba}) = U_a^* U_b - U_b^* U_a.$$

The equality $U_{ab} = U_{ba}$ follows from smoothness of the frame. Substitution of the formula for U_a proves the curvature identity.

Differentiation in both parameters gives

$$(P_0 - \lambda)U_{ab} = -P_{ab} U - P_a U_b - P_b U_a + U_a \mathsf{L}_b + U_b \mathsf{L}_a + U\mathsf{L}_{ab}.$$

For $a, b \in \ker D\mathsf{L}(0)$ the two terms containing first derivatives of L vanish. Left multiplication by U^* proves

$$\mathsf{L}_{ab} = U^* P_{ab} U - U^* P_a R_\lambda P_b U - U^* P_b R_\lambda P_a U.$$

Pairing with C gives (2.56). For a source change $v = \theta(w)$, the additional second-order term is $D\mathsf{L}(0)D^2\theta(0)[a, b]$ and is annihilated by C. For a frame change $U(v)O(v)$ with $O(0) = I_m$, differentiation of $O(v)^T\mathsf{L}(v)O(v)$ contributes only terms containing L_a or L_b, or the second derivative of $\lambda O^T O = \lambda I_m$. All these terms vanish on the indicated kernel. □

Write $\Pi = \Pi(0)$ and

$$R_\lambda = (P - \lambda)^{-1}|_{E_\lambda^\perp}, \qquad R_\lambda|_{E_\lambda} = 0.$$

Write $\mathcal{R}$ for this real self-adjoint reduced resolvent. The Riesz–Gram construction gives

$$U^* DU(0) = 0, \qquad \Pi DU(0) = 0, \qquad \omega(0) = 0. \tag{2.57}$$

Differentiating $(P + M_f)U(f) = U(f)\mathsf{L}(f)$ at zero and projecting to the orthogonal complement yields

$$K_f := (I - \Pi)DU(0)[f] = -\mathcal{R} M_f U. \tag{2.58}$$

This holds for every direction f, whether or not $D\mathsf{L}(0)f = 0$. The term $UD\mathsf{L}(0)f$ vanishes after complement projection.

Differentiation of $\omega = U^*\,dU$, together with (2.57), gives at zero

$$\begin{aligned}\Omega(f,h) &= K_f^* K_h - K_h^* K_f \\ &= U^* M_f \mathcal{R}^2 M_h U - U^* M_h \mathcal{R}^2 M_f U.\end{aligned} \tag{2.59}$$

Equivalently, $\Omega(f,h) = U^*[\Pi'(f), \Pi'(h)]U$. These are formulas for the intrinsic projection connection. A change of orthonormal frame conjugates the curvature and conjugates based holonomy by its value at the base point.

For a skew matrix $Z \in \mathfrak{so}(m)$, define

$$b_Z(x,y) = \sum_{i,j} Z_{ij} u_i(x) u_j(y), \qquad T_Z^{(s)} = \sum_{i,j} Z_{ij} M_{u_i} \mathcal{R}^s M_{u_j} \quad (s = 1, 2). \tag{2.60}$$

Then $b_Z(y,x) = -b_Z(x,y)$ and $(T_Z^{(s)})^* = -T_Z^{(s)}$. With $\langle Z, \Omega\rangle = \sum Z_{ij}\Omega_{ij}$, exchanging indices in the second term of (2.59) gives

$$\boxed{\langle Z, \Omega(f,h)\rangle = 2\langle f, T_Z^{(2)} h\rangle_{L^2}.} \tag{2.61}$$

The curvature contraction contains $\mathcal{R}^2$, whereas the Hessian contraction in (2.56) contains $\mathcal{R}$.

3. Diagonal vanishing and reduced resolvents

Fix $P = \Delta_g + V$ with one of the realizations of Section 2. Let $u_1, \dots, u_m$ be an orthonormal basis of $\ker(P - \lambda)$ and set

$$R = (P - \lambda)^{-1}|_{E_\lambda^\perp}, \qquad R|_{E_\lambda} = 0.$$

For a real matrix M, define

$$b_M(x,y) = \sum_{i,j} M_{ij} u_i(x) u_j(y), \qquad T_M^{(s)} = \sum_{i,j} M_{ij} M_{u_i} R^s M_{u_j} \quad (s = 1, 2). \tag{3.1}$$

Then $(T_M^{(s)})^* = T_{M^T}^{(s)}$ and the kernel of $T_M^{(s)}$ is $b_M(x,y)R^s(x,y)$, with respect to $dv_g(y)$. Here $(M_f u)(x) = f(x)u(x)$.

3.1. The first nonzero diagonal term.

Lemma 3.1 (Parity and harmonicity). *Let $O \subset M^\circ$ be nonempty and open, and let $M \neq 0$ satisfy $M^T = \epsilon M$, where $\epsilon \in \{1, -1\}$. If $\epsilon = 1$, assume $b_M(x,x) = 0$ on O. There is a least finite transverse diagonal order ℓ whose coefficient is not identically zero on O. On an interior coordinate neighborhood,*

$$b_M(x, x+z) = P_x(z) + E(x,z), \qquad P_x(z) = \sum_{|\alpha|=\ell} \frac{z^\alpha}{\alpha!} \sum_{i,j} M_{ij} u_i(x) \partial^\alpha u_j(x). \tag{3.2}$$

The polynomial P_x is homogeneous of degree ℓ, is nonzero for some x, and satisfies $g^{ab}(x)\partial_{z_a}\partial_{z_b} P_x = 0$. Moreover $(-1)^\ell = \epsilon$. Thus $\ell \geq 2$ is even in the symmetric case, and $\ell \geq 1$ is odd in the skew case.

Proof. Suppose all transverse jets vanish at every (x,x) with $x \in O$. For fixed x, the function $y \mapsto b_M(x,y)$ solves $(P_y - \lambda)b_M(x,y) = 0$ and vanishes to infinite order at $y = x$. The scalar unique-continuation theorem specified in Section 2 makes it zero throughout M°, and continuity gives zero on M. Its inner product with $u_j(y)$ gives

$$\sum_i M_{ij} u_i(x) = 0 \qquad (x \in O).$$

Each left side solves the same scalar eigenfunction equation. A second application of unique continuation and then orthonormality gives every $M_{ij} = 0$. This contradiction proves the existence of ℓ.

All coefficient functions of lower order vanish on O. On a smaller product chart Taylor's formula gives

$$E(x,z) = (\ell+1)\sum_{|\alpha|=\ell+1}\frac{z^\alpha}{\alpha!}\int_0^1 (1-t)^\ell(\partial_2^\alpha b_M)(x,x+tz)\,dt.$$

For every fixed finite list of derivatives in x, differentiating this identity at most twice in z gives

$$|\partial_x^\alpha\partial_z^\beta E(x,z)| \le C_{\alpha\beta}|z|^{\ell+1-|\beta|}, \qquad |\beta|\le 2, \tag{3.3}$$

on a compact subchart. Since $b_M(y,x) = \epsilon\, b_M(x,y)$,

$$P_x(z) = \epsilon P_{x+z}(-z) + O(|z|^{\ell+1}) = \epsilon P_x(-z) + O(|z|^{\ell+1}).$$

Comparison of homogeneous terms at a point where $P_x \neq 0$ proves $(-1)^\ell = \epsilon$. The diagonal term is zero in both cases, so $\ell > 0$.

In coordinates write

$$P_y - \lambda = -g^{ab}(y)\partial_{y_a}\partial_{y_b} + b^j(y)\partial_{y_j} + V(y) - \lambda.$$

For $\ell \ge 2$, the term of degree $\ell-2$ in its application to (3.2) is

$$-g^{ab}(x)\partial_{z_a}\partial_{z_b}P_x.$$

The variation of $g^{ab}(x+z)$ contributes order $\ell-1$; so do the first-order part and the differentiated remainder in (3.3). The zeroth-order part contributes order ℓ. Hence the displayed homogeneous polynomial vanishes. For $\ell = 1$, every second derivative of P_x is zero. □

Under a change $z = L\widetilde{z}+O(|\widetilde{z}|^2)$, the vanishing of all smaller jets makes the first nonzero polynomial transform as $P_x(L\widetilde{z})$. Its degree and its leading symmetric tensor are therefore independent of the coordinate chart.

Lemma 3.2 (A finite order bound at a fixed reference). *Let $\mathscr{M}$ be a finite-dimensional space of matrices to which Lemma 3.1 applies. There are finitely many transverse jet evaluations $\ell_1,\dots,\ell_q$ on O and constants $c, C > 0$ such that*

$$c\|M\| \le \left(\sum_\nu |\ell_\nu(M)|^2\right)^{1/2} \le C\|M\| \qquad (M\in\mathscr{M}).$$

In particular, the orders of these evaluations have a finite maximum.

Proof. If $\mathscr{M} = \{0\}$, the inequalities hold with any single transverse jet evaluation and any $c, C > 0$. Suppose $\mathscr{M} \neq \{0\}$. All jet evaluations are linear functionals on $\mathscr{M}$. Their common kernel is zero by the first paragraph of the preceding proof. They span $\mathscr{M}^*$: otherwise a nonzero vector would annihilate their span. Select a basis from them. The resulting map to $\mathbb{R}^q$ is injective. Its norm on the unit sphere has a positive minimum and a finite maximum. Homogeneity proves the inequalities. □

The integer and constants depend on the fixed reference metric and eigenvalue.

3.2. **Reduced inverses in an interior chart.** Put $A = P - \lambda$. Its coefficients are smooth, and its principal symbol $|\xi|_g^2$ is positive for $\xi \neq 0$. The elliptic parametrix theorem [34, Section 18.1] gives a two-sided parametrix Q of order -2 with principal symbol $|\xi|_g^{-2}$.

For the interior kernel comparison, choose $U_0 \Subset U_1 \Subset M^\circ$ and nested cutoffs. Conjugate by the square root of the smooth density so that kernels are measured with respect to Lebesgue measure. The spectral equations give

$$AR = RA = I - \Pi.$$

On $U_0\times U_0$, if K_R is the reduced kernel and K_Q the local parametrix kernel, then

$$(A_x + A_y^t)(K_R - K_Q) \in C^\infty.$$

The principal symbol of $A_x + A_y^t$ is $|\xi|^2_{g_x} + |\eta|^2_{g_y}$, positive whenever $(\xi, \eta) \neq 0$. Interior elliptic regularity [24, Theorem 15.1] gives $K_R - K_Q \in C^\infty$. The same equations away from the diagonal give smoothness of K_R there. Derivatives of the cutoffs have support outside $U_0 \times U_0$.

The equations

$$AR^2 = R^2A = R$$

and $AQ^2 = Q$, $Q^2A = Q$ modulo smoothing terms give the same conclusion for $R^2 - Q^2$. Indeed the right sides differ by smooth kernels after localization, and the same product operator is elliptic. Thus R^s, $s = 1, 2$, has local order $-2s$ and principal symbol $|\xi|_g^{-2s}$, modulo a smooth kernel.

3.3. **The first nonzero symbol.** For left Kohn–Nirenberg symbols, with $D_x = i^{-1}\partial_x$, the composition expansion is

$$\sigma(P_1P_2) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha \sigma(P_1))(\partial_x^\alpha \sigma(P_2)).$$

After N terms its remainder has order lower by N; the cutoffs make the operators properly supported. Applied to (3.1), it gives

$$\sigma(T_M^{(s)}) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}\left(\sum_{i,j} M_{ij}u_i\partial^\alpha u_j\right)\partial_\xi^\alpha \sigma(R^s).$$

Every coefficient of order below ℓ vanishes. The degree $-2s-\ell$ therefore involves only P_x and the principal symbol of R^s.

Let $P(\xi) = A_{i_1\dots i_\ell}\xi_{i_1}\cdots\xi_{i_\ell}$ be homogeneous and harmonic, so that A is symmetric and each contraction $A_{jji_3\dots i_\ell}$ is zero. If f is a smooth function of one variable, repeated differentiation gives

$$\partial_{i_1}\cdots\partial_{i_\ell} f(|\xi|^2) = \sum_\pi 2^{|\pi|} f^{(|\pi|)}(|\xi|^2)\prod_{\{a\}\in\pi}\xi_{i_a}\prod_{\{a,b\}\in\pi}\delta_{i_ai_b},$$

where π runs over the partitions of $\{1,\dots,\ell\}$ into blocks of size one or two. Contraction with A annihilates every summand containing a block of size two. Taking $f(t) = t^{-s}$ gives

$$P(\partial_\xi)|\xi|^{-2s} = (-2)^\ell(s)_\ell\frac{P(\xi)}{|\xi|^{2s+2\ell}}, \qquad (s)_\ell = s(s+1)\cdots(s+\ell-1). \tag{3.4}$$

Since $f^{(\ell)}(t) = (-1)^\ell(s)_\ell t^{-s-\ell}$, the composition formula gives

$$\sigma_{-2s-\ell}(T_M^{(s)})(x,\xi) = i^{-\ell}(-2)^\ell(s)_\ell\frac{P_x(\xi^{\#_g})}{|\xi|_g^{2s+2\ell}}. \tag{3.5}$$

It is nonzero at some point and covector. In the symmetric case it is real and even; in the skew case it is purely imaginary and odd. Density conjugation multiplies the leading symbol by mutually inverse scalar factors and does not change it.

Choose coordinates orthonormal at a point where $P_x \neq 0$. Harmonicity, the divergence theorem, and Euler's identity give

$$0 = \int_{B^n}\sum_a \partial_{z_a}^2 P_x\,dz = \int_{S^{n-1}}\partial_\nu P_x\,dS = \ell\int_{S^{n-1}} P_x\,dS.$$

Here $\ell > 0$. The continuous function $P_x|_{S^{n-1}}$ is nonzero and has integral zero, hence takes both positive and negative values. The symmetric symbol therefore has both signs.

3.4. **Oscillations and the derivative kernel.**

Lemma 3.3. *Let T be a real, self-adjoint classical pseudodifferential operator of order $d \le 0$ on a coordinate neighborhood, with even principal symbol $t_d(x,\xi)$. If $a \in C_c^\infty$ is real and $\xi_0 \neq 0$ is constant in these coordinates, then*

$$\langle T(a\cos(Nx\cdot\xi_0)), a\cos(Nx\cdot\xi_0)\rangle = \frac{N^d}{2}\int a^2 t_d(x,\xi_0)\,dv_g + O(N^{d-1}). \tag{3.6}$$

For two amplitudes with disjoint compact supports, the cross pairing is $O(N^{-L})$ for every L.

Proof. Conjugate first by the positive density factor, as in the density conjugation in the preceding subsection, and absorb its smooth square root into the amplitude. It suffices to work with Lebesgue measure. Insert fixed cutoffs equal to one near the amplitude supports. A smoothing remainder contributes $O(N^{-L})$ in every fixed C^j norm, for every L, by integration by parts in its smooth kernel.

For the left symbol t of the remaining properly supported operator,

$$e^{-iNx\cdot\xi_0}T(e^{iNx\cdot\xi_0}a) = (2\pi)^{-n}\int e^{ix\cdot\eta}t(x,N\xi_0+\eta)\widehat{a}(\eta)\,d\eta. \tag{3.7}$$

On $|\eta| \le N|\xi_0|/2$, homogeneity of t_d and the order-$d-1$ remainder imply, for each fixed multi-index α,

$$\left|\partial_x^\alpha[t(x,N\xi_0+\eta) - N^d t_d(x,\xi_0)]\right| \le C_\alpha N^{d-1}(1+|\eta|).$$

The segment between $N\xi_0$ and $N\xi_0+\eta$ has length at most $N|\xi_0|/2$ and remains in $|\xi| \ge N|\xi_0|/2$; the bound follows from $|\partial_\xi \partial_x^\alpha t| \le C(1+|\xi|)^{d-1}$. When derivatives also act on $e^{ix\cdot\eta}$, they add a fixed power of $|\eta|$.

On the complementary region, every fixed x derivative of t is bounded because $d \le 0$. For all integers $j, L \ge 0$, rapid Fourier decay gives

$$\int_{|\eta|>N|\xi_0|/2}(1+|\eta|)^j|\widehat{a}(\eta)|\,d\eta = O(N^{-L}).$$

The same estimate applies to the subtracted principal term. Integrating the two regions proves, on the fixed compact set in the range variable,

$$\left\|e^{-iNx\cdot\xi_0}T(e^{iNx\cdot\xi_0}a) - N^d t_d(x,\xi_0)a\right\|_{C^j} \le C_j N^{d-1}. \tag{3.8}$$

Apply (3.8) with $-\xi_0$ in place of ξ_0 to obtain the corresponding estimate for $ae^{-iNx\cdot\xi_0}$. Put $u_\pm = ae^{\pm iNx\cdot\xi_0}$ and extend the L^2 pairing sesquilinearly. Equation (3.8), with $j=0$, gives

$$(Tu_\pm, u_\pm)_{L^2} = N^d\int a^2 t_d(x,\pm\xi_0)\,dx + O(N^{d-1}).$$

The other two pairings have principal terms

$$N^d\int e^{\pm 2iNx\cdot\xi_0}a^2 t_d(x,\pm\xi_0)\,dx.$$

More precisely, put $b_{\pm,N} = e^{\mp iNx\cdot\xi_0}T(u_\pm)$. Equation (3.8) gives $\|b_{\pm,N}\|_{C^j} \le C_j N^d$ for every fixed j. Consequently the full mixed pairing, including its remainder, satisfies

$$\left|\int e^{\pm 2iNx\cdot\xi_0}a(x)b_{\pm,N}(x)\,dx\right| \le C_j N^{d-j}.$$

This follows by j integrations with $(\xi_0\cdot\partial_x)/(\pm 2iN|\xi_0|^2)$; a has compact support. Choosing j arbitrarily large proves rapid decay. Since $a\cos(Nx\cdot\xi_0) = (u_+ + u_-)/2$ and $t_d(x,-\xi_0) = t_d(x,\xi_0)$, the two diagonal pairings prove (3.6). Undoing density conjugation replaces a^2dx by a^2dv_g. Let $K_T(x,y)$ be the distribution kernel, and let the two compact amplitude supports be disjoint. Pseudolocality of the

properly supported operator, included in the parametrix and composition results cited above, makes K_T smooth on a neighborhood of their product. Each cross pairing is a sum of four integrals

$$\iint e^{iN(\epsilon y-\epsilon' x)\cdot\xi_0} a_2(y)a_1(x)K_T(x,y)\,dx\,dy, \qquad \epsilon,\epsilon' \in \{-1,1\}.$$

Integrate L times in x with $(\xi_0\cdot\partial_x)/(-i\epsilon' N|\xi_0|^2)$. There are no boundary terms because the amplitudes have compact support. The integral is bounded by $C_L N^{-L}$, where C_L is a finite sum of L^1 norms of derivatives of the smooth amplitude-kernel product. Hence the integral is $O(N^{-L})$ for every L. The same calculation applies to smoothing remainders. □

Theorem 3.4 (Infinite inertia). *For the allowed potential or conformal cluster map, every nonzero contracted cokernel Hessian has infinite positive and negative index on the allowed derivative kernel. The parameter spaces are localized potentials on every nonempty interior open set; global conformal factors on a closed manifold; and interior-supported conformal factors on a dense allowed region described in Section 2.3. More locally, whenever $C \neq 0$ satisfies $F_C = 0$ on an open ball, the same conclusion holds for conformal perturbations supported in that ball and in their derivative kernel, provided $\lambda > 0$. The conclusion is preserved by every finite collection of linear constraints on the kernel.*

Proof. For potentials the contracted Hessian is $-2\langle f, B_C f\rangle$, with $B_C = T_C^{(1)}$ and $F_C = 0$ on the support region. For conformal factors it is $-2\langle \mathcal{A}f, B_C\mathcal{A}f\rangle$, by (2.51). If supp f lies in a ball on which $F_C = 0$, the direct second-variation contraction is also zero, since it is an integral of F_C times a function supported in that ball. Thus the representing operator is $T = -2B_C$ or $T = -2\mathcal{A}B_C\mathcal{A}$.

Lemma 3.1 and (3.5) give a nonzero real even principal symbol with both signs. Let σ be the order of T on the allowed open set. Then $\sigma = -2-\ell$ for potentials and surface conformal factors, and $\sigma = 2-\ell \leq 0$ for conformal factors in dimension at least three. Fix a constant coordinate covector ξ_0 and a relatively compact open set on which one sign is strict. Given N_0, choose nonzero real amplitudes $a_1,\dots,a_{N_0}$ with disjoint compact supports in that set, and put $f_{i,N} = a_i\cos(Nx\cdot\xi_0)$. Lemma 3.3 gives diagonal pairings of the chosen sign with absolute value at least cN^σ, and off-diagonal pairings $O(N^{-L})$ for every L. Set

$$G_N = (\langle T f_{i,N}, f_{j,N}\rangle)_{i,j=1}^{N_0}.$$

For the negative sign, choose a common frequency large enough that

$$(G_N)_{ii} \leq -cN^\sigma, \qquad |(G_N)_{ij}| \leq \frac{cN^\sigma}{2N_0} \quad (i\neq j).$$

For $a\in\mathbb{R}^{N_0}$, the inequality $2|a_ia_j| \leq a_i^2+a_j^2$ gives

$$a^T G_N a \leq -cN^\sigma\sum_i a_i^2 + \frac{cN^\sigma}{2N_0}(N_0-1)\sum_i a_i^2 \leq -\frac{c}{2}N^\sigma|a|^2.$$

For the positive sign, the same estimate applies to $-G_N$.

Let E_0 be the allowed function space; for the local conformal assertion, take $E_0 = C_c^\infty(B)$ in the ball B on which $F_C = 0$. Choose coordinates on $D\mathsf{L}(0)(E_0)$, denote their compositions with $D\mathsf{L}(0)$ by $\ell_1,\dots,\ell_b$, and choose $e_j\in E_0$ with $\ell_i(e_j) = \delta_{ij}$. The first-variation formulas and integration by parts express each ℓ_i as integration against a fixed smooth function. For an oscillatory test f_N, put

$$\widehat{f_N} = f_N - \sum_{j=1}^{b}\ell_j(f_N)e_j. \tag{3.9}$$

Then $D\mathsf{L}(0)[\widehat{f_N}] = 0$ and the correction is $O(N^{-L})$ in every fixed smooth norm, for every L. In the local conformal case, choose a compactly supported cutoff in B equal to one on all the supports of the a_i and e_j. In the other conformal cases $F_C = 0$ globally. The corresponding localized operator T has order at most zero and is bounded on L^2 by [34, Section 18.1]. For potentials, the global L^2 bound for B_C suffices.

Writing $e_N = \widehat{f_N} - f_N$, the change in a pairing is bounded by

$$\|T\|_{2\to 2}\big(\|e_{i,N}\|_2\|f_{j,N}\|_2 + \|f_{i,N}\|_2\|e_{j,N}\|_2 + \|e_{i,N}\|_2\|e_{j,N}\|_2\big).$$

Choose $L > 1 - \sigma$. This change is $o(N^\sigma)$, so the corrected Gram matrix is still definite. Definiteness also proves linear independence. The opposite sign gives the other index. Since N_0 was arbitrary, both are infinite.

If H has codimension b in the derivative kernel, its intersection with an $(N + b)$-dimensional definite subspace has dimension at least N. The restriction retains definiteness. The index is therefore infinite on H. □

3.5. **Alternating rank.**

Proposition 3.5. *For localized scalar potentials and every nonzero $Z \in \mathfrak{so}(m)$, the alternating form*

$$(f, h) \longmapsto \langle Z, \Omega_0(f, h)\rangle$$

has infinite algebraic rank on $C_c^\infty(O)$. It retains infinite rank after every finite-codimensional restriction.

Proof. For $Z \neq 0$, choose an interior cutoff χ equal to one near a point and covector where (3.5), with $M = Z$ and $s = 2$, is nonzero. Put $T = \chi T_Z^{(2)} \chi$. Then $T^* = -T$, and T maps smooth test functions to smooth compactly supported functions. Suppose rank $T = N < \infty$. Choose a basis $f_1, \dots, f_N$ of its image consisting of images of test functions. Continuity and finite-dimensionality give distributions η_j such that its kernel is

$$K_T(x, y) = \sum_{j=1}^{N} f_j(x)\eta_j(y).$$

There are tests ϕ_i with $\int \phi_i f_j = \delta_{ij}$: the linear functionals defined by the independent smooth functions f_j are independent, so their map onto $\mathbb{R}^N$ is surjective. Pairing K_T in x against ϕ_i gives $\eta_i = T^t\phi_i$. The transpose maps tests to smooth functions; hence all η_i are smooth. Thus K_T is smooth. This contradicts its nonzero principal symbol of order $-4 - \ell$ in (3.5). Equation (2.61) now gives infinite rank of the contracted curvature form.

Let $H \subset E$ have codimension $b < \infty$, and write $E = H \oplus Z_0$, $\dim Z_0 = b$. An alternating form ω defines a map $E \to E^*$ whose matrix in this splitting is

$$\begin{pmatrix} \omega_{HH} & \omega_{HZ_0} \\ \omega_{Z_0H} & \omega_{Z_0Z_0} \end{pmatrix}.$$

The difference from $\mathrm{diag}(\omega_{HH}, 0)$ is a sum of a map with domain Z_0 and a map with range in Z_0^*. Its rank is at most $2b$. Finite rank of $\omega|_{H\times H}$ would therefore imply finite rank of ω, contrary to the infinite-rank conclusion above. □

3.6. **Conformal curvature.** The operators in this subsection are identified by the half-density map of Section 2. Put $R = (\Delta_g - \lambda)^{-1}$ on the eigenspace complement and retain (3.1).

3.6.1. *Surfaces.* For $g_a = (1 + a)g$, the generalized eigenvalue equation is

$$\Delta_g\Phi(a) = (1 + a)\Phi(a)\mathsf{L}(a), \qquad U(a) = (1 + a)^{1/2}\Phi(a).$$

After differentiating and projecting away from the reference cluster,

$$K_f = \big(\lambda\mathcal{R} + \tfrac{1}{2}\Pi^\perp\big)\, M_f U. \tag{3.10}$$

Set $S = \lambda\mathcal{R} + \frac{1}{2}\Pi^\perp$. Its square is

$$S^2 = \lambda^2\mathcal{R}^2 + \lambda\mathcal{R} + \tfrac{1}{4}\Pi^\perp.$$

The contracted curvature is twice the alternating operator pairing with

$$\mathscr{T}_Z = \lambda^2 T_Z^{(2)} + \lambda T_Z^{(1)} + \tfrac{1}{4}\sum Z_{ij} M_{u_i}\Pi^\perp M_{u_j}. \tag{3.11}$$

Since $\sum Z_{ij}u_iu_j = 0$, the identity part of the last sum is zero; its projection part is smoothing. The order-$-2-\ell$ principal symbol of (3.11) is

$$i^{-\ell}(-2)^\ell \ell!\lambda \frac{P_x(\xi^{\sharp_g})}{|\xi|_g^{2\ell+2}}. \tag{3.12}$$

It is nonzero because $\lambda > 0$.

3.6.2. *Dimensions at least three.* Let $\beta = 4/(n-2)$ and $\mathscr{D} = \Delta_g + \beta\lambda$. The pencil in (2.37) is

$$A_a = \Delta_g - (1+a)^{-1}\Delta_g(1+a), \qquad B_a = (1+a)^\beta.$$

Here the subtracted term is multiplication by the displayed scalar function. The fixed-Hilbert frame is $U(a) = B_a^{1/2}\Phi(a)$. The differentiated pencil, followed by complement projection, gives

$$K_f = \mathcal{R}M_{\mathscr{D}f}U + \tfrac{\beta}{2}\Pi^\perp M_f U. \tag{3.13}$$

The second summand is the orthogonal projection of $\partial_t B_{tf}^{1/2}\big|_{t=0} U = (\beta/2)M_f U$.

Expanding $K_f^*K_h - K_h^*K_f$, contracting with Z, and integrating $\mathscr{D}$ by parts on the closed manifold gives

$$\begin{aligned} \tfrac{1}{2}\langle Z, \Omega(f,h)\rangle &= \langle f, \mathscr{T}_Z h\rangle, \\ \mathscr{T}_Z &= \mathscr{D}T_Z^{(2)}\mathscr{D} + \tfrac{\beta}{2}\left(\mathscr{D}T_Z^{(1)} + T_Z^{(1)}\mathscr{D}\right) + \tfrac{\beta^2}{4}\sum Z_{ij}M_{u_i}\Pi^\perp M_{u_j}. \end{aligned} \tag{3.14}$$

Again the last term is smoothing. By (3.5), the first two terms have order $-\ell$ with joint principal symbol

$$\boxed{\sigma_{-\ell}(\mathscr{T}_Z) = i^{-\ell}(-2)^\ell \ell!((\ell+1)+\beta)\frac{P_x(\xi^{\sharp_g})}{|\xi|_g^{2\ell}}.} \tag{3.15}$$

The coefficient is $\ell!((\ell+1)+\beta) > 0$ before multiplication by $i^{-\ell}(-2)^\ell$.

Proposition 3.6. *For the full conformal class on a closed manifold, every nonzero skew contraction of the projection curvature has infinite alternating rank on smooth conformal directions and after any finite-codimensional restriction.*

Proof. Let T_Z be the skew-adjoint operator in (3.11) or (3.14), and choose an interior cutoff χ equal to one near a point where (3.12) or (3.15) is nonzero. If $\chi T_Z \chi$ had finite rank, its range and its transpose range on smooth functions would be finite-dimensional spaces of smooth functions. The kernel argument in the proof of Proposition 3.5 would make its distribution kernel smooth. Its nonzero homogeneous symbol excludes this possibility. Thus its alternating form has infinite rank. If its restriction to a subspace of codimension b had rank $r < \infty$, adjoining a b-dimensional complement would give full rank at most $r + 2b$, a contradiction. □

For the full conformal parameter space on a closed manifold, Theorem 3.4 and Proposition 3.6 give, on the same derivative kernel, infinite symmetric indices and infinite alternating rank. These are the hypotheses of Theorem 4.10.

4. Quadratic maps and local right inverses

Let E be a real Banach space, let $E_0 \subset E$ be a linear subspace, and let F be smooth near zero with values in a finite-dimensional Euclidean space W, with $F(0) = 0$. All sections and germs in this section take values in finite-dimensional subspaces of E_0. It is therefore sufficient that E be locally convex and that the restrictions of F to finite-dimensional subspaces be smooth. Put

$$A = DF(0)|_{E_0}, \qquad R = A(E_0), \quad C = R^\perp, \quad K = \ker A, \qquad a = \dim R, \quad d = \dim C. \tag{4.1}$$

The intrinsic quadratic map and its polarization are

$$Q(v) = \tfrac{1}{2}P_C D^2F(0)[v,v], \qquad B(v,w) = \tfrac{1}{2}P_C D^2F(0)[v,w] \quad (v,w \in K). \tag{4.2}$$

The infinite-index hypothesis is

$$\text{for every } c \in C^* \setminus \{0\}, \quad \operatorname{ind}_-(cQ) = \infty. \tag{4.3}$$

Replacing c with $-c$ gives $\operatorname{ind}_+(cQ) = \infty$.

4.1. **Regular zeros of quadratic maps.** For regular-zero criteria for quadratic maps, see [3, 8] and [4, Proposition 9.4]. Let E be a real vector space and let $Q : E \to \mathbb{R}^d$ have quadratic-form coordinates. Its polarization is the symmetric bilinear map

$$B(x,y) = \tfrac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big), \qquad DQ(x)y = 2B(x,y). \tag{4.4}$$

Assume $d \geq 1$ and that every scalar form $c \cdot Q$, $0 \neq c \in \mathbb{R}^d$, has infinite negative index. Replacing c by $-c$ also gives infinite positive index.

Theorem 4.1. *Under this hypothesis, Q maps every finite-codimensional subspace of E onto $\mathbb{R}^d$. There exist $x, y \in E$ and $w_1, \dots, w_d \in E$ such that*

$$Q(x+y) = Q(x-y) = 0, \qquad Q(y) = -Q(x) \neq 0, \tag{4.5}$$

and both derivatives $DQ(x+y)$ and $DQ(x-y)$ map the span of the w_i isomorphically onto $\mathbb{R}^d$. In particular, Q has a nonzero regular zero in a finite-dimensional subspace.

4.1.1. *Uniform index on a finite-dimensional restriction.* If $H \subset E$ has finite codimension q, an $(N+q)$-dimensional negative-definite subspace for a scalar form meets H in dimension at least N. Thus the hypothesis remains true on H. It suffices first to prove surjectivity on E. For each unit vector $c \in \mathbb{R}^d$, choose a $(d+2)$-dimensional subspace V_c on which $c \cdot Q$ is negative definite. With any norm on V_c, there is $\eta_c > 0$ such that

$$c \cdot Q(v) \leq -2\eta_c |v|^2 \qquad (v \in V_c).$$

Since $c' \mapsto c' \cdot Q|_{V_c}$ is continuous, the bound $c' \cdot Q(v) \leq -\eta_c|v|^2$ holds for c' in a neighborhood of c. A finite subcover of the dual unit sphere gives subspaces $V_{c_1}, \dots, V_{c_l}$. Put $V = \sum_{j=1}^{l} V_{c_j}$ and choose an inner product on V, as in [4, Lemma 9.2]. Let $A(c)$ be the symmetric matrix of $c \cdot Q$ on V, with eigenvalues $\lambda_1(A(c)) \leq \cdots \leq \lambda_{\dim V}(A(c))$. The min–max principle gives

$$|\lambda_j(A(c)) - \lambda_j(A(c'))| \leq \|A(c) - A(c')\|_{\mathrm{op}}.$$

The right side tends to zero as $c' \to c$, because $A(c)$ is linear in c. Every unit c has at least $d+2$ negative eigenvalues on V. Therefore

$$\eta = -\max_{|c|=1} \lambda_{d+2}(A(c)) > 0.$$

For $c \neq 0$, the span of eigenvectors corresponding to the first $d+2$ eigenvalues of $A(c)$ satisfies

$$c \cdot Q(v) \leq -\eta |c|\,|v|^2. \tag{4.6}$$

4.1.2. *Surjectivity.* Fix $a \in \mathbb{R}^d \setminus \{0\}$ and choose $R_0^2 > |a|/\eta$. The function $F(z) = \tfrac{1}{2}|Q(z) - a|^2$ attains a minimum on $\{|z| \leq R_0\} \subset V$. Put $p = Q(z) - a$ at a minimizing point. Suppose $p \neq 0$. At an interior minimum, set $\nu = 0$; then $DF(z) = 0$. At a boundary minimum, restriction to the sphere gives $\nabla F(z) = \alpha z$. The inward derivative satisfies

$$DF(z)[-z] = -\alpha R_0^2 \geq 0,$$

so $\alpha \leq 0$. Put $\nu = -\alpha/2 \geq 0$. Thus $\nabla F(z) + 2\nu z = 0$. The inner product with z, together with $DQ(z)z = 2Q(z)$, gives

$$\nu R_0^2 = -p \cdot Q(z) = -p \cdot (a+p) \leq |p|\,|a| \tag{4.7}$$

at a boundary minimum. Let $W_p \subset V$ be a subspace of dimension at least $d+2$ on which (4.6) holds with $c = p$. The linear map

$$W_p \longrightarrow \mathbb{R}^d \oplus \mathbb{R}, \qquad v \longmapsto (DQ(z)v, \langle v, z\rangle)$$

has a nonzero kernel, since its target has dimension $d+1$. Fix a nonzero vector in that kernel. Then $DQ(z)v = 0$, $v \perp z$, and $p \cdot Q(v) \le -\eta|p||v|^2$. At a boundary minimum it is tangent to the sphere. The Hessian of the Lagrangian on such a tangent vector is nonnegative. For a boundary minimizer z, consider the curve

$$z(t) = R_0 \frac{z + tv}{|z + tv|}.$$

Since $|z| = R_0$ and $v \perp z$, it has $z'(0) = v$ and $z''(0) = -|v|^2 z/R_0^2$. Thus

$$0 \le (F \circ z)''(0) = D^2F(z)[v, v] + 2\nu|v|^2,$$

by $DF(z)[w] = -2\nu\langle z, w\rangle$. At an interior minimum the unconstrained Hessian is nonnegative. Since $D^2Q[v, v] = 2Q(v)$,

$$\begin{aligned} D^2F(z)[v, v] + 2\nu|v|^2 &= |DQ(z)v|^2 + 2p \cdot Q(v) + 2\nu|v|^2 \\ &\le 2|p| \left(-\eta + \frac{|a|}{R_0^2}\right) |v|^2 < 0. \end{aligned} \tag{4.8}$$

At an interior minimum, $\nu = 0$ gives the same strict contradiction. Thus $p = 0$ and $Q(z) = a$. The target zero is attained at zero. This proves $Q(V) = \mathbb{R}^d$. Replacing E by H gives $Q(H) = \mathbb{R}^d$ for every finite-codimensional H.

4.1.3. *A regular point.* Suppose the maximum rank of $DQ|_V$ were $r < d$, and choose a point attaining it. If $r > 0$, a nonzero maximal minor stays nonzero in a neighborhood, and maximality prevents higher rank. If $r = 0$, the derivative is zero throughout V. In both cases the rank is locally constant. Choose $0 \ne c$ annihilating the image of DQ at that point. For every $v \in \ker DQ$, Lemma A.4 supplies a smooth curve in its local level set with initial velocity v. If the curve is $\zeta(t)$, with $\zeta(0) = z$ and $\zeta'(0) = v$, then

$$0 = \frac{d^2}{dt^2} Q(\zeta(t))\Big|_{t=0} = 2Q(v) + DQ(z)\zeta''(0).$$

Pairing with c, which annihilates $\operatorname{im} DQ(z)$, gives $c \cdot Q(v) = 0$. Every negative-definite subspace meets $\ker DQ$ only at zero. Its dimension is therefore at most r. This contradicts the negative index at least $d+2$ in (4.6). There is a regular point x. It can be chosen with $Q(x) \ne 0$: a restriction to a d-dimensional derivative-invertible subspace and Lemma A.2 put an open neighborhood of $Q(x)$ in the image. A nonzero $d \times d$ minor stays nonzero nearby. Choose a nonzero value in that open image and its preimage on the slice.

4.1.4. *Two regular zeros.* Choose $w_1, \dots, w_d \in V$ such that $DQ(x)w_i$ are a basis of $\mathbb{R}^d$. Define

$$H = \{y \in E \,:\, B(x, y) = 0,\ B(w_i, y) = 0\ (1 \le i \le d)\}. \tag{4.9}$$

The defining map $y \mapsto (B(x, y), B(w_1, y), \dots, B(w_d, y))$ is linear into $(\mathbb{R}^d)^{d+1}$, so $\operatorname{codim} H \le d(d+1)$. Each nonzero scalar form still has infinite negative index on H. Since $Q(H) = \mathbb{R}^d$, there is $y \in H$ with $Q(y) = -Q(x)$. The vectors $x, y, w_1, \dots, w_d$ span a finite-dimensional subspace of E. Equation (4.4) gives $Q(x \pm y) = Q(x) + Q(y) \pm 2B(x, y) = 0$, and $DQ(x \pm y)w_i = DQ(x)w_i \pm 2B(y, w_i) = DQ(x)w_i$. Also $Q(y) = -Q(x) \ne 0$. Neither $x + y$ nor $x - y$ is zero: either equality would give $B(x, y) = \pm Q(x) \ne 0$, contradicting (4.9). Both zeros and the common derivative complement belong to $\operatorname{span}\{x, y, w_1, \dots, w_d\}$.

The vectors $w_1, \dots, w_d$ are independent: applying $DQ(x)$ to a relation between them gives the same relation between a basis of $\mathbb{R}^d$. Therefore each restriction in (4.5) is an isomorphism. This proves Theorem 4.1.

Corollary 4.2 (A finite regular zero). *Under* (4.3)*, if $d > 0$, there are $z_0 \neq 0$ and a d-dimensional space $W_1 \subset K$ such that*

$$Q(z_0) = 0, \qquad DQ(z_0)|_{W_1} : W_1 \longrightarrow C \text{ is an isomorphism.}$$

The sum $\mathbb{R}z_0 \oplus W_1$ is direct. These choices can be made after imposing any finite collection of linear constraints on K.

Proof. Apply Theorem 4.1 to the constrained kernel and take $z_0 = x + y$. Euler's identity gives $DQ(z_0)z_0 = 2Q(z_0) = 0$. Since $DQ(z_0)$ is injective on W_1, one has $z_0 \notin W_1$. □

4.2. **Symmetric and alternating Gram data.**

Lemma 4.3 (Mixed Gram prescription). *Let $B : E_0 \times E_0 \to C$ be symmetric bilinear, $Q(v) = B(v, v)$, and let $\omega : E_0 \times E_0 \to \mathfrak{g}$ be alternating bilinear, with finite-dimensional targets. Suppose every nonzero scalar contraction of Q has infinite negative index and every nonzero scalar contraction of ω has infinite rank. A zero target imposes no hypothesis.*

For every N and every two arrays

$$S_{ij} = S_{ji} \in C, \qquad H_{ij} = -H_{ji} \in \mathfrak{g},$$

there are vectors $v_1, \dots, v_N \in E_0$ such that

$$B(v_i, v_j) = S_{ij}, \qquad \omega(v_i, v_j) = H_{ij}. \tag{4.10}$$

The Gram arrays are also realizable in every finite-codimensional subspace.

Proof. If the space of prescribed Gram arrays is zero, take $v_i = 0$. A scalar alternating form η of infinite rank has the following property. Choose x_1, y_1 with $\eta(x_1, y_1) = 1$ and put

$$H_1 = \{v : \eta(v, x_1) = \eta(v, y_1) = 0\}.$$

Then $E_0 = \operatorname{span}\{x_1, y_1\} \oplus H_1$ is an η-orthogonal sum, and $\eta|_{H_1}$ has infinite rank. Induction gives x_i, y_j with $\eta(x_i, y_j) = \delta_{ij}$ for $1 \leq i, j \leq N$. On the subspaces

$$(x, y) = \left(\sum_i a_i x_i, \ \pm \sum_i a_i y_i \right)$$

of $E_0 \oplus E_0$, the quadratic form $\eta(x, y)$ is $\pm \sum_i a_i^2$. Thus it has both indices infinite.

Consider the quadratic map on E_0^N comprising all entries $B(v_i, v_j)$ and $\omega(v_i, v_j)$. A scalar contraction is specified by a symmetric array $c_{ij} \in C^*$ and an alternating array $z_{ij} \in \mathfrak{g}^*$. If the first array is nonzero, there is $\zeta \in \mathbb{R}^N$ with

$$c_\zeta = \sum_{i,j} \zeta_i \zeta_j c_{ij} \neq 0.$$

Otherwise polarization would give $c_{ij} = 0$ for all i, j. Restrict to $v_i = \zeta_i v$. Every alternating term vanishes, and the contraction is $c_\zeta Q(v)$, with infinite negative index.

If the symmetric array is zero, choose a nonzero z_{ij} and set every entry except $v_i = x, v_j = y$ equal to zero. The contraction is a nonzero scalar multiple of $z_{ij}\omega(x, y)$. The first paragraph supplies infinite negative index. Theorem 4.1 therefore makes the amplified map surjective, proving (4.10). Scalar indices survive finite-codimensional restriction by intersection; alternating ranks survive it by the rank bound in Proposition 3.5. □

Corollary 4.4 (Symmetric Gram prescription). *Under the scalar index hypothesis, every finite symmetric C-valued Gram array is realized. In particular, if $d = \dim C > 0$ and $r \geq 1$, there is an injective linear map*

$$J : \mathbb{R}^r \oplus C^r \longrightarrow E_0, \qquad Q(J(t, w)) = \sum_{i=1}^{r} t_i w_i. \tag{4.11}$$

Under the additional alternating hypothesis, the values $\omega(J(e_i, 0), J(e_j, 0))$ may be prescribed independently, and every pairing involving a w coordinate may be set to zero.

Proof. Apply Lemma 4.3, first with $\mathfrak{g} = 0$ if only symmetric data are required. Prescribe the Gram array of $(t, w) \mapsto \sum_i t_i w_i$. Its common bilinear radical is zero: pairing against arbitrary w' forces $t = 0$, and pairing against arbitrary t' then forces $w = 0$. A vector in $\ker J$ would lie in this radical. Hence J is injective. The alternating entries in the same finite array can be prescribed independently by the lemma. □

4.3. **A local section from a regular quadratic zero.**

Theorem 4.5 (Finite-dimensional section principle). *In* (4.1)–(4.2)*, suppose either $d = 0$, or Q has a nonzero regular zero z_0 with a d-dimensional derivative complement. Then F has a local right inverse through zero, taking values in a fixed finite-dimensional subspace of E_0. It is pairwise $1/2$-Hölder and smooth off zero. If $d > 0$, there is a smooth curve $\gamma(s) = sz_0 + O(s^2)$ in $F^{-1}(0)$ whose points with $s \neq 0$ are regular on that subspace.*

If $d = 0$, $\dim W$ perturbations suffice and the section is smooth at zero with a linear estimate. If $d > 0$, $\dim W + 1$ perturbations suffice. Analytic finite-dimensional restrictions give analytic solutions off zero and analytic exact-level curves. The conclusion applies in particular under (4.3)*.*

Proof. Choose $X \subset E_0$ of dimension a with $A|_X : X \to R$ invertible. If $d = 0$, apply the finite inverse theorem, proved in Appendix A, to $F|_X$.

Suppose $d > 0$. Let W_1 be the derivative complement at z_0 and put $Z = \mathbb{R}z_0 \oplus W_1$. Set

$$Q_R(z) = \tfrac{1}{2} P_R D^2 F(0)[z, z], \qquad \mathcal{P}(x, z) = \big(Ax + Q_R(z), Q(z)\big).$$

Taylor's formula gives

$$\|F(x + z) - \mathcal{P}(x, z)\| \leq C\big(|x|^2 + |x||z| + |z|^3\big).$$

With weight two on X and weight one on Z, each target coordinate has weight two. The three remainder terms have weights four, three, and three, respectively, and are bounded by finite sums of absolute monomials of those weights. At

$$h_* = \big(-(A|_X)^{-1} Q_R(z_0), z_0\big)$$

the polynomial $\mathcal{P}$ vanishes, and its derivative on $X \oplus W_1$ is the invertible block matrix

$$\begin{pmatrix} A|_X & DQ_R(z_0)|_{W_1} \\ 0 & DQ(z_0)|_{W_1} \end{pmatrix}.$$

Thus $\mathcal{P}$ is a regular weighted truncation in the sense of [11, Definitions 2.1–2.2]. The corresponding rescaled implicit equation is [11, proof of Theorem 3.3, equations (4.19)–(4.22)]. For the range component, solve

$$P_R F(x + z) = y, \qquad x = x(z, y) \in X,$$

by the parameter-dependent inverse theorem. At zero, $x = 0$, $D_z x = 0$, and $x(z, y) = O(|z|^2 + |y|)$. The reduced map is

$$G(z, y) = P_C F(x(z, y) + z) = Q(z) + O(|z|^3 + |z||y| + |y|^2). \tag{4.12}$$

The second derivative of x does not contribute to Q, because it is multiplied by $P_C A = 0$.

Write a target as $Y = Y_R + Y_C$ and substitute $z = s(z_0 + w)$, $Y = s^2 D$. The equation is

$$s^{-2} G(s(z_0 + w), s^2 D_R) = D_C, \qquad w \in W_1. \tag{4.13}$$

It extends smoothly at $s = 0$. Indeed $G(0) = DG(0) = 0$, and Taylor's integral formula gives

$$s^{-2} G(su, s^2 y) = \int_0^1 (1 - t) D^2 G(tsu, ts^2 y)[(u, sy), (u, sy)]\, dt.$$

At $s = 0$ this equals $Q(u)$. The derivative of (4.13) in w at $(s, w, D) = (0, 0, 0)$ is the isomorphism $DQ(z_0)|_{W_1}$. The inverse theorem gives $w = w(s, D)$. Define

$$\Psi(s, D) = x(s(z_0 + w(s, D)), s^2 D_R) + s(z_0 + w(s, D)). \tag{4.14}$$

Then

$$\Psi(0,D)=0, \qquad F(\Psi(s,D))=s^2D. \tag{4.15}$$

On the closure of a smaller parameter product, for every continuous seminorm p on the allowed perturbations,

$$p(\partial_s\Psi)\le C_p, \qquad p(D_D\Psi[H])\le C_p|s|\,|H|.$$

The second estimate follows by differentiating $\Psi(0,D)=0$ and integrating the mixed derivative. These are finite-dimensional estimates on a single compact parameter product.

Fix L so large that L^{-2} lies in the permitted D-ball. For $Y\neq 0$, put

$$S(Y)=\Psi\left(L\sqrt{\|Y\|},\frac{Y}{L^2\|Y\|}\right), \qquad S(0)=0. \tag{4.16}$$

It is continuous at zero, has $F(S(Y))=Y$, and is smooth off zero. Differentiation of the Euclidean norm and the preceding estimates give

$$p(DS(Y)[H])\le C_p\|Y\|^{-1/2}\|H\|.$$

Let $H=Y'-Y\neq 0$ and $t_0=-\langle Y,H\rangle/\|H\|^2$. Orthogonal projection gives

$$\|Y+tH\|\ge\|H\||t-t_0|, \qquad \int_0^1|t-t_0|^{-1/2}\,dt\le 4.$$

Integration proves $p(S(Y')-S(Y))\le 4C_p\|Y'-Y\|^{1/2}$. If the segment meets zero, integrate on the two truncated segments and pass to the limit by continuity.

At $D=0$, $\Psi(s,0)=sz_0+O(s^2)$ and $F(\Psi(s,0))=0$. The reduced derivative on W_1 is

$$sDQ(z_0)|_{W_1}+O(s^2),$$

which is invertible for small $s\neq 0$. The range derivative remains invertible. If the full derivative has blocks $\left(\begin{smallmatrix}A_0 & B_0\\ C_0 & D_0\end{smallmatrix}\right)$, the reduced derivative is $D_0-C_0A_0^{-1}B_0$. Solving

$$(D_0-C_0A_0^{-1}B_0)z=b-C_0A_0^{-1}a, \qquad x=A_0^{-1}(a-B_0z)$$

produces every target (a,b). Thus the full derivative is onto on $X\oplus Z$, of dimension $a+d+1$.

If F is analytic on the selected finite-dimensional subspace, the range solution and the rescaled inverse are analytic by the analytic implicit function theorem. Finally, Corollary 4.2 supplies the required zero under (4.3). □

Corollary 4.6 (Essential value and optimal exponent). *Under Theorem 4.5, zero is an essential value of F in every sufficiently small source neighborhood. If $d>0$ and F is twice continuously differentiable in a Banach norm, every continuous local right inverse S through zero with values in E_0 satisfies*

$$\|S(tc)\|\ge c_0\sqrt{t} \qquad (0<t<t_0)$$

for each unit vector $c\in C$ and suitable constants $c_0,t_0>0$.

Proof. Given a source neighborhood U, choose a closed target ball $\overline{B}_\delta$ whose section image is in U. If $\widetilde{F}$ is continuous with $\sup_U\|\widetilde{F}-F\|<\delta/2$, then

$$H_t(Y)=Y+t(\widetilde{F}(S(Y))-F(S(Y)))$$

has no zero on ∂B_δ. Homotopy invariance of Brouwer degree gives $\deg(H_1,B_\delta,0)=1$, so $\widetilde{F}$ attains zero. For the exponent, choose a unit $c\in C$. Taylor's formula gives

$$\langle c,F(0)\rangle=0, \qquad D\langle c,F\rangle(0)|_{E_0}=0, \qquad |\langle c,F(v)\rangle|\le C_0\|v\|^2 \quad (v\in E_0,\ \|v\|<\rho).$$

For $v=S(tc)$ the last inequality is $t\le C_0\|S(tc)\|^2$. An estimate $\|S(Y)\|\le C\|Y\|^\alpha$ with $\alpha>1/2$ would give $1\le C_0C^2t^{2\alpha-1}$ for all sufficiently small $t>0$, which is impossible. □

4.4. **An exact correction of a bilinear equation.**

Lemma 4.7 (Taylor factorization and an implicit equation). *Let C be finite dimensional and let*

$$H(t,w)=\sum_{i=1}^{r} t_i w_i+\mathcal{E}(t,w), \qquad t\in\mathbb{R}^r, \quad w\in C^r,$$

be smooth near $(0,0)$. Suppose all derivatives of $\mathcal{E}$ of total order at most two vanish there. Then there is a smooth $w(t)=O(|t|^2)$ with $H(t,w(t))=0$. Real-analytic $\mathcal{E}$ gives real-analytic w.

Proof. Let $S=(S_{ij})\in \mathrm{Sym}_r(C)$ be an auxiliary symmetric array, and write

$$(St)_i=\sum_j S_{ij}t_j, \qquad \widehat{\mathcal{E}}(t,S)=\mathcal{E}(t,St).$$

For every S near zero, $j^2_{t=0}\widehat{\mathcal{E}}(t,S)=0$. Taylor's integral formula gives

$$\widehat{\mathcal{E}}(t,S)=\sum_{i,j} t_i t_j\, K_{ij}(t,S), \tag{4.17}$$

$$K_{ij}(t,S)=\int_0^1 (1-s)\left(D_t^2\widehat{\mathcal{E}}\right)(st,S)[e_i,e_j]\,ds. \tag{4.18}$$

The array K is symmetric and smooth, and $K(0,S)=0$ for all small S.

Consider the finite-dimensional equation

$$S+K(t,S)=0. \tag{4.19}$$

Its derivative in S at $(0,0)$ is the identity because $K(0,S)$ vanishes identically. The ordinary implicit function theorem gives $S=S(t)$, with $S(0)=0$ and $S(t)=O(|t|)$. Put $w(t)=S(t)t$. Then $w(t)=O(|t|^2)$, and (4.17)–(4.19) give

$$H(t,w(t))=\sum_{i,j} t_i t_j\big(S_{ij}(t)+K_{ij}(t,S(t))\big)=0.$$

If $\mathcal{E}$ is real analytic, the integral in (4.18) is real analytic on a smaller neighborhood, as follows either from its convergent power series or a uniform complexification. The analytic implicit function theorem then gives an analytic solution. □

Remark 4.8. For $H(t,w)=\sum_i t_i w_i+O(|(t,w)|^3)$, Lemma 4.7 gives a smooth solution of the form $w(t)=S(t)t$, with $S(t)\in\mathrm{Sym}_r(C)$ and $S(0)=0$.

4.5. **A single perturbation space for compatible germs.**

Theorem 4.9 (Finite-parameter germ realization). *Suppose $d>0$ and* (4.3) *holds. For every integer $r\geq 1$, there is a linear subspace $V_r\subset E_0$ of dimension*

$$\dim V_r=a+r(d+1) \tag{4.20}$$

with the following property. Every smooth germ $f:(\mathbb{R}^r,0)\to(W,0)$ satisfying

$$Df(0)(\mathbb{R}^r)\subset R \tag{4.21}$$

admits a smooth germ $h:(\mathbb{R}^r,0)\to(V_r,0)$ such that

$$F(h(t))=f(t). \tag{4.22}$$

The germ h may be chosen to be an embedding. For every sufficiently small $t\neq 0$,

$$DF(h(t))|_{V_r}:V_r\longrightarrow W \quad \text{is surjective.} \tag{4.23}$$

For fixed norms on V_r, W, the smallest possible norm of a right inverse is bounded above and below by positive constants times $|t|^{-1}$.

If the restriction of F to V_r and the target f are real analytic, the lift is real analytic. The space V_r is selected from F, its allowed perturbations, and r, before f is specified. Its domain of realization and all estimates may depend on f.

Condition (4.21) is necessary for any differentiable lift by the chain rule. If $d = 0$, the finite-dimensional implicit function theorem gives realization without the singular analysis. An embedding can also be arranged in that case when sufficiently many independent kernel perturbations are available.

The derivative satisfies $A \circ Dh(0) = Df(0)$. Conditions for a prescribed $Dh(0)$ are given in Section 6.6, and relative extension theorems in Section 8.

Proof. Choose $X \subset E_0$ of dimension a such that $A|_X : X \to R$ is an isomorphism, and write $L : R \to X$ for its inverse. Restrict K to

$$K_1 = \{v \in K : P_C D^2F(0)[v, Ly] = 0 \text{ for every } y \in R\}. \tag{4.24}$$

This is a finite-codimensional subspace of K; hence (4.3) survives there. Corollary 4.4 supplies an injection

$$J : \mathbb{R}^r \oplus C^r \longrightarrow K_1, \qquad Q(J(t, w)) = \sum_i t_i w_i.$$

Define

$$V_r = X \oplus J(\mathbb{R}^r \oplus C^r).$$

The sum is direct since the second summand lies in $\ker A$ and $A|_X$ is injective. This proves (4.20). The space V_r is independent of the target germ f.

4.5.1. *The range equation and the exact two-jet.* The equation

$$P_R F(x + J(t, w)) = y$$

has a unique small solution $x = \zeta(t, w, y) \in X$ by the implicit function theorem. At the origin,

$$\zeta(0) = 0, \qquad D_{t,w}\zeta(0) = 0, \qquad D_y\zeta(0) = L.$$

Set

$$G(t, w, y) = P_C F\big(\zeta(t, w, y) + J(t, w)\big).$$

Because $P_C DF(0)$ vanishes on all selected perturbations, derivatives of order two of ζ do not contribute to the two-jet of G. The mixed kernel–range terms vanish by (4.24). Consequently

$$G(t, w, y) = \sum_i t_i w_i + b(y) + \mathcal{E}_0(t, w, y), \qquad b(y) = \frac{1}{2} P_C D^2F(0)[Ly, Ly], \tag{4.25}$$

where $\mathcal{E}_0$ has zero two-jet.

4.5.2. *The target equation.* Write $f = y + z$ according to $W = R \oplus C$. Compatibility means $Dz(0) = 0$. Let y_1 be the degree-one Taylor term of y and z_2 the degree-two Taylor term of z. Choose a linear map $\ell : \mathbb{R}^r \to C^r$ satisfying

$$\sum_i t_i \ell_i(t) = z_2(t) - b(y_1(t)). \tag{4.26}$$

If the right-hand side is $\sum_{i,j} t_i t_j c_{ij}$ with $c_{ij} = c_{ji} \in C$, set $\ell_i(t) = \sum_j c_{ij} t_j$.

Substitute $w = \ell(t) + v$ and $y = y(t)$ in (4.25), and subtract $z(t)$. Equations (4.25)–(4.26) show that the resulting equation is

$$\sum_i t_i v_i + \mathcal{E}_1(t, v) = 0,$$

where $\mathcal{E}_1$ has zero two-jet. Lemma 4.7 gives $v(t) = O(|t|^2)$ solving it exactly. Therefore

$$h(t) = \zeta\big(t, \ell(t) + v(t), y(t)\big) + J\big(t, \ell(t) + v(t)\big) \tag{4.27}$$

satisfies both components of $F(h(t)) = f(t)$.

Its derivative at zero has kernel component $J(t, \ell(t))$. This map is injective because J is injective and its first coordinate is t. The independent range component lies in X, so $Dh(0)$ is injective. The finite-dimensional constant-rank theorem, followed by restriction to a smaller neighborhood, makes h an embedding.

4.5.3. *Regularity away from the central parameter.* Differentiate (4.25) in w, holding t, y fixed. Along the selected family all three variables are $O(|t|)$, so

$$D_w G\big(t, \ell(t) + v(t), y(t)\big)\xi = \sum_i t_i \xi_i + O(|t|^2)\,\|\xi\|\,. \tag{4.28}$$

For $t \neq 0$, the first map has right inverse

$$S_t c = \left(\frac{t_i}{|t|^2} c\right)_{i=1}^{r}, \qquad \|S_t\| = |t|^{-1}.$$

The error in (4.28) composed with S_t is $O(|t|)$. A Neumann series gives a right inverse for $D_w G$ of norm at most $2|t|^{-1}$ for small nonzero t. The range derivative remains invertible. The Schur-complement formula for the full derivative of $F|_{V_r}$ thus proves (4.23) and the bound $C|t|^{-1}$.

For the lower bound, $P_C DF(0)|_{V_r} = 0$ and $h(t) = O(|t|)$ imply

$$\big\|P_C DF(h(t))|_{V_r}\big\| \le C_0|t|.$$

If T_t is any right inverse and $c \in C$ has norm one, then

$$1 = \|P_C DF(h(t))T_t c\| \le C_0|t|\,\|T_t\|\,.$$

Consequently the infimum of the right-inverse norms is bounded above and below by positive multiples of $|t|^{-1}$. The lift is real analytic when $F|_{V_r}$ and f are real analytic, by the analytic implicit function theorem.

□

4.6. **Simultaneous germs and curvature.**

Theorem 4.10. *Retain the notation* (4.1)–(4.2)*. Let a rank-m real metric bundle, $m \ge 2$, carry a smooth metric connection near zero. Fix an orthonormal identification of its fiber at zero with $\mathbb{R}^m$, so that its curvature there is an $\mathfrak{so}(m)$-valued alternating form Ω_0. Assume* (4.3) *when $d > 0$, and assume that $\langle Z, \Omega_0\rangle|_{K\times K}$ has infinite alternating rank for every nonzero $Z \in \mathfrak{so}(m)$.*

For every smooth $f : (\mathbb{R}^r, 0) \to (W, 0)$ with $Df(0)(\mathbb{R}^r) \subset R$, and every array $\mathscr{R}_{ij} = -\mathscr{R}_{ji} \in \mathfrak{so}(m)$, there is an embedded smooth germ h in a finite-dimensional allowed space V satisfying

$$F(h(t)) = f(t), \qquad (h^*\Omega)_0(e_i, e_j) = \mathscr{R}_{ij}.$$

If $d > 0$, $\dim V = a + r(d+1)$ suffices, with surjective restricted derivative away from zero. If $d = 0$, $\dim V = \dim W + r$ suffices. Analytic finite-dimensional data give analytic h. The space V may depend on the prescribed data.

Proof. Choose a range complement X and its inverse $L : R \to X$. Restrict the kernel to

$$K_1 = \{v \in K \,:\, P_C D^2F(0)[v, Ly] = 0,\ \Omega_0(v, Ly) = 0 \text{ for all } y \in R\}.$$

There are finitely many linear constraints. Both index and alternating-rank hypotheses survive. Write $f = y + z$ and $y_1(t) = Dy(0)t$.

Suppose $d > 0$. By Lemma 4.3, choose $J : \mathbb{R}^r \oplus C^r \to K_1$ with the bilinear identity (4.11). Prescribe every alternating pairing involving a w coordinate to be zero, and prescribe

$$\Omega_0(J(e_i, 0), J(e_j, 0)) = \mathscr{R}_{ij} - \Omega_0(Ly_1(e_i), Ly_1(e_j)).$$

The map J is injective by Corollary 4.4. Solve the range equation on $X \oplus \operatorname{im} J$, as in Theorem 4.9. The reduced equation is

$$G(t, w, y) = \sum_i t_i w_i + b(y) + E(t, w, y), \qquad j^2 E = 0, \quad b(y) = \tfrac{1}{2} P_C D^2F(0)[Ly, Ly].$$

The mixed terms vanish by the definition of K_1. Choose linear ℓ with $\sum_i t_i \ell_i(t) = z_2(t) - b(y_1(t))$. Lemma 4.7 solves the residual equation with $w(t) = \ell(t) + O(|t|^2)$. The resulting lift has

$$Dh(0)t = Ly_1(t) + J(t, \ell(t)).$$

Put $a_i = Ly_1(e_i)$, $b_i = J(e_i, 0)$, and $c_i = J(0, \ell(e_i))$. Then $Dh(0)e_i = a_i + b_i + c_i$. The definition of K_1 gives $\Omega_0(a_i, b_j) = \Omega_0(a_i, c_j) = 0$; the prescribed alternating array gives $\Omega_0(b_i, c_j) = \Omega_0(c_i, c_j) = 0$. Hence

$$\Omega_0(Dh(0)e_i, Dh(0)e_j) = \Omega_0(a_i, a_j) + \Omega_0(b_i, b_j) = \mathscr{R}_{ij}.$$

Together with $F(h) = f$, this gives both identities. Injectivity, surjectivity off zero, and the dimension count are the same as in the preceding theorem: the explicit t coordinate is injective, and $D_w G = \big(\xi \mapsto \sum_i t_i \xi_i\big) + O(|t|^2)$.

If $d = 0$, restrict K only by $\Omega_0(K_1, X) = 0$. Choose independent $v_i \in K_1$ with pairings equal to $\mathscr{R}_{ij} - \Omega_0(LDf(0)e_i, LDf(0)e_j)$. Independence can be imposed in the Gram prescription by adjoining auxiliary vectors z_j with $\Omega_0(v_i, z_j) = \delta_{ij} Z_0$, for one fixed $Z_0 \neq 0$. The auxiliary vectors need not belong to V. Solve $F(x + \sum_i t_i v_i) = f(t)$ for $x \in X$. The derivative is $LDf(0)t + \sum_i t_i v_i$, and the same pairing calculation proves the curvature identity. The derivative of F remains onto near zero. If the finite-dimensional data are analytic, all inverse maps and the resulting germ h are analytic. □

Corollary 4.11. *In classes* P *and* C*, a compatible matrix germ and arbitrary reference curvature are simultaneously realized as in Theorem B. For the constant germ, the family lies in the exact level* $\mathsf{L} = \lambda I_m$.

Proof. Theorem 3.4 proves (4.3). Proposition 3.5 and the conformal calculation in Section 3.6 prove the alternating hypothesis, also on the finite-codimensional derivative kernel. Apply Theorem 4.10. The fixed finite family remains in the allowed support or conformal class. The Riesz contour retains rank m. □

5. Localized perturbations and fixed volume density

Let W be a finite-dimensional subspace of $C_c^\infty(O; E)$, where E is a smooth vector bundle and $O \subset M^\circ$ is open. If $w_1, \dots, w_N$ is a basis of W, then

$$K = \bigcup_{j=1}^{N} \operatorname{supp} w_j \Subset O, \qquad \operatorname{supp} w \subset K \quad (w \in W).$$

5.1. Scalar potentials and conformal metrics.

Theorem 5.1. *Theorem A holds in classes* P *and* C*. In class* P*, the compact set containing the supports of all perturbations lies inside the prescribed open set O. Every real reference eigenvalue is allowed. In class* C*, the reference eigenvalue is positive and M is closed.*

Proof. The first-variation cokernels are given by $F_C = 0$ on O for potentials, and by $F_C = 0$ on M for global conformal variations; see Section 2. Theorem 3.4 proves (4.3) on their derivative kernels. Corollary 4.2 and Theorem 4.5 give a section and exact-level regular curve on $D + 1$ perturbations when the derivative is singular, and on D perturbations otherwise.

In the potential case each chosen direction lies in $C_c^\infty(O)$. Its finite support union is compact in O; enlarge it slightly to $K \Subset O$. In the conformal case every parameter produces a metric in the original conformal class. The fixed finite perturbation space is smooth, so one parameter ball gives the estimates in every fixed C^k norm. The isolating contour excludes all other spectral values and retains rank m, proving exact multiplicity. The exponent statement follows from Corollary 4.6. The metric chart and its inverse have locally comparable C^k norms. □

5.2. **Conformal perturbations in a dense interior region.**

Theorem 5.2. *Let $\lambda > 0$ be an eigenvalue of the closed, Dirichlet, or Neumann Laplacian on M, and let $Z \subset M$ be closed with $Z \cap M^\circ$ of empty interior. There is a compact $K \Subset M^\circ \setminus Z$ for which the conformal cluster map has a local section and exact-level regular approximation as in Theorem A, with conformal factor equal to one on $M \setminus K$.*

Proof. Put $O = M^\circ \setminus Z$ and let C annihilate the derivative on $C_c^\infty(O)$. Equation (2.40) gives $\mathcal{A}F_C = 0$ on O, where $\mathcal{A}$ is defined by (2.39). Since O is dense in M° and $\mathcal{A}F_C$ is smooth, $\mathcal{A}F_C = 0$ on M°. For Dirichlet data, $F_C|_{\partial M} = 0$; for Neumann data, $\partial_\nu F_C = 0$. If $n = 2$, the equation is $\lambda F_C = 0$. If $n \geq 3$, integration by parts gives

$$0 = \langle \mathcal{A}F_C, F_C \rangle = \|dF_C\|_2^2 + \frac{4\lambda}{n-2}\|F_C\|_2^2.$$

Thus $F_C = 0$. Theorem 3.4 gives infinite positive and negative index for every nonzero cokernel contraction on the supported derivative kernel. Theorem 4.5 supplies a section and a curve of regular points in the level $\mathsf{L} = \lambda I_m$ in one finite-dimensional subspace of $C_c^\infty(O)$. Its basis has support in some $K \Subset O$. The resulting metrics equal g near ∂M, so their Dirichlet and Neumann domains are the fixed domains. □

Here K depends on g, E_λ, and O. For $n \geq 3$, Theorem 5.14 gives a smooth reference and a nonempty open set O for which the scalar cluster map has a strict maximum at the reference.

5.3. **Pointwise preservation of the volume density.** Let M be a compact connected smooth manifold, possibly with smooth boundary, and let Δ_g be the positive Laplace–Beltrami operator. Fix the closed realization, or the Dirichlet or Neumann realization. All perturbations in the boundary cases are supported in the interior, so the boundary operator and domain remain fixed. The notation dv_g denotes the Riemannian volume density, and the volume form when M is oriented; orientability is not assumed. Let $\lambda > 0$ be an isolated eigenvalue of multiplicity m, choose an orthonormal basis $u_1, \dots, u_m$, and define the corresponding Riesz–Gram trivialization of its cluster. Denote the symmetric cluster matrix by L; thus $\mathsf{L}(g) = \lambda I_m$.

Theorem 5.3 (Prescribed support and pointwise fixed volume). *Assume* $\dim M = n \geq 3$. *For every nonempty open $O \subset M^\circ$, there exist a compact $K \Subset O$, a finite-dimensional space*

$$W \subset C_c^\infty(O; \mathrm{Sym}_0(TM, g)),$$

a number $\delta > 0$, and a map $S : B_\delta(0) \subset \mathrm{Sym}_m(\mathbb{R}) \to W$ such that, with

$$g_Y(X, Z) = g\big(e^{S(Y)}X, Z\big),$$

one has

$$S(0) = 0, \qquad \mathsf{L}(g_Y) = \lambda I_m + Y, \qquad dv_{g_Y} = dv_g, \qquad g_Y = g \text{ on } M \setminus K.$$

Here the endomorphisms in W are g-self-adjoint and trace-free. The map is smooth away from $Y = 0$, and for every fixed nonnegative integer k,

$$\|g_Y - g_{Y'}\|_{C^k} \leq C_k \|Y - Y'\|^{1/2}.$$

There are $H_j \in W$, tending to zero in C^∞, such that

$$\mathsf{L}(ge^{H_j}) = \lambda I_m \quad \text{and} \quad D\big[H \mapsto \mathsf{L}(ge^H)\big]_{H_j}|_W \text{ is onto } \mathrm{Sym}_m(\mathbb{R}).$$

If the derivative at the reference is already onto on the allowed trace-free perturbations, the section can be smooth at zero with a linear estimate. If it is not onto, an estimate at zero with exponent greater than $1/2$, in a fixed C^k norm in which the cluster map is twice differentiable (for example $k \geq 4$), cannot realize every target direction.

The space W, the compact set K, and δ depend on (g, E_λ, O).

Corollary 5.4 (Absence of local eigenvalue extrema). *If $n \geq 3$, g is smooth, and $\lambda_j(g) > 0$, then for every nonempty interior open set O and every C^∞ neighborhood of g, there are metrics g_+ and g_- in that neighborhood with*

$$dv_{g_\pm} = dv_g, \qquad g_\pm = g \text{ off a compact subset of } O, \qquad \lambda_j(g_-) < \lambda_j(g) < \lambda_j(g_+).$$

The statement includes multiple eigenvalues and the fixed Dirichlet and Neumann cases.

Proof. Let λ be the reference eigenvalue containing the ordered index j, and let its full multiplicity be m. Choose $a < \lambda < b$ so that (a, b) contains no other reference eigenvalue and a, b do not belong to the spectrum. If p is the first index of this cluster, then $p \leq j \leq p + m - 1$. Norm-resolvent continuity and nonnegativity preserve the number of eigenvalues below a and the rank m of the intervening projection, by Riesz contours around the corresponding finite spectral intervals. For $s > 0$ with $\sqrt{m}\, s$ in the target ball of Theorem 5.3, put $Y_\pm = \pm s I_m$. The resulting metrics satisfy

$$\mathsf{L}(g_\pm) = (\lambda \pm s) I_m, \qquad \lambda_i(g_\pm) = \lambda \pm s \quad (p \leq i \leq p + m - 1).$$

Take also $s < \min(\lambda - a, b - \lambda)$. Thus $\lambda_j(g_-) < \lambda_j(g) < \lambda_j(g_+)$. The section tends to the reference in every spatial C^k norm. Its metric family has the same volume density and is equal to g outside a compact subset of the prescribed O. □

For the first positive eigenvalue on a closed manifold, the absence of local minima under fixed total volume is already proved by El Soufi–Ilias [25, Corollary 1.1(ii)]. Corollary 5.4 concerns every positive index, excludes local maxima as well, and retains the pointwise density and prescribed support. First-order criticality is distinct from a local extremum; see [26, 37].

Corollary 5.5 (Localized spectral bundles). *Under Theorem 5.3, every smooth rank-r subbundle of a trivial real m-plane bundle over a compact smooth base, $1 \leq r < m$, is realized as a lower spectral subbundle of nearby metrics with the same volume density and perturbations supported in K. The family can be arbitrarily small in C^∞. Any strict open C^k geometric condition satisfied by the reference persists after taking the family sufficiently small.*

Proof. Let P_b be the orthogonal projection onto the given subbundle and set

$$Y_b = s\left(\frac{r}{m} I_m - P_b\right), \qquad s > 0.$$

The two eigenvalues of Y_b are $-s(m - r)/m$ and sr/m, with eigenspaces $\operatorname{im} P_b$ and $\ker P_b$. Their difference is s, and

$$\|Y_b\|_{\mathrm{HS}} = s\sqrt{\frac{r(m - r)}{m}} > 0.$$

For sufficiently small s, all Y_b belong to the punctured domain of the section in Theorem 5.3. The map $b \mapsto g_{Y_b}$ is therefore smooth. Its lower spectral subbundle is identified with $\operatorname{im} P_b$ by the Riesz–Gram frame. For every fixed k, $\sup_b \|g_{Y_b} - g\|_{C^k} \leq C_k s^{1/2}$, which also gives the asserted preservation of open conditions. □

If $\mathscr{P}$ denotes the lower spectral projection in the Riesz–Gram frame, then

$$\mathscr{P}\big(g_{s(rI_m/m - P)}\big) = P, \qquad P \in \mathrm{Gr}_r(\mathbb{R}^m).$$

Thus this family is a right inverse of $\mathscr{P}$, as in Section 8.

5.4. **The fixed-volume Hessian and its transverse symbol.** Write a symmetric covariant variation as gH, with H self-adjoint, and work in the exact volume-density chart

$$g_H = g(e^H \cdot, \cdot), \qquad \operatorname{tr} H = 0.$$

The Hilbert space $L^2(M, dv_g)$ is fixed. In divergence form,

$$P_H = -\operatorname{div}_g(e^{-H} \nabla_g).$$

For a direction h, the first and second derivatives at zero are

$$P'_h = \operatorname{div}_g(h\nabla_g), \qquad P''_{h,h} = -\operatorname{div}_g(h^2\nabla_g).$$

The formulas hold with fixed boundary realization because h vanishes near the boundary.

For $C \in \operatorname{Sym}_m(\mathbb{R})$ put

$$F_C = \sum_{i,j} C_{ij} u_i u_j, \qquad T_C = \sum_{i,j} C_{ij}\, du_i \otimes du_j.$$

The first variation is

$$D\mathsf{L}(0)[h]_{ij} = -\int_M \langle h\, du_i, du_j\rangle\, dv_g.$$

Consequently, on trace-free perturbations supported in O,

$$C \in \operatorname{coker} D\mathsf{L}(0) \quad\Longleftrightarrow\quad T_C = t_C g \text{ on } O, \qquad t_C = \frac{1}{n}\operatorname{tr}_g T_C. \tag{5.1}$$

Let Π be the orthogonal projection onto the reference eigenspace and let

$$R = (\Delta_g - \lambda)^{-1}\big|_{\operatorname{im}(I-\Pi)}, \qquad R\Pi = 0.$$

For $h \in \ker D\mathsf{L}(0)$, differentiation of the cluster equation gives

$$\langle C, D^2\mathsf{L}(0)[h,h]\rangle = \sum_{i,j} C_{ij}\left(\langle u_i, P''_{h,h}u_j\rangle - 2\langle P'_h u_i, RP'_h u_j\rangle\right). \tag{5.2}$$

The terms containing the component of the differentiated frame in $\operatorname{im}\Pi$ are multiples of $\Pi P'_h\Pi = D\mathsf{L}(0)[h] = 0$.

Lemma 5.6 (Transverse symbol). *The right side of* (5.2)*, extended as a quadratic form on arbitrary compactly supported trace-free tensor directions, is represented locally by an order-zero matrix pseudo-differential operator. If C satisfies* (5.1)*, its principal quadratic symbol at (x,ξ), $\xi \neq 0$, is*

$$t_C(x)\left(\operatorname{tr}(H^2) - 2|He|^2\right), \qquad e = \xi^\sharp/|\xi|.$$

Writing, in the splitting $\mathbb{R}e \oplus e^\perp$,

$$H = \begin{pmatrix} \alpha & b^T \\ b & D \end{pmatrix}, \qquad \alpha + \operatorname{tr} D = 0,$$

this is exactly

$$\boxed{t_C(x)\big(\operatorname{tr}(D^2) - (\operatorname{tr} D)^2\big).} \tag{5.3}$$

Proof. The direct term in (5.2) is $\int T_C : h^2 = \int t_C \operatorname{tr}(h^2)$. As a map from tensors h to functions, $h \mapsto P'_h u_i$ has order one and principal symbol $i\,\xi(h\,du_i)$. The reduced resolvent has the same interior principal symbol as an elliptic inverse, namely $|\xi|^{-2}$; the finite-rank spectral subtraction is smoothing. Hence the resolvent term has order zero and principal quadratic symbol

$$-2\frac{T_C(H\xi^\sharp, H\xi^\sharp)}{|\xi|^2} = -2t_C|He|^2.$$

Finally

$$\operatorname{tr}(H^2) - 2|He|^2 = \alpha^2 + 2|b|^2 + \operatorname{tr}(D^2) - 2(\alpha^2 + |b|^2) = \operatorname{tr}(D^2) - (\operatorname{tr} D)^2.$$

Local cutoffs reduce the boundary cases to the same interior computation. □

Lemma 5.7 (Nonzero stress gives infinite inertia). *Suppose $n \geq 3$, $C \neq 0$ satisfies* (5.1)*, and t_C is nonzero somewhere in O. Then the contracted Hessian in* (5.2) *has infinite positive and negative index on* $\ker D\mathsf{L}(0)$.

Proof. Shrink to a coordinate ball on which t_C has one strict sign. The transverse space has dimension $n-1 \geq 2$. A nonzero trace-free D gives $\operatorname{tr}(D^2) > 0$, whereas $D = I_{n-1}$ gives

$$\operatorname{tr}(D^2) - (\operatorname{tr} D)^2 = (n-1) - (n-1)^2 < 0.$$

These choices lift to smooth trace-free H's on a smaller ball. If T denotes the self-adjoint tensor operator representing (5.2), the scalar compression

$$T_H a = \langle H, T(aH) \rangle_g$$

is real and self-adjoint of order zero, with even principal symbol $\sigma_0(T)(x,\xi)[H,H]$. Thus Lemma 3.3 applies to tensors of the form aH.

Fix $N \geq 1$. Choose tensor amplitudes $h_1, \dots, h_N$ with disjoint compact supports in the sign region and the coordinate-linear phase $\varphi(x) = x \cdot \xi_0$, for a fixed $\xi_0 \neq 0$. Put $h_{j,\tau} = h_j \cos(\tau\varphi)$, and let G_τ be their contracted Hessian matrix. Lemma 3.3, applied to these scalar compressions, gives

$$(G_\tau)_{jj} = d_j + O(\tau^{-1}), \qquad d_j = \frac{1}{2}\int \sigma_0(x, d\varphi)[h_j, h_j]\, dv_g, \qquad (G_\tau)_{ij} = O(\tau^{-L}) \quad (i \neq j)$$

for every L. The nonzero d_j have the prescribed common sign. If $c = \min_j |d_j|$, choose τ so large that $\|G_\tau - \operatorname{diag}(d_1, \dots, d_N)\|_{\mathrm{op}} < c/2$. For the negative sign this gives $a^T G_\tau a \leq -c|a|^2/2$; for the positive sign it gives $a^T G_\tau a \geq c|a|^2/2$.

Choose finitely many fixed test tensors realizing the range of $D\mathsf{L}(0)$, and subtract from each oscillatory tensor its first-variation coordinates in these tests. Those coordinates are $O(\tau^{-N_0})$ for every N_0, since they integrate oscillations against smooth weights. The extended quadratic form in (5.2) is bounded on local L^2 tensor fields, being order zero. If $e_{j,\tau}$ is the correction and T is a localized order-zero operator representing the Hessian, each altered matrix entry differs by at most

$$\|T\|_{L^2 \to L^2}\big(\|e_{i,\tau}\|_2 \|h_{j,\tau}\|_2 + \|h_{i,\tau}\|_2 \|e_{j,\tau}\|_2 + \|e_{i,\tau}\|_2 \|e_{j,\tau}\|_2\big) = o(1).$$

Thus the corrected matrix differs from G_τ in operator norm by less than $c/4$ for sufficiently large τ. Its quadratic form has the same sign and a bound $c|a|^2/4$ in absolute value. The corrected tensors belong to $\ker D\mathsf{L}(0)$ and are linearly independent. Both signs and every integer N are admissible. □

5.5. **Conformal directions and infinitesimal diffeomorphisms.** Suppose $t_C = 0$ on O. Then $T_C = t_C g = 0$ on O.

Lemma 5.8 (Vanishing stress forces the quadratic relation). *If $T_C = 0$ on a nonempty open set and $\lambda > 0$, then $F_C = 0$ on that set.*

Proof. The common eigenfunction equation gives

$$\operatorname{div} T_C = \frac{1}{2} d(\operatorname{tr}_g T_C) - \frac{\lambda}{2} dF_C, \qquad \Delta_g F_C = 2\lambda F_C - 2 \operatorname{tr}_g T_C.$$

Thus $dF_C = 0$, and the second identity implies $2\lambda F_C = 0$. □

The local conformal Hessian. If $C \neq 0$ and $F_C = 0$ on a nonempty open ball B, the conformal cluster Hessian has infinite positive and negative index on its first-derivative kernel for perturbations in $C_c^\infty(B)$, and this remains true after any finite collection of linear constraints. For $n \geq 3$, in the chart $g_a = (1+a)^{4/(n-2)} g$, put

$$\mathcal{A} = \Delta_g + \frac{4\lambda}{n-2}, \qquad B_C = \sum_{i,j} C_{ij} M_{u_i} R M_{u_j}.$$

On the kernel its contraction is

$$-2\langle \mathcal{A}a, B_C \mathcal{A}a \rangle. \tag{5.4}$$

The direct second derivative of the pencil is paired with F_C on $\operatorname{supp} a \subset B$ and is therefore zero. By Lemma 3.1, there are $x_0 \in B$, a ball $B_0 \Subset B$ containing x_0, and an even integer $\ell \geq 2$ such that the

transverse diagonal jets of degree less than ℓ vanish on B_0, whereas the degree-ℓ polynomial P_x is harmonic and $P_{x_0} \neq 0$. Equation (3.5) gives

$$\sigma_{2-\ell}(-2\mathcal{A}B_C\mathcal{A})(x,\xi) = -2i^{-\ell}(-2)^\ell \ell!\, \frac{P_x(\xi^\sharp)}{|\xi|^{2\ell-2}}.$$

At x_0, its restriction to the unit cotangent sphere is a nonzero harmonic polynomial of positive degree, so it has mean zero and both signs. The oscillatory construction of Theorem 3.4, with amplitudes supported in B_0, gives infinite positive and negative index. Intersection with finitely many linear kernels preserves both indices.

Lemma 5.9 (Compact divergence correction). *In a coordinate ball B, every $f \in C_c^\infty(B)$ with $\int_B f\, dv_g = 0$ has a smooth compactly supported vector field X in B satisfying* $\operatorname{div}_g X = f$. *The solution can be chosen linearly on a fixed coordinate box containing the support.*

Proof. Choose coordinates identifying B with $\mathbb{R}^n$; an ordinary ball chart followed by a diffeomorphism from the open ball to $\mathbb{R}^n$ has this property. The image of $\operatorname{supp} f$ is compact. Choose a product box $I_1 \times \cdots \times I_n$ containing it, with compact closure in this chart. Write $dv_g = \rho\, dx$ and $a = \rho f$. Then a has compact support in that box and $\int a\, dx = 0$.

The equation $a = \sum_i \partial_i Y^i$ admits the following inductive solution with compact support in the box. For $n = 1$, set $Y^1(x) = \int_{-\infty}^{x} a(s)\, ds$. It vanishes to the left and to the right of the support, the latter because $\int a = 0$. For $n > 1$, put

$$b(x') = \int_{\mathbb{R}} a(s, x')\, ds, \qquad \chi \in C_c^\infty(I_1), \quad \int \chi = 1,$$

and define

$$Y^1(x_1, x') = \int_{-\infty}^{x_1} \big(a(s, x') - \chi(s) b(x')\big)\, ds.$$

The integrand has integral zero in s for every x'. Hence Y^1 has compact support in the box and $\partial_1 Y^1 = a - \chi b$. Fubini's theorem gives $\int b\, dx' = 0$. By induction, choose $Z^2, \dots, Z^n$ with compact support in $I_2 \times \cdots \times I_n$ and $\sum_{i=2}^n \partial_i Z^i = b$. Set $Y^i(x_1, x') = \chi(x_1) Z^i(x')$ for $i \geq 2$. Then

$$\sum_{i=1}^n \partial_i Y^i = a - \chi b + \chi b = a.$$

Finally, put $X^i = Y^i/\rho$ in the chart and extend X by zero. Its support is compact in B and $\operatorname{div}_g X = \rho^{-1} \sum_i \partial_i(\rho X^i) = f$. With the boxes and functions χ fixed, every operation is linear in f. □

Lemma 5.10 (Gauge invariance on the kernel). *Suppose C annihilates the full metric derivative for perturbations supported in B. For $h \in C_c^\infty(B; S^2T^*M) \cap \ker D\mathsf{L}(g)$ and $X \in C_c^\infty(B; TM)$, put $q_C(v) = \langle C, D^2\mathsf{L}(g)[v,v]\rangle$ on its derivative kernel. Then*

$$q_C(h - \mathcal{L}_X g) = q_C(h).$$

Moreover $D\mathsf{L}(g)[\mathcal{L}_X g] = 0$.

Proof. Consider the affine family of metrics $g_s = g + sh$ and let ϕ_t be the flow of X. Both h and X vanish outside a compact subset of B. Pullback gives a unitary equivalence of the Laplacians, so their Riesz–Gram matrices satisfy

$$\mathsf{L}(\phi_t^* g_s) = O(s,t)^T \mathsf{L}(g_s) O(s,t), \qquad O(s,t)^T O(s,t) = I_m, \qquad O(0,0) = I_m.$$

At $s = 0$ the right side is λI_m for every t; consequently $D\mathsf{L}(g)[\mathcal{L}_X g] = 0$. Since $D\mathsf{L}(g)[h] = 0$, the s derivative of the right side at $s = 0$ is zero for every t: the terms differentiating $O^T O$ sum to zero and the middle term is $O(0,t)^T D\mathsf{L}(g)[h] O(0,t) = 0$. Its mixed derivative is therefore zero. The chain rule on the left gives

$$D^2\mathsf{L}(g)[h, \mathcal{L}_X g] + D\mathsf{L}(g)[\mathcal{L}_X h] = 0.$$

Since $\mathcal{L}_X h$ is supported in B and C annihilates the corresponding derivative range,

$$\langle C, D^2\mathsf{L}(g)[h, \mathcal{L}_X g]\rangle = 0.$$

Twice differentiating $\mathsf{L}(\phi_t^* g) = \lambda I_m$ gives

$$\langle C, D^2\mathsf{L}(g)[\mathcal{L}_X g, \mathcal{L}_X g]\rangle = -\langle C, D\mathsf{L}(g)[\mathcal{L}_X^2 g]\rangle = 0,$$

because $\mathcal{L}_X^2 g$ is supported in B. Polarization gives

$$q_C(h - \mathcal{L}_X g) = q_C(h) - 2\langle C, D^2\mathsf{L}(g)[h, \mathcal{L}_X g]\rangle + q_C(\mathcal{L}_X g) = q_C(h).$$

For a different chart on the supported metric space, the second-derivative difference is $D\mathsf{L}(g)$ applied to a supported acceleration tensor. Its contraction with C is zero. Hence q_C is intrinsic on the supported derivative kernel. □

Proposition 5.11 (Infinite inertia for every fixed-volume obstruction). *For $n \geq 3$, every nonzero cokernel matrix of the supported fixed-volume-density cluster map has infinite positive and negative index on its derivative kernel.*

Proof. If $t_C \not\equiv 0$, apply Lemma 5.7. Otherwise $T_C = 0$ on O, and Lemma 5.8 gives $F_C = 0$ there. These two relations imply that C annihilates the *full* supported metric derivative: its stress pairing is

$$-T_C + \frac{1}{2}(\mathrm{tr}_g\, T_C - \lambda F_C)g = 0.$$

Choose a ball $B \Subset O$. Equation (5.4) produces arbitrarily large definite spaces of conformal directions fg in the full derivative kernel. Impose also $\int_B f\, dv_g = 0$; this adds one linear equation, so infinite inertia persists. For each such f, solve $\mathrm{div}_g X = nf/2$ in B by Lemma 5.9. Then

$$h = fg - \mathcal{L}_X g, \qquad \mathrm{tr}_g\, h = nf - 2\,\mathrm{div}_g\, X = 0.$$

It lies in the derivative kernel and has the same contracted Hessian by Lemma 5.10. On a definite subspace this correction is injective, since a vector mapped to zero has zero quadratic value. Thus it preserves arbitrarily large positive and negative definite subspaces. All fields have compact support in B. For each N, the construction gives an N-dimensional definite subspace of smooth tensors supported in B. □

Proof of Theorem 5.3. Proposition 5.11 gives the index hypothesis of Theorem 4.5 in the trace-free exponential chart. Every selected endomorphism is smooth and compactly supported in O. Their finite support union is contained in some $K \Subset O$. Since $\mathrm{tr}\, H = 0$, one has $\det(e^H) = 1$ pointwise; hence $dv_{g_H} = dv_g$. Matrix exponential and its local logarithm are smooth in every fixed C^k norm. The estimates and the optimal exponent therefore pass between H and $g_H - g$. Exact-level regularity is obtained on the same finite-dimensional space. This proves class V of Theorem A. □

5.6. **Dimension and the choice of allowed metric changes.**

Theorem 5.12 (Arbitrary support on smooth surfaces). *On every smooth compact connected surface, with the closed, Dirichlet, or Neumann realization, for each isolated $\lambda > 0$ and every nonempty $O \subset M^\circ$, the full cluster matrix admits a local pairwise $1/2$-Hölder section with conformal perturbations supported in one compact $K \Subset O$. Exact-level regular approximation also holds.*

Proof. For $g_a = (1 + a)g$, the generalized eigenproblem is $\Delta_g \Phi = \theta(1 + a)\Phi$. Hence

$$D\mathsf{L}(0)[a]_{ij} = -\lambda \int a u_i u_j\, dv_g.$$

A local cokernel matrix satisfies $F_C = 0$ on O. There is no direct second-order pencil term, and on the derivative kernel the contracted Hessian is

$$-2\lambda^2\langle a, B_C a\rangle.$$

For $C \neq 0$, Theorem 3.4 gives infinite positive and negative index for this form on the supported derivative kernel. Theorem 4.5 then gives the section and the regular curve in one finite-dimensional subspace of $C_c^\infty(O)$. The union of the supports of its basis is contained in some $K \Subset O$. □

Combining this with Theorem 5.3 gives local matrix realization by unrestricted metric perturbations supported in a prescribed open set in all dimensions $n \geq 2$. Universal fixed-density realization fails in dimension two: on the round S^2, a positive scalar shift of its first cluster at fixed area contradicts Hersch's inequality $\lambda_1 \operatorname{Area} \leq 8\pi$ [33].

Proposition 5.13 (Analytic references and conformal localization). *If $n \geq 3$ and the reference metric is real analytic in the connected interior, arbitrary-open-supported conformal sections also exist for $\lambda > 0$, with the same conclusions as Theorem 5.12.*

Proof. The local annihilator equation is

$$\left(\Delta_g + \frac{4\lambda}{n-2}\right) F_C = 0 \quad \text{on } O.$$

In analytic coordinates, $\Delta_g - \lambda$ is elliptic with real-analytic coefficients. Analytic elliptic regularity [45] makes every u_i, and hence F_C and the displayed expression, real analytic in the interior. The analytic identity theorem extends the equation throughout the connected interior. Integration by parts, with the inherited Dirichlet or Neumann condition on F_C, gives

$$\int |dF_C|^2 + \frac{4\lambda}{n-2} \int F_C^2 = 0.$$

Thus $F_C = 0$. Theorems 3.4 and 4.5, on $C_c^\infty(O)$, give the section and the regular curve with common compact support in O. □

Theorem 5.14 gives, for each $n \geq 3$, a smooth metric g and an open set O for which every sufficiently small conformal perturbation supported in O satisfies $\lambda_1(g_H) \leq \lambda_1(g)$, with equality only at $H = 1$.

5.7. **A closed smooth conformal obstruction.**

Theorem 5.14 (A strict local conformal maximum on a closed torus). *For every $n \geq 3$, there are a smooth metric g on T^n, a nonempty open set $O \Subset T^n$, and $\eta > 0$, such that $\lambda_1(g) > 0$ is simple and*

$$\lambda_1(H^{4/(n-2)} g) < \lambda_1(g)$$

for every smooth $H > 0$, $H \neq 1$, with $\operatorname{supp}(H-1) \subset O$ and $\|H-1\|_\infty < \eta$.

Proof. Put $T_0 = \pi/2$ and $w_\delta(t) = 1 + \delta \cos 2t$ on $\mathbb{R}/(2\pi\mathbb{Z})$, with $0 < \delta < 1/2$. Let $A_\delta u = -w_\delta^{-1}(w_\delta u')'$ on periodic functions, in $L^2(w_\delta\, dt)$. The first-variation matrix of the energy and mass forms on $\operatorname{span}\{\cos t, \sin t\}$ is diagonal, with entries

$$\frac{\int_0^{2\pi} \cos 2t\, (|u'|^2 - u^2)\, dt}{\int_0^{2\pi} u^2\, dt} = -1 \quad (u = \cos t), \qquad +1 \quad (u = \sin t).$$

The mixed entry is zero because its integrand is odd under $t \mapsto -t$. Analyticity of these forms and the two-dimensional cluster reduction give two distinct eigenvalues $1 - \delta + O(\delta^2)$ and $1 + \delta + O(\delta^2)$. Fix $\delta > 0$ sufficiently small that the first is the simple first positive eigenvalue λ; the next unperturbed positive level is 4, so the isolating contours preserve this ordering. Reflection about 0 and about T_0 commutes with A_δ. The branch issuing from $\cos t$ is even for the first reflection and odd for the second, since these eigenvalues of the reflections are discrete and continuous on the simple branch. Its restriction to $[0, T_0]$ is the first Neumann–Dirichlet eigenfunction: at $\delta = 0$ that mixed problem has first two eigenvalues $1, 9$, and form continuity preserves the separating gap. A minimizer of its Rayleigh quotient can be replaced by its absolute value. The resulting nonnegative solution cannot vanish in $(0, T_0)$, since at an interior zero its derivative is zero and uniqueness of the second-order equation would give the zero solution. Normalize this eigenfunction by $u(0) = 1$. Then $u > 0$ on $[0, T_0)$ and

$$w_\delta(t) u'(t) = -\lambda \int_0^t w_\delta(s) u(s)\, ds < 0 \quad (0 < t \leq T_0).$$

In particular $u'(T_0) < 0$.

Set $p = 2n/(n-2)$, $c = \sqrt{p\lambda/2}$, and $v = -u'/u$ on $(0, T_0)$. The preceding endpoint conditions give $v(t) \to 0$ as $t \downarrow 0$ and $v(t) \to +\infty$ as $t \uparrow T_0$. Choose t_0 with $v(t_0) = c$. For small $\varepsilon > 0$, take $0 \le \chi_\varepsilon \le 1$ supported in $(t_0 - \varepsilon, t_0 + \varepsilon)$ and equal to one on $J = (t_0 - \varepsilon/2, t_0 + \varepsilon/2)$. Put

$$\widetilde{v} = v + \chi_\varepsilon(c - v), \qquad \widetilde{u}(t) = \exp\left(-\int_0^t \widetilde{v}(s)\, ds\right).$$

On the modification interval $v, \widetilde{v}$ are bounded above and below by positive constants independent of ε, and $\|\widetilde{v} - v\|_\infty \le C\varepsilon$. Define $\widetilde{w} = w_\delta$ before that interval and, within and after it, define it by

$$\log \frac{\widetilde{w}(t)}{w_\delta(t)} = -\log \frac{\widetilde{v}(t)}{v(t)} + \int_0^t \left(\widetilde{v} - v + \lambda\left(\frac{1}{\widetilde{v}} - \frac{1}{v}\right)\right) ds.$$

The integrand is set to zero off the modification interval. This formula is smooth, matches the zero logarithm before that interval, and has supremum $O(\varepsilon)$; its integral term is $O(\varepsilon^2)$. Differentiating it gives

$$\frac{\widetilde{w}'}{\widetilde{w}} = \widetilde{v} - \frac{\widetilde{v}'}{\widetilde{v}} + \frac{\lambda}{\widetilde{v}}.$$

Since $\widetilde{u}' = -\widetilde{v}\widetilde{u}$, it follows that $-\widetilde{u}'' - (\widetilde{w}'/\widetilde{w})\widetilde{u}' = \lambda\widetilde{u}$. Near 0 the functions agree with u, w_δ; near T_0 each is a positive constant multiple of its original value. Even reflection of $\widetilde{u}$ at 0, odd reflection at T_0, and even reflection of $\widetilde{w}$ at both endpoints therefore produce smooth functions, with $\widetilde{w}$ periodic of period π and $\widetilde{u}$ antiperiodic of period π.

Let $\theta_\varepsilon = \|\widetilde{w}/w_\delta - 1\|_\infty \to 0$. For every nonzero periodic $a \in H^1$, comparison of both integrals in the quotient gives

$$\frac{1-\theta_\varepsilon}{1+\theta_\varepsilon}\frac{\int w_\delta |a'|^2}{\int w_\delta |a|^2} \le \frac{\int \widetilde{w}|a'|^2}{\int \widetilde{w}|a|^2} \le \frac{1+\theta_\varepsilon}{1-\theta_\varepsilon}\frac{\int w_\delta |a'|^2}{\int w_\delta |a|^2}.$$

Min–max gives the same inequalities for every ordered eigenvalue. If $\lambda_2(A_\delta) > \lambda$ is the next positive eigenvalue, choose ε with $\theta_\varepsilon < (\lambda_2(A_\delta) - \lambda)/(\lambda_2(A_\delta) + \lambda)$. Then $\lambda_2(A_{\widetilde{w}}) > \lambda$. The exact eigenfunction at the positive eigenvalue λ and the simple zero eigenvalue of constants imply that λ is still the simple first positive eigenvalue. On J one has

$$|\widetilde{u}'|^2 = c^2\widetilde{u}^2 = \frac{p}{2}\lambda\widetilde{u}^2. \tag{5.5}$$

Take a flat metric h on T^{n-1} and $g = dt^2 + \tau^2\widetilde{w}(t)^{2/(n-1)}h$. The volume density is a constant multiple of $\widetilde{w}(t)\, dt\, dv_h$. On functions independent of the torus variable, its Laplacian is $A_{\widetilde{w}}$. For a fiber eigenfunction with eigenvalue $\kappa > 0$, the separated radial quotient has the additional term $\tau^{-2}\kappa\widetilde{w}^{-2/(n-1)}$. It is bounded below by $\tau^{-2}\lambda_1(h)(\max \widetilde{w})^{-2/(n-1)}$. Choose τ so small that this bound exceeds λ. Fourier decomposition in the torus variable is orthogonal for both energy and mass; hence the function $u(t) = \widetilde{u}(t)$ is the simple first positive eigenfunction of g. Its weighted mean is zero, either by antiperiodicity or by integrating its positive-eigenvalue equation.

For $H > 0$ with $H = 1$ off $O \Subset J \times T^{n-1}$, write

$$N_H = \int H^2|du|^2, \quad D_H = \int H^p u^2, \quad I_H = \int H^p u, \quad V_H = \int H^p,$$

where the measure is dv_g. The function $u - I_H/V_H$ has zero $H^p\, dv_g$ mean. Its energy for $H^{4/(n-2)}g$ is N_H and its mass is $D_H - I_H^2/V_H > 0$, the strict inequality following because u is not constant. Equation (5.5) on O and the identity $\int u = 0$ give

$$N_H - \lambda(D_H - I_H^2/V_H) = \lambda\left[-\int_O u^2\phi(H) + \frac{I_H^2}{V_H}\right], \qquad \phi(s) = s^p - 1 - \frac{p}{2}(s^2 - 1).$$

Since $p > 2$, strict convexity of $z^{p/2}$ gives $\phi(s) > 0$ for $s \neq 1$. Choose $0 < \eta < 1/2$ so that $\phi''(s) \geq p(p-2)/2$ for $|s-1| \leq \eta$. Then

$$\phi(s) \geq c_0(s-1)^2, \quad c_0 = p(p-2)/4, \qquad |s^p - 1| \leq C_0|s-1|, \quad C_0 = p(1+\eta)^{p-1}.$$

Cauchy–Schwarz gives $I_H^2 \leq C_0^2 \operatorname{Vol}_g(O) \int_O u^2(H-1)^2$, and $V_H \geq (1-\eta)^p \operatorname{Vol}_g(M)$. Choose the nonempty O with

$$\operatorname{Vol}_g(O) < \frac{c_0(1-\eta)^p \operatorname{Vol}_g(M)}{2C_0^2}.$$

The preceding difference is then at most $-\lambda c_0 \int_O u^2(H-1)^2/2 < 0$ for every admissible $H \neq 1$, because $u > 0$ on O. The Rayleigh characterization of the first positive eigenvalue proves the strict inequality. □

For the metric and open set in Theorem 5.14, put

$$\mathcal{U} = \{H > 0 : \operatorname{supp}(H-1) \subset O,\ \|H-1\|_\infty < \eta\}, \qquad F(H) = \lambda_1(H^{4/(n-2)}g) - \lambda_1(g).$$

Then $F(\mathcal{U}) \subset (-\infty, 0]$, and $F(H) = 0$ exactly at $H = 1$. For every $\epsilon > 0$, the map $F - \epsilon$ has empty zero set and is uniformly ϵ-close to F. Thus 0 is not an essential value of F.

Proposition 5.15 (Strict decrease of the Dirichlet ground eigenvalue). *For every $n \geq 3$, there are a compact smooth warped cylinder (M, g) with $\lambda_1^D(g) = 1$ and a nonempty open $O \Subset M^\circ$ such that*

$$\lambda_1^D(H^{4/(n-2)}g) < 1$$

for every smooth $H > 0$ with $\operatorname{supp}(H-1) \subset O$ and $H \not\equiv 1$.

Proof. Put $p = 2n/(n-2)$, $T = 4$, and $a = \sqrt{n/(n-2)}$. Choose disjoint endpoint neighborhoods in $(0, T)$ on which $\tan x$ and $\cot(T-x)$ are positive. Smooth cutoffs supported there and equal to one near the endpoints give a positive convex interpolation v equal to these respective functions and equal to a on a nonempty interior interval J. Set

$$u(x) = \exp\left(-\int_0^x v(s)\,ds\right), \qquad \frac{w'}{w} = v - \frac{v'}{v} + \frac{1}{v},$$

with a fixed positive initial weight. Near zero, $u = \cos x$ and w is constant; near T, $u = C\sin(T-x)$ and w is constant. Thus even reflection at zero gives smooth u, w on $[-T, T]$, with $u > 0$ in the interior and $u(\pm T) = 0$. Direct differentiation gives $-(wu')' = wu$.

The first Dirichlet minimizer of $\int w|f'|^2 / \int w|f|^2$ can be chosen nonnegative by replacing it with its absolute value. Its equation and uniqueness for the initial-value problem make it positive in the interior. Its inner product with u is positive, so self-adjointness forces its eigenvalue to be 1. The radial eigenvalue is simple: solutions with the left Dirichlet condition form a one-dimensional space for fixed eigenvalue. On $[-T, T] \times T^{n-1}$ with $g = dx^2 + w^{2/(n-1)}h$, where h is flat, each nonconstant fiber Fourier mode adds a strictly positive potential $\kappa w^{-2/(n-1)}$ to the radial operator. Its first eigenvalue is at least $1 + \kappa(\max w)^{-2/(n-1)} > 1$. Hence u is the simple Dirichlet ground state of the product metric. On J, one has $|du|^2 = a^2u^2 = (p/2)u^2$. For every positive smooth H equal to one off a nonempty $O \Subset J \times T^{n-1}$,

$$E_{H^{4/(n-2)}g}(u) - \|u\|^2_{L^2(H^{4/(n-2)}g)} = -\int_O u^2\left(H^p - 1 - \frac{p}{2}(H^2-1)\right) dv_g.$$

The integrand in parentheses is positive wherever $H \neq 1$. If $H \not\equiv 1$, its nonzero set is open, $u > 0$ there, and the displayed integral is strictly negative. The Dirichlet Rayleigh quotient of u is therefore less than 1. □

5.8. **Open geometric inequalities.**

Corollary 5.16. *Suppose the reference satisfies a strict C^k-open geometric inequality on a compact neighborhood of the support of the realizing perturbations. The realizations in Theorem A satisfy the same inequality after restriction to a sufficiently small target ball. In particular,* $\mathrm{Ric}_g > cg$ *is preserved by sufficiently small C^2 realizations.*

Proof. An open inequality holds in some C^k neighborhood of the reference. The finite-family estimates place the constructed metrics in that neighborhood. For Ricci curvature the explicit estimate and lower margin are proved in Lemma A.5. Outside the support the metrics agree exactly. □

6. Strong stability and singular spectral levels

For a positive eigenvalue on a closed connected manifold, Proposition 2.1 gives

$$\operatorname{coker} D\mathsf{L}(g) = \{C \in \mathrm{Sym}_m(\mathbb{R}) \,:\, F_C = 0,\ T_C = 0\}.$$

6.1. **Signature of a metric obstruction.**

Theorem 6.1 (Signature bound). *Let $u_1, \dots, u_m$ be linearly independent real λ-eigenfunctions on a connected manifold, with $\lambda > 0$. If a nonzero $C \in \mathrm{Sym}_m(\mathbb{R})$ satisfies*

$$\sum C_{ij} u_i u_j = 0, \qquad \sum C_{ij}\, du_i \otimes du_j = 0, \tag{6.1}$$

then C has at least four positive and four negative eigenvalues. Consequently every positive eigenvalue of multiplicity at most seven on a closed connected manifold satisfies the Strong Arnold Hypothesis for unrestricted metric variations. Multiplicity eight admits failure.

6.1.1. *Spherical immersions.*

Lemma 6.2. *Let $F : O \subset S^2 \to S^q$ be a smooth isometric immersion between unit spheres. Let* II *be its second fundamental form inside S^q. Suppose that, at each point of O, a positive-definite contravariant quadratic form Q on TO satisfies* $\mathrm{tr}_Q \mathrm{II} = 0$. *Then* $\mathrm{II} = 0$. *Near each point, $F(y) = Ay$ for a constant linear isometric inclusion $A : \mathbb{R}^3 \to \mathbb{R}^{q+1}$.*

Proof. Choose an orthonormal tangent basis and an orthonormal normal basis at a point. Write A_α for the symmetric 2×2 matrix of the normal component of II. The hypothesis is $\mathrm{tr}(QA_\alpha) = 0$. The symmetric matrix $Q^{1/2} A_\alpha Q^{1/2}$ has eigenvalues $b_\alpha, -b_\alpha$. Hence

$$\det A_\alpha = -\frac{b_\alpha^2}{\det Q} \le 0, \tag{6.2}$$

with equality only if $A_\alpha = 0$. Write D for the Euclidean connection of $\mathbb{R}^{q+1}$. For tangent fields X, Y on O,

$$D_X(dF(Y)) = dF(\nabla_X Y) + \mathrm{II}(X, Y) - \langle X, Y\rangle F.$$

The radial term follows by differentiating $\langle dF(Y), F\rangle = 0$. Put $\mathbb{B}(X, Y) = \mathrm{II}(X, Y) - \langle X, Y\rangle F$. Commuting the Euclidean covariant derivatives and taking tangent components gives, on an orthonormal pair e_1, e_2,

$$K_O(e_1, e_2) = \langle \mathbb{B}(e_1, e_1), \mathbb{B}(e_2, e_2)\rangle - |\mathbb{B}(e_1, e_2)|^2 = 1 + \sum_\alpha \det A_\alpha.$$

Here II is perpendicular to F and $|F| = 1$. Since O has the unit-sphere metric, $K_O = 1$, and the determinant sum is zero. Equation (6.2) makes every summand nonpositive; thus each $A_\alpha = 0$. If the normal space is zero-dimensional, $\mathrm{II} = 0$ by its definition. Fix $y_0 \in O$ and define A on $\mathbb{R}y_0 \oplus T_{y_0}S^2$ by $Ay_0 = F(y_0)$ and $A|_{T_{y_0}S^2} = dF_{y_0}$. These orthogonal prescriptions make A an isometric inclusion. A sufficiently short unit-speed great circle γ from y_0 satisfies $\gamma'' = -\gamma$. Since $\mathrm{II} = 0$, $F \circ \gamma$ satisfies the same equation in $\mathbb{R}^{q+1}$, with the initial data of $A\gamma$. Uniqueness of this linear differential equation gives $F \circ \gamma = A\gamma$. Such segments cover a neighborhood of y_0. □

6.1.2. *Quadratic relations.*

Proof of Theorem 6.1. Diagonalize C by a constant orthogonal matrix. Multiply each resulting eigenfunction by the square root of the absolute value of its nonzero coefficient, and omit zero coefficients. The resulting functions remain linearly independent. They form two vector-valued eigenfunctions $U : M \to \mathbb{R}^p$ and $V : M \to \mathbb{R}^q$ satisfying

$$|U|^2 = |V|^2, \qquad dU^T dU = dV^T dV, \qquad \Delta_g U = \lambda U, \quad \Delta_g V = \lambda V. \tag{6.3}$$

Both p, q are positive, since a sum of nonzero squares cannot vanish identically. A constant linear combination of the components in (6.3) is a λ-eigenfunction. If it vanishes on a nonempty open set, Aronszajn's theorem [7], with the scalar ellipticity and coefficient hypotheses stated in Section 2, makes it zero on M. Linear independence therefore excludes every such local relation with nonzero coefficients. Suppose that $p \leq 3$. On a nonempty open set where $r = |U| = |V| > 0$, put $\phi = U/r$ and $\psi = V/r$. Their values lie in S^{p-1} and S^{q-1}. Since $\phi \cdot d\phi = \psi \cdot d\psi = 0$,

$$dU^T dU = dr \otimes dr + r^2 d\phi^T d\phi, \qquad \phi^* g_{S^{p-1}} = \psi^* g_{S^{q-1}}. \tag{6.4}$$

Consequently $\ker d\phi = \ker d\psi$ and their ranks agree. Choose a point where this rank is maximal on the chosen open set. If the maximal rank is positive, one minor of that order is nonzero on a neighborhood; maximality excludes larger rank there. If the maximal rank is zero, both differentials vanish on the chosen open set. The common rank is constant and belongs to $\{0, 1, 2\}$. At rank zero, ϕ and ψ are constant on a connected smaller neighborhood. All components of U, V are constant multiples of r. Since $p + q \geq 2$, this contradicts the absence of a local constant relation. At rank one, the coordinates of Lemma A.4 for ϕ have the form (t, z), and the fibres with fixed t may be taken connected. Equality of kernels makes ψ constant on these fibres. Thus $\phi = \gamma(t)$ and $\psi = \delta(t)$. Reparametrizing by the arc length of γ gives $|\gamma'| = 1$. Equation (6.4) then gives $|\delta'| = 1$, and $dt \neq 0$. The equation for $U = r\gamma(t)$ is

$$(\Delta r - \lambda r)\gamma + (r\Delta t - 2\langle dr, dt\rangle)\gamma' - r|dt|^2\gamma'' = 0. \tag{6.5}$$

The identities $|\gamma| = |\gamma'| = 1$ imply $\gamma \cdot \gamma' = 0$, $\gamma \cdot \gamma'' = -1$, and $\gamma' \cdot \gamma'' = 0$. Taking inner products of (6.5) with γ and γ' gives

$$\Delta r - \lambda r + r|dt|^2 = 0, \qquad r\Delta t - 2\langle dr, dt\rangle = 0. \tag{6.6}$$

Since $r|dt|^2 > 0$, substitution yields $\gamma'' + \gamma = 0$. The equation for $V = r\delta(t)$ has the same scalar coefficients, so it gives $\delta'' + \delta = 0$. There are constant vectors a, b, c, d with $\gamma = a\cos t + b\sin t$ and $\delta = c\cos t + d\sin t$. All components of U, V lie in the span of $r\cos t$ and $r\sin t$. Rank one requires $p, q \geq 2$, so $p + q \geq 4$. A nonzero constant relation follows, a contradiction. At rank two, $p = 3$. The map ϕ is a submersion onto an open subset $O \subset S^2$. Apply Lemma A.4 to obtain product coordinates with connected fibres. Equality of kernels implies

$$\psi = F \circ \phi, \qquad F : O \longrightarrow S^{q-1}. \tag{6.7}$$

The definition of F is independent of the chosen lift: in a product neighborhood the fibre is connected, and the derivative of ψ vanishes on each tangent vector to it. Integration along paths inside the fibre makes ψ constant there. If $\xi, \eta \in T_{\phi(x)}S^2$ and $d\phi(X) = \xi$, $d\phi(Y) = \eta$, then

$$\langle dF(\xi), dF(\eta)\rangle = \langle d\psi(X), d\psi(Y)\rangle = \langle d\phi(X), d\phi(Y)\rangle = \langle \xi, \eta\rangle.$$

Thus F is an isometric immersion. Fix a point x. Choose a local g-orthonormal frame satisfying $\nabla_{e_i} e_j(x) = 0$. All identities in the following calculation are evaluated at x. Write $\xi_i = d\phi(e_i)$ and let ∇^ϕ be the connection on $\phi^* TS^2$. The Euclidean derivative of F gives

$$\begin{aligned} D_{e_i} D_{e_i}(rF(\phi)) = {} & e_i e_i(r) F(\phi) + 2e_i(r) dF(\xi_i) \\ & + r\, dF(\nabla^\phi_{e_i} \xi_i) + r\, \mathrm{II}(\xi_i, \xi_i) - r|\xi_i|^2 F(\phi). \end{aligned}$$

Let pr_{N^S} denote projection onto the normal space of $F(O)$ inside S^{q-1}. The first and last terms belong to $\mathbb{R}F(\phi(x))$; the second and third belong to $dF(T_{\phi(x)}O)$. Therefore

$$\mathrm{pr}_{N^S}\,\Delta_g(rF\circ\phi) = -r\sum_i \mathrm{II}(\xi_i,\xi_i).$$

The normal projection of $\lambda rF(\phi)$ is zero. Since $r>0$, the common eigenfunction equation implies

$$\sum_{i=1}^{n}\mathrm{II}\big(d\phi(e_i),d\phi(e_i)\big)=0. \tag{6.8}$$

Choose a smooth local section $s: O\to M$ of the submersion ϕ, after decreasing O. Define

$$Q_y=\sum_{i=1}^{n} d\phi_{s(y)}(e_i)\otimes d\phi_{s(y)}(e_i),\qquad y\in O,$$

where e_i is any g-orthonormal basis at $s(y)$. This tensor is independent of that basis and is smooth in y. If $0\neq\xi\in T^*_yS^2$, surjectivity of $d\phi_{s(y)}$ gives

$$Q_y(\xi,\xi)=\sum_i\big(\xi(d\phi_{s(y)}e_i)\big)^2>0.$$

Equation (6.8), evaluated at $s(y)$, gives $\mathrm{tr}_{Q_y}\mathrm{II}_y=0$. Lemma 6.2 applies. The lemma gives $F(y)=Ay$ on an open set. Therefore $V=rF(\phi)=AU$ there. Each component of $V-AU$ is a constant linear combination of the independent components of U,V, with a nonzero coefficient on a component of V. Unique continuation contradicts their independence. All ranks have been excluded when $p\le 3$. Interchanging U,V gives the exclusion when $q\le 3$. Hence $p,q\ge 4$. By Proposition 2.1, every positive eigenvalue of multiplicity at most seven has surjective unrestricted metric derivative. □

Example 6.3. On $\mathbb{R}^2/(2\pi\mathbb{Z})^2$, the shell $|k|^2=5$ is $\{\pm(1,2),\pm(2,-1),\pm(1,-2),\pm(2,1)\}$. These are all solutions, since each coordinate square is at most five and the only sum of two integer squares equal to five is $1+4$. The real eigenspace has dimension eight. Put

$$U=(\cos(x+2y),\sin(x+2y),\cos(2x-y),\sin(2x-y)),$$
$$V=(\cos(x-2y),\sin(x-2y),\cos(2x+y),\sin(2x+y)).$$

Distinct Fourier frequencies give independence, while

$$|U|^2=|V|^2=2,\qquad dU^T\,dU=dV^T\,dV=5(dx^2+dy^2). \tag{6.9}$$

Thus $\mathrm{diag}(I_4,-I_4)$ is a nonzero cokernel element. This is the known sharp torus example; see [28, Section 4.2].

6.2. **A rank-sensitive refinement.**

Lemma 6.4. *Let $F: U\subset S^r\to S^q$ be an isometric immersion of unit spheres. If a positive-definite contravariant form A satisfies $\mathrm{tr}_A\,\mathrm{II}=0$, then $\mathrm{II}=0$. Near each point F is the restriction of a linear isometric inclusion.*

Proof. At a point choose orthonormal tangent and normal frames. Let $(S_\alpha)_{ij}$ be the components of the spherical second fundamental form. Equality of the intrinsic curvature with the ambient sectional-curvature-one tensor gives

$$0=\sum_\alpha\big((S_\alpha)_{ij}(S_\alpha)_{k\ell}-(S_\alpha)_{i\ell}(S_\alpha)_{kj}\big).$$

Contract this identity with $A^{ij}A^{k\ell}$. Since S_α and A are symmetric,

$$0=\sum_\alpha\big((\mathrm{tr}(AS_\alpha))^2-\mathrm{tr}(AS_\alpha AS_\alpha)\big).$$

The assumption gives $\operatorname{tr}(AS_\alpha) = 0$. Also

$$\operatorname{tr}(AS_\alpha AS_\alpha) = \operatorname{tr}\big((A^{1/2}S_\alpha A^{1/2})^2\big) = \|A^{1/2}S_\alpha A^{1/2}\|^2_{\mathrm{HS}}.$$

Each summand is nonnegative, so every $S_\alpha = 0$. Hence $\mathrm{II} = 0$. At $x_0 \in U$ define a linear map $L : \mathbb{R}^{r+1} \to \mathbb{R}^{q+1}$ by $Lx_0 = F(x_0)$ and $L|_{T_{x_0}S^r} = dF_{x_0}$. These orthogonal prescriptions make L an isometry onto its image. For every short unit-speed great circle γ from x_0, the Euclidean derivatives satisfy

$$(F \circ \gamma)'' = -F \circ \gamma, \qquad (L\gamma)'' = -L\gamma.$$

Their values and first derivatives at zero agree. Uniqueness of the linear differential equation gives $F \circ \gamma = L\gamma$. Such segments fill a neighborhood of x_0. □

Proposition 6.5. *Let C be a nonzero unrestricted metric obstruction, with signature (p, q) and nullity ℓ. Normalize its active positive and negative eigenmaps to $\phi : U \to S^{p-1}$ and $\psi : U \to S^{q-1}$ on the open set where their common norm is positive. Then*

$$\operatorname{rank} d\phi = \operatorname{rank} d\psi \le \min(p, q) - 2. \tag{6.10}$$

The common rank is at least two on an open dense subset of that open set. For the full projective eigenfunction map $\Phi = [u_1 : \cdots : u_m]$, wherever defined,

$$\operatorname{rank} d\Phi \le \min(p, q) - 2 + \ell \le m - 6.$$

Consequently an immersive full projective eigenmap can be unstable only if $m \ge n + 6$.

Proof. The equal-pullback identity (6.4) gives equal kernels and equal ranks. Suppose the common rank is $p - 1$ at one point. It remains $p - 1$ on a neighborhood; there ϕ is a submersion onto an open subset of S^{p-1}. Connected local fibers and the common kernel give $\psi = F \circ \phi$, where F is an isometric immersion. The normal projection of the common eigenfunction equation is exactly (6.8), now with $p - 1$ tangent directions. A local section of ϕ supplies the positive-definite contraction tensor. Lemma 6.4 makes F linear, producing a nonzero constant linear relation among the active eigenfunctions. Unique continuation contradicts their independence. Thus the rank is at most $p - 2$. Interchange the two eigenmaps to obtain (6.10).

If the set of rank at most one contained an open subset, select on it a neighborhood of maximal constant rank. The rank-zero and rank-one arguments in the proof of Theorem 6.1 apply there and give the same contradiction. The rank-at-least-two locus is open, and its complement has empty interior; it is therefore dense.

A constant invertible change of homogeneous coordinates preserves rank $d\Phi$. In coordinates diagonalizing C, write the active eigenmaps as $r\phi, r\psi$, and let w comprise the ℓ null-coordinate eigenfunctions. On $\{r > 0\}$, division by r gives the projective coordinates $(\phi, \psi, w/r)$. The first two have common differential kernel, so their combined rank is the rank of $d\phi$. Adjoining ℓ coordinates increases it by at most ℓ. This gives the first projective bound. The active set is dense, by unique continuation. Vanishing of all minors above this bound passes by continuity to the whole domain of Φ. Finally,

$$\min(p, q) - 2 + \ell = m - \max(p, q) - 2 \le m - 6,$$

since $p, q \ge 4$. If Φ immerses an n-manifold, its rank is n. □

6.3. **The multiplicity locus.**

Theorem 6.6. *Let g be a smooth metric on a closed connected manifold and let $\lambda > 0$ be an isolated eigenvalue of multiplicity m. In the full metric space or the full conformal class of g, put*

$$\Gamma(g') = \mathsf{L}(g') - \frac{\operatorname{tr}\mathsf{L}(g')}{m} I_m, \qquad \mathcal{Z} = \Gamma^{-1}(0).$$

In any fixed C^k chart, $k \ge 4$, the set $\mathcal{Z}$ is a C^1 embedded submanifold near g if and only if $D\mathsf{L}(g)$ is onto $\mathrm{Sym}_m(\mathbb{R})$. In that case $\mathcal{Z}$ is smooth and has codimension $m(m + 1)/2 - 1$.

Proof. Scaling gives $D\mathsf{L}(g)[g] = -\lambda I_m$, so $D\Gamma(g)$ is onto Sym^0_m if and only if $D\mathsf{L}(g)$ is onto Sym_m. In this case a finite right inverse splits the derivative, and the parameter-dependent inverse theorem expresses $\mathcal{Z}$ as a graph over its kernel. The codimension is $\dim \mathrm{Sym}^0_m$.

Suppose the derivative is not onto. Let C be the cokernel of $D\Gamma(g)$ in Sym^0_m and let $K = \ker D\Gamma(g)$. Every $c \in C$ is a trace-free annihilator of $D\mathsf{L}(g)$. Its contracted Hessian has infinite indices on K: Theorem 3.4 gives this conclusion on the conformal subspace $\ker D\mathsf{L} \subset K$. Every unrestricted annihilator also annihilates the conformal derivative, so the same definite subspaces are contained in the full metric kernel. Define $Q = \frac{1}{2}P_C D^2\Gamma(g)|_K$.

Theorem 4.1 gives $x, y \in K$ such that $z_+ = x + y$ and $z_- = x - y$ are regular zeros of Q, whereas $Q(y) \neq 0$. Choose a finite-dimensional subspace $V \subset K$ containing x, y and subspaces on which $DQ(z_+)$ and $DQ(z_-)$ are onto C. Let X be a finite-dimensional complement on which $D\Gamma(g)$ maps isomorphically to its range. Solving the range equation gives $\psi : V \to X$ with $\psi(0) = D\psi(0) = 0$. The remaining equation is

$$G(z) := P_C\Gamma\big(g + z + \psi(z)\big) = Q(z) + O(|z|^3), \qquad z \in V,$$

where addition denotes the chosen chart. Taylor's integral formula gives a smooth extension of

$$\widetilde{G}(s, z) = s^{-2}G(sz), \qquad \widetilde{G}(0, z) = Q(z).$$

Since $D_z\widetilde{G}(0, z_\pm) = DQ(z_\pm)$ is onto, the inverse function theorem gives smooth $z_\pm(s)$ with $z_\pm(0) = z_\pm$ and $\widetilde{G}(s, z_\pm(s)) = 0$. Thus

$$c_\pm(s) = g + sz_\pm(s) + \psi(sz_\pm(s)) \in \mathcal{Z}, \qquad c'_\pm(0) = x \pm y.$$

In the chart centered at g, put $H(v) = P_C\Gamma(g + v)$. Then $H(0) = 0$ and $DH(0) = 0$. Any C^1 curve in $\mathcal{Z}$ with velocity v has expansion $tv + o(t)$ in the chart, so Taylor's formula gives

$$0 = H(tv + o(t)) = \tfrac{1}{2}t^2 D^2H(0)[v, v] + o(t^2).$$

Thus its velocity must satisfy $Q(v) = 0$. A C^1 submanifold has a linear tangent space, and each tangent vector is the velocity of a C^1 curve in it. The two constructed velocities would therefore imply that $2y$ is also such a velocity. But $Q(2y) = 4Q(y) \neq 0$, a contradiction. □

6.4. **Prescribed jets and all tangent directions.** Let E be Banach and let $F : E \to W$ be smooth near zero, with $F(0) = 0$. Take the decomposition (4.1) with $E_0 = E$ and suppose (4.3) holds. If a dense linear subspace $E_\infty \subset E$ is specified, then $A(E_\infty) = A(E)$: the former is a dense linear subspace of the finite-dimensional space $A(E)$. A range complement may therefore be chosen in E_∞. In the spectral applications, the finite-dimensional negative subspaces and derivative complements may also be chosen smooth, by continuity of their finite Gram matrices.

Theorem 6.7 (Exact prescribed-jet correction). *Assume $d > 0$ and* (4.3). *Let $k \geq 1$, let $p : (\mathbb{R}^r, 0) \to (E, 0)$ be a polynomial, and let $f : (\mathbb{R}^r, 0) \to (W, 0)$ be smooth. Suppose*

$$j_0^{2k}(F \circ p - f) = 0. \tag{6.11}$$

There is a map h, smooth off 0 and $C^{k,1/2}$ near 0, such that

$$F(h(t)) = f(t), \qquad j_0^k h = j_0^k p. \tag{6.12}$$

It takes values in a finite-dimensional linear space V_* containing the coefficients of p. If their span has dimension N_p, one may arrange

$$\dim V_* \leq N_p + \dim W + 1.$$

For $0 < |t|$ small, the restricted derivative $DF(h(t))|_{V_*}$ is onto and has a right inverse of norm $O(|t|^{-k-1/2})$. For every multi-index β with $|\beta| \leq k + 1$, off the origin,

$$\big\|D^\beta(h - p)(t)\big\| \leq C_\beta |t|^{k+1/2-|\beta|}. \tag{6.13}$$

The space V_* and the constants may depend on p and f.

The exponent $1/2$ is optimal under these hypotheses; see Proposition 6.9.

Theorem 6.8 (Prescribed derivative and the full tangent cone). *Assume* (4.3)*, and let $L : \mathbb{R}^r \to E$ be linear. A smooth target germ f with $f(0) = 0$ has a C^1 exact lift h with $h(0) = 0$ and $Dh(0) = L$ if and only if*

$$AL = Df(0), \tag{6.14}$$

$$P_C D^2 f(0)[u, v] = P_C D^2 F(0)[Lu, Lv] \quad (u, v \in \mathbb{R}^r). \tag{6.15}$$

When these hold, the lift may be chosen $C^{1,1/2}$, smooth off 0, and regular on fixed finite perturbations at every nonzero parameter. If L is injective, it may be restricted to a C^1 embedded disk.

For the zero fiber $Z = F^{-1}(0)$, its Bouligand contingent tangent cone is

$$T_0 Z = \{v \in K \,:\, Q(v) = 0\}. \tag{6.16}$$

Every vector on the right is the velocity of a two-sided $C^{1,1/2}$ curve in Z whose nonzero points are regular on a fixed finite space of perturbations.

For $d = 0$, the split implicit function theorem gives the lift and the tangent cone $\ker DF(0)$.

6.5. **Exact correction of a prescribed finite jet.**

Proof of Theorem 6.7. Choose a range complement X of dimension a. For $z \in K$ solve

$$P_R F(p(t) + z + x(t, z)) = f_R(t), \qquad x(t, z) \in X.$$

At $(0, 0)$ this is the ordinary parameter-dependent implicit function theorem. $x(0, 0) = 0$, $D_z x(0, 0) = 0$, and $x(t, 0) = O(|t|^{2k+1})$ by (6.11). Define

$$g(t, z) = P_C F(p(t) + z + x(t, z)) - f_C(t).$$

Then $g(t, 0) = O(|t|^{2k+1})$, $D_z g(0, 0) = 0$, and $\frac{1}{2} D_z^2 g(0, 0) = Q$ on K.

Impose the finitely many linear conditions

$$H = \bigcap_{|\alpha| \le k} \ker\big(\partial_t^\alpha D_z g(0, 0) \,:\, K \to C\big). \tag{6.17}$$

Hypothesis (4.3) survives on H. Choose the $d + 1$-dimensional block from (4.11) with $r = 1$ inside H, let Z_* be its image, and let z_0, W_0 be its regular zero and derivative complement. On this finite space,

$$g(t, z) = g(t, 0) + L(t)z + Q(z) + E(t, z), \tag{6.18}$$

$$L(t)|_{Z_*} = O(|t|^{k+1}), \qquad E(t, z) = O((|t| + |z|)|z|^2).$$

More precisely,

$$E(t, z) = \int_0^1 (1 - s)\big(D_z^2 g(t, sz) - D_z^2 g(0, 0)\big)[z, z]\, ds.$$

Taylor's formula for the smooth coefficients in this integral gives the corresponding differentiated bounds.

On $\mathbb{R}^r \oplus Z_*$ set

$$\mathfrak{F}(\tau, z) = \big(\tau, g(\tau, z) - g(\tau, 0)\big), \qquad \mathfrak{P}(\tau, z) = \big(\tau, Q(z)\big).$$

Give the τ-coordinates weight 2 and the z-coordinates weight $2k + 1$. The corresponding target weights are 2 and $4k + 2$. By (6.18), the remainder in the second component is bounded by absolute monomials of weights at least $4k + 3$, $4k + 4$, and $6k + 3$. Hence $\mathfrak{P}$ is a weighted truncation. Moreover,

$$\mathfrak{P}(0, z_0) = 0, \qquad D\mathfrak{P}(0, z_0)(\tau, \zeta) = \big(\tau, DQ(z_0)\zeta\big)$$

is onto, and restricts to an isomorphism on $\mathbb{R}^r \oplus W_0$. The equation

$$\mathfrak{F}(\tau, z) = \big(t, -g(t,0)\big)$$

is equivalent to $\tau = t$ and $g(t,z) = 0$. It therefore has the weighted inverse form of [11, Theorem 3.3 and equations (4.19)–(4.22)]. The parameter coordinates in this augmentation add no perturbation directions to $X \oplus Z_*$.

Write $t = \varepsilon^2\theta$, $\theta \in S^{r-1}$, and $z = \varepsilon^{2k+1}w$. Then

$$\Psi(\varepsilon, \theta, w) = \varepsilon^{-4k-2} g(\varepsilon^2\theta, \varepsilon^{2k+1}w) \tag{6.19}$$

extends smoothly across $\varepsilon = 0$, for w in any fixed bounded subset. Taylor factorization gives a factor ε^{4k+2} from $g(t,0)$; the mixed linear term has exponent at least

$$2(k+1) + (2k+1) = 4k+3;$$

and the nonlinear quadratic remainder has exponent at least $4k+4$. Cubic terms in z have still higher exponent because $k \geq 1$. Hence

$$\Psi(0, \theta, w) = P_{2k+1}(\theta) + Q(w), \tag{6.20}$$

where P_{2k+1} is the degree-$2k+1$ Taylor term of $g(t,0)$.

The restriction of Q to $z_0 + W_0$ has a smooth local inverse σ near target zero, with $\sigma(0) = z_0$. Choose a constant M large enough that $-P_{2k+1}(\theta)/M^2$ lies in a relatively compact subset of this one inverse neighborhood for all θ. Then

$$w_0(\theta) = M\sigma(-P_{2k+1}(\theta)/M^2)$$

is smooth, lies in $Mz_0 + W_0$, satisfies $Q(w_0) = -P_{2k+1}(\theta)$, and has invertible derivative on W_0. On the compact graph of w_0, set $L_\theta = D_w\Psi(0, \theta, w_0(\theta))|_{W_0}$. The continuous function $\theta \mapsto L_\theta^{-1}$ has a finite norm bound M_0. For $\xi \in W_0$, put $T_{\varepsilon,\theta}(\xi) = \xi - L_\theta^{-1}\Psi(\varepsilon, \theta, w_0(\theta) + \xi)$. Uniform continuity on the compact graph gives $\rho, \varepsilon_0 > 0$ such that

$$\sup_{|\varepsilon|<\varepsilon_0,\, \theta,\, |\xi|\leq\rho} \|D_\xi T_{\varepsilon,\theta}\| \leq \tfrac{1}{2}, \qquad |T_{\varepsilon,\theta}(0)| \leq \rho/2.$$

The fixed-point theorem gives a smooth family

$$w(\varepsilon, \theta) \in Mz_0 + W_0, \qquad \Psi(\varepsilon, \theta, w(\varepsilon, \theta)) = 0,$$

for all sufficiently small $|\varepsilon|$. The fixed point lies in $\overline{B}_\rho(0) \subset W_0$ and is unique there for each (ε, θ). Differentiating the fixed-point equation in local sphere coordinates, with $(I - D_\xi T)^{-1}$ bounded by 2, gives smoothness and agreement on coordinate overlaps.

For $t \neq 0$ set

$$z(t) = |t|^{k+1/2} w(\sqrt{|t|}, t/|t|), \qquad h(t) = p(t) + z(t) + x(t, z(t)), \tag{6.21}$$

and set $h(0) = 0$. Both components of $F(h) = f$ now hold exactly. All values belong to the coefficient span of p, plus $X + Z_*$, which proves the dimension bound.

The expression $h - p$ is of the form

$$|t|^{k+1/2} U(\sqrt{|t|}, t/|t|)$$

with U smooth on a compact polar cylinder. Indeed, $x(t,0) = O(|t|^{2k+1})$ and

$$x(t,z) - x(t,0) = O((|t| + |z|)|z|),$$

with corresponding smooth Taylor factorizations. Put $\rho = |t|$, $\varepsilon = \sqrt{\rho}$, and $\theta = t/\rho$. Let V_j be the tangential vector field $e_j - \theta_j\theta$ on S^{r-1}. For $\gamma \in \mathbb{R}$,

$$\partial_{t_j}\big(\rho^\gamma U(\sqrt{\rho}, \theta)\big) = \rho^{\gamma-1}\left(\gamma\theta_j U + \tfrac{1}{2}\varepsilon\theta_j\partial_\varepsilon U + V_j U\right).$$

For $r = 1$, $S^0 = \{-1, 1\}$ and $V_1 = 0$. Induction gives

$$D^\beta(h - p)(t) = \rho^{k+1/2-|\beta|} U_\beta(\sqrt{\rho}, \theta),$$

with U_β smooth on a compact cylinder. This proves (6.13). For $|\beta| \le k$, the right side tends to zero. Applying the definition of a derivative successively at zero shows that $h - p$ is C^k there and $j_0^k(h - p) = 0$.

For a multi-index α with $|\alpha| = k$, write $R(t) = D^\alpha(h - p)(t)$ and set $R(0) = 0$. The derivative bounds give $|R(t)| \le C|t|^{1/2}$ and $\|DR(t)\| \le C|t|^{-1/2}$ off zero. Let $\rho = \max(|t|, |t'|)$ and $d = |t - t'|$. If $d \ge \rho/2$, then $|R(t) - R(t')| \le 2C\rho^{1/2} \le 2\sqrt{2}Cd^{1/2}$. If $0 < d < \rho/2$, every point on the joining segment has norm at least $\rho/2$, and integration gives

$$|R(t) - R(t')| \le \sqrt{2}C\rho^{-1/2}d \le \sqrt{2}Cd^{1/2}.$$

Since p is polynomial, on a fixed compact parameter ball

$$|D^\alpha p(t) - D^\alpha p(t')| \le C_p|t - t'|.$$

After restricting to a ball of diameter at most one, this is at most $C_p|t - t'|^{1/2}$. Adding the estimate for R proves that $D^\alpha h$ is pairwise $1/2$-Hölder.

Finally, along (6.21), the reduced derivative on W_0 is

$$D_z g(t, z(t))|_{W_0} = |t|^{k+1/2} DQ(w(\sqrt{|t|}, t/|t|))|_{W_0} + O(|t|^{k+1}).$$

After division by $|t|^{k+1/2}$ the error tends to zero. Invertibility on the compact limiting branch gives a right inverse bounded by $C|t|^{-k-1/2}$. The range block remains invertible, and its Schur complement is precisely this reduced derivative. Therefore the unreduced derivative on V_* has a right inverse of norm at most $C|t|^{-k-1/2}$. □

The image of $h - p$ is contained in the fixed space $X \oplus Z_*$.

6.6. **Tangent cones and sharpness.**

Proof of Theorem 6.8. If $d = 0$, the chain rule makes $AL = Df(0)$ necessary, and the second compatibility condition is empty. Assume the first condition. Choose $E = X \oplus K$ with $A|_X : X \to W$ invertible and write $L = L_X + L_K$ according to this decomposition. Solve

$$F(x(t) + L_K t) = f(t), \qquad x(t) \in X,$$

by the implicit function theorem. Differentiation at zero gives $A\,Dx(0) = Df(0) = AL_X$, hence $Dx(0) = L_X$. Thus $h(t) = x(t) + L_K t$ has derivative L; its restricted derivative on X remains onto. If L is injective, projection onto its finite-dimensional image and the inverse function theorem make h a local embedded graph. The same coordinates identify the zero fibre with a smooth graph over K, whose tangent cone is K. Henceforth suppose $d > 0$.

If $h(t) = Lt + o(|t|)$ and $F(h(t)) = f(t)$, the first chain rule gives (6.14). Since $P_C DF(0) = 0$ on the entire source,

$$P_C F(h(t)) = \tfrac{1}{2} P_C D^2 F(0)[Lt, Lt] + o(|t|^2).$$

Comparison with the smooth Taylor expansion of f and polarization gives (6.15). C^1 regularity of h suffices for this expansion.

Conversely, let $L_R : R \to X$ be a right inverse of A. Define

$$p(t) = Lt + \tfrac{1}{2} L_R P_R \big(D^2 f(0)[t, t] - D^2 F(0)[Lt, Lt]\big).$$

The two compatibility conditions imply $j^2(F \circ p - f) = 0$. Apply Theorem 6.7 with $k = 1$. The resulting lift has derivative L at zero and is $C^{1,1/2}$. If L is injective, a bounded linear projection to its finite-dimensional tangent image and the ordinary inverse function theorem show that the resulting family is a local embedded graph.

For the contingent cone, suppose $x_n \in Z$, $t_n \downarrow 0$, and $x_n/t_n \to v$ in the Banach norm. Dividing $F(x_n) = 0$ by t_n gives $Av = 0$. Dividing its cokernel Taylor expansion by t_n^2 gives $Q(v) = 0$. Conversely, $Av = Q(v) = 0$ meets the compatibility conditions of Theorem 6.8 for the constant target and $L(t) = tv$. The produced two-sided curve realizes v, and its positive-time restriction gives a contingent sequence. This proves (6.16). □

Proposition 6.9 (Optimal parameter exponent). *For every integer $k \geq 1$ there is a real-analytic map satisfying* (4.3)*, a prescribed polynomial p, and a target satisfying* (6.11)*, for which no exact $C^{k,\alpha}$ lift with the same k-jet exists for any $\alpha \in (1/2, 1]$.*

Proof. On $E = \mathbb{R} \oplus \ell^2 \oplus \ell^2$, define

$$F_k(s, x, y) = s^{2k+1} + \|x\|^2 - \|y\|^2, \qquad p(t) = (t, 0, 0), \qquad f(t) = 0.$$

The derivative vanishes at zero and the intrinsic quadratic form is $\|x\|^2 - \|y\|^2$, with both indices infinite. The residual $F_k(p(t)) = t^{2k+1}$ has zero $2k$-jet.

If an exact lift with the same k-jet were $C^{k,\alpha}$, Taylor's estimate would give

$$s(t) = t + O(|t|^{k+\alpha}), \qquad x(t), y(t) = O(|t|^{k+\alpha}).$$

For $\alpha > 1/2$, substitution yields $F_k(h(t)) = t^{2k+1} + o(|t|^{2k+1})$, a contradiction. The exponent $1/2$ is attained by $s = t$, $y = t^{k+1/2}e_1$, $x = 0$ for $t \geq 0$, and $x = (-t)^{k+1/2}e_1$, $y = 0$ for $t < 0$. □

Under (4.3), the regular zeros of Q are dense in $Q^{-1}(0)$. For $d = 0$ every point is regular. For $d > 0$, fix $v \in K$ with $Q(v) = 0$ and put

$$H_v = \{w \in K \,:\, B(v, w) = 0\}.$$

Its codimension is at most d, so Corollary 4.2 gives $z \in H_v$ and $W_0 \subset H_v$ with $Q(z) = 0$ and $DQ(z)|_{W_0}$ onto C. For $\varepsilon \neq 0$,

$$Q(v + \varepsilon z) = 0, \qquad DQ(v + \varepsilon z)|_{W_0} = \varepsilon DQ(z)|_{W_0}.$$

Thus $v + \varepsilon z$ is regular and tends to v. The tangent-cone identity (6.16) also follows from this density and the 2-regularity theorem of [36, Theorem 2.1 and Remark 2.1].

6.6.1. *The spectral tangent-cone statement.* In a fixed spectral chart let $F = \mathsf{L} - \lambda I_m$ and let C be the cokernel of its allowed derivative. By Theorem 3.4, at a smooth reference the exact-level tangent cone is

$$T_g\{\mathsf{L} = \lambda I_m\} = \left\{ h \,:\, D\mathsf{L}_g h = 0,\ \left\langle c, D^2\mathsf{L}_g[h, h] \right\rangle = 0 \text{ for every } c \in C \right\}. \tag{6.22}$$

For smooth h, the realizing curve may consist of smooth metrics or potentials and be $C^{1,1/2}$ in the parameter with values in each spatial C^s norm.

For the multiplicity locus, put

$$\Gamma = \mathsf{L} - \tfrac{\operatorname{tr}\mathsf{L}}{m} I_m \in \operatorname{Sym}^0_m .$$

The identical conclusion holds with Γ in (6.22) and its own derivative cokernel. Infinite inertia is still available: an annihilator of $D\Gamma$ is a trace-free annihilator of $D\mathsf{L}$, and $\ker D\mathsf{L} \subset \ker D\Gamma$. For $\lambda > 0$, scaling in the full conformal or full metric space gives $D\mathsf{L}_g[g] = -\lambda I_m$.

7. Spectral bundles and holonomy

Let Π_v be the projection onto an isolated spectral cluster and let U_v be its Riesz–Gram frame. The connection $\nabla = \Pi_v d$ has connection form $\omega = U_v^* dU_v$ and curvature $\Omega = d\omega + \omega \wedge \omega$. For a separated part of the cluster, the corresponding matrix spectral projection defines a subbundle of the trivial bundle with frame U_v.

7.1. **Holonomy on an exact spectral level.** Consider either the localized potential family of Theorem 5.1, or the full conformal class of a closed manifold at a positive eigenvalue. Assume that the eigenspace has dimension $m \geq 2$. Let a and d be the dimensions of the derivative range and cokernel.

Theorem 7.1 (Full local holonomy at the reference). *There is one finite-dimensional perturbation space $W \subset E_0$ with the following property. In every sufficiently small parameter neighborhood of zero, every $T \in \mathrm{SO}(m)$ is the holonomy of a smooth loop $v(s) \in W$, based at zero, satisfying*

$$\mathsf{L}(v(s)) = \lambda I_m \qquad (0 \leq s \leq 1). \tag{7.1}$$

The loops can be chosen inside an embedded m-parameter family through the reference satisfying (7.1). *They are nullhomotopic within that same exact spectral level. In the potential case their support is in a single compact $K_O \Subset O$. One can take*

$$\dim W = a + m(d+1) \quad (d > 0), \qquad \dim W = \dim \mathrm{Sym}_m + m \quad (d = 0).$$

The restricted holonomy of the connection on every sufficiently small ball in this parameter family is $\mathrm{SO}(m)$.

Since U_v is a real orthonormal frame, $\omega(v)[w]^T = -\omega(v)[w]$. Parallel transport is therefore orthogonal; its determinant is 1 along a path beginning at I_m. For $m = 1$, $\omega = 0$ and the local holonomy is trivial.

Lemma 7.2 (Small rectangles). *Let a smooth metric connection on a trivial rank-m bundle over a ball in $\mathbb{R}^r$ have curvature values $\Omega_{ij}(0)$. Let $H_{ij}(a,b)$ be its parallel transport around the coordinate rectangle based at zero with sides ae_i and be_j, with one fixed orientation. Then*

$$H_{ij}(a,b) = I - ab\Omega_{ij}(0) + O(|ab|(|a|+|b|)). \tag{7.2}$$

In particular, $H_{ij}(a,0) = H_{ij}(0,b) = I$ and

$$a^{-1}\partial_b H_{ij}(a,0) \longrightarrow -\Omega_{ij}(0).$$

The sign reverses upon reversing orientation.

Proof. Write $\omega = \sum_k \omega_k(x)\,dx_k$. For a segment from x to $x + ce_k$, let $V_k(c;x)$ be the value at $t = 1$ of

$$\frac{d}{dt}V(t) = -c\omega_k(x + tce_k)V(t), \qquad V(0) = I_m.$$

Iteration of this integral equation gives the convergent series of ordered integrals. On a compact parameter set its term of order n has norm at most $(M|c|)^n/n!$. Differentiating any fixed finite number of times gives a locally uniformly convergent series as well. Thus V_k is smooth in (c,x), with

$$V_k(0;x) = I_m, \qquad \partial_c V_k(0;x) = -\omega_k(x).$$

For the oriented boundary $0 \to ae_i \to ae_i + be_j \to be_j \to 0$,

$$H_{ij}(a,b) = V_j(b;0)^{-1}V_i(a;be_j)^{-1}V_j(b;ae_i)V_i(a;0).$$

This formula gives $H_{ij}(a,0) = H_{ij}(0,b) = I_m$. At $(a,b) = (0,0)$ the mixed derivatives of the middle two factors contribute $\partial_j\omega_i - \partial_i\omega_j$. The products of their first derivatives and those of the outer factors sum to $\omega_j\omega_i - \omega_i\omega_j$. Therefore

$$\partial_a\partial_b H_{ij}(0,0) = -\partial_i\omega_j + \partial_j\omega_i - [\omega_i,\omega_j] = -\Omega_{ij}(0).$$

The two axis identities imply the exact formula

$$H_{ij}(a,b) - I_m = ab\int_0^1\int_0^1 (\partial_a\partial_b H_{ij})(sa,tb)\,ds\,dt.$$

A bound on its third derivatives now gives $\|H_{ij}(a,b) - I_m + ab\Omega_{ij}(0)\| \leq C|ab|(|a|+|b|)$. Differentiating at $b = 0$ gives $a^{-1}\partial_b H_{ij}(a,0) = -\Omega_{ij}(0) + O(|a|)$. The reverse loop has inverse transport; its leading mixed term has the opposite sign. □

Proof of Theorem 7.1. In Corollary 4.11, take $r = m$ and prescribe $\mathscr{R}_{ij} = E_{ij} - E_{ji}$ for $i < j$. Let h be the resulting embedded family with $F(h(t)) = 0$. Fix a parameter ball B_ρ whose image lies in the required source neighborhood and in the domain of the isolating contour. Put $p = m(m-1)/2$ and order the pairs $i < j$. For $0 < |a| < \rho/4$, let $\Phi_a(b)$ be the product, in that order, of the rectangle holonomies with sides ae_i and $b_{ij}e_j$. For all sufficiently small b these rectangles lie in B_ρ and $\Phi_a(0) = I_m$. Lemma 7.2 gives

$$a^{-1}D_b(\log\Phi_a)(0)[\xi] = -\sum_{i<j}\xi_{ij}(E_{ij} - E_{ji}) + O(|a|)|\xi|.$$

The first map on the right is an isomorphism $\mathbb{R}^p \to \mathfrak{so}(m)$. Choose a so small that the error composed with its inverse has norm less than $1/2$. Then $D_b(\log\Phi_a)(0)$ is invertible. The finite inverse theorem gives a neighborhood $\mathcal{N}$ of I_m all of whose elements are holonomies of the indicated finite concatenations.

Every $T \in SO(m)$ has a real skew logarithm B: orthogonal decomposition writes it as planar rotation blocks and identity blocks, pairing the even number of -1 eigenvalues into rotations of angle π. Choose N with $\exp(B/N) \in \mathcal{N}$. Concatenating N realizing loops gives holonomy $\exp(B/N)^N = T$. All parameter loops remain in B_ρ. The map $(u, s) \mapsto h((1-u)\gamma(s))$ contracts each such based loop in the exact spectral level.

Reparametrize each segment by

$$\vartheta(t) = \frac{\displaystyle\int_0^t e^{-1/(s(1-s))}\,ds}{\displaystyle\int_0^1 e^{-1/(s(1-s))}\,ds}, \qquad 0 \le t \le 1,$$

where the integrand is extended by zero at the endpoints. This function is increasing on $(0,1)$ and all its positive-order derivatives vanish at $0, 1$. Consequently all derivatives agree at concatenation points. The transport equation is invariant under this increasing reparametrization, so the holonomy is unchanged. Every loop takes values in W, and $\operatorname{supp} v(s) \subset K_O$ in the potential case. The dimension of W is the one specified in the theorem. Finally, $H^TH = I_m$ along every transport and $H(0) = I_m$ imply $H \in SO(m)$. Consequently the restricted holonomy group is $\mathrm{SO}(m)$. □

7.2. **Amplitude and total variation of realizing loops.** Write $\|\cdot\|_E = \|\cdot\|_{L^\infty}$ for potential perturbations and $\|\cdot\|_E = \|\cdot\|_{C^k}$, with a fixed sufficiently large $k \ge 4$, for conformal parameters. For a smooth loop v, let

$$\mathrm{Var}_E(v) = \int_0^1 \|\dot v(s)\|_E \; ds.$$

For a fixed $T \in \mathrm{SO}(m) \setminus \{I\}$, define $\ell_T(\epsilon)$ as the infimum of this variation over smooth based loops satisfying

$$\sup_s \|v(s)\|_E \le \epsilon, \qquad F(v(s)) = 0, \qquad \mathrm{Hol}(v) = T.$$

Let $\ell_T^W(\epsilon)$ denote the same infimum with the additional condition $v(s) \in W$, where W is the space in Theorem 7.1. Then $\ell_T(\epsilon) \le \ell_T^W(\epsilon)$.

Theorem 7.3 (Amplitude and total variation). *For each fixed nonidentity T, there are constants $c_T, C_T, \epsilon_T > 0$ such that*

$$\frac{c_T}{\epsilon} \le \ell_T(\epsilon) \le \ell_T^W(\epsilon) \le \frac{C_T}{\epsilon} \qquad (0 < \epsilon < \epsilon_T). \tag{7.3}$$

The constants depend on the reference operator, the support condition, T, and the norm $\|\cdot\|_E$.

Proof. On a sufficiently small Banach parameter ball, the connection form is continuously differentiable. The Riesz–Gram frame satisfies $U_0^*DU(0) = 0$ by the proof of Lemma 2.3, so $\omega(0) = 0$. By

continuity of $D\omega$ at zero, choose a smaller ball with $\|D\omega\| \le M$ there. Integration on the segment from 0 to v yields

$$\|\omega(v)[w]\|_{\mathrm{op}} \le M\|v\|_E\|w\|_E.$$

For a loop of amplitude at most ϵ, orthogonality of its transport gives

$$\|T - I_m\|_{\mathrm{op}} \le \int_0^1 \|\dot{H}(s)\|_{\mathrm{op}}\, ds \le M\epsilon\, \mathrm{Var}_E(v). \tag{7.4}$$

Thus one may take $c_T = \|T - I_m\|_{\mathrm{op}}/M > 0$.

For the upper bound, let h be the family in Theorem 7.1, with the specified curvature basis. For each pair take rectangle side lengths δ and δs_{ij}, and let $\Phi(\delta, s)$ be the product of their holonomies. The double integral formula in the proof of Lemma 7.2 writes each factor as $I_m + \delta^2 F_{ij}(\delta, s_{ij})$, with F_{ij} smooth. A finite product has the form $I_m + \delta^2 F_0(\delta, s)$. The convergent series

$$\log(I_m + X) = \sum_{n\ge1} \frac{(-1)^{n+1}}{n} X^n$$

therefore shows that $L(\delta, s) = \delta^{-2}\log\Phi(\delta, s)$ extends smoothly to $\delta = 0$. Its limiting map is

$$L(0, s) = -\sum_{i<j} s_{ij}(E_{ij} - E_{ji}) =: L_0 s, \qquad L(\delta, 0) = 0.$$

Choose $r > 0$ and $\delta_0 > 0$ such that

$$\sup_{|s|\le r,\ |\delta|\le\delta_0} \|I - L_0^{-1} D_s L(\delta, s)\| \le \tfrac{1}{2}.$$

This follows from continuity at the compact set $\{0\} \times \overline{B}_r$, since the derivative there is exactly L_0. Lemma A.2 gives a fixed $c > 0$ such that every $Z \in \mathfrak{so}(m)$ with $|Z| < c$ equals $L(\delta, s)$ for some $|s| < r$, for every $0 < \delta < \delta_0$. Thus $\exp(\delta^2 Z)$ is attained exactly.

Let $p = m(m-1)/2$ and bound $\|Dh\|_{\mathbb{R}^m \to E}$ by M_h on the chosen parameter ball. Each concatenation has parameter amplitude at most $\delta\sqrt{1+r^2}$ and parameter length at most $2p\delta(1+r)$. Its image under h therefore has amplitude at most $A_0\delta$, $A_0 = M_h\sqrt{1+r^2}$, and variation at most $B_0\delta$, $B_0 = 2pM_h(1+r)$. Fix a skew logarithm B of T and set

$$N_\delta = 1 + \left\lceil \frac{2\|B\|}{c\delta^2} \right\rceil.$$

Then $Z = B/(N_\delta\delta^2)$ has norm less than c, so a block loop realizes $\exp(B/N_\delta)$. Repeating it N_δ times realizes T, without increasing amplitude, and with variation bounded by

$$B_0\delta N_\delta \le B_0\left(2\delta + \frac{2\|B\|}{c\delta}\right).$$

For sufficiently small ϵ, take $\delta = \epsilon/(2\max(A_0, 1))$. Thus the total variation is at most C_T/ϵ. The increasing endpoint-flat reparametrization preserves $\int \|\dot{v}\|_E$ by change of variables, and hence produces smooth loops with the same bound. □

7.3. **Two reference eigenspaces.**

7.3.1. *Nine localized potential perturbations on the round two-sphere.* Let $P = \Delta_{S^2}$ for the unit-round metric and consider its eigenvalue 2, of multiplicity three. For every preassigned nonempty open $O \subset S^2$, the derivative of the cluster map on $C_c^\infty(O)$ is onto $\mathrm{Sym}_3(\mathbb{R})$. Take the normalized coordinate eigenfunctions $u_i = cx_i$. A symmetric annihilator would satisfy $x^T Cx = 0$ on O. This polynomial is real analytic on the sphere, hence vanishes on the whole sphere; evaluation on unit vectors gives $C = 0$.

Thus $D = 6$ and $d = 0$. Proposition 3.5 and Theorem 4.10 give a nine-dimensional space

$$W \subset C_c^\infty(O)$$

and an analytic embedded three-parameter family with the eigenvalue 2 of multiplicity exactly three, with all SO(3) holonomies on arbitrarily small based loops. The nine perturbations have support in a common compact subset of O.

For every $\ell \geq 1$, Lemma C.1 gives a surjective potential derivative on the degree-ℓ eigenspace. Theorem 4.10 therefore gives an m-parameter family with prescribed curvature in a space of dimension $m(m+3)/2$, where $m = 2\ell + 1$.

7.3.2. *An unstable eight-dimensional eigenspace of the square torus.* For the square $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the eigenvalue 5 has multiplicity eight. The conformal derivative has corank three, by (B.3). Hence $D = 36$, $a = 33$, $d = 3$. Taking $r = m = 8$ gives

$$\dim W = 33 + 8(3+1) = 65.$$

By Theorems 3.4 and 7.1, a fixed 65-dimensional conformal perturbation space contains an analytic eight-parameter exact-level family through the flat reference, whose local holonomy is all of SO(8).

7.4. **Fixed volume density on round spheres.** For $n \geq 2$, the centered first-cluster map on the round S^n is a submersion under metric perturbations with fixed volume density.

7.4.1. *A centered submersion with fixed volume density.* Let $n \geq 2$, $m = n+1$, and let γ be the unit-round metric on $S^n \subset \mathbb{R}^m$. Fix $a > 0$ and put $g_* = a\gamma$. Its first positive eigenvalue is $\lambda_* = n/a$, with eigenspace spanned by

$$u_i = \left(\frac{m}{\operatorname{Vol}(S^n, g_*)}\right)^{1/2} x_i, \qquad 1 \leq i \leq m. \tag{7.5}$$

The next eigenvalue is $2(n+1)/a$. For a homogeneous harmonic polynomial $r^\ell Y(\omega)$, the separation formula $\sum_{j=1}^{n+1} \partial_{x_j}^2 = \partial_r^2 + nr^{-1}\partial_r - r^{-2}\Delta_\gamma$ gives $\Delta_\gamma Y = \ell(\ell+n-1)Y$. The Euclidean operator on the left is the sum of second partial derivatives; it is the negative of the positive Euclidean Laplacian.

These eigenfunctions span a dense subspace of $L^2(S^n)$. If P is homogeneous of degree ℓ, let $D = \sum_j \partial_{x_j}^2$, $r^2 = \sum_j x_j^2$, and set

$$H = \sum_{j=0}^{\lfloor \ell/2 \rfloor} a_j r^{2j} D^j P, \quad a_0 = 1, \quad a_j = -\frac{a_{j-1}}{2j(2\ell+n-1-2j)}. \tag{7.6}$$

The identity

$$D(r^{2j}D^jP) = 2j(2\ell+n-1-2j)r^{2j-2}D^jP + r^{2j}D^{j+1}P$$

shows $DH = 0$ by cancellation of consecutive terms. The difference $P - H$ is divisible by r^2; induction on ℓ decomposes the restriction of P into restrictions of harmonic homogeneous polynomials. Polynomial restrictions are dense in $C(S^n)$ by Stone–Weierstrass, and hence in L^2. There are no other eigenspaces. Indeed, an eigenfunction with eigenvalue ν different from every $\ell(\ell+n-1)$ is orthogonal, by self-adjointness, to every restricted harmonic polynomial. Density would make it zero. If $\nu = \ell(\ell+n-1)$, subtract its orthogonal projection onto that finite-dimensional polynomial space. The remainder is still a ν-eigenfunction, is orthogonal to that space by construction and to every other polynomial eigenspace by self-adjointness, and is consequently zero. Thus these are the complete eigenspaces. The values $\ell(\ell+n-1)$ increase strictly for integers $\ell \geq 0$. Scaling the metric by a divides them by a, giving the first two positive eigenvalues and their multiplicities. Reflection in a coordinate hyperplane gives $\int x_i x_j = 0$ for $i \neq j$. Coordinate permutations and $\sum_i x_i^2 = 1$ give $\int x_i^2 = \operatorname{Vol}(S^n, g_*)/m$, which gives the orthonormality in (7.5). For $h \in C_c^\infty(O; S_0^2 T^* S^n)$, where $O \subset S^n$ is nonempty and open, (2.22) becomes

$$D\mathsf{L}(g_*)[h]_{ij} = -\int_{S^n} h(\nabla u_i, \nabla u_j)\, dv_{g_*}. \tag{7.7}$$

Moreover, $\sum_i du_i \otimes du_i = m\gamma/\operatorname{Vol}(S^n, g_*)$ is a constant multiple of g_*. Therefore $\operatorname{tr} D\mathsf{L}(g_*)[h] = 0$, and $D\Gamma(g_*)[h] = D\mathsf{L}(g_*)[h]$.

Lemma 7.4. *The map $D\Gamma(g_*) : C_c^\infty(O; S_0^2 T^*S^n) \to \mathrm{Sym}_m^0$ is onto for every nonempty open O.*

Proof. If $C \in \mathrm{Sym}_m^0$ annihilates the map, set $T_C = \sum C_{ij} du_i \otimes du_j$. Testing (7.7) gives $T_C^{\mathrm{TF}} = 0$ on O. Up to the fixed normalization in (7.5), this says that the restriction of the ambient form C to $x^\perp$ is scalar for every $x \in O$. Let $P_x = I - x \otimes x$. The tensor condition is

$$P_x C P_x - \frac{\operatorname{tr} C - x^T C x}{n} P_x = 0. \tag{7.8}$$

Its entries are polynomial restrictions to S^n. Fix a point of O. Along a great circle through that point each entry is a trigonometric polynomial, vanishes on an interval, and thus vanishes on the whole circle. Every point of S^n lies on a great circle through the fixed point. Hence (7.8) holds on all of S^n. For distinct i, j, choose $k \neq i, j$; this is possible because $m \geq 3$. At $x = e_k$, the vectors e_i, e_j are tangent. Equation (7.8) gives $C_{ij} = 0$ and $C_{ii} = C_{jj}$. Thus $C = cI_m$. Its trace is zero, so $C = 0$. A map to a finite-dimensional space with trivial annihilator is onto. □

Choose $\chi \in C_c^\infty(O)$, $\chi \geq 0$, positive on some nonempty open subset. On Sym_m^0 define the operator G by

$$\langle GC, D \rangle = \int_{S^n} \chi \left\langle T_C^{\mathrm{TF}}, T_D^{\mathrm{TF}} \right\rangle_{g_*} dv_{g_*}. \tag{7.9}$$

If its value on (C, C) is zero, T_C^{TF} vanishes on a nonempty open set and the preceding proof gives $C = 0$. Thus $G > 0$. Put

$$R(C) = -\chi T^{\mathrm{TF}}_{G^{-1}C}. \tag{7.10}$$

For every $D \in \mathrm{Sym}_m^0$, contraction with (7.7) gives $\langle D\Gamma(g_*)R(C), D\rangle = \langle C, D\rangle$; hence $D\Gamma(g_*)R(C) = C$. For a small $B \in \mathrm{Sym}_m^0$, define

$$g(B)(X, Y) = g_*\big(e^{g_*^{-1}R(B)}X, Y\big). \tag{7.11}$$

The exponent is self-adjoint and trace-free. Its exponential is positive and self-adjoint, and $\det e^{g_*^{-1}R(B)} = e^{\operatorname{tr}(g_*^{-1}R(B))} = 1$. Consequently $dv_{g(B)} = dv_{g_*}$. Outside $\operatorname{supp}\chi$ the exponent is zero and $g(B) = g_*$. Proposition A.1 gives smoothness, support in $\operatorname{supp}\chi$, and the derivative at zero. The map $B \mapsto \Gamma(g(B))$ has derivative I at zero. Lemma A.2 gives a smooth local inverse and hence a family

$$A \longmapsto \widehat{g}(A), \qquad \Gamma(\widehat{g}(A)) = A, \qquad dv_{\widehat{g}(A)} = dv_{g_*}, \qquad \widehat{g}(A) = g_* \text{ off } \operatorname{supp}\chi. \tag{7.12}$$

It is defined on a full ball around zero in Sym_m^0. Its image converges to g_* in every C^k norm as $A \to 0$.

Theorem 7.5. *Let B be compact and smooth and $E \subset B \times \mathbb{R}^{n+1}$ a smooth rank-r subbundle, $1 \leq r \leq n$. For every sufficiently small fixed $\varepsilon > 0$, the metrics*

$$g_b = \widehat{g}\big(\varepsilon(rI_m - mP_b)\big) \tag{7.13}$$

have a rank-r spectral bundle isomorphic to E, corresponding to the first r positive eigenvalues counted with multiplicity, where P_b is the orthogonal projection onto E_b. They have the fixed volume density and support properties of (7.12).

Proof. The cluster matrix is

$$\mathsf{L}(g_b) = c(b)I_m + \varepsilon(rI_m - mP_b). \tag{7.14}$$

Its eigenvalues are $c(b) - (m-r)\varepsilon$ with multiplicity r and $c(b) + r\varepsilon$ with multiplicity $m - r$. Compactness of the set of rank-r projections gives a common parameter range. The original cluster is the first m positive eigenvalues and is separated from zero and from $2(n+1)/a$. For sufficiently small parameters it retains those properties.

Choose $\eta > 0$ smaller than half of both n/a and $(n+2)/a$. Choose two rectangular contours with real intersections $(-1, n/a - \eta)$ and $(-1, n/a + \eta)$. Their reference ranks are 1 and $m+1$. The resolvent construction in Section 2 retains these ranks for small parameters. The Laplacian is nonnegative, and

$$\Delta_{g_b} u = 0 \implies \int_{S^n} |du|^2_{g_b}\, dv_{g_b} = 0 \implies u \text{ is constant.}$$

Thus precisely one eigenvalue, the simple zero eigenvalue, lies below the chosen cluster. The cluster still occupies indices $1, \dots, m$. The frame (2.11) identifies its lower eigenspaces with $\operatorname{im} P_b = E_b$, smoothly over B. □

Every smooth real vector bundle of positive finite rank over a compact smooth manifold embeds in a finite-dimensional trivial bundle. To construct an embedding, choose finitely many local trivializations and smooth functions ϕ_i supported in their domains, with $\sum_i \phi_i^2 = 1$. Such functions are obtained by normalizing a finite family of subordinate bump functions with positive sum of squares. Send a vector to the tuple of its local coordinates multiplied by ϕ_i, extending each component by zero. At least one multiplier is nonzero at every point, so this map is fibrewise injective. Adding zero coordinates if necessary gives ambient dimension at least three and strictly larger than the rank. Theorem 7.5 therefore realizes any such bundle after increasing the sphere dimension.

7.4.2. *Equivariance.* Let $G_0 \subset O(n+1)$ be a compact group preserving χ. For $Q \in G_0$, the metric action is $g \mapsto (Q^{-1})^* g$, and the function action is $\mathcal{T}_Q f = f \circ Q^{-1}$. If $U_* v = \big(m/\operatorname{Vol}(S^n, g_*)\big)^{1/2} \langle v, x \rangle$, then

$$(\mathcal{T}_Q U_* v)(x) = \left(\frac{m}{\operatorname{Vol}(S^n, g_*)} \right)^{1/2} \langle v, Q^{-1} x \rangle = (U_* Q v)(x).$$

Thus $\mathcal{T}_Q U_* = U_* Q$ as operators from $\mathbb{R}^m$ to $L^2(S^n, dv_{g_*})$. For the metric action $(Q^{-1})^*$, the identity $\mathcal{T}_Q U_* = U_* Q$ gives $(Q^{-1})^* T_C = T_{QCQ^{-1}}$. It commutes with the trace-free projection because Q is an isometry of g_*. Invariance of χ and the measure in (7.9) gives $G(QCQ^{-1}) = QG(C)Q^{-1}$. Its inverse has the same property. Hence $R(QCQ^{-1}) = (Q^{-1})^* R(C)$, and the exponential family (7.11) is equivariant. Volume densities are fixed in this family, so all functions belong to the same L^2. The Laplacian and its projection transform by $\mathcal{T}_Q$. Substitution of $\mathcal{T}_Q U_* = U_* Q$ into (2.9) gives

$$U_{(Q^{-1})^* g} = \mathcal{T}_Q U_g Q^{-1}, \qquad \mathsf{L}((Q^{-1})^* g) = Q \mathsf{L}(g) Q^{-1}. \tag{7.15}$$

The Gram square root satisfies $(QBQ^{-1})^{-1/2} = QB^{-1/2}Q^{-1}$ for positive B, which follows by diagonalizing B. Choose invariant matrix balls on which $B \mapsto \Gamma(g(B))$ is injective and its image contains a smaller invariant ball.

Such balls exist because the derivative at zero is the identity and the Frobenius norm is invariant under orthogonal conjugation. First choose a Frobenius ball within the inverse domain, then choose a smaller target ball contained in its open image. Every conjugate of a source point in the first ball remains in that ball. If B is the preimage of A, equivariance makes QBQ^{-1} the preimage of QAQ^{-1}. Uniqueness gives

$$\widehat{g}(QAQ^{-1}) = (Q^{-1})^* \widehat{g}(A). \tag{7.16}$$

For $O = S^n$ and $\chi \equiv 1$, the family is $O(n+1)$-equivariant. Since $O(n+1)$ acts transitively on the rank-r orthogonal projections, the metrics associated with any such projection family are pairwise isometric and have the same spectrum, including multiplicities.

7.4.3. *An isometry family with fixed exterior on the four-sphere.* Write $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$, with coordinates (z, s), and put $g_* = \frac{1}{2}\gamma$. Then

$$\operatorname{Ric}_{g_*} = 6 g_*, \qquad \lambda_1(g_*) = \cdots = \lambda_5(g_*) = 8, \qquad \lambda_6(g_*) = 20. \tag{7.17}$$

Fix any polar cap $O = \{s > a\}$, $-1 < a < 1$. Choose a nonnegative nonzero cutoff $\chi(s)$ with compact support in O. The action of $U(2)$ on z, fixing s, preserves χ and g_*. Its real representation is contained in $SO(5)$. For $\ell \in \mathbb{CP}^1$, let $p_\ell : \mathbb{C}^2 \to \ell$ be Hermitian orthogonal projection. Regard it as a real rank-two projection and put

$$P_\ell = p_\ell^{\mathbb{R}} \oplus I_{\mathbb{R}}, \qquad A_{\varepsilon,\ell} = \varepsilon(3I_5 - 5P_\ell), \qquad g_{\varepsilon,\ell} = \widehat{g}(A_{\varepsilon,\ell}). \tag{7.18}$$

The target has trace zero, eigenvalue -2ε on $\operatorname{im} P_\ell$, and eigenvalue 3ε on its orthogonal complement. The family is smooth in ℓ, since the projection in homogeneous coordinates is $zz^*/|z|^2$. The group

$U(2)$ acts transitively on $\mathbb{CP}^1$ and satisfies $P_{Q\ell} = QP_\ell Q^{-1}$. Equation (7.16) gives $g_{\varepsilon,Q\ell} = (Q^{-1})^* g_{\varepsilon,\ell}$. For fixed ε the metrics are therefore pairwise isometric, and $\operatorname{spec}\Delta_{g_{\varepsilon,\ell}}$, including multiplicities, is independent of ℓ. The scalar part $c(\varepsilon) = \operatorname{tr}\mathsf{L}(g_{\varepsilon,\ell})/5$ is independent of ℓ by conjugacy. Since its first derivative vanishes for trace-free metric variations by (7.7),

$$\begin{aligned} c(\varepsilon) &= 8 + O(\varepsilon^2), \\ \lambda_1 = \lambda_2 = \lambda_3 &= c(\varepsilon) - 2\varepsilon, \\ \lambda_4 = \lambda_5 &= c(\varepsilon) + 3\varepsilon. \end{aligned} \tag{7.19}$$

Choose contours with real intersections $(-1, 4)$ and $(-1, 16)$, avoiding the reference spectrum. Their ranks at g_* are 1 and 6, respectively: the reference eigenvalues below 16 are 0 and 8, with multiplicities 1 and 5. The uniform resolvent construction of Section 2 retains these ranks and excludes spectrum on the endpoints for all sufficiently small parameters. On a closed connected manifold the Laplacian is nonnegative and its zero eigenspace consists exactly of the constants, since $\langle \Delta u, u\rangle = \|du\|_2^2$. Thus there is no additional positive eigenvalue below the first-five cluster, and the next eigenvalue satisfies $\lambda_6 > 16$. Compactness of the projection orbit makes one choice of $\varepsilon_0 > 0$ valid for every ℓ. Let $K = \operatorname{supp}\chi \Subset O$. Equation (7.12) gives

$$g_{\varepsilon,\ell} = g_* \text{ on } S^4 \setminus K, \qquad dv_{g_{\varepsilon,\ell}} = dv_{g_*}. \tag{7.20}$$

By (7.17), $\operatorname{Ric}_{g_*} - 3g_* = 3g_*$. The finite family satisfies

$$\sup_{\ell\in\mathbb{CP}^1} \|g_{\varepsilon,\ell} - g_*\|_{C^2} \le C_2|\varepsilon|.$$

Apply Lemma A.5 with $c = 3$ and $\delta = 3$. If $C_2|\varepsilon|$ is smaller than both its admissible radius and $3/(2(C+3))$, then

$$\operatorname{Ric}_{g_{\varepsilon,\ell}} - 3g_{\varepsilon,\ell} \ge \tfrac{3}{2} g_* > 0$$

for every ℓ. The frame $U_{g_{\varepsilon,\ell}}$ is a smooth isometry from the trivial five-plane bundle onto the sum of the first five positive eigenspaces. Equation (7.18) identifies the lower eigenspace with $\operatorname{im} P_\ell$. Therefore the first-three eigenspace bundle is

$$\mathcal{E}_\varepsilon \cong L_{\mathbb{R}} \oplus \underline{\mathbb{R}}, \tag{7.21}$$

where L is the tautological complex line. Its orthogonal complement in the trivial first-five eigenspace bundle has rank two. Choose unit frames of L on the two coordinate disks of $\mathbb{CP}^1$. On their equator the transition can be chosen $e^{-i\theta}$. The bundle in (7.21) has clutching loop $\operatorname{diag}(\operatorname{Rot}_{-\theta}, 1) \in SO(3)$. Its lift through the double covering $\operatorname{Spin}(3) \to SO(3)$ starts at 1 and ends at -1, since a one-plane rotation of angle 2π lifts to a half-angle path. Thus it does not lift to a closed loop. The obstruction is

$$\left\langle w_2(\mathcal{E}_\varepsilon), [S^2] \right\rangle = 1. \tag{7.22}$$

Equivalently, $w_2(L_{\mathbb{R}}) = c_1(L) \bmod 2$ and $c_1(L) = -1$ on the chosen orientation; the sign does not affect (7.22). These clutching descriptions of w_2 are also given in [44, Sections 4 and 14].

7.4.4. *Topology of the gap space.* Let $\mathcal{G}_{\mathrm{gap}}$ consist of smooth metrics on this fixed S^4 with $\operatorname{Ric} > 3g$, $dv_g = dv_{g_*}$, $g = g_*$ off K, and $\lambda_3(g) < \lambda_4(g)$. For fixed $0 < \varepsilon < \varepsilon_0$, (7.18) defines a map $f_\varepsilon : S^2 \to \mathcal{G}_{\mathrm{gap}}$.

Corollary 7.6. *The map f_ε is not nullhomotopic. In particular it represents a nonzero element of the based π_2 of its component of $\mathcal{G}_{\mathrm{gap}}$.*

Proof. A nullhomotopy extends the family to a continuous family over D^3. The boundary bundle has the nonzero class (7.22); compare [47, Lemma 1.6]. The first-three projections are locally continuous by resolvent continuity and the gap, and define a rank-three bundle. Compactness of the parameter ball gives a uniform positive minimum for the third-to-fourth gap; one may also construct the bundle with local contours. Homotopy invariance of pullback bundles [44, Section 3], applied to the contraction of D^3 to a point, makes this bundle trivial. Naturality of Stiefel–Whitney classes [44, Section 4] then gives zero w_2 on its restriction to S^2, contrary to (7.22). Choosing a basepoint on the original sphere family gives the based conclusion. □

Corollary 7.7. *Fix $0 < \varepsilon < \varepsilon_0$. Let $g_{t,b}$, $(t,b) \in [0,1] \times S^2$, be a continuous family of smooth metrics, with the C^∞ topology, such that $g_{0,b} = g_{\varepsilon,b}$ and*

$$\lambda_3(g_{t,b}) < \lambda_4(g_{t,b}) \qquad \text{for all } (t,b).$$

Then for some $b \in S^2$,

$$\lambda_1(g_{1,b}) = \lambda_2(g_{1,b}) \quad \text{or} \quad \lambda_2(g_{1,b}) = \lambda_3(g_{1,b}).$$

Proof. Local Riesz projections define a bundle on $S^2 \times [0,1]$. The two endpoint inclusions are homotopic, so their pullback bundles are isomorphic by [44, Section 3]. Suppose the final three eigenvalues are distinct at every parameter. Their increasing order defines three continuous rank-one Riesz projections, and hence a splitting into real line bundles. Real line bundles are classified by their first Stiefel–Whitney class [44, Sections 4–5]; each is trivial because $H^1(S^2; \mathbb{Z}/2) = 0$. The Whitney product formula [44, Section 4] gives zero second Stiefel–Whitney class for their sum, contradicting (7.22). □

Equation (7.16) gives a constant map from S^2 to the set of isometry classes, while

$$[f_\varepsilon] \neq 0 \quad \text{in } \pi_2(\mathcal{G}_{\mathrm{gap}}), \qquad \lambda_4(g_{\varepsilon,\ell}) - \lambda_3(g_{\varepsilon,\ell}) = 5\varepsilon.$$

7.5. **Bundles on the fixed round two-sphere.** Let γ be the unit-round metric on S^2. By Lemma C.1, the matrix derivative of every round cluster is onto for potentials supported in any prescribed nonempty open set.

Proposition 7.8. *Fix $\ell \geq 1$, $m = 2\ell + 1$, $\lambda = \ell(\ell+1)$, and a nonempty open set $O \subset S^2$. There are $K \Subset O$, a finite-dimensional $W \subset \{q \in C^\infty(S^2; \mathbb{R}) : \operatorname{supp} q \subset K\}$, and a smooth local section $\mathscr{S}_\ell$ at zero such that*

$$\mathsf{L}(\mathscr{S}_\ell(Y)) = \lambda I_m + Y, \qquad \mathscr{S}_\ell(0) = 0,$$

and, on a sufficiently small target ball,

$$\|\mathscr{S}_\ell(Y) - \mathscr{S}_\ell(Y')\|_{C^k} \leq C_{k,\ell} \|Y - Y'\| \qquad (k \geq 0). \tag{7.23}$$

Proof. Choose $\chi \in C_c^\infty(O)$ nonnegative and positive on a nonempty open set. In an orthonormal eigenbasis put $F_C = \sum C_{ij} u_i u_j$ and define the matrix-space Gram operator G by

$$\langle GC, D \rangle = \int_{S^2} \chi F_C F_D \, dv_\gamma.$$

Lemma C.1 shows that G is positive definite. Define

$$R(C) = \chi F_{G^{-1}C}.$$

Equation (2.31) gives $D\mathsf{L}(0)R(C) = C$. Hence $C \mapsto \mathsf{L}(R(C)) - \lambda I_m$ has derivative the identity. Its finite-dimensional local inverse defines $\mathscr{S}_\ell$ with values in $\operatorname{im} R$ and support in $\operatorname{supp} \chi$. On a smaller convex ball the derivatives into every C^k space are bounded, since the image is finite-dimensional and consists of fixed smooth functions. Integration along segments proves (7.23). □

Theorem 7.9. *Let B be a compact smooth manifold and let $E \to B$ be a smooth real vector bundle of rank $r > 0$. Fix a nonempty open disk $O \subset S^2$. There are an integer $\ell \geq 1$, $m = 2\ell + 1 > r$, a compact set $K \Subset O$, and a smooth family of real potentials*

$$q_{\varepsilon,b} \in C^\infty(S^2; \mathbb{R}), \quad \operatorname{supp} q_{\varepsilon,b} \subset K, \qquad b \in B, \quad |\varepsilon| < \varepsilon_0,$$

with $q_{0,b} = 0$, for which the $\ell(\ell+1)$-cluster of $\Delta_\gamma + q_{\varepsilon,b}$ has, for $0 < \varepsilon < \varepsilon_0$, exactly the eigenvalues

$$\ell(\ell+1) - (m-r)\varepsilon \quad \text{with multiplicity } r, \qquad \ell(\ell+1) + r\varepsilon \quad \text{with multiplicity } m - r. \tag{7.24}$$

The bundle of eigenspaces for the first value is isomorphic to E. For every fixed k, the family is $O(\varepsilon)$ in $C^k(S^2)$, uniformly in b.

Proof. Choose finitely many trivializing neighborhoods for E and smooth functions ϕ_i with supports in them and $\sum_i \phi_i^2 = 1$. The map sending a vector to the direct sum of its local coordinates multiplied by the ϕ_i is a smooth fibrewise injection into a finite trivial bundle. The terms extend by zero because the supports lie inside the trivializing neighborhoods. At each point at least one ϕ_i is nonzero, which proves injectivity. If the finite trivial bundle has rank N, choose an odd $m = 2\ell + 1 \geq N$ with $m > r$, and compose the injection with the coordinate inclusion into $B \times \mathbb{R}^m$. Let P_b be the Euclidean orthogonal projection onto the image of E_b. Define

$$Y_{\varepsilon,b} = \varepsilon(rI_m - mP_b), \qquad q_{\varepsilon,b} = \mathscr{S}_\ell(Y_{\varepsilon,b}).$$

Compactness of B permits one ε_0. Proposition 7.8 gives a family smooth also at $\varepsilon = 0$ and the stated estimates. The matrix $Y_{\varepsilon,b}$ acts by $-(m-r)\varepsilon$ on $\operatorname{im} P_b$ and by $r\varepsilon$ on its orthogonal complement. The exact section identity therefore proves (7.24). Let $U_q : \mathbb{R}^m \to L^2(S^2, dv_\gamma)$ be the Riesz–Gram frame. The map

$$(b, v) \longmapsto U_{q_{\varepsilon,b}} v, \qquad v \in \operatorname{im} P_b,$$

is a smooth fibrewise isomorphism onto the required eigenspace bundle. The Hilbert space is the fixed space $L^2(S^2, dv_\gamma)$. Spectral isolation from the other round clusters persists for small potentials.

For an explicit isolation bound, let d_ℓ be the distance from $\ell(\ell+1)$ to the excluded round spectrum and choose $0 < \eta < d_\ell/3$. The variational characterization of the ordered eigenvalues gives

$$|\lambda_j(\Delta_\gamma + q) - \lambda_j(\Delta_\gamma)| \leq \|q\|_\infty \quad \text{for every } j :$$

the Rayleigh quotients differ by at most $\|q\|_\infty$, and taking the infimum and supremum preserves the bound. Take the parameter range so that $\|q_{\varepsilon,b}\|_\infty < \eta$ and $m|\varepsilon| < \eta$ uniformly in b. The displayed two eigenvalues then remain in the isolated cluster, and no other eigenvalue has either value. □

The levels preceding $\ell(\ell+1)$ have total multiplicity $\sum_{j=0}^{\ell-1}(2j+1) = \ell^2$. Thus, with eigenvalues numbered from zero, the rank-r bundle occupies indices $\ell^2, \dots, \ell^2 + r - 1$ for sufficiently small $\varepsilon > 0$.

Corollary 7.10. *There is a smooth family of arbitrarily small potentials on the fixed round S^2, supported in any prescribed nonempty open disk and parametrized by $\mathbb{CP}^1$, for which the degree-two cluster has the two constant eigenvalues*

$$6 - 3\varepsilon \quad \textit{of multiplicity } 2, \qquad 6 + 2\varepsilon \quad \textit{of multiplicity } 3.$$

The lower rank-two eigenbundle has Euler number -1 for the complex orientation. Fix a sufficiently small $\varepsilon > 0$. Let $q_{t,b}$, $(t, b) \in [0,1] \times \mathbb{CP}^1$, be a continuous map into $C^\infty(S^2; \mathbb{R})$ beginning at this family. Order the complete spectrum as $\nu_0(t,b) \leq \nu_1(t,b) \leq \cdots$. If

$$\nu_3(t,b) < \nu_4(t,b), \qquad \nu_5(t,b) < \nu_6(t,b) \quad \textit{for all } (t,b),$$

then $\nu_4(1,b) = \nu_5(1,b)$ for some $b \in \mathbb{CP}^1$. The two inequalities are the gaps immediately below and above the selected rank-two subcluster.

Proof. Apply Theorem 7.9 to the real plane bundle underlying the tautological complex line over $\mathbb{CP}^1$, embedded in $\mathbb{C}^2 \subset \mathbb{R}^5$, and take $\ell = 2$ and $r = 2$. Its clutching function is $e^{-i\theta}$, so its Euler number is -1. At the unperturbed sphere, the levels below 6 have total multiplicity $1 + 3 = 4$. The isolation estimate in Theorem 7.9 therefore places the selected rank-two eigenspace at indices $4, 5$. Under the two gap inequalities in the statement, local Riesz projections onto these two eigenvalues agree on overlaps, and define a rank-two bundle over $\mathbb{CP}^1 \times [0,1]$. Its endpoint restrictions are isomorphic. If the two eigenvalues were simple at every parameter, the endpoint bundle would split into two real line bundles. Each real line bundle over S^2 is trivial because $H^1(S^2; \mathbb{Z}/2) = 0$. Their sum has Euler class zero, a contradiction. Homotopy invariance and the classification of real line bundles are as in [44, Sections 3–5]. □

8. Parameter families and relative lifting

A section of the matrix map gives spectral families parametrized by Grassmannians. For a closed parameter subset A_0, a relative lift $\bar{\alpha}$ satisfies $\bar{\alpha}|_{A_0} = \alpha$ for the prescribed lift α.

8.1. **Grassmannian retractions.** For the topology of finite-dimensional matrix families, see [2, 14]. For spectral-bundle obstructions in Dirac families, see [47, Lemma 1.6 and Section 4 of the arXiv version].

Let $\mathcal{M}$ be a metric parameter space on which a spectral-cluster frame is fixed on a neighborhood $\mathcal{U}$ of g_0. Suppose that a continuous map

$$s : B_\delta(0) \subset \mathrm{Sym}^0_m \longrightarrow \mathcal{U}, \qquad s(0) = g_0, \quad \Gamma(s(A)) = A \tag{8.1}$$

is given and is smooth on the punctured matrix ball. A centered submersion gives a section smooth also at zero. At an arbitrary reference of Theorem A, Theorem 4.5 gives (8.1) by restricting the full matrix section to trace-free targets. The latter section need not be differentiable at zero. Fix $1 \le r < m$. Let $\mathcal{U}_r \subset \mathcal{U}$ be the metrics for which the r-th and $(r+1)$-st ordered eigenvalues of L differ. Define

$$p_r : \mathcal{U}_r \longrightarrow \mathrm{Gr}_r(\mathbb{R}^m) \tag{8.2}$$

by the lower rank-r eigenspace of L. The Grassmannian is identified with real orthogonal rank-r projections. The finite-dimensional Riesz formula [38, Chapter II, Section 1.4] makes the map smooth. At any metric in $\mathcal{U}_r$, choose a contour containing precisely the lower r matrix eigenvalues. The strict internal gap keeps that contour disjoint from the nearby matrix spectra; the resolvent series of Section 2 differentiates its integral. For a rank-r orthogonal projection P, put

$$A_\varepsilon(P) = \varepsilon(rI_m - mP), \qquad \sigma_\varepsilon(P) = s(A_\varepsilon(P)). \tag{8.3}$$

This matrix is trace-free. Its eigenvalues are $-(m-r)\varepsilon$ on $\operatorname{im} P$ and $r\varepsilon$ on $\ker P$. Its Frobenius norm is independent of P:

$$\|A_\varepsilon(P)\|^2 = \varepsilon^2\big(r(m-r)^2 + (m-r)r^2\big) = \varepsilon^2 mr(m-r) > 0. \tag{8.4}$$

For a fixed sufficiently small $\varepsilon > 0$, every target therefore belongs to the punctured ball $B_\delta(0) \setminus \{0\}$. The restriction of s to this family is smooth.

Proposition 8.1. *The family* (8.3) *lies in $\mathcal{U}_r$ and satisfies*

$$p_r \circ \sigma_\varepsilon = \mathrm{Id}_{\mathrm{Gr}_r(\mathbb{R}^m)}. \tag{8.5}$$

Its image is a smooth embedded copy of the Grassmannian, and $\sigma_\varepsilon \circ p_r$ is a retraction onto that image. The lower spectral bundle restricts to the tautological bundle.

Proof. The full matrix $\mathsf{L}(s(A))$ differs from A by a scalar multiple of I_m, which does not change its eigenspaces. The eigenvalue difference in (8.3) is $m\varepsilon > 0$. This proves (8.5). It also gives injectivity and injectivity of the derivative of σ_ε. The Grassmannian is compact and the metric space is Hausdorff, so the injective continuous map is a homeomorphism onto its image. Differentiation of $p_r\sigma_\varepsilon = \mathrm{Id}$ gives a left inverse for its tangent map. In a C^k metric chart, the finite-dimensional tangent image is complemented: compose that left inverse with $D\sigma_\varepsilon$ to obtain a bounded projection onto the image. The local inverse argument in Section A therefore expresses the image as a smooth graph over that finite-dimensional subspace. Thus σ_ε is a smooth embedding. The composite $\sigma_\varepsilon p_r$ has image in that embedded copy and is the identity there; it is a retraction.

Let γ_r be the tautological bundle. For $g' \in \mathcal{U}_r$, let $V_{g'} = J_{g'}^{-1} U_{g'}$ be the frame of (2.11). The map

$$(g', v) \longmapsto (g', V_{g'} v), \qquad v \in \operatorname{im} p_r(g'),$$

is a smooth fibrewise isometry from $p_r^*\gamma_r$ onto the lower spectral bundle of $\Delta_{g'}\,\mathcal{E}_r$. Its inverse is the adjoint of $V_{g'}$ restricted to that bundle. Thus this identification is valid even when the Riemannian density varies. Pulling back by σ_ε gives

$$\sigma_\varepsilon^* \mathcal{E}_r \cong \gamma_r. \tag{8.6}$$

□

With compatible basepoints, (8.5) gives

$$(p_r)_*(\sigma_\varepsilon)_* = \mathrm{Id} \quad \text{on } \pi_k(\mathrm{Gr}_r(\mathbb{R}^m)), \qquad \sigma_\varepsilon^* p_r^* = \mathrm{Id} \quad \text{on } H^*(\mathrm{Gr}_r(\mathbb{R}^m); A)$$

for every abelian coefficient group A. Hence $(\sigma_\varepsilon)_*$ and p_r^* are split injections.

Corollary 8.2. *Let B be a compact smooth manifold, let $m \geq 2$ and $1 \leq r < m$, and let $E \subset B \times \mathbb{R}^m$ be a smooth rank-r subbundle. Its orthogonal projections P_b are smooth, and the family $b \mapsto \sigma_\varepsilon(P_b)$ has lower spectral bundle isomorphic to E.*

Proof. In a local smooth frame represented by a full-rank matrix $V(b)$, the orthogonal projection is $V(V^TV)^{-1}V^T$, hence is smooth. These local formulas agree. The bundle E is the pullback of γ_r by $b \mapsto P_b$. Apply (8.6). □

Corollary 8.3. *Fix a smooth reference metric with a positive eigenvalue λ of multiplicity $m \geq 2$, and let $1 \leq r < m$. In every sufficiently small neighborhood with its whole-cluster frame, the internal gap space contains the Grassmannian sections of Proposition 8.1, within the reference conformal class. They can be chosen with*

$$\mathsf{L}(\sigma_\varepsilon(P)) = \lambda I_m + \varepsilon(rI_m - mP), \qquad \|\sigma_\varepsilon(P) - g_0\|_{C^k} \leq C_k\sqrt{\varepsilon}. \tag{8.7}$$

Proof. Let $\mathscr{S}$ be the full matrix section of Theorem 4.5, and set $\sigma_\varepsilon(P) = \mathscr{S}(\varepsilon(rI_m - mP))$. The target matrix has nonzero constant Frobenius norm for fixed $\varepsilon > 0$, so the section is smooth along the entire Grassmannian. The target identity gives the first equality in (8.7), and the Hölder estimate at zero gives the second, uniformly over the compact set of projections. For small ε the image lies in the prescribed neighborhood. Proposition 8.1 applies with the original cluster frame. □

A differentiable right inverse at a reference with nonsurjective $D\mathsf{L}$ is impossible, since the chain rule would give $D\mathsf{L} \circ DS = \mathrm{Id}_{\mathrm{Sym}_m}$.

8.2. **The exact quadratic model.** Let E be a real Banach space, or a metrizable locally convex space, and let $Q : E \to \mathbb{R}^d$ be continuous quadratic, with polarization B. Assume every nonzero scalar contraction $c \cdot Q$ has infinite negative index. The positive index is then infinite as well.

Theorem 8.4 (Open inverse images). *For every open $U \subset \mathbb{R}^d$, the map $Q : Q^{-1}(U) \to U$ is a weak homotopy equivalence.*

Proof. If $d = 0$, the nonempty inverse image is the contractible space E. Suppose $d \geq 1$. For a finite-dimensional subspace $V \subset E$, put

$$H_V = \{h \in E : B(h, v) = 0 \text{ for every } v \in V\}.$$

It has codimension at most $d \dim V$ and retains the index hypothesis. Corollary 4.2 and a finite-dimensional inverse give a map σ on a neighborhood of a closed ball $\overline{B}_\eta(0) \subset \mathbb{R}^d$, with $Q\sigma(a) = a$, $\sigma(0) \neq 0$, and image in one finite-dimensional subspace of H_V. Define

$$s_V(0) = 0, \qquad s_V(a) = \sqrt{|a|/\eta}\,\sigma(\eta a/|a|) \quad (a \neq 0).$$

Homogeneity gives $Qs_V(a) = a$. For every continuous seminorm p on E, compactness of the sphere gives $p(s_V(a)) \leq C_p|a|^{1/2}$, which proves continuity at zero.

If $x, y : P \to E$ take values in finite-dimensional linear subspaces and $Qx = Qy = a$, choose a finite-dimensional V containing both images and put

$$h_t = (1-t)x + ty + s_V\big(a - Q((1-t)x + ty)\big). \tag{8.8}$$

The first summand lies in V and the second in H_V. Therefore

$$Q(h_t) = Q((1-t)x + ty) + Q\big(s_V(a - Q((1-t)x + ty))\big) = a.$$

At $t = 0, 1$ the second summand vanishes. It also vanishes at every parameter where $x = y$, so the homotopy is relative to that set.

Let Z be a finite polyhedron, $z_0 \in Z$, and let $x : Z \to Q^{-1}(U)$ be continuous. Compactness of $x(Z)$ and openness of $Q^{-1}(U)$ give a balanced convex neighborhood V_0 of zero such that $x(Z)+V_0 \subset Q^{-1}(U)$. Choose a finite open cover U_i of Z and points z_i such that $x(z_i) - x(z) \in V_0$ for $z \in U_i$. The cover can be chosen with $z_0 = z_1 \in U_1$ and $z_0 \notin U_i$ for $i > 1$; first choose U_1 and then cover its compact complement by sets avoiding z_0. For a subordinate partition of unity ϕ_i, put

$$x_f(z) = \sum_i \phi_i(z)x(z_i).$$

Then $x_f(z_0) = x(z_0)$, x_f has finite-dimensional image, and $x_f(z) - x(z) \in V_0$ by convexity. The homotopy $x(z) + t(x_f(z) - x(z))$ stays in $Q^{-1}(U)$ and fixes z_0.

Fix $x_0 \in Q^{-1}(U)$ and set

$$s_{x_0}(a) = x_0 + s_{\mathbb{R}x_0}(a - Qx_0).$$

Orthogonality gives $Qs_{x_0}(a) = a$ and $s_{x_0}(Qx_0) = x_0$. Composition with s_{x_0} lifts every based sphere map into U, proving surjectivity on each based homotopy group. For injectivity, suppose $x : (S^k, z_0) \to (Q^{-1}(U), x_0)$ has Qx nullhomotopic in U. Replace x by the based finite-dimensional approximation x_f above. Then Qx_f is still nullhomotopic. Both x_f and $s_{x_0}(Qx_f)$ have finite-dimensional images, have the same Q-value, and agree at z_0. Equation (8.8) joins them by a based homotopy in $Q^{-1}(U)$. Composing a based nullhomotopy of Qx_f with s_{x_0} contracts the second map to x_0. Thus x is nullhomotopic.

For path components, surjectivity follows from $Qs_{\{0\}} = \mathrm{id}$. If Qx_0 and Qx_1 are joined by a path a in U, then $s_{x_0} \circ a$ joins x_0 to $s_{x_0}(Qx_1)$. The latter point and x_1 have the same Q-value. Apply (8.8) to these two constant maps to join them in that fiber. Thus Q induces a bijection on path components and isomorphisms on all based homotopy groups. □

Let A be closed in a compact metrizable space P, and let $a : P \to \mathbb{R}^d$ and $x_A : A \to V$ be continuous, where V is finite dimensional and $Qx_A = a|_A$. For $A \neq \emptyset$, Lemma 8.6 gives a continuous extension $\bar{x} : P \to V$; for $A = \emptyset$, take $\bar{x} = 0$. Then

$$x = \bar{x} + s_V(a - Q\bar{x})$$

is an exact lift agreeing with x_A.

Suppose E is normed and U is invariant under positive scalar multiplication. For $\rho > 0$, the map

$$c(x) = \frac{\rho}{\rho + \|x\|}x$$

sends $Q^{-1}(U)$ into $Q^{-1}(U) \cap B_\rho$. For $0 \le s \le 1$, multiplication by $(1 - s) + s\rho/(\rho + \|x\|) > 0$ gives a homotopy from the identity to c. It remains in $Q^{-1}(U)$ by homogeneity, and on B_ρ its factor is at most one. Thus inclusion of the intersection with B_ρ is a homotopy equivalence. For the split map

$$P(r, z) = (r, Q(z)),$$

equip the source with the product norm $\|(r, z)\| = |r| + \|z\|$ and set $\tau(r, z) = \rho/(\rho + |r| + \|z\|)$. The map $(r, z) \mapsto (\tau^2 r, \tau z)$ sends the inverse image of an open cone into its intersection with the radius-ρ ball and multiplies its target by τ^2. Replacing τ by $(1 - s) + s\tau$ gives the two required homotopies, with no increase of the norm on that ball. For

$$G_r = \{A \in \mathrm{Sym}_m : \mu_r(A) < \mu_{r+1}(A)\},$$

let $P_r(A)$ be the lower rank-r spectral projection. The homotopy

$$H_s(A) = (1 - s)A + s\big((r/m)I - P_r(A)\big)$$

has gap $(1 - s)(\mu_{r+1}(A) - \mu_r(A)) + s > 0$ and fixes every matrix $(r/m)I - P$. It is a deformation retraction onto $\{(r/m)I - P : P \text{ is a rank-}r \text{ projection}\} \cong \mathrm{Gr}_r(\mathbb{R}^m)$.

The same correction formula applies to the map $(r, z) \mapsto (r, Q(z))$ on $\mathbb{R}^a \oplus E$: preserve its first coordinate and correct only the second one. For a finite-rank source family, the orthogonality space

in the proof is chosen from the finitely many z coordinates. A homotopy of target pairs is lifted by adjoining its first coordinate and the corresponding quadratic section in the second. This proves the weak equivalence on every open target set for that split model.

Thus, for the split quadratic map with target $\mathrm{Sym}_m(\mathbb{R})$, the inverse image of G_r in any source ball has the weak homotopy type of $\mathrm{Gr}_r(\mathbb{R}^m)$.

8.3. **Relative lifting in corank one.** Let $F : E \supset U_0 \to W$ be smooth, $F(0) = 0$, with E Banach and W finite-dimensional. Take the decomposition (4.1)–(4.2) with $E_0 = E$. In this subsection assume $\dim C = 1$ and that the scalar quadratic form Q has infinite positive and negative indices.

Theorem 8.5 (Relative lifting in corank one). *Suppose $d = 1$ and (4.3) holds. For every neighborhood U of 0 in the domain and every integer $r \geq 0$, there are a neighborhood V of 0, $V \subset U$, and a target neighborhood D of 0 such that the following holds. Let P be a compact metrizable space of covering dimension at most r, let $A_0 \subset P$ be closed, and let*

$$f : P \longrightarrow D, \qquad \alpha : A_0 \longrightarrow V$$

be continuous maps with $F \circ \alpha = f|_{A_0}$. Then there is a continuous map

$$\bar{\alpha} : P \longrightarrow U, \qquad \bar{\alpha}|_{A_0} = \alpha, \qquad F \circ \bar{\alpha} = f. \tag{8.9}$$

The neighborhoods are selected before P, A_0, f, α once U and r are fixed.

8.4. **A relative extension lemma for an indefinite quadratic form.** For a metrizable space X, covering dimension satisfies

$$\dim X \leq n \quad \Longleftrightarrow \quad S^n \in AE(X) \tag{8.10}$$

where $S^n \in AE(X)$ means that every continuous map from a closed subset of X to S^n extends continuously to X. Moreover, $\dim U \leq \dim X$ for each open subset $U \subset X$ [35, 22].

Lemma 8.6 (Convex extension; [23]). *Let P be metrizable, let $A \subset P$ be nonempty and closed, and let E be a locally convex topological vector space. Every continuous $f : A \to E$ has a continuous extension $\bar{f} : P \to E$ with image in $\mathrm{conv}(f(A))$. If E is normed and $\|f\| \leq c$ on A, the extension satisfies $\|\bar{f}\| \leq c$ on P.*

Proof. Fix a compatible metric d. If $A = P$, take $\bar{f} = f$. Otherwise cover $P \setminus A$ by $B(x, d(x, A)/2)$, and choose a locally finite refinement U_i and a subordinate partition of unity ϕ_i on $P \setminus A$. For each i, choose x_i with $U_i \subset B(x_i, d_i/2)$, $d_i = d(x_i, A)$, and choose $a_i \in A$ with $d(x_i, a_i) < 2d_i$. Set

$$\bar{f}(x) = \sum_i \phi_i(x) f(a_i) \quad (x \notin A), \qquad \bar{f}|_A = f.$$

The sum is locally finite off A, so it is continuous there and belongs to $\mathrm{conv}(f(A))$. If $x \in U_i$ and $a \in A$, then

$$d_i \leq 2d(x, A) \leq 2d(x, a), \qquad d(a_i, a) < 2d_i + \tfrac{1}{2} d_i + d(x, a) \leq 6d(x, a).$$

For any convex neighborhood V of zero in E, continuity of f at a gives $\eta > 0$ such that $f(a') - f(a) \in V$ whenever $a' \in A$ and $d(a', a) < 6\eta$. If $d(x, a) < \eta$, every index with $\phi_i(x) \neq 0$ satisfies this condition on a_i. Convexity gives $\bar{f}(x) - f(a) \in V$, proving continuity at a. The norm bound follows from $\|\bar{f}(x)\| \leq \sum_i \phi_i(x) \|f(a_i)\| \leq c$. For an empty prescribed subset, a constant map supplies an extension. □

Lemma 8.7 (Quadratic extension with a norm bound). *Let*

$$q_N(u, v) = |u|^2 - |v|^2, \qquad (u, v) \in \mathbb{R}^N \oplus \mathbb{R}^N.$$

Let P be compact metrizable with $\dim P \leq N - 1$ and let $A_0 \subset P$ be closed. Suppose $b : P \to \mathbb{R}$ and $z_A = (u_A, v_A) : A_0 \to \mathbb{R}^{2N}$ are continuous and $q_N(z_A) = b|_{A_0}$. Then z_A extends to a continuous $z : P \to \mathbb{R}^{2N}$ with $q_N(z) = b$. If, for some $\delta > 0$,

$$|b| \leq \delta^2 \text{ on } P, \qquad |z_A| \leq \delta \text{ on } A_0,$$

then one can arrange $|z| \leq \sqrt{3}\, \delta$ on P.

Proof. If no numerical bound is specified, compactness permits a positive δ satisfying both stated bounds. It therefore suffices to work under those bounds. On A_0 put

$$s_A = |u_A|^2 + |v_A|^2, \qquad e_A = s_A - |b|.$$

Since $b = |u_A|^2 - |v_A|^2$ there, $e_A = 2\min(|u_A|^2, |v_A|^2) \in [0, \delta^2]$. Extend e_A to $e : P \to [0, \delta^2]$ by Lemma 8.6. For a compatible metric d_P on P, when $\emptyset \neq A_0 \neq P$ take

$$\tau(p) = \delta^2 \frac{d_P(p, A_0)}{1 + d_P(p, A_0)}.$$

Take $\tau = \delta^2/2$ and $e = 0$ if $A_0 = \emptyset$, and $\tau = 0$ if $A_0 = P$. Set

$$s = |b| + e + \tau, \qquad U_+ = \sqrt{(s+b)/2}, \qquad U_- = \sqrt{(s-b)/2}. \tag{8.11}$$

Then $s \geq |b|$, $s \leq 3\delta^2$, and on A_0 the radii are $|u_A|, |v_A|$.

For each sign define $P_\pm = \{U_\pm > 0\}$. These are metrizable open subspaces of dimension at most $N-1$. The maps $u_A/|u_A|$ on $A_0 \cap P_+$ and $v_A/|v_A|$ on $A_0 \cap P_-$ have closed domains in P_+ and P_-, respectively. By (8.10), they extend to maps $e_\pm : P_\pm \to S^{N-1}$. If the prescribed domain for either sign is empty, choose a constant unit vector; if $P_\pm$ is empty no map on it is required. Put

$$u(p) = \begin{cases} U_+(p) e_+(p), & p \in P_+, \\ 0, & p \notin P_+, \end{cases} \qquad v(p) = \begin{cases} U_-(p) e_-(p), & p \in P_-, \\ 0, & p \notin P_-. \end{cases}$$

At a point p_0 with $U_\pm(p_0) = 0$, the corresponding vector has norm exactly $U_\pm(p)$, tending to zero as $p \to p_0$. This proves continuity across the zero sets. On A_0 the formulas give the prescribed vectors, including their zero values. Finally,

$$|u|^2 - |v|^2 = U_+^2 - U_-^2 = b, \qquad |u|^2 + |v|^2 = U_+^2 + U_-^2 = s \leq 3\delta^2.$$

□

For $b > 0$, the projection $(u, v) \mapsto u/|u|$ is defined on $q_N^{-1}(b)$ and restricts to the identity on $\{(\sqrt{b}\, u, 0) : u \in S^{N-1}\}$. An extension of this sphere to D^N in $q_N^{-1}(b)$ would therefore give a retraction $D^N \to S^{N-1}$, which is impossible.

8.5. **Finite nondegenerate Morse restrictions and relative lifting.**

Lemma 8.8 (Finite Morse block with Banach parameters). *Assume $d = 1$. If the scalar form Q has at least N positive and N negative directions, there are a Banach space K_1, a smooth local coordinate map Θ at the origin, and a smooth scalar function b_0 such that*

$$F(\Theta(x, y, u, v)) = (y, b_0(x, y) + |u|^2 - |v|^2), \tag{8.12}$$

where $x \in K_1$, $y \in R$, and $u, v \in \mathbb{R}^N$. The map Θ is a local diffeomorphism from a neighborhood of zero in $K_1 \oplus R \oplus \mathbb{R}^N \oplus \mathbb{R}^N$ onto a source neighborhood. Here $b_0(0,0) = 0$ and $Db_0(0,0) = 0$.

Proof. Let B_Q be the polarization of Q. Choose an N-dimensional subspace Z_+ on which Q is positive definite, and let $p_+ : K \to Z_+$ be its B_Q-orthogonal projection. In a basis e_i of Z_+, this projection is

$$p_+ v = \sum_{i,j} e_i (G_+^{-1})_{ij} B_Q(v, e_j), \qquad (G_+)_{ij} = B_Q(e_i, e_j).$$

It is bounded. If Z'_- is any N-dimensional negative-definite subspace, then

$$Q(v - p_+ v) = Q(v) - Q(p_+ v) < 0 \quad (0 \neq v \in Z'_-).$$

Thus $I - p_+$ is injective on Z'_- and its image Z_- is negative definite and orthogonal to Z_+. Set $Z = Z_+ \oplus Z_-$. Its form is nondegenerate of signature (N, N). The same finite Gram formula defines a bounded projection p_Z on K. Hence $K = K_1 \oplus Z$, where $K_1 = \ker p_Z$ is a closed Banach subspace.

Choose X with $E = X \oplus K$ and $A|_X : X \to R$ invertible. Solving the range equation in the X variable gives local coordinates $(x, z, y) \in K_1 \oplus Z \oplus R$ in which $F = (y, G(x, z, y))$. After a fixed linear change on Z, the z Hessian at zero is $2J$, where $J = \mathrm{diag}(I_N, -I_N)$. The equation $D_z G(x, z, y) = 0$ has a unique smooth solution $z_c(x, y)$ near zero, because its z derivative is $2J$. Put $b_0(x, y) = G(x, z_c(x, y), y)$. Taylor's integral formula gives

$$G(x, z_c + \xi, y) - b_0(x, y) = \xi^T H(x, y, \xi)\xi, \quad H = \int_0^1 (1 - t) D_z^2 G(x, z_c + t\xi, y)\, dt,$$

so H is symmetric and $H(0) = J$.

For H near J, the binomial series defines $S(H) = (JH)^{1/2}$ and gives $S(J) = I$. Since $(JH)^T J = J(JH)$, every power of $JH - I$ has the same J-symmetry. Consequently

$$S(H)^T J = JS(H), \qquad S(H)^T JS(H) = JS(H)^2 = H.$$

The map $\xi \mapsto S(H(x, y, \xi))\xi$ has derivative I at the origin and therefore a smooth inverse with the Banach parameters x, y. Writing its new coordinates as $(u, v) \in \mathbb{R}^N \oplus \mathbb{R}^N$ gives (8.12). Each coordinate change is locally invertible on the entire source. Finally, the derivative of $P_C F$ at zero vanishes on all of E, so the chain rule gives $b_0(0, 0) = 0$ and $Db_0(0, 0) = 0$. □

Lemma 8.6 applies to the Banach coordinate space K_1. It also applies to the locally convex space of smooth functions with a fixed compact support, with its family of C^s seminorms.

Proof of Theorem 8.5. Choose $N \geq r+1$ and the coordinates of Lemma 8.8. Inside their domain choose a product neighborhood whose image lies in U. Choose $\delta > 0$ so small that all coordinate points with sufficiently small x, y and $|(u, v)| \leq \sqrt{3}\delta$ stay in this product. By continuity of b_0, choose $\eta > 0$ so that $\|x\| \leq \eta$, $|y| \leq \eta$ imply $|b_0(x, y)| \leq \delta^2/2$ and stay in the coordinate domain. Make the target neighborhood D so small that

$$|f_R| < \eta, \qquad |f_C| < \delta^2/2.$$

Choose $V \subset U$ so that every point of V has coordinates with $\|x\| < \eta$ and $|(u, v)| < \delta$, after decreasing η. All these choices depend only on U, r, F.

Write the partial lift in coordinates as $(x_A, f_R|_{A_0}, u_A, v_A)$. Lemma 8.6 gives an extension $x : P \to K_1$ with $\|x\| \leq \eta$. For empty A_0, take $x = 0$. Define

$$c(p) = f_C(p) - b_0(x(p), f_R(p)).$$

Then $|c| \leq \delta^2$, and on A_0 it equals $|u_A|^2 - |v_A|^2$. Lemma 8.7 extends (u_A, v_A) to (u, v) with $|u|^2 - |v|^2 = c$ and $|(u, v)| \leq \sqrt{3}\delta$. Finally set

$$\bar{\alpha}(p) = \Theta(x(p), f_R(p), u(p), v(p)).$$

The coordinate bounds place this map in U. Equation (8.12) proves its exact target identity, and every coordinate agrees with the prescribed lift on A_0. □

Corollary 8.9 (Local finite-dimensional connectivity). *For every neighborhood U and every $k \geq 0$, a sufficiently small neighborhood V has the property that every map $S^k \to V \cap F^{-1}(0)$ extends to a map $D^{k+1} \to U \cap F^{-1}(0)$.*

Proof. Apply Theorem 8.5 with $r = k + 1$ and $f \equiv 0$. □

8.6. **Internal spectral gaps in corank one.** Fix a spectral cluster of multiplicity m at a smooth reference and a whole-cluster Riesz–Gram frame. In a local parameter chart put

$$F(v) = \mathsf{L}(v) - \lambda I_m \in \mathrm{Sym}_m(\mathbb{R}).$$

Suppose its allowed derivative has corank one and the scalar intrinsic Hessian has infinite positive and negative indices. For $1 \leq j < m$, define

$$\mathcal{G}_j = \{Y \in \mathrm{Sym}_m(\mathbb{R}) : \mu_j(Y) < \mu_{j+1}(Y)\}, \qquad U_j = \{v \in U : F(v) \in \mathcal{G}_j\}.$$

Here μ_i are increasingly ordered eigenvalues. Let $p_j : U_j \to \mathrm{Gr}_j(\mathbb{R}^m)$ be the lower j-plane map in the fixed frame.

Theorem 8.10 (Gap obstructions after shrinking). *For every sufficiently small source neighborhood U and every $k \geq 0$, there is a neighborhood $V \subset U$ such that, for every continuous $\alpha : S^k \to V_j$,*

$$\alpha \text{ extends to } D^{k+1} \to U_j \quad \Longleftrightarrow \quad p_j \circ \alpha \text{ is nullhomotopic in } \mathrm{Gr}_j(\mathbb{R}^m). \tag{8.13}$$

In addition, in every U_j there is a continuous family parametrized by $\mathrm{Gr}_j(\mathbb{R}^m)$ whose lower-plane map is the identity.

Proof. If α extends to $D^{k+1} \to U_j$, composition with the continuous lower-plane map gives a nullhomotopy of $p_j\alpha$. Conversely, apply Theorem 8.5 with parameter-dimension bound $k+1$ and source neighborhood U. Choose a round target ball D contained in the resulting target neighborhood, and shrink the source neighborhood V so that $F(V) \subset D$. Put $Y(x) = F(\alpha(x))$ and $P(x) = p_j(\alpha(x))$ on S^k. A nullhomotopy is an extension $\widetilde{P} : D^{k+1} \to \mathrm{Gr}_j(\mathbb{R}^m)$ with $\widetilde{P}|_{S^k} = P$. For sufficiently small $\epsilon > 0$ all matrices $T_\epsilon(P) = \epsilon((j/m)I_m - P)$ lie in D. Define $\widetilde{Y}$ on the disk by

$$\widetilde{Y}(rx) = \begin{cases} (2r-1)Y(x) + (2-2r)T_\epsilon(P(x)), & 1/2 \leq r \leq 1, \\ T_\epsilon(\widetilde{P}(2rx)), & 0 \leq r \leq 1/2, \end{cases} \qquad x \in S^k.$$

The formulas agree at $r = 1/2$ and the inner formula is continuous at the center. For the outer formula, $Y(x)$ commutes with $P(x)$. Its gap between the two invariant blocks is

$$(2r-1)(\mu_{j+1}(Y(x)) - \mu_j(Y(x))) + (2-2r)\epsilon > 0.$$

The inner formula has gap ϵ. Convexity of D puts the outer formula in D, and the choice of ϵ does the same for the inner formula. Thus $\widetilde{Y} : D^{k+1} \to D \cap \mathcal{G}_j$ extends $F\alpha$. The relative theorem, with prescribed lift α on the entire boundary, gives an extension $\bar{\alpha} : D^{k+1} \to U$ satisfying $F\bar{\alpha} = \widetilde{Y}$. The strict gap makes its image lie in U_j.

For the Grassmannian family, apply the same relative theorem with empty prescribed subset and parameter space $\mathrm{Gr}_j(\mathbb{R}^m)$, of dimension $j(m-j)$. Choose ϵ small enough for its target neighborhood and prescribe the target $T_\epsilon(P)$. The lifted family's lower projection equals P at every parameter. □

8.6.1. *Two explicit unstable references.* On the flat square torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2$, the eigenspace at $\lambda = 1$ has basis proportional to

$$\cos x, \ \sin x, \ \cos y, \ \sin y.$$

The sole conformal product relation is a multiple of

$$\cos^2 x + \sin^2 x - \cos^2 y - \sin^2 y = 0.$$

The shell formula in Appendix B gives conformal corank one. By Theorem 3.4, Theorem 8.10 applies in the conformal class with target Grassmannians $\mathrm{Gr}_j(\mathbb{R}^4)$, $j = 1, 2, 3$. For localized scalar potentials in any prescribed nonempty open set, the same relation gives corank one: a product relation on that open set is a trigonometric polynomial and hence extends globally by analyticity.

At $\lambda = 5$, the shell consists of

$$\pm(1,2), \quad \pm(2,-1), \quad \pm(1,-2), \quad \pm(2,1).$$

Its real multiplicity is eight. The unrestricted metric cokernel has dimension $4 - 3 = 1$, whereas the conformal cokernel has dimension $4 - 1 = 3$. Thus the relative theorem applies here to *unrestricted metric variations*, with $\mathrm{Gr}_j(\mathbb{R}^8)$ for $1 \leq j \leq 7$. If c annihilates the unrestricted derivative, it also annihilates the conformal derivative, and $\ker D\mathsf{L}|_{\mathrm{conf}} \subset \ker D\mathsf{L}$. Theorem 3.4 therefore gives infinite positive and negative indices for its contraction on the unrestricted kernel.

8.6.2. *Smooth spatial perturbations and support.* Given a neighborhood U in the C^∞ topology, choose s_0 sufficiently large and a neighborhood U_0 in a C^{s_0} chart such that $U_0 \cap C^\infty \subset U$. At a smooth reference, choose the range complement X and the Morse subspace Z in C^∞. The projections onto X and Z are finite sums of continuous linear functionals multiplied by fixed smooth coefficients, and are continuous in C^∞. The coordinate maps in Lemma 8.8 have the form

$$(x, y, u, v) \longmapsto x + \sum_{j=1}^{l} a_j(x, y, u, v) e_j, \qquad e_j \in C^\infty,$$

where the finitely many scalar functions a_j are continuous in the C^{s_0} norm. Consequently these maps and their inverses preserve smooth coefficients and are continuous for all the C^s seminorms together. Lemma 8.6, applied with those seminorms, gives a C^∞-continuous extension of the x coordinate. Thus Theorem 8.5 holds in the Fréchet topology of smooth coefficients; compare [28, Appendix A].

In the potential case, fix a nonempty open set O and select the finite-dimensional spaces $X, Z \subset C_c^\infty(O)$. Let $K_0 \Subset O$ contain their supports. For a compact $K \Subset O$ with $K_0 \subset K$, put

$$E_K = \{q \in C^\infty(M) \,:\, \operatorname{supp} q \subset K\}.$$

If the prescribed partial lift satisfies $\alpha(A_0) \subset E_K$, the convex extension of its residual coordinate takes values in E_K. Every subsequent correction belongs to $X \oplus Z \subset E_K$. Hence $\bar\alpha(P) \subset E_K$. More generally, if $\alpha(A_0) \subset E_{K_A}$ for a compact $K_A \Subset O$, then $\bar\alpha(P) \subset E_{K_0 \cup K_A}$.

8.7. **Germs with prescribed graph coordinates.** For $d \geq 1$, the bilinear coordinates (4.11) give smooth relative extensions of families with fixed first coordinate $t \in \mathbb{R}^r$. The precise statement is Theorem D.2.

9. Invariant perturbations and common symmetries

For a group G acting on an eigenspace E_λ, invariant perturbations give cluster matrices in the self-adjoint commutant $\operatorname{Sym}_G(E_\lambda)$.

9.1. **The self-adjoint commutant.** Let G be a finite group acting effectively and smoothly on a compact connected manifold M, with $\dim M \geq 2$. The metric and the background potential are invariant. In the potential case, $O \subset M^\circ$ is nonempty, open, and invariant, and the allowed parameters belong to $C_c^\infty(O)^G$. In the metric case, M is closed, $\lambda > 0$, and the allowed parameters are the invariant functions in the full conformal class. For an isolated eigenspace E_λ, put

$$\mathcal{A} = \operatorname{Sym}_G(E_\lambda) = \{Y = Y^* \,:\, Y\rho(g) = \rho(g)Y \ (g \in G)\}, \qquad D = \dim \mathcal{A}. \tag{9.1}$$

The scalar matrix I_m belongs to $\mathcal{A}$. Write

$$A = D\mathsf{L}(0)|_{E_0}, \quad \mathcal{R} = A(E_0), \quad \mathcal{C} = \mathcal{R}^\perp \subset \mathcal{A}, \quad \mathcal{K} = \ker A, \tag{9.2}$$

where E_0 is the indicated invariant space. Orthogonality is the trace pairing.

Theorem 9.1 (Invariant local realization). *In either setting, the cluster map has a local section $S : B_\delta(0) \subset \mathcal{A} \to W \subset E_0$ such that*

$$S(0) = 0, \qquad \mathsf{L}(S(Y)) = \lambda I_m + Y, \qquad \|S(Y) - S(Y')\|_{C^k} \leq C_k \|Y - Y'\|^{1/2}.$$

Here W is finite-dimensional, consists of smooth invariant parameters, and in the potential case has a common compact support in O. The section belongs to $C^\omega(B_\delta(0) \setminus \{0\}; W)$. If $d = \dim \mathcal{C} > 0$, one may take $\dim W = D + 1$; there is also a nonconstant analytic curve $v(s) = sz_0 + O(s^2)$ in W with

$$\mathsf{L}(v(s)) = \lambda I_m, \qquad D\mathsf{L}(v(s))|_W \,:\, W \longrightarrow \mathcal{A} \text{ onto for } 0 < |s| \ll 1.$$

If $d = 0$, D perturbations suffice and the section is analytic at zero with a linear estimate. At a singular reference no exponent greater than $1/2$ gives a uniform realization estimate in all target directions in a norm controlling the second derivative.

9.2. **Frames, free orbits, and invariant annihilators.**

Lemma 9.2 (A free ball). *The points with trivial stabilizer form an open dense subset of M°. Every nonempty invariant interior open set contains a ball with pairwise disjoint translated closures.*

Proof. Let $\bar{g}$ be the average of any smooth metric under G. Every $\gamma \in G$ is an isometry of $\bar{g}$. If an isometry equals the identity on a nonempty interior open set, the set

$$Z_\gamma = \{x \in M^\circ \,:\, \gamma x = x,\ D\gamma_x = I_{T_xM}\}$$

is nonempty and closed in M°. It is also open: for $x \in Z_\gamma$, the identity $\gamma \exp_x(v) = \exp_{\gamma x}(D\gamma_x v)$ gives equality with the identity on an exponential neighborhood. Connectedness of M° gives $Z_\gamma = M^\circ$, and continuity extends equality to the boundary. Effectiveness excludes this for $\gamma \neq 1$. Hence each fixed-point set of a nonidentity element is closed with empty interior. The complement of their finite union is open and dense and consists exactly of points with trivial stabilizer.

If $G = \{1\}$, take any ball compactly contained in the prescribed open set. Suppose $G \neq \{1\}$ and choose a point x with trivial stabilizer in that set. The nonempty finite set $\{d_{\bar{g}}(x, \gamma x) \,:\, \gamma \neq 1\}$ has a positive minimum d_0. Choose a ball $B = B_{\bar{g}}(x, \rho)$ with compact closure in that open set and $3\rho < d_0$. For $\gamma \neq \eta$, the centers $\gamma x, \eta x$ have distance at least d_0, and the balls $\gamma B, \eta B$ have radius ρ. Their closures are disjoint. The same compact containment holds for every translated ball, since the prescribed open set is invariant. □

An ineffective action may be replaced by its effective quotient. Its eigenspace representations factor through that quotient. All subsequent actions are effective.

Let $T_\gamma f = f \circ \gamma^{-1}$. On a fixed invariant L^2 space these are orthogonal operators. The reference spectral projection Π and the perturbed projection Π_v commute with them. Choose an isometric intertwiner $U : \mathbb{R}^m \to E_\lambda$. The Gram frame is

$$U_v = \Pi_v U (U^* \Pi_v U)^{-1/2}. \tag{9.3}$$

The finite Gram matrix commutes with $\rho(G)$, as does its inverse square root; hence U_v is an intertwiner. For conformal metrics, identify the Hilbert spaces by the positive square root of the density ratio. Its multiplication factor is invariant, so it also commutes with the group. The spectral matrix therefore lies in (9.1).

The resolvent Neumann series and the finite Gram formula make this matrix analytic on every finite perturbation space. For potentials it is analytic in a small L^∞ ball. For conformal metrics, the conformal coefficients and their inverses are analytic functions of v in a sufficiently high C^k Banach algebra; the fixed-domain operator, resolvent, and cluster matrix inherit that analyticity. The analytic dependence follows from the resolvent expansion and Gram formula of Section 2 [38].

For $C \in \mathcal{A}$, put

$$F_C(x) = \sum_{i,j} C_{ij} u_i(x) u_j(x), \qquad b_C(x, y) = \sum_{i,j} C_{ij} u_i(x) u_j(y). \tag{9.4}$$

The intertwining identities imply

$$F_C(\gamma x) = F_C(x), \qquad b_C(\gamma x, \gamma y) = b_C(x, y). \tag{9.5}$$

Lemma 9.3 (Annihilator after restriction to invariant perturbations). *For potentials, the cokernel in $\mathcal{A}$ is*

$$\mathcal{C} = \{C \in \mathcal{A} \,:\, F_C = 0 \text{ on } O\}. \tag{9.6}$$

For conformal metrics, it is $\{C \in \mathcal{A} \,:\, F_C = 0 \text{ on } M\}$.

Proof. For potentials, the first derivative is

$$D\mathsf{L}(0)[f]_{ij} = \int_M f u_i u_j \, dv_g. \tag{9.7}$$

If C annihilates all invariant perturbations, test with $\sum_\gamma T_\gamma \phi$, where $\phi \in C_c^\infty(O)$ is arbitrary. Invariance of F_C and the density gives

$$0 = \int_M \Big(\sum_\gamma T_\gamma \phi \Big) F_C \, dv_g = |G| \int_M \phi F_C \, dv_g.$$

Thus $F_C = 0$ on O; the converse follows by substituting $F_C = 0$ in (9.7).

For conformal metrics, define

$$\mathscr{D} = \begin{cases} \lambda \,\mathrm{Id}, & n = 2, \\ \Delta_g + \beta\lambda, & n \geq 3, \quad \beta = 4/(n-2). \end{cases} \tag{9.8}$$

The pencil calculation gives

$$D\mathsf{L}(0)[f]_{ij} = -\int_M (\mathscr{D} f) u_i u_j \, dv_g. \tag{9.9}$$

The operator $\mathscr{D}$ commutes with G. Averaging arbitrary global tests gives $\mathscr{D} F_C = 0$. If $n = 2$, positivity of λ suffices; if $n \geq 3$, integration gives

$$0 = \int_M |dF_C|^2 \, dv_g + \beta\lambda \int_M F_C^2 \, dv_g.$$

Thus $F_C = 0$. □

Lemma 9.4. *For $0 \neq C \in \operatorname{coker} D\mathsf{L}(0)$ and any free ball in the allowed region, the contracted Hessian is represented, on a smaller ball, by a real self-adjoint operator T_C with a nonzero even homogeneous principal symbol taking both signs. Its order is at most zero; its kernel is smooth off the diagonal.*

Proof. Lemma 9.3 gives $F_C = 0$ on the allowed region. Theorem 3.4 identifies the representing operator as $-2B_C$ for potentials, $-2\lambda^2 B_C$ for surface conformal parameters, and $-2\mathscr{D} B_C \mathscr{D}$ in higher dimension, where $\mathscr{D} = \Delta_g + 4\lambda/(n-2)$. Lemma 3.1 and (3.5) apply on each free ball. Their orders are $-2-\ell$ or $2-\ell \leq 0$. The sign conclusion follows from harmonicity of the first nonzero even polynomial. Since C commutes with the group, its two-point coefficient, the reduced resolvent, and T_C are equivariant. □

9.3. **Oscillations on disjoint orbits.**

Lemma 9.5 (Orbit-sum oscillatory identity). *Let $\mathcal{T}$ be a real, self-adjoint, G-equivariant operator whose localized kernel is pseudodifferential of order $\sigma \leq 0$ near a free ball B and its translates, and smooth off the diagonal. For $a \in C_c^\infty(B;\mathbb{R})$ and a nonzero constant coordinate covector ξ, put*

$$f_N(x) = a(x)\cos(Nx\cdot\xi), \qquad F_N = \sum_{\gamma\in G} T_\gamma f_N.$$

If t_σ is the real even principal symbol, then

$$\langle \mathcal{T} F_N, F_N \rangle = \frac{|G| N^\sigma}{2} \int_B a^2 t_\sigma(x,\xi)\, dv_g + O(N^{\sigma-1}). \tag{9.10}$$

For a fixed finite family of amplitudes with disjoint compact supports in B, all mixed pairings between their orbit sums are $O(N^{-L})$ for every L.

Proof. On one ball the ordinary oscillatory formula is

$$\langle \mathcal{T} f_N, f_N \rangle = \frac{N^\sigma}{2} \int a^2 t_\sigma(x,\xi)\, dv_g + O(N^{\sigma-1}).$$

After conjugating the density, write a localized symbol as $t(x,\eta)$. Applied to $ae^{iNx\cdot\xi}$ its demodulated expression is the inverse Fourier integral of $t(x, N\xi+\eta)\widehat{a}(\eta)$. For $|\eta| \leq N|\xi|/2$, expand the symbol at $N\xi$, obtaining $N^\sigma t_\sigma(x,\xi)a + O(N^{\sigma-1})$ in every fixed finite differentiability norm. On the complement the Fourier transform of a decays faster than every power. The two exponential signs give the cosine formula; their mixed phase integrals are rapidly decaying by integration by parts. This is Lemma 3.3.

Equivariance and unitarity give

$$\langle \mathcal{T}\, T_\gamma f_N, T_\gamma f_N\rangle = \langle T_\gamma \mathcal{T} f_N, T_\gamma f_N\rangle = \langle \mathcal{T} f_N, f_N\rangle.$$

Thus the diagonal orbit sum is $|G|$ times the single-ball pairing. For $\gamma \neq \eta$, the two compact supports are disjoint and the kernel on their product is smooth. In local coordinates on the first support, choose a smooth vector field X with $X\varphi = 1$ for its oscillatory phase. Repeated integration by parts with the identity $e^{iN\varphi} = (iN)^{-1}X(e^{iN\varphi})$ gives, for every integer L, an absolute bound $C_L N^{-L}$. The same estimate applies to both signs of the cosine phases. The finite sum over $\gamma \neq \eta$ has that bound. For distinct base amplitudes, their compact supports and all their corresponding translates remain disjoint. The identical integration formula therefore bounds each mixed pairing by $C_L N^{-L}$. □

Theorem 9.6 (Infinite inertia in invariant perturbations). *For every nonzero $C \in \mathcal{C}$, the form $f \mapsto \langle C, D^2\mathsf{L}(0)[f,f]\rangle$ has infinite positive and negative indices on $\mathcal{K}$. It retains these indices on every finite-codimensional linear subspace of $\mathcal{K}$.*

Proof. Fix $N_0 \geq 1$. By Lemma 9.4, a free ball contains a smaller ball and a covector $\xi \neq 0$ on which the principal symbol of T_C has a strict fixed sign. Choose nonzero amplitudes $a_1, \dots, a_{N_0}$ with disjoint compact supports there. Let $F_{j,N}$ be their orbit sums, and put

$$d_j = \frac{|G|}{2}\int a_j^2 t_\sigma(x,\xi)\, dv_g, \qquad c = \min_j |d_j| > 0.$$

Lemma 9.5 gives the quadratic matrix

$$M_N = \big(\langle T_C F_{i,N}, F_{j,N}\rangle\big)_{i,j} = N^\sigma \operatorname{diag}(d_1, \dots, d_{N_0}) + O(N^{\sigma-1}).$$

For sufficiently large N, its error has operator norm at most $cN^\sigma/2$. Hence M_N is definite with the prescribed sign and $|z^T M_N z| \geq cN^\sigma |z|^2/2$.

The exact first-derivative constraints are imposed as follows. Put $a = \dim \mathcal{R}$ and choose invariant smooth preimages $b_1, \dots, b_a$ of a basis of $\mathcal{R}$. Let $\ell_1, \dots, \ell_a$ be the corresponding independent derivative coordinates, so $\ell_i(b_j) = \delta_{ij}$. Equations (9.7)–(9.9), with integration by parts for conformal parameters, express each coordinate as integration against a fixed smooth function. Consequently $\ell_i(F_{j,N}) = O(N^{-L})$ for every L and every i, j. Put

$$\widehat{F}_{j,N} = F_{j,N} - \sum_{i=1}^{a} \ell_i(F_{j,N}) b_i. \tag{9.11}$$

These functions are invariant, are in $\mathcal{K}$, and differ from $F_{j,N}$ by $O(N^{-L})$ in every fixed smooth norm. For potentials all supports remain compactly inside O; B_C is globally L^2 bounded because R_λ is bounded and the u_i are smooth. For conformal metrics the global condition $F_C = 0$ makes $T_C = -2\mathscr{D} B_C \mathscr{D}$ have order at most zero globally, hence it is L^2 bounded. In both cases the correction changes every entry of the fixed quadratic matrix by $O(N^{-L})$ for every L. Taking $L > 1 - \sigma$ makes this difference $o(N^\sigma)$, so the definite matrix estimate persists.

The opposite symbol sign proves the opposite index. Since N_0 was arbitrary, both indices are infinite. A subspace of codimension b intersects any $(N_0 + b)$-dimensional definite subspace in dimension at least N_0, so its restriction still has infinite index. □

Remark 9.7. Suppose $G \neq \{1\}$ and let B be a free ball. Choose $\gamma \neq 1$ and nonzero functions f_j with pairwise disjoint compact supports in B. The averaging projection

$$P_G = \frac{1}{|G|}\sum_{g\in G} T_g$$

satisfies $P_G(f_j - T_\gamma f_j) = 0$. Restriction to B shows that the functions $f_j - T_\gamma f_j$ are linearly independent. Thus $\ker P_G$ is infinite-dimensional and the invariant perturbation space has infinite codimension.

Proof of Theorem 9.1. The preceding theorem gives infinite positive and negative index for every nonzero cokernel contraction in the invariant derivative kernel. Corollary 4.2 supplies a nonzero regular quadratic zero. Theorem 4.5 gives the section, the curve, and the dimension $D + 1$; if $d = 0$, the inverse function theorem applies on a D-dimensional subspace. All selected vectors lie in E_0. Their finite support union is a compact subset of O in the potential case. The invariant Gram frame takes values in (9.1), and its fixed contour preserves the cluster rank. The eigenvalue along the scalar level is therefore exactly λ with multiplicity exactly m. The lower bound at a singular reference is the cokernel Taylor estimate in Corollary 4.6. □

9.4. **Generic irreducibility.**

Theorem 9.8. *For every effective finite smooth action on a closed connected manifold of dimension at least two, a residual subset of every invariant conformal class has irreducible real Laplace eigenspaces. The same conclusion holds in the space of all invariant smooth metrics. For invariant potentials, fix a free ball B with disjoint translates and set $K = \bigcup_{g\in G} g\overline{B} \Subset O$. In the Fréchet space of invariant smooth potentials supported in K, irreducibility of every real eigenspace is residual.*

Lemma 9.9 (Equivariant matrix separation)**.** *If E is a finite-dimensional real orthogonal G-module, choose an orthogonal decomposition $E = E_1 \oplus \cdots \oplus E_s$ into irreducible invariant subspaces. For pairwise distinct real numbers $c_1, \dots, c_s$, the symmetric operator*

$$Y = \sum_{\alpha=1}^{s} c_\alpha P_{E_\alpha} \tag{9.12}$$

commutes with G and has exactly those subspaces as its eigenspaces. It may be arbitrarily small.

Proof. Invariant orthogonal complements give complete reducibility by finite-dimensional induction. Each orthogonal projection commutes with G, and (9.12) has the stated eigenspaces by distinctness of its coefficients. □

Proof of Theorem 9.8. At any reference, apply Theorem 9.1 to (9.12). The equivariant Gram frame identifies the new matrix eigenspaces with the corresponding operator eigenspaces as G-modules. Thus a single isolated cluster can be split into irreducible eigenspaces by an arbitrarily small allowed perturbation.

For openness, suppose a finite cluster is irreducible. In a small invariant neighborhood its spectral projection has the same rank and an equivariant Gram isomorphism to the reference representation. Any split into distinct eigenvalues would give a nontrivial invariant orthogonal decomposition, contradicting irreducibility. Exterior gaps remain open. Thus the property that all eigenspaces meeting the first N ordered eigenvalues are irreducible is open, including the whole final cluster if it crosses the index cutoff.

For density at a finite cutoff, isolate all the finitely many clusters concerned. Split a reducible current cluster by a perturbation small enough to preserve every exterior gap and every irreducible cluster already obtained. At each stage preserve all existing gaps within the chosen finite spectral interval. Every strict splitting increases the number of its distinct eigenvalues, which is bounded by the fixed total rank of its spectral projections. After finitely many splittings every eigenspace in that interval is irreducible. Consequently the finite-cutoff property is dense.

The smooth invariant metric space is an open cone in a Fréchet space and is Baire. The positive invariant conformal-factor space is likewise Baire. The countable intersection of the open dense finite-cutoff sets is residual in each. On a closed connected manifold, the zero Laplace eigenspace consists of constants, an irreducible trivial representation.

For potentials, let K be as in Theorem 9.8 and put

$$\mathcal{E}_K = \{q \in C^\infty(M;\mathbb{R})^G \,:\, \operatorname{supp} q \subset K\}.$$

This is a closed Fréchet subspace and hence is Baire. At each reference in it, apply the local theorem with the nonempty invariant open set $\operatorname{int} K$, with additional perturbations in $C_c^\infty(\operatorname{int} K)^G \subset \mathcal{E}_K$. The

same finite-cutoff argument applies. The numerical eigenvalues are continuous in L^∞ by the min–max principle, and the Riesz projections are continuous in the same small neighborhoods.

Finally, for a finite normal cover $p : M \to M_0$, pullback identifies smooth metrics on M_0 with invariant smooth metrics on M. It also identifies each conformal class with the invariant conformal class upstairs. Thus the inverse image of the residual set under pullback is residual among metrics on the base; this is the generic irreducibility statement formulated in [64]. □

Real, complex, and quaternionic multiplicities. Writing the real representation as $\bigoplus_\rho V_\rho^{\oplus n_\rho}$, its self-adjoint commutant is the orthogonal product of Hermitian multiplicity spaces over $\mathbb{D}_\rho = \mathrm{End}_G(V_\rho) \cong \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Their real dimensions are respectively

$$\frac{n_\rho(n_\rho+1)}{2}, \qquad n_\rho^2, \qquad n_\rho(2n_\rho-1).$$

In particular,

$$\mathrm{Sym}_G(E_\lambda) \cong \bigoplus_\rho \mathrm{Herm}_{n_\rho}(\mathbb{D}_\rho)$$

as real vector spaces.

9.5. **Finite covers and a quaternionic example.**

9.5.1. *The generic spectrum of an intermediate cover.* Let $\widetilde{M} \to M_0$ be a connected normal finite cover with group G, and let $H \le G$. A metric from the base gives the intermediate manifold $M_H = \widetilde{M}/H$. Pullback identifies

$$E_\lambda(M_H) \cong E_\lambda(\widetilde{M})^H.$$

For metrics in the residual set above, $E_\lambda(\widetilde{M})$ is an irreducible real V_ρ, so

$$\mathrm{mult}_{M_H}(\lambda) = \dim_{\mathbb{R}} V_\rho^H, \tag{9.13}$$

with the value zero meaning that λ does not occur downstairs. The same residual base set works for every subgroup, since there are only finitely many subgroups.

Choose normalized smooth functions ϕ_j with pairwise disjoint compact supports in a free ball. The spaces

$$V_j = \mathrm{span}\{T_g\phi_j : g \in G\}$$

are mutually orthogonal copies of the real regular representation. Every irreducible real representation therefore has an infinite-dimensional isotypic subspace in $L^2(\widetilde{M})$. Its orthogonal projection commutes with the Laplacian. Since each Laplace eigenspace is finite-dimensional and the spectral projections have dense sum, each real irreducible type occurs at infinitely many distinct eigenvalues. It follows that the intermediate cover has generically simple spectrum exactly when

$$\dim_{\mathbb{R}} V_\rho^H \le 1 \quad \text{for every irreducible real } V_\rho. \tag{9.14}$$

For the permutation representation $\mathbb{R}[G/H]$, real Frobenius reciprocity gives

$$\dim_{\mathbb{R}} V_\rho^H = m_\rho \dim_{\mathbb{R}} \mathrm{End}_G(V_\rho)$$

when m_ρ is the multiplicity in $\mathbb{R}[G/H]$. Hence (9.14) is equivalent to this permutation representation being multiplicity-free and having only real Schur types. For $\dim M \ge 2$, Theorem 9.8 gives this criterion. Under the additional representation-dimension hypothesis it is Zelditch's criterion [64, Theorem B].

9.5.2. *A degree-eight cover of a genus-two surface.* Let

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}, \qquad i^2 = j^2 = k^2 = ijk = -1.$$

For the closed oriented genus-two surface, define a homomorphism

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle \longrightarrow Q_8$$

by sending both a_1, a_2 to i and both b_1, b_2 to j. Since $[i, j] = -1$, the relator maps to one; the homomorphism is onto. The corresponding connected regular cover has degree eight, and

$$\chi(\widetilde{\Sigma}) = 8\chi(\Sigma_2) = -16, \qquad \widetilde{\Sigma} \cong \Sigma_9.$$

The deck action is free.

The real irreducible representations of Q_8 are its four one-dimensional characters and the four-dimensional left action on $\mathbb{H}$. The evaluations at the four characters together with $\mathbb{R}[Q_8] \to \mathbb{H}$ give an isomorphism of real algebras

$$\mathbb{R}[Q_8] \cong \mathbb{R}^4 \oplus \mathbb{H}.$$

For injectivity, write a group-algebra element with coefficients c_g and put $s_u = c_u + c_{-u}$ and $d_u = c_u - c_{-u}$ for $u = 1, i, j, k$. Its four character evaluations are

$$s_1 + \epsilon s_i + \eta s_j + \epsilon\eta s_k, \qquad (\epsilon, \eta) \in \{\pm 1\}^2.$$

If they vanish, summing after multiplication by $1, \epsilon, \eta, \epsilon\eta$ gives $s_1 = s_i = s_j = s_k = 0$. Its quaternionic evaluation is $d_1 + d_i i + d_j j + d_k k$; vanishing gives $d_1 = d_i = d_j = d_k = 0$. Hence all coefficients vanish. Both real algebras have dimension eight, so the homomorphism is an isomorphism. The quaternionic module is irreducible: if an invariant real subspace contains $q \neq 0$, it contains q, iq, jq, kq, which are a basis because right multiplication by q is invertible.

Consequently, within the pullback of *every* smooth conformal class on Σ_2, a residual set of metrics on Σ_9 has all real eigenspace dimensions in

$$\{1, 4\}. \tag{9.15}$$

Both dimensions occur infinitely often by the regular-representation decomposition. Here $\dim M = 2 < 4 = \max_\rho \dim_{\mathbb{R}} V_\rho$. The parameter space consists of all smooth metrics in the indicated conformal class, without a constant-curvature condition.

9.6. **The common symmetry of an admissible family.** For a compact linear group $G \subset \mathrm{O}(n)$ define

$$\widehat{G} = \{h \in \mathrm{O}(n) \,:\, hx \in Gx \text{ for every } x \in \mathbb{R}^n\}. \tag{9.16}$$

Thus $\widehat{G}$ preserves each G-orbit setwise.

Lemma 9.10. *The set $\widehat{G}$ is a compact subgroup of $\mathrm{O}(n)$ containing G, and $\widehat{G}x = Gx$ for every x. Every G-invariant domain and every G-invariant scalar function are $\widehat{G}$-invariant. If G is finite, then $\widehat{G} = G$.*

Proof. If h_1, h_2 satisfy (9.16), then $h_1 h_2 x \in G(h_2 x) = Gx$. Given h satisfying (9.16), set $y = h^{-1}x$. Since $x = hy \in Gy$, $Gy = Gx$ and hence $h^{-1}x \in Gx$. Closedness follows from compactness of every orbit. Every invariant domain is a union of G-orbits, and every invariant scalar function is constant on each such orbit.

If G is finite, fix $h \in \widehat{G}$. For every x, some $g \in G$ satisfies $(h - g)x = 0$, so

$$\mathbb{R}^n = \bigcup_{g \in G} \ker(h - g).$$

A finite union of proper linear subspaces cannot cover $\mathbb{R}^n$; for instance, each has Lebesgue measure zero. Thus $h = g$ for some $g \in G$. □

Proposition 9.11 (Common-symmetry obstruction). *Let a norm-resolvent-continuous family of self-adjoint operators on a fixed real Hilbert space commute with a fixed orthogonal representation of a compact group $H \supset G$. Let E be a finite-dimensional isolated reference eigenspace. In the equivariant Riesz–Gram frame, its local cluster map takes values in*

$$\mathrm{Sym}_H(E) \subseteq \mathrm{Sym}_G(E).$$

If this inclusion is strict, the cluster map has no local section onto a neighborhood of the scalar reference matrix in $\mathrm{Sym}_G(E)$*. The scalar matrix is not an essential value relative to arbitrary small continuous perturbations with values in that larger target.*

Proof. Let Π_p be the cluster projection and let $U : \mathbb{R}^m \to E$ be an orthonormal reference frame. The projection commutes with H. Its Gram normalization

$$U_p = \Pi_p U (U^* \Pi_p U)^{-1/2}$$

intertwines the same reference representation of H, since both the Gram matrix and its inverse square root commute with that representation. Hence $\mathsf{L}(p) = U_p^* P_p U_p$ belongs to $\mathrm{Sym}_H(E)$. This is the equivariant version of the frame formula in Section 2.

Choose $A \in \mathrm{Sym}_G(E) \setminus \mathrm{Sym}_H(E)$. No nonzero ϵA can be the displacement of the cluster map from a scalar matrix. Moreover,

$$\widetilde{\mathsf{L}}(p) = \mathsf{L}(p) + \epsilon A$$

is uniformly arbitrarily close to L as $\epsilon \to 0$ and never attains the reference scalar matrix. Indeed, attainment would imply $\epsilon A \in \mathrm{Sym}_H(E)$. □

For a G-invariant Euclidean domain Ω, every $h \in \widehat{G}$ preserves both Ω and the Euclidean metric. Pullback by h therefore commutes with the Dirichlet and Neumann Laplacians and with every G-invariant scalar potential.

Theorem 9.12. *Let $G = \mathrm{Sp}(1)$ act on $\mathbb{H} \cong \mathbb{R}^4$ by left multiplication. Every shell*

$$\Omega_{a,b} = \{q : a < |q| < b\}, \qquad 0 < a < b,$$

is a bounded connected smooth domain with a free G-action. It has infinitely many distinct positive Neumann eigenvalues whose real G-eigenspaces are reducible. Every sufficiently small G-invariant C^2 normal deformation of the shell has reducible eigenspaces in angular degree two.

9.7. **The quaternionic degree-two eigenspace.** Write $q = x_0 + x_1 i + x_2 j + x_3 k$. Unit quaternions act orthogonally and effectively by $q \mapsto aq$. If $q \neq 0$, then $aq = q$ implies $a = 1$, so the action is free away from the origin. It is transitive on every centered sphere: for $|q| = |q'| \neq 0$, the quaternion $q' q^{-1}$ is a unit and sends q to q'. Therefore

$$\widehat{\mathrm{Sp}(1)} = \mathrm{O}(4). \tag{9.17}$$

Every invariant domain is a union of centered spheres. In particular, any invariant domain sufficiently close to a shell is another concentric shell.

Let $e_1 = i, e_2 = j, e_3 = k$ and define

$$P_{ba}(q) = \langle e_b, q e_a \overline{q} \rangle.$$

The corresponding matrix is

$$P(q) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}. \tag{9.18}$$

Lemma 9.13. *The nine entries of P are a basis of the real harmonic homogeneous quadratics on $\mathbb{R}^4$. Restricted to S^3, they have Laplace eigenvalue 8 and satisfy*

$$P(q)^T P(q) = P(q)P(q)^T = |q|^4 I_3, \qquad P(aq) = P(a)P(q). \tag{9.19}$$

Under the left G-action on functions, their span decomposes as

$$\mathcal{H}_2(S^3)\big|_{\mathrm{Sp}(1)} \cong \mathrm{Ad} \oplus \mathrm{Ad} \oplus \mathrm{Ad}\,. \tag{9.20}$$

Each summand has real dimension three.

Proof. Every entry in (9.18) has zero Euclidean Laplacian. The three diagonal entries are independent trace-free diagonal quadratics. The other six entries span all six mixed monomials, by adding and subtracting the paired entries. The space of harmonic quadratics has dimension $10 - 1 = 9$, so the nine entries form a basis.

Quaternionic multiplication implies the covariance identity in (9.19). Conjugation by a unit quaternion preserves lengths on the imaginary quaternions; scaling gives the other identities. The Euclidean polar formula in dimension four sends a harmonic homogeneous polynomial of degree ℓ to a spherical eigenfunction with eigenvalue $\ell(\ell+2)$. For $\ell = 2$ this is 8.

For a fixed column a, pullback by left multiplication gives

$$P_{ba}(s^{-1}q) = \sum_c P_{bc}(s^{-1})P_{ca}(q).$$

Thus each column spans an invariant three-plane. Conjugations by unit quaternions give all rotations of the imaginary quaternions: conjugation by $\cos(t/2) + e\sin(t/2)$ is rotation through angle t about the unit imaginary axis e. The real three-dimensional rotation representation is irreducible, since the orbit of a nonzero vector spans $\mathbb{R}^3$. Independence of the nine entries makes the three-column sum direct. □

For normalized spherical measure the Gram matrix is

$$\int_{S^3} P_{ba}(\omega)P_{dc}(\omega)\, d\mu(\omega) = \frac{1}{3}\delta_{bd}\delta_{ac}. \tag{9.21}$$

For normalized spherical measure, put $A = \int x_0^4$ and $B = \int x_0^2 x_1^2$. Orthogonal invariance and reflection symmetry give

$$A = \frac{1}{4}\int (x_0 + x_1)^4 = \frac{1}{2}A + \frac{3}{2}B, \qquad 1 = \int \Big(\sum_{i=0}^{3} x_i^2\Big)^2 = 4A + 12B.$$

Hence $A = 1/8$ and $B = 1/24$; every monomial with an odd exponent has integral zero. Substitution into (9.18) gives (9.21).

Right multiplication rotates the columns, whereas left multiplication rotates the rows. Consequently the self-adjoint commutants on this nine-plane are

$$\mathrm{Sym}_G(\mathcal{H}_2) = \mathrm{Sym}_3(\mathbb{R}) \otimes I_3, \qquad \mathrm{Sym}_{\mathrm{SO}(4)}(\mathcal{H}_2) = \mathbb{R} I_9. \tag{9.22}$$

Here the first tensor factor is the multiplicity space indexed by the columns. Indeed, a real matrix commuting with all three-dimensional rotations is scalar: rotations fixing a unit vector force its image to lie on that vector's line, and linearity makes the scalar independent of the vector. Every block of a left-commuting operator is therefore scalar. Requiring right rotations as well makes its matrix on the columns scalar. An invariant proper subspace of an orthogonal representation would have a nontrivial commuting orthogonal projection; hence (9.22) also proves irreducibility under the larger group.

9.8. **Separation of variables and an isolated multiplicity-nine cluster.** On $\Omega_{a,b}$ write $q = r\omega$, $\omega \in S^3$. The nonnegative Laplacian is

$$\Delta = -\partial_r^2 - \frac{3}{r}\partial_r + \frac{1}{r^2}\Delta_{S^3}.$$

In angular degree ℓ, the radial operator is

$$T_\ell R = -r^{-3}(r^3 R')' + \frac{\ell(\ell+2)}{r^2}R, \qquad R'(a) = R'(b) = 0, \tag{9.23}$$

on $L^2((a,b), r^3\,dr)$. It is associated with the closed quadratic form

$$\int_a^b \left(|R'|^2 + \frac{\ell(\ell+2)}{r^2}|R|^2 \right) r^3\,dr$$

on $H^1(a,b)$. Since $a > 0$, this is a regular Sturm–Liouville problem. Compactness of the interval Sobolev embedding gives discrete spectrum tending to infinity and a complete eigenbasis. Each radial eigenvalue is simple: for a fixed eigenvalue, the Neumann initial condition at a leaves a one-dimensional space of ODE solutions, since a solution with both $R(a) = R'(a) = 0$ is zero.

Every radial eigenfunction of T_2 gives the nine Neumann eigenfunctions

$$u_{ba}(r,\omega) = R(r)P_{ba}(\omega). \tag{9.24}$$

The boundary conditions follow from $R'(a) = R'(b) = 0$. These functions are independent and contain the three invariant column subspaces in (9.20). Any full eigenspace containing them is reducible, regardless of coincidences with other angular degrees. The T_2 eigenvalues are positive because their Rayleigh quotients are at least $8/b^2$. There are infinitely many distinct such eigenvalues.

A C^2 normal graph over either spherical boundary component is G-invariant precisely when its displacement is constant, because G acts transitively on that component. Hence a neighborhood of the shell in the space of invariant domains is identified with an open set of pairs (a', b') satisfying $0 < a' < b'$. Every domain in this neighborhood has the three-column eigenspaces above. This proves Theorem 9.12.

9.8.1. *Rational spectral isolation on one explicit shell.* Let $\mu_{\ell,j}$, $j = 0, 1, \dots$, be the radial eigenvalues of (9.23). Put $\kappa_\ell = \ell(\ell+2)$. Directly from the Rayleigh quotient,

$$\frac{\kappa_\ell}{b^2} \le \mu_{\ell,0} \le \frac{2\kappa_\ell}{a^2+b^2}. \tag{9.25}$$

The constant trial function gives the upper bound. Also, for $j \ge 1$,

$$\mu_{\ell,j} \ge \mu_{0,1} \ge \left(\frac{a}{b}\right)^3 \frac{\pi^2}{(b-a)^2}. \tag{9.26}$$

Indeed, the weighted kinetic Rayleigh quotient is at least $(a/b)^3$ times the unweighted quotient for every function. Min–max and the first positive Neumann interval eigenvalue give (9.26); the angular potential is nonnegative.

Take $a = 1$ and $b = 11/10$. Equations (9.25)–(9.26) give

$$\mu_{0,0} = 0, \qquad \mu_{1,0} \le \frac{600}{221} < 4, \tag{9.27}$$

$$\frac{800}{121} \le \mu_{2,0} \le \frac{1600}{221}, \qquad \mu_{\ell,0} \ge \frac{1500}{121} > 10 \quad (\ell \ge 3), \tag{9.28}$$

$$\mu_{\ell,j} > \frac{900000}{1331} > 10 \qquad (j \ge 1), \tag{9.29}$$

since $\pi^2 > 9$. The spherical harmonics of all degrees give a complete orthogonal decomposition: homogeneous polynomial harmonic decomposition and density of polynomial restrictions imply completeness. Therefore $(4, 10)$ contains exactly the angular-degree-two radial-ground eigenvalue. Its total multiplicity is exactly nine, and its real G-module is $\mathrm{Ad}^{\oplus 3}$.

Corollary 9.14 (Persistence of the scalar nine-dimensional cluster). *On $1 < |q| < 11/10$, let V be any smooth G-invariant potential with $\|V\|_\infty < 1$. The interval $(4, 10)$ contains exactly one eigenvalue of $\Delta + V$, of multiplicity nine, with G-module $\mathrm{Ad}^{\oplus 3}$.*

Proof. An invariant potential is radial. Every radial eigenvalue changes by at most $\|V\|_\infty$ by the min–max principle. The displayed bounds remain strictly separated from 4 and 10 after these shifts: $600/221 + 1 < 4$, $800/121 - 1 > 4$, $1600/221 + 1 < 10$, and $1500/121 - 1 > 10$. The radial ground state remains simple by the ODE argument. The same nine spherical harmonics therefore account for the whole cluster. □

9.9. **Free actions on domains in higher dimension.** Let $k \geq 1$, write $\mathbb{R}^{4+k} = \mathbb{H} \times \mathbb{R}^k$, and let G act by

$$a \cdot (q, y) = (aq, y).$$

Let $D \Subset (0, \infty) \times \mathbb{R}^k$ be any bounded connected smooth domain and set

$$\Omega_D = \{(q, y) \,:\, (|q|, y) \in D\}. \tag{9.30}$$

This is diffeomorphic to $S^3 \times D$, is smooth and bounded, and has a free G-action. For $h \in \mathrm{O}(4)$, the equality $|hq| = |q|$ gives $(hq, y) \in \Omega_D$ exactly when $(q, y) \in \Omega_D$; hence the domain is $\mathrm{O}(4)$-invariant in the q-variable.

Theorem 9.15. *Every domain* (9.30) *has infinitely many distinct positive Neumann eigenvalues with reducible G-eigenspaces. The same reducibility conclusion holds with Dirichlet boundary conditions, or with an arbitrary invariant scalar potential in $C^\infty(\overline{\Omega_D}; \mathbb{R})$ added (for infinitely many positive eigenvalues).*

Proof. The identity $dq\, dy = r^3\, dr\, dv_{S^3}\, dy$ and the orthogonal spherical-harmonic decomposition give the unitary isomorphism

$$L^2(\Omega_D, dq\, dy) \cong \widehat{\bigoplus}_{\ell \geq 0} L^2(D, r^3\, dr\, dy) \otimes \mathcal{H}_\ell(S^3, dv_{S^3}).$$

In degree ℓ, the meridional operator is

$$T_{\ell,D} = -r^{-3}\partial_r(r^3\partial_r) - \Delta_y^{\mathrm{Eucl}} + \frac{\ell(\ell+2)}{r^2}, \tag{9.31}$$

where $-\Delta_y^{\mathrm{Eucl}}$ is nonnegative. It is associated with the form

$$\int_D \left(|\partial_r A|^2 + |\nabla_y A|^2 + \frac{\ell(\ell+2)}{r^2}|A|^2 \right) r^3 dr\, dy$$

on $H^1(D)$, or $H_0^1(D)$ in the Dirichlet case. The weight is smooth and bounded above and below by positive numbers because $\overline{D}$ stays away from $r = 0$. Compact embedding gives infinitely many distinct eigenvalues with finite multiplicities and smooth eigenfunctions.

If A is a meridional degree-two eigenfunction, every $A(r, y)P_{ba}(\omega)$ is an eigenfunction on Ω_D. Let (ν_r, ν_y) be the meridional outward unit normal. The full outward normal is $(\nu_r\omega, \nu_y)$, and

$$\partial_{\nu_{\Omega_D}}(AP_{ba}) = (\nu_r\partial_r A + \nu_y \cdot \nabla_y A)P_{ba}.$$

Thus the weighted natural Neumann condition is exactly the required full Neumann condition. For Dirichlet data, $A|_{\partial D} = 0$ implies $(AP_{ba})|_{\partial\Omega_D} = 0$. For each fixed $a \in \{1, 2, 3\}$, the functions $AP_{1a}, AP_{2a}, AP_{3a}$ span a three-dimensional invariant subspace by the covariance identity $P(aq) = P(a)P(q)$. The three subspaces form a direct sum because the nine angular functions are independent and $A \not\equiv 0$. An invariant scalar potential is a function of (r, y), so it preserves this separation and adds multiplication by V to $T_{2,D}$. If ν_j and $\nu_j(V)$ are the ordered meridional eigenvalues, the Rayleigh quotient gives

$$\nu_j - \|V\|_\infty \leq \nu_j(V) \leq \nu_j + \|V\|_\infty.$$

Since $\nu_j \to +\infty$, infinitely many distinct perturbed meridional eigenvalues are positive; each associated full eigenspace contains the three displayed invariant subspaces. □

For an explicit connected-boundary example in $\mathbb{R}^5$, take

$$\Omega_* = \{(q, y) : (|q| - 2)^2 + y^2 < 1/16\}. \tag{9.32}$$

Its quotient is a disk with $7/4 < r < 9/4$. The boundary is diffeomorphic to $S^3 \times S^1$ and is connected; the domain is simply connected because it is diffeomorphic to S^3 times a disk. A sufficiently small G-invariant normal deformation of $\partial\Omega_*$ is determined by a function on the quotient circle. Its domain still has the form (9.30), with r bounded away from zero. The quadratic-form decomposition in Theorem 9.15 holds also for C^2 boundaries of D. Thus every domain in a C^2 neighborhood of Ω_* in the space of G-invariant domains has infinitely many positive eigenvalues with reducible real G-eigenspaces.

9.10. **Failure of local sections in the larger commutant.** Consider the fixed shell in Corollary 9.14. Its invariant scalar perturbations are precisely the radial functions. Let $\lambda_* = \mu_{2,0}$ at $V = 0$ and choose the nine-dimensional reference frame from (9.24), normalized according to (9.21). For every sufficiently small invariant potential, the exact whole-cluster matrix is

$$\mathsf{L}(V) = \mu_2(V) I_9. \tag{9.33}$$

This equality is independent of the choice of orthonormal frame, since the entire cluster is one eigenspace. In particular,

$$\mathrm{Sym}_G(E) = \mathrm{Sym}_3(\mathbb{R}) \otimes I_3, \qquad \dim_{\mathbb{R}} \mathrm{Sym}_G(E) = 6, \qquad \mathsf{L}(\mathcal{U}) \subset \mathbb{R} I_9, \tag{9.34}$$

where $\mathcal{U}$ is a sufficiently small ball of invariant smooth potentials. Since

$$\mu_2(V + t) = \mu_2(V) + t,$$

the image contains an interval in the scalar line. If perturbations are restricted to a nonempty invariant subannulus, choose a nonzero nonnegative radial f supported there. For the normalized positive radial ground state R,

$$D\mu_2(0)[f] = \int_a^b f(r) R(r)^2 r^3 \, dr > 0.$$

The restricted image also contains an interval in the scalar line.

For every $A \in \mathrm{Sym}_3(\mathbb{R})$ with $\operatorname{tr} A = 0$, let $C = A \otimes I_3$. Then

$$\langle C, \mathsf{L}(V) - \lambda_* I_9 \rangle = 0 \tag{9.35}$$

identically on the full invariant potential neighborhood. Consequently, for every $k \geq 1$ and invariant directions $V_1, \dots, V_k$,

$$\langle C, D^k \mathsf{L}(V)[V_1, \dots, V_k] \rangle = 0.$$

The space $\{A \otimes I_3 : A \in \mathrm{Sym}_3(\mathbb{R}),\ \operatorname{tr} A = 0\}$ has dimension five.

There is also a pointwise version. If R is normalized in $L^2((a, b), r^3 dr)$ and $d\omega$ is the unnormalized round measure, an orthonormal basis is $u_{ba} = \sqrt{3/\operatorname{Vol}(S^3)} R(r) P_{ba}(\omega)$ by (9.21). Consequently

$$\sum_{ba,dc} C_{(ba),(dc)} u_{ba}(r, \omega) u_{dc}(r, \omega) = \frac{3R(r)^2}{\operatorname{Vol}(S^3)} \operatorname{tr}\big(AP(\omega)^T P(\omega)\big) = \frac{3R(r)^2}{\operatorname{Vol}(S^3)} \operatorname{tr} A = 0.$$

Thus $F_C = 0$ pointwise. For $C \neq 0$, Theorem 3.4 gives infinite positive and negative index on the derivative kernel of the unrestricted potential map. On radial directions, every contraction $\langle C, D^2 \mathsf{L}(V) \rangle$ is zero by (9.35).

Corollary 9.16 (Trivial projection-connection holonomy on the isolated eigenspace). *On the fixed shell, over the ball of smooth radial potentials with $\|V\|_\infty < 1$, the nine-dimensional cluster bundle in Corollary 9.14 has a global parallel orthonormal frame for the connection $\nabla = \Pi\, d$. Every smooth loop of these potentials has holonomy I_9 on this block.*

Proof. Choose the positive normalized radial ground state R_V of $T_2 + V$. A ground state has one sign by minimizing the Rayleigh quotient with its absolute value; an interior zero of a nonnegative solution would also have zero derivative and would force the ODE solution to vanish. Positivity fixes the sign globally. Simplicity and the fixed-domain spectral projection show that R_V depends smoothly on V. Choose a fixed orthonormal angular basis $Y_1, \dots, Y_9$ from (9.21). The frame $u_j(V) = R_V Y_j$ is globally defined and

$$\langle u_i(V), du_j(V)\rangle = \delta_{ij}\langle R_V, dR_V\rangle_{L^2(r^3\,dr)} = 0,$$

since $\|R_V\|_{L^2(r^3\,dr)} = 1$. Thus its connection matrix is zero. □

For the Euclidean domains in Theorems 9.12 and 9.15, the set of domains with irreducible real G-eigenspaces is disjoint from a nonempty open set of invariant C^2 domains. A sufficiently small neighborhood is parametrized by an open subset of the Banach space of invariant C^2 normal displacements, and is therefore Baire. The irreducibility set is consequently not residual. This gives a counterexample to [41, Conjecture 1].

The corresponding invariant tensor metrics form a larger space. For example, let $\sigma_1, \sigma_2, \sigma_3$ be a left-invariant coframe on the unit quaternions and let a_1, a_2, a_3 be distinct positive constants. The metric

$$dr^2 + r^2\left(a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2\right)$$

on a quaternionic shell is invariant under left multiplication. Right multiplication rotates the coframe by the adjoint representation; invariance under every right multiplication would require $\mathrm{diag}(a_1, a_2, a_3)$ to commute with all of $SO(3)$, and hence to be scalar. Thus this metric is not invariant under the full right action.

10. Simultaneous spectral-cluster equations

Let $\lambda_1, \dots, \lambda_s$ be distinct isolated eigenvalues, of multiplicities $m_1, \dots, m_s$. The joint cluster map has target $\bigoplus_{a=1}^{s} \mathrm{Sym}_{m_a}(\mathbb{R})$ and derivative kernel $\bigcap_{a=1}^{s} \ker D\mathsf{L}_a(0)$.

10.1. **The joint derivative and its Hessian.** Let $\mathcal{H}_j$ denote the real spherical harmonics of degree j on a unit round sphere. In this subsection $d\mu$ denotes the unnormalized Riemannian measure. On S^n their eigenvalue is $\lambda_j = j(j+n-1)$. For a real potential q sufficiently close to zero, let $\mathsf{L}_j(q)$ be the matrix of the cluster issuing from $\mathcal{H}_j$, in its Riesz–Gram frame based at a fixed real orthonormal basis of $\mathcal{H}_j$. Thus $\mathsf{L}_j(0) = \lambda_j I$.

For several clusters, the first derivative is

$$D\mathsf{L}_a(0)[f]_{ij} = \int f u_{a,i} u_{a,j}\, d\mu. \tag{10.1}$$

Consequently a tuple $C = (C_a)$ annihilates the joint derivative precisely when

$$\sum_a \sum_{i,j} (C_a)_{ij} u_{a,i} u_{a,j} = 0. \tag{10.2}$$

Let K be the joint derivative kernel and let R_a be $(\Delta - \lambda_a)^{-1}$ on $\mathcal{H}_a^{\perp}$, extended by zero on $\mathcal{H}_a$. Differentiation of the cluster equation gives, for $f \in K$,

$$\sum_a \langle C_a, D^2\mathsf{L}_a(0)[f,f]\rangle = -2\sum_a \sum_{i,j} (C_a)_{ij}\langle f u_{a,i}, R_a(f u_{a,j})\rangle. \tag{10.3}$$

For $b \neq a$, the restriction of R_a to $\mathcal{H}_b$ is multiplication by $(\lambda_b - \lambda_a)^{-1}$. Thus every selected eigenspace occurs in the quadratic expression (10.3).

10.1.1. *Two clusters on S^n.*

Lemma 10.1. *For $n \geq 2$, the joint map $(\mathsf{L}_0, \mathsf{L}_1)$ has a local section through zero taking values in a finite-dimensional space of smooth real potentials. The section is smooth away from zero and is pairwise $1/2$-Hölder in every C^k norm. Its source space may be chosen with dimension $2 + (n+1)(n+2)/2$, one more than the target dimension.*

Proof. Put $v_n = \mathrm{Vol}(S^n)$. The derivative range is isomorphic to $\mathcal{H}_0 \oplus \mathcal{H}_2$, and its cokernel is one-dimensional. A cokernel functional is

$$F(q) = \frac{\operatorname{tr} \mathsf{L}_1(q)}{n+1} - \mathsf{L}_0(q) - n.$$

The joint derivative kernel is $K = (\mathcal{H}_0 \oplus \mathcal{H}_2)^\perp$. Its scalar Hessian is

$$D^2F(0)[f,f] = -\frac{8}{v_n} \sum_{k \neq 0,2} \frac{\|f_k\|_2^2}{\lambda_k(\lambda_k - 2(n+1))}, \qquad f = \sum_k f_k \in K. \tag{10.4}$$

An orthonormal basis of $\mathcal{H}_1$ is $\sqrt{(n+1)/v_n}\, x_i$, $1 \leq i \leq n+1$. For a homogeneous harmonic polynomial p of degree k,

$$x_i p = h_{k+1,i} + \frac{|x|^2}{2k+n-1} \partial_i p, \qquad \Delta_{\mathbb{R}^{n+1}} h_{k+1,i} = 0.$$

The two terms restrict to orthogonal spherical harmonics. Moreover,

$$\sum_i \|\partial_i p\|_2^2 = \big(k^2 + k(k+n-1)\big)\|p\|_2^2 = k(2k+n-1)\|p\|_2^2.$$

Thus the summed squared norms of the lower and upper components of $x_i p$ are respectively

$$\frac{k}{2k+n-1}\|p\|_2^2, \qquad \frac{k+n-1}{2k+n-1}\|p\|_2^2.$$

For $k \neq 0, 2$, the normalized first-cluster resolvent coefficient is

$$\begin{aligned} a_1(k) &= \frac{k}{(2k+n-1)(k-2)(k+n-1)} + \frac{k+n-1}{(2k+n-1)k(k+n+1)} \\ &= \frac{\lambda_k - 2(n-1)}{\lambda_k(\lambda_k - 2(n+1))}. \end{aligned}$$

The ground-state coefficient is $1/\lambda_k$, and their difference is $4/[\lambda_k(\lambda_k - 2(n+1))]$. Rotation invariance makes distinct $\mathcal{H}_k$ orthogonal for this scalar Hessian. Equation (10.3) proves (10.4).

The positive index of (10.4) is exactly $n+1$, and its negative index is infinite. Choose real unit vectors $f_1 \in \mathcal{H}_1$, $f_3 \in \mathcal{H}_3$. The vector

$$f_* = f_1 + \sqrt{\frac{3(n+4)}{n}}\, f_3$$

is a nonzero zero of the scalar quadratic form; its derivative in the f_1 direction is nonzero. Add $\mathcal{H}_0 \oplus \mathcal{H}_2$ to $\operatorname{span}\{f_1, f_3\}$. The derivative on $\mathcal{H}_0 \oplus \mathcal{H}_2$ is an isomorphism onto the joint range. Moreover,

$$D^2F(0)[f_*, f_1] = \frac{8}{v_n n(n+2)} \neq 0.$$

Theorem 4.5, with quadratic kernel space $\operatorname{span}\{f_1, f_3\}$, gives the section. The dimension is

$$\dim(\mathcal{H}_0 \oplus \mathcal{H}_2) + 2 = 2 + \frac{(n+1)(n+2)}{2}.$$

□

10.1.2. *The three-cluster statement.* From now on the sphere is S^2, $v = 4\pi$, and $\lambda_j = j(j+1)$. Write $z = x_3$ and let P_j be the Legendre polynomial normalized by $P_j(1) = 1$. Define

$$R_0 = \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4, \tag{10.5}$$

$$V_0 = \operatorname{span}\{P_1(z), P_3(z), P_5(z), P_7(z), x_1, x_2, z(x_1^2 - x_2^2), 2zx_1x_2\}, \tag{10.6}$$

$$W = R_0 \oplus V_0. \tag{10.7}$$

Here $\dim R_0 = 15$, $\dim V_0 = 8$, and $\dim W = 23$. Every member of W is the restriction of a polynomial of degree at most seven.

Theorem 10.2. *There are $\delta > 0$ and a map*

$$\mathscr{S} \,:\, B_\delta(0) \subset \mathbb{R} \times \operatorname{Sym}_3(\mathbb{R}) \times \operatorname{Sym}_5(\mathbb{R}) \longrightarrow W$$

such that $\mathscr{S}(0) = 0$ and, for $Y = (Y_0, Y_1, Y_2)$,

$$\mathsf{L}_0(\mathscr{S}(Y)) = Y_0, \qquad \mathsf{L}_1(\mathscr{S}(Y)) = 2I_3 + Y_1, \qquad \mathsf{L}_2(\mathscr{S}(Y)) = 6I_5 + Y_2. \tag{10.8}$$

The section is smooth away from zero. For every integer $k \geq 0$,

$$\|\mathscr{S}(Y) - \mathscr{S}(Y')\|_{C^k} \leq C_k \|Y - Y'\|^{1/2}. \tag{10.9}$$

There are nonzero $q_j \in W$ tending to zero in C^∞ for which

$$(\mathsf{L}_0(q_j), \mathsf{L}_1(q_j), \mathsf{L}_2(q_j)) = (0, 2I_3, 6I_5)$$

and the joint derivative on W is surjective. No exponent greater than $1/2$ is possible in a uniform realization estimate at zero for all joint matrix directions.

10.1.3. *The joint cokernel.* Multiplication gives the rotation-equivariant isomorphism

$$\operatorname{Sym}^2 \mathcal{H}_l \longrightarrow \bigoplus_{j=0}^{l} \mathcal{H}_{2j}.$$

For $l = 0, 1, 2$ it follows either by direct multiplication of harmonic polynomials or from the spherical-harmonic product formula. Hence the joint derivative has range dimension 15, its restriction to R_0 is an isomorphism onto that range, and its kernel is $R_0^\perp$. The cokernel is isomorphic to

$$\mathbb{R}^2 \oplus \mathcal{H}_2. \tag{10.10}$$

All of V_0 belongs to the joint derivative kernel, since its elements are odd under the antipodal map.

Two scalar cokernel functionals are

$$F_1(q) = \frac{1}{3} \operatorname{tr} \mathsf{L}_1(q) - \mathsf{L}_0(q) - 2, \qquad F_2(q) = \frac{1}{5} \operatorname{tr} \mathsf{L}_2(q) - \mathsf{L}_0(q) - 6. \tag{10.11}$$

For $h \in \mathcal{H}_2$, let

$$J_l(h)_{ij} = \int_{S^2} h u_{l,i} u_{l,j} \, d\mu, \qquad l = 1, 2.$$

Rotation invariance on $\mathcal{H}_2$, followed by evaluation at $h = h' = P_2$, gives

$$\operatorname{tr}(J_l(h) J_l(h')) = \kappa_l \int_{S^2} h h' \, d\mu, \qquad \kappa_1 = \frac{6}{5v}, \quad \kappa_2 = \frac{10}{7v}. \tag{10.12}$$

Indeed, in the bases ordered by $|m| = 0, 1$ and $|m| = 0, 1, 2$,

$$J_1(P_2) = \operatorname{diag}\left(\frac{2}{5}, -\frac{1}{5}, -\frac{1}{5}\right), \quad J_2(P_2) = \operatorname{diag}\left(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, -\frac{2}{7}, -\frac{2}{7}\right), \tag{10.13}$$

and $\|P_2\|_2^2 = v/5$. Define

$$\Phi_h(q) = \frac{5}{v} \left(\kappa_1^{-1} \operatorname{tr}(J_1(h) \mathsf{L}_1(q)) - \kappa_2^{-1} \operatorname{tr}(J_2(h) \mathsf{L}_2(q)) \right). \tag{10.14}$$

Its value and derivative vanish at zero. Equations (10.12) and the orthogonality of harmonic degrees show that these five functionals, together with F_1, F_2, give the entire cokernel in (10.10).

10.1.4. *Three exact quadratic equations.* For an axisymmetric potential direction f, let $H_{lm}(f)$ be the second derivative in a normalized harmonic of degree l and order m. The real cosine and sine harmonics have the same value when $m > 0$. By (10.11) and (10.13),

$$D^2F_1(0)[f,f] = \frac{H_{10}(f) + 2H_{11}(f)}{3} - H_{00}(f), \tag{10.15}$$

$$D^2F_2(0)[f,f] = \frac{H_{20}(f) + 2H_{21}(f) + 2H_{22}(f)}{5} - H_{00}(f), \tag{10.16}$$

$$D^2\Phi_{P_2}(0)[f,f] = \frac{5}{3}\big(H_{10}(f) - H_{11}(f)\big) - H_{20}(f) - H_{21}(f) + 2H_{22}(f). \tag{10.17}$$

An axisymmetric element of $\mathcal{H}_2$ is a multiple of P_2. Vanishing of these three expressions therefore implies vanishing of all seven cokernel components at an axisymmetric direction.

Lemma 10.3. *For $f = aP_1 + bP_3 + cP_5 + dP_7$, the three expressions* (10.15)–(10.17) *are w^TA_iw, where $w = (a,b,c,d)^T$,*

$$A_1 = \operatorname{diag}\left(\frac{1}{3}, -\frac{1}{63}, -\frac{1}{990}, -\frac{1}{5250}\right), \tag{10.18}$$

$$A_2 = \operatorname{diag}\left(\frac{1}{3}, \frac{1}{42}, -\frac{1}{150}, -\frac{2}{2625}\right), \tag{10.19}$$

$$A_3 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{7} & 0 & 0 \\ -\frac{1}{7} & -\frac{59}{378} & \frac{25}{2079} & 0 \\ 0 & \frac{25}{2079} & \frac{149}{7722} & \frac{7}{7150} \\ 0 & 0 & \frac{7}{7150} & \frac{31}{16575} \end{pmatrix}. \tag{10.20}$$

Proof. Let the associated Legendre polynomials be $P_r^m = (-1)^m(1-z^2)^{m/2}(d/dz)^mP_r$ and set

$$N_{rm} = \int_{-1}^{1} (P_r^m)^2\,dz = \frac{2(r+m)!}{(2r+1)(r-m)!}.$$

Define

$$c_r(j;l,m) = N_{rm}^{-1}\int_{-1}^{1} P_jP_l^mP_r^m\,dz.$$

Orthogonality gives $P_jP_l^m = \sum_{r=m}^{j+l} c_r(j;l,m)P_r^m$. These coefficients satisfy

$$zP_r^m = \frac{r-m+1}{2r+1}P_{r+1}^m + \frac{r+m}{2r+1}P_{r-1}^m, \tag{10.21}$$

$$(j+1)P_{j+1} = (2j+1)zP_j - jP_{j-1}. \tag{10.22}$$

The lower term in (10.21) is absent at $r = m$. Equation (10.3) gives the (i,j) entry of the coefficient matrix for H_{lm} as

$$-2\sum_{\substack{r\ge m\\ r\ne l}} \frac{c_r(i;l,m)c_r(j;l,m)N_{rm}}{N_{lm}\big(r(r+1) - l(l+1)\big)}, \qquad i,j \in \{1,3,5,7\}. \tag{10.23}$$

The term $r = l$ is omitted because R_l vanishes on $\mathcal{H}_l$; all remaining terms have $r \le l+7$. Applying (10.21)–(10.22) in (10.23), followed by the three linear combinations (10.15)–(10.17), gives (10.18)–(10.20).

The same recurrence gives the first-derivative columns for the zonal potentials P_0, P_2, P_4:

(l,m)	P_0	P_2	P_4
$(0,0)$	1	0	0
$(1,0)$	1	$2/5$	0
$(1,1)$	1	$-1/5$	0
$(2,0)$	1	$2/7$	$2/7$
$(2,1)$	1	$1/7$	$-4/21$
$(2,2)$	1	$-2/7$	$1/21$

.

The 3×6 matrix of the linear forms in (10.15)–(10.17) annihilates this 6×3 matrix and has rank three. □

10.1.5. *An isolated zero of the axisymmetric equations.* Set $a = 1$ and $d = t$. The first two equations $w^T A_i w = 0$, $i = 1, 2$, are equivalent to

$$b(t) = \sqrt{\frac{392}{27} - \frac{13}{3375}t^2}, \qquad c(t) = \sqrt{\frac{2750}{27} - \frac{121}{945}t^2}, \tag{10.24}$$

when the positive square roots are chosen. Let

$$G(t) = \frac{1}{3} - \frac{2}{7}b(t) - \frac{59}{378}b(t)^2 + \frac{50}{2079}b(t)c(t) + \frac{149}{7722}c(t)^2 + \frac{14}{7150}c(t)t + \frac{31}{16575}t^2. \tag{10.25}$$

Lemma 10.4. *There is a unique $t_0 \in [282/25, 1129/100]$ such that $G(t_0) = 0$. It lies in the interior of the interval and satisfies $G'(t_0) > 0$. The three quadratic equations have derivative rank three at $(1, b(t_0), c(t_0), t_0)$, on the directions (b, c, d).*

Proof. Both radicands in (10.24) are positive throughout the stated interval. The following rational bounds hold:

$$-\frac{11}{10^6} < G(282/25) < -\frac{10}{10^6}, \tag{10.26}$$

$$\frac{3}{10^6} < G(1129/100) < \frac{4}{10^6}, \tag{10.27}$$

$$\frac{12}{10^4} < G'(t) < \frac{15}{10^4} \quad \left(\frac{282}{25} \le t \le \frac{1129}{100}\right). \tag{10.28}$$

For the integers B_t, C_t in the following table,

$$B_t < 10^{18} b(t) < B_t + 1, \qquad C_t < 10^{18} c(t) < C_t + 1 \ :$$

t	$10^{18} b(t)$	$10^{18} c(t)$
$282/25$	3745452583046787398	9249862209755932330
$1129/100$	3745336525452537874	9248299938152511391

.

Each entry follows by squaring its two rational endpoints and comparing with (10.24). Substitution in (10.25) gives (10.26)–(10.27). The derivatives are

$$b' = -\frac{13t}{3375b}, \qquad c' = -\frac{121t}{945c},$$

and

$$G' = -\frac{2}{7}b' - \frac{59}{189}bb' + \frac{50}{2079}(b'c + bc') + \frac{149}{3861}cc' + \frac{14}{7150}(c't + c) + \frac{62}{16575}t.$$

Both b, c decrease on the interval. Their endpoint enclosures and $282/25 \le t \le 1129/100$, substituted by rational interval arithmetic in this last expression, give the sharper enclosure

$$\frac{1264}{10^6} < G'(t) < \frac{1415}{10^6},$$

which implies (10.28). All interval denominators are positive. In particular, G' is positive throughout the closed interval.

The intermediate value theorem and (10.28) give existence and uniqueness of t_0. The first two quadratic equations have invertible derivative in (b, c), since $b, c > 0$ and

$$\det \begin{pmatrix} -1/63 & -1/990 \\ 1/42 & -1/150 \end{pmatrix} \neq 0.$$

After these variables are eliminated, the derivative of the third equation is $G'(t_0) > 0$. The full derivative therefore has rank three. □

Write $b_0 = b(t_0)$, $c_0 = c(t_0)$ and put

$$f_* = P_1 + b_0 P_3 + c_0 P_5 + t_0 P_7. \tag{10.29}$$

Its full cokernel quadratic value is zero by axisymmetry and Lemmas 10.3–10.4.

10.1.6. *The four transverse derivative directions.*

Lemma 10.5. *The reduced joint quadratic map has derivative rank seven at f_*. Its derivative is an isomorphism on*

$$\operatorname{span}\{P_3, P_5, P_7, x_1, x_2, z(x_1^2 - x_2^2), 2zx_1x_2\}. \tag{10.30}$$

Proof. Under rotations fixing the z axis, the target $\mathbb{R}^2 \oplus \mathcal{H}_2$ is the direct sum of a three-dimensional weight-zero space and real two-dimensional spaces of weights one and two. The derivative at the axisymmetric f_* preserves weights. Its rank on the weight-zero space is three by Lemma 10.4.

For each pair of harmonic degrees j, k, a rotation-equivariant bilinear map from $\mathcal{H}_j \times \mathcal{H}_k$ to $\mathcal{H}_2$ is a scalar multiple of projection of the product to $\mathcal{H}_2$, whenever this projection is nonzero. This is the multiplicity-one spherical-harmonic product decomposition.[1] Its scalar is determined here by the corresponding entry of A_3. Only the pairs with $|j - k| \leq 2$ contribute.

The required ratios are obtained from elementary polynomial integrals:

$$\frac{\int_{-1}^{1} P_j P_k^m P_2^m \, dz}{\int_{-1}^{1} P_j P_k P_2 \, dz} = \begin{cases} 3, & (j,k,m) = (1,1,1), \\ -2, & (j,k,m) = (3,1,1), \\ 40, & (j,k,m) = (1,3,2), \\ -60, & (j,k,m) = (3,3,2), \\ 12, & (j,k,m) = (5,3,2). \end{cases} \tag{10.31}$$

The denominators are nonzero. Substitute $P_r^m = (-1)^m (1 - z^2)^{m/2} P_r^{(m)}$ and integrate the resulting polynomials. For instance, $P_1^1 = -\sqrt{1 - z^2}$, $P_2^1 = -3z\sqrt{1 - z^2}$, $P_2^2 = 3(1 - z^2)$, and $P_3^2 = 15z(1 - z^2)$. The azimuthal integration for either a cosine or sine of nonzero weight introduces the same nonzero factor for all terms of that weight.

For weight one, take the potential directions $P_1^1(z)\cos\varphi$ and $P_1^1(z)\sin\varphi$, which span $\operatorname{span}\{x_1, x_2\}$. Equations (10.20) and (10.31) show that their derivative factor is a nonzero normalization multiple of

$$\frac{1}{3}\, 3 + \left(-\frac{1}{7}\right) b_0 (-2) = 1 + \frac{2b_0}{7} > 0.$$

For weight two, take $P_3^2(z)\cos(2\varphi)$ and $P_3^2(z)\sin(2\varphi)$. They span $\operatorname{span}\{z(x_1^2 - x_2^2), 2zx_1x_2\}$. Their derivative factor is a nonzero normalization multiple of

$$-\frac{1}{7}\, 40 - \frac{59}{378} b_0 (-60) + \frac{25}{2079} c_0\, 12 = -\frac{40}{7} + \frac{590}{63} b_0 + \frac{100}{693} c_0 > 0.$$

Indeed $b_0 > 37/10$ and $c_0 > 0$, so this coefficient is greater than

$$-\frac{40}{7} + \frac{590}{63}\frac{37}{10} = \frac{1823}{63} > 0.$$

The weight-one and weight-two restrictions are therefore invertible. Their ranks are two each; with the rank-three weight-zero restriction, the total rank is seven. This proves the asserted isomorphism. □

[1] The product and Gaunt formulas are given in the NIST Digital Library of Mathematical Functions, equations (34.3.20)–(34.3.22), https://dlmf.nist.gov/34.3.

10.1.7. *The local section.*

Proof of Theorem 10.2. Put

$$\mathcal{F}(q) = (\mathsf{L}_0(q), \mathsf{L}_1(q) - 2I_3, \mathsf{L}_2(q) - 6I_5).$$

This map is analytic on a neighborhood of zero in W. Its derivative restricts to an isomorphism from R_0 onto its range, of dimension 15. The cokernel has dimension seven. By Lemmas 10.4 and 10.5, the cokernel quadratic map has the nonzero zero f_* and an invertible derivative on (10.30). Theorem 4.5 gives (10.8), (10.9), and an analytic curve $q(s) = sf_* + O(s^2)$ such that

$$\mathcal{F}(q(s)) = 0, \qquad D\mathcal{F}(q(s))|_W \text{ is onto} \quad (0 < |s| < \epsilon).$$

Every C^k norm is bounded by a constant times a fixed Euclidean norm on W.

Let c be a unit cokernel vector. Analyticity of the potential cluster map in L^∞ and $D\langle c, \mathcal{F}\rangle(0) = 0$ give

$$|\langle c, \mathcal{F}(q)\rangle| \le C\|q\|_{L^\infty}^2.$$

Thus $\mathcal{F}(q) = tc$, $t > 0$, implies $\|q\|_{L^\infty} \ge C^{-1/2}\sqrt{t}$. An estimate with exponent greater than $1/2$ is impossible in any norm that bounds L^∞. Shrinking the neighborhood keeps the three Riesz projections at ranks $1, 3, 5$. □

10.2. **Simultaneous second-order equations for conformal metrics.** For conformal metric variations on the unit-round sphere, put $d\mu = dv_\gamma/(4\pi)$. The spaces $\mathcal{H}_j$ and the projectors Π_j refer to this fixed Hilbert space.

The multiplication isomorphism of Colin de Verdière [18], proved in Lemma C.1, is

$$\operatorname{Sym}^2 \mathcal{H}_\ell \longrightarrow \bigoplus_{j=0}^{\ell} \mathcal{H}_{2j}, \qquad u \odot v \longmapsto uv, \quad \text{is an isomorphism.} \tag{10.32}$$

For the L^2 and trace pairings, its dual is the isomorphism

$$\mathcal{M}_\ell : \bigoplus_{j=0}^{\ell} \mathcal{H}_{2j} \longrightarrow \operatorname{Sym}(\mathcal{H}_\ell), \qquad a \longmapsto \Pi_\ell M_a \Pi_\ell. \tag{10.33}$$

Here Π_ℓ is orthogonal projection onto $\mathcal{H}_\ell$. In particular, compression is scalar precisely when every positive-degree component of a in (10.33) is zero. The compression of a constant is that constant times the identity.

Let f be antipodally odd and let

$$g(t) = (1 + tf + t^2 h + o(t^2))\gamma \tag{10.34}$$

be a C^2 parameter family, with the remainder understood in a fixed sufficiently high C^k norm. The coefficients f, h are smooth in the spatial variable. Products of two degree-ℓ harmonics are even, so $\Pi_\ell M_f \Pi_\ell = 0$. Define the reduced resolvent

$$R_\ell = (\Delta - \lambda_\ell)^{-1}\big|_{\mathcal{H}_\ell^\perp}, \qquad R_\ell\big|_{\mathcal{H}_\ell} = 0. \tag{10.35}$$

The coefficient of t^2 in the cluster matrix is

$$B_\ell(f, h) = -\lambda_\ell \Pi_\ell M_h \Pi_\ell - \lambda_\ell^2 \Pi_\ell M_f R_\ell M_f \Pi_\ell. \tag{10.36}$$

Thus $L_\ell''(0) = 2B_\ell(f, h)$.

Indeed write $\Phi(t) = U + t\Phi_1 + t^2\Phi_2 + o(t^2)$ and $L(t) = \lambda_\ell I + t^2 B + o(t^2)$ in the generalized eigenvalue equation

$$\Delta\Phi(t) = (1 + tf + t^2 h + o(t^2))\Phi(t)L(t).$$

The coefficients of t and t^2 satisfy

$$(\Delta-\lambda_\ell)\Phi_1=\lambda_\ell M_f U,\qquad (\Delta-\lambda_\ell)\Phi_2=UB+\lambda_\ell M_f\Phi_1+\lambda_\ell M_h U.$$

Consequently $(I-\Pi_\ell)\Phi_1=\lambda_\ell R_\ell M_f U$. Since $\Pi_\ell M_f\Pi_\ell=0$, left multiplication of the second equation by U^* gives

$$B=-\lambda_\ell U^*M_hU-\lambda_\ell^2U^*M_fR_\ell M_fU,$$

which is (10.36).

Define the quadratic, rotation-equivariant map

$$\mathcal{S}_\ell(f)=-\lambda_\ell\mathcal{M}_\ell^{-1}(\Pi_\ell M_fR_\ell M_f\Pi_\ell)\in\bigoplus_{j=0}^{\ell}\mathcal{H}_{2j}. \tag{10.37}$$

It follows that $B_\ell(f,h)$ is scalar if and only if

$$h_{2j}=(\mathcal{S}_\ell(f))_{2j},\qquad 1\le j\le\ell. \tag{10.38}$$

The constant component changes only the scalar part of B_ℓ. If the multiplicities of all clusters with indices in I are preserved to second order, then

$$h_{2j}=(\mathcal{S}_\ell(f))_{2j}\qquad(\ell\in I,\ 1\le j\le\ell).$$

10.3. **A sharper obstruction for real cubic harmonics.** The top-degree obstruction concerns clusters d and $d+1$. For a real cubic harmonic, the obstruction already occurs between clusters 2 and 3.

Theorem 10.6 (Cubic incompatibility). *For every nonzero real $f\in\mathcal{H}_3$ there is no smooth h such that both $B_2(f,h)$ and $B_3(f,h)$ are scalar. Thus a C^2 conformal family through the round metric with first variation in $\mathcal{H}_3\setminus\{0\}$ cannot keep the multiplicity-five and multiplicity-seven clusters unsplit.*

For every $f\in\mathcal{H}_3$,

$$(\mathcal{S}_2(f))_4=\frac{5}{24}(f^2)_4,\qquad(\mathcal{S}_3(f))_4=-\frac{5}{4}(f^2)_4. \tag{10.39}$$

There is only one copy of $\mathcal{H}_4$ in $\mathrm{Sym}^2\mathcal{H}_3=\mathcal{H}_0\oplus\mathcal{H}_2\oplus\mathcal{H}_4\oplus\mathcal{H}_6$. Thus each degree-four component in (10.37) is a scalar multiple of $(f^2)_4$. For $f=xyz$,

$$(f^2)_4=-\frac{1}{55}Z_4,\qquad(\mathcal{S}_2(f))_4=-\frac{1}{264}Z_4,\qquad(\mathcal{S}_3(f))_4=\frac{1}{44}Z_4, \tag{10.40}$$

where

$$Z_4=x^4+y^4+z^4-3(x^2y^2+x^2z^2+y^2z^2). \tag{10.41}$$

Put

$$Z_6=x^6+y^6+z^6-\frac{15}{2}\sum_{i\ne j}x_i^4x_j^2+90x^2y^2z^2.$$

Both Z_4 and Z_6 are harmonic. The square of xyz is invariant under permutations and sign changes of the coordinates. Its Fischer decomposition is

$$x^2y^2z^2=\frac{2}{231}Z_6-\frac{q}{55}Z_4+\frac{q^3}{105},\qquad q=x^2+y^2+z^2.$$

Orthogonal transformations commute with Π_ℓ and R_ℓ, so the same invariance holds for $\mathcal{S}_\ell(xyz)$. An invariant polynomial of even degree at most six is a symmetric polynomial in x^2,y^2,z^2. Imposing harmonicity gives the invariant harmonic spaces $\mathbb{R}$, 0, $\mathbb{R}Z_4$, and $\mathbb{R}Z_6$ in degrees $0,2,4,6$. Hence

$$\mathcal{S}_2(xyz)=c_0+c_4Z_4,\qquad\mathcal{S}_3(xyz)=d_0+d_4Z_4+d_6Z_6.$$

Set $Z_0=1$. Pairing the compression identities with u,u, first for

$$u=x^2-y^2,\ xy\quad(\ell=2),$$

and then for

$$u = xyz,\ x(y^2 - z^2),\ x(2x^2 - 3y^2 - 3z^2) \quad (\ell = 3),$$

gives the systems

$$\begin{pmatrix} 4/15 & 8/105 \\ 1/15 & -4/315 \end{pmatrix} \begin{pmatrix} c_0 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1/3465 \\ 1/20790 \end{pmatrix},$$

$$\begin{pmatrix} 1/105 & -4/1155 & 16/3003 \\ 4/105 & -8/3465 & -16/1001 \\ 4/7 & 8/77 & 400/3003 \end{pmatrix} \begin{pmatrix} d_0 \\ d_4 \\ d_6 \end{pmatrix} = \begin{pmatrix} -1/45045 \\ -2/9009 \\ 34/9009 \end{pmatrix}.$$

The coefficient of the degree-a unknown is $\int Z_a u^2\, d\mu$, and the right-hand side is $-\lambda_\ell \int (fu) R_\ell(fu)\, d\mu$. Put $D_{\mathrm{E}} = \partial_x^2 + \partial_y^2 + \partial_z^2$ and, for $n = \ell + 3$, $\mathfrak{D}_n = n(n+1)I - qD_{\mathrm{E}}$ on homogeneous polynomials of degree n. With $I_n = \{n, n-2, \dots, n - 2\lfloor n/2 \rfloor\}$, Fischer decomposition gives

$$R_\ell(p|_{S^2}) = \sum_{k \in I_n} \frac{1}{k(k+1) - \lambda_\ell} \prod_{\substack{a \in I_n \\ a \neq k}} \frac{\mathfrak{D}_n - a(a+1)I}{k(k+1) - a(a+1)} p \Bigg|_{S^2}.$$

All denominators are nonzero, since n and ℓ have opposite parity. The matrix entries and right-hand sides follow from this identity and

$$\int x^{2a} y^{2b} z^{2c}\, d\mu = \frac{(2a-1)!!(2b-1)!!(2c-1)!!}{(2a+2b+2c+1)!!}, \qquad (-1)!! = 1;$$

monomials with an odd exponent have integral zero. The determinants of the two coefficient matrices are $-8/945$ and $256/2837835$. Solving the two systems gives

$$(c_0, c_4) = (0, -1/264), \qquad (d_0, d_4, d_6) = (0, 1/44, 7/660).$$

This proves (10.40), and hence (10.39) by rotation equivariance.

The degree-four component is nonzero for every nonzero real cubic, by the following lemma.

Lemma 10.7. *If $f \in \mathcal{H}_3$ is real and nonzero, then $(f^2)_4 \neq 0$.*

Proof. Take homogeneous polynomial representatives and put $q = x^2 + y^2 + z^2$. Fischer decomposition gives

$$f^2 = H_6 + qH_4 + q^2 H_2 + q^3 H_0. \tag{10.42}$$

Since $D_{\mathrm{E}} f = 0$, one has $D_{\mathrm{E}}(f^2) = 2\sum_i (\partial_i f)^2$. On the complex null cone $q = 0$, Fischer decomposition gives

$$\sum_i (\partial_i f)^2 = 11 H_4. \tag{10.43}$$

Parametrize the null cone by

$$\xi(s,t) = \left(\frac{s^2 - t^2}{2}, \frac{s^2 + t^2}{2i}, st \right), \qquad F(s,t) = f(\xi(s,t)). \tag{10.44}$$

At $(s,t) = (1,0)$, the derivatives have the following form. Put $x_* = (1/2, -i/2, 0)$, $g = \nabla f(x_*)$, and $H = D^2 f(x_*)$. Euler's identities and harmonicity give

$$g \cdot x_* = 3 f(x_*), \qquad H x_* = 2g, \qquad \operatorname{tr} H = 0.$$

The derivatives of ξ at $(1,0)$ are $\xi_s = 2x_*$, $\xi_t = e_3$, $\xi_{ss} = 2x_*$, $\xi_{st} = e_3$, and $\xi_{tt} = (-1, -i, 0)$. Consequently

$$F_{ss} = 30 f(x_*), \qquad F_{st} = 5 g_3, \qquad F_{tt} = H_{33} - g_1 - i g_2 = -5(g_1 + i g_2).$$

Indeed, the first two components of $Hx_* = 2g$ imply $H_{11} + H_{22} = 4(g_1 + ig_2)$, and $6f(x_*) = g_1 - ig_2$. It follows that

$$(F_{ss}F_{tt} - F_{st}^2)(1,0) = -25(g_1^2 + g_2^2 + g_3^2).$$

To extend this equality, identify $x \in \mathbb{C}^3$ with the symmetric matrix

$$B(x) = \begin{pmatrix} x_1 + ix_2 & x_3 \\ x_3 & -x_1 + ix_2 \end{pmatrix}, \qquad \det B(x) = -(x_1^2 + x_2^2 + x_3^2).$$

For $A \in \mathrm{SL}_2(\mathbb{C})$, the relation $B(R_A x) = AB(x)A^T$ defines a complex orthogonal map R_A, and $\xi(Az) = R_A\xi(z)$ for $z = (s,t)^T$. The polynomial $f \circ R_A$ is again harmonic. The Hessian determinant of $F \circ A$ is $(\det A)^2 \det D^2F(Az) = \det D^2F(Az)$, and the squared complex-bilinear norm of its corresponding gradient is unchanged by R_A. Since every nonzero $z \in \mathbb{C}^2$ is $A(1,0)^T$ for some $A \in \mathrm{SL}_2(\mathbb{C})$, the identity holds for every nonzero z, hence also at zero. Thus

$$F_{ss}F_{tt} - F_{st}^2 = -25\left(\sum_i (\partial_i f)^2\right) \circ \xi = -275H_4 \circ \xi. \tag{10.45}$$

If $H_4 = 0$, the binary sextic F has zero Hessian determinant. Such a nonzero binary sextic must be a sixth power of a linear form. Write $F(s,t) = s^6 G(t/s)$. Its Hessian determinant is $5s^8(6GG'' - 5(G')^2)$. At a root of G of multiplicity $m > 0$, the leading coefficient in this expression is a nonzero constant times $m(m-6)$, unless $m = 6$. The fundamental theorem of algebra therefore implies G is constant or is a scalar times a sixth power of one linear polynomial. Homogenization proves $F = c(as + bt)^6$.

The restriction map from harmonic cubics to binary sextics is injective. Indeed, division by the monic polynomial $q = x^2 + y^2 + z^2$ in the variable z writes a polynomial vanishing on $q = 0$ as

$$qR + a(x,y)z + b(x,y).$$

For $x^2 + y^2 \neq 0$, substitution of the two roots $z = \pm\sqrt{-x^2 - y^2}$ gives $a(x,y) = b(x,y) = 0$. The polynomials a, b therefore vanish identically. Thus q divides the polynomial. Fischer decomposition implies that a harmonic cubic divisible by q is zero. Since the spaces of harmonic cubics and binary sextics both have dimension seven, the restriction map is an isomorphism. A square $(as + bt)^2$ is the pullback under ξ of a linear form $v \cdot x$ with $v \cdot v = 0$. Its cube is harmonic. Uniqueness therefore gives $f = c(v \cdot x)^3$ for a nonzero complex null vector v.

Since f has real coefficients, unique factorization forces the line spanned by v to equal its complex conjugate. Multiplication by a complex scalar makes v a nonzero real vector. But a nonzero real vector cannot satisfy $v \cdot v = 0$. This contradiction proves the lemma. □

Proof of Theorem 10.6. If both matrices are scalar, equations (10.38) and (10.39) give

$$h_4 = \frac{5}{24}(f^2)_4 = -\frac{5}{4}(f^2)_4, \qquad \frac{35}{24}(f^2)_4 = 0.$$

Lemma 10.7 then gives $f = 0$, a contradiction. □

The nonvanishing conclusion of Lemma 10.7 fails over $\mathbb{C}$: if $0 \neq v \in \mathbb{C}^3$ satisfies $v \cdot v = 0$, then $(v \cdot x)^3$ is harmonic and the degree-four component of its square is zero.

For every $\ell \geq 1$, the individual derivative $DL_\ell(0) = -\lambda_\ell \mathcal{M}_\ell$ is surjective on $\bigoplus_{j=0}^{\ell} \mathcal{H}_{2j}$. The simultaneous second-order equations additionally require equality of the corresponding components of $\mathcal{S}_\ell(f)$.

11. Multiplicity rigidity on finite harmonic subspaces

Let γ be the unit-round metric on S^2, and put $d\mu = dv_\gamma/(4\pi)$. The nonnegative Laplacian is denoted by Δ. The space of real spherical harmonics of degree j is $\mathcal{H}_j$, with eigenvalue $j(j+1)$ and dimension $2j+1$. Each spherical harmonic is identified with its homogeneous harmonic polynomial; the identities extend complex-bilinearly. For a function u, write u_j for its $\mathcal{H}_j$ component.

For an integer $N \geq 2$, define

$$V_N = \bigoplus_{j=2}^{N} \mathcal{H}_j, \qquad g_v = (1+v)\gamma, \qquad v \in V_N. \tag{11.1}$$

For each integer $k \geq 0$ there is $C_{N,k}$ such that $\|v\|_{C^k} \leq C_{N,k}\|v\|_{L^2(d\mu)}$ on V_N. In particular $1 + v > 0$ if $\|v\|_{L^2} < C_{N,0}^{-1}$. By harmonic orthogonality,

$$\int v \, d\mu = 0, \qquad \int v x_i \, d\mu = 0 \quad (1 \leq i \leq 3). \tag{11.2}$$

Since $dv_{g_v} = (1+v)dv_\gamma$, the area of g_v is 4π. The degree-one orthogonality conditions remove the tangent directions of the nonisometric conformal transformations at γ.

Put $\lambda_\ell = \ell(\ell+1)$. On a common sufficiently small neighborhood, for each $1 \leq \ell \leq N+1$ choose the Riesz–Gram matrix $L_\ell(v)$ of the isolated λ_ℓ cluster, based on a fixed real orthonormal basis of $\mathcal{H}_\ell$. It is real analytic in the finite parameters. Define

$$\Gamma_\ell(v) = L_\ell(v) - \frac{\operatorname{tr} L_\ell(v)}{2\ell+1} I, \qquad w_\ell(v) = \max \operatorname{spec} L_\ell(v) - \min \operatorname{spec} L_\ell(v). \tag{11.3}$$

Thus $w_\ell = 0$ means that the cluster has not split. Its common eigenvalue is free to change. With the complete spectrum indexed starting at zero, these are the indices $\ell^2, \dots, (\ell+1)^2 - 1$ in a sufficiently small neighborhood.

Theorem 11.1 (Finite-mode multiplicity rigidity). *For every $N \geq 2$ there are $c_N > 0$ and $\epsilon_N > 0$ such that*

$$\max_{1 \leq \ell \leq N+1} w_\ell(v) \geq c_N \, \|v\|^2_{L^2(d\mu)} \qquad (v \in V_N, \ \|v\|_{L^2} < \epsilon_N). \tag{11.4}$$

In particular, in this neighborhood the only $v \in V_N$ for which all these clusters are unsplit is $v = 0$.

For $N \geq 3$, no uniform positive linear lower bound in $\|v\|_{L^2}$ can replace (11.4): if $f \in \mathcal{H}_3$ is nonzero, every selected cluster satisfies $\Gamma_\ell(tf) = O(t^2)$. The constants c_N and ϵ_N in the statement may depend on N.

11.1. **A top-degree obstruction**.

Proposition 11.2. *Let $f = f_1 + f_3 + \cdots + f_d$ be a real odd spherical polynomial, where $d \geq 3$ is odd and $f_d \neq 0$. There is no smooth h for which both $B_d(f,h)$ and $B_{d+1}(f,h)$ are scalar. Consequently no family* (10.34) *with this first variation keeps both clusters unsplit to second order.*

Proof. The rotation tensor-product decomposition gives

$$(\mathcal{S}_\ell(f))_{2d} = a_{\ell,d}(f_d^2)_{2d}, \qquad \ell \geq d. \tag{11.5}$$

Here the top degree $2d$ occurs once in $\operatorname{Sym}^2 \mathcal{H}_d$. No mixed product with a smaller degree can contribute to it. By Appendix C, $\mathcal{H}_j^{\mathbb{C}}$ has weights $-j, -j+1, \dots, j$, and weight $2d$ in $\operatorname{Sym}^2 \mathcal{H}_d^{\mathbb{C}}$ occurs only as the square of the weight-d vector. Successively subtracting the weight characters gives the bound $j + k$ for the largest degree in $\mathcal{H}_j \otimes \mathcal{H}_k$. Multiplication is nonzero on that top summand since $(x+iy)^{2d} \neq 0$.

Moreover,

$$(f_d^2)_{2d} \neq 0 \quad \text{whenever } f_d \neq 0. \tag{11.6}$$

Indeed, in homogeneous polynomials write $q = x^2 + y^2 + z^2$. The top harmonic component of a degree-$2d$ polynomial vanishes exactly when it is divisible by q, by the Fischer decomposition. Over $\mathbb{C}$, a reducible quadratic form in three variables is a product of two linear forms and has rank at most two. Thus q, of rank three, is irreducible and hence prime in $\mathbb{C}[x,y,z]$. If it divided f_d^2, it would divide f_d. A nonzero homogeneous harmonic polynomial is not divisible by q, again by uniqueness of Fischer decomposition. This proves (11.6), also for complex f_d.

Set

$$a_d = a_{d,d}, \qquad b_d = a_{d+1,d}. \tag{11.7}$$

The coefficient inequality of Sections 11.2–11.3 is

$$b_d - a_d > 0 \qquad \text{for every odd } d \geq 3. \tag{11.8}$$

Then (10.38) for the degree-$2d$ component would require

$$h_{2d} = a_d (f_d^2)_{2d} = b_d (f_d^2)_{2d}, \tag{11.9}$$

contradicting (11.6) and (11.8). □

11.2. **Exact Legendre reduction and ladder identity.** Complexify the quadratic identities and put $\zeta = x + iy$. For the coefficient $a_{\ell,d}$, take the highest-weight polynomial $f = \zeta^d$. For $\ell = d$, pair the compressed identity with $u = \bar{\zeta}^d$ on both sides, with the complex-bilinear extension of the real pairing. Then $fu = (1 - z^2)^d$. For $\ell = d + 1$, take $u = z\bar{\zeta}^d$, which is a degree-$d + 1$ harmonic. On axisymmetric functions the Laplacian is the operator

$$L = -\frac{d}{dz}\left((1 - z^2)\frac{d}{dz}\right), \qquad \langle a, b\rangle = \frac{1}{2}\int_{-1}^{1} a(z)b(z)\, dz. \tag{11.10}$$

Write

$$p = (1 - z^2)^d, \quad \lambda = d(d + 1), \quad N_0 = \langle p, p\rangle, \quad N_1 = \langle zp, zp\rangle = \frac{N_0}{4d + 3}, \tag{11.11}$$

and

$$I_0 = \langle p, (L - \lambda)^{-1} p\rangle, \qquad I_1 = \langle zp, (L - (d + 1)(d + 2))^{-1} zp\rangle. \tag{11.12}$$

For odd d, parity excludes the resonant eigenspace in each expression. The coefficients are exactly

$$a_d = -d(d + 1)\frac{I_0}{N_0}, \qquad b_d = -(d + 1)(d + 2)\frac{I_1}{N_1}. \tag{11.13}$$

This follows because the left compression pairings of ζ^{2d} with u, u are N_0 and N_1, respectively.

Lemma 11.3 (Ladder relation). *For every positive odd d,*

$$b_d = \frac{(d + 2)(4d + 3)}{9d}\, a_d - \frac{(d + 2)(4d + 2)}{9(d + 1)}. \tag{11.14}$$

Proof. Let

$$A_d = (z^2 - 1)\frac{d}{dz} + (d + 1)z. \tag{11.15}$$

Expansion of both sides gives the differential identity

$$(L - (d + 1)(d + 2))A_d = A_{d+2}(L - d(d + 1)). \tag{11.16}$$

Since $A_{d+2}p = 3(d + 1)zp$, if $u = (L - \lambda)^{-1}p$, then

$$(L - (d + 1)(d + 2))^{-1}(zp) = \frac{A_d u}{3(d + 1)}. \tag{11.17}$$

All functions are polynomials and parity excludes the respective kernels. Under the displayed integral pairing,

$$A_d^* = (1 - z^2)\frac{d}{dz} + (d - 1)z, \qquad A_d^*(zp) = [1 - (d + 2)z^2]p. \tag{11.18}$$

The integration-by-parts boundary term vanishes because $z^2 - 1 = 0$ at the endpoints. If $J = \langle z^2 p, u\rangle$, then

$$I_1 = \frac{I_0 - (d + 2)J}{3(d + 1)}. \tag{11.19}$$

A second direct differentiation gives

$$(L - d(d+1))[(1-z^2)p] = (d+1)[(2-d) - 3(d+2)z^2]p. \tag{11.20}$$

Self-adjointness and pairing with u give

$$N_0 - N_1 = (d+1)[(2-d)I_0 - 3(d+2)J]. \tag{11.21}$$

Eliminating J gives

$$I_1 = \frac{I_0}{9} + \frac{N_0 - N_1}{9(d+1)^2}. \tag{11.22}$$

Substitution in (11.13), with $N_1/N_0 = (4d+3)^{-1}$, proves (11.14). □

11.3. **The coefficient gap in every odd degree.** From (11.14),

$$b_d - a_d = \frac{2(2d^2+d+3)}{9d}(a_d - T_d), \qquad T_d = \frac{d(d+2)(2d+1)}{(d+1)(2d^2+d+3)}. \tag{11.23}$$

11.3.1. *A moment estimate for all odd degrees at least 41.* Expand p in the even Legendre polynomials P_{2k}, $0 \le k \le d$. After normalizing squared coefficients by N_0, let $w_k \ge 0$ be the spectral weights and set $s_k = 2k(2k+1)$. Then

$$\sum_k w_k = 1, \qquad a_d = \sum_k w_k \frac{\lambda}{\lambda - s_k}, \qquad \lambda = d(d+1). \tag{11.24}$$

Put $m_j = \sum_k w_k s_k^j$. Direct integration gives

$$m_1 = d, \qquad m_2 = \frac{6d^3}{2d-1} > 3d^2, \tag{11.25}$$
$$m_4 = \frac{12d^3(35d^3 - 45d^2 - 9d + 1)}{(2d-3)(2d-1)} \le 140d^4 \quad (d \ge 3).$$

These moments follow from

$$Lp = 2d[1 - (2d+1)z^2](1-z^2)^{d-1}, \tag{11.26}$$
$$L^2p = 4d^2[(2d+1)^2z^4 - 12dz^2 + 3](1-z^2)^{d-2}, \tag{11.27}$$

since

$$m_1 = \frac{\langle p, Lp\rangle}{N_0}, \qquad m_2 = \frac{\langle Lp, Lp\rangle}{N_0}, \qquad m_4 = \frac{\langle L^2p, L^2p\rangle}{N_0}.$$

Each monomial term is a beta integral

$$\frac{1}{2}\int_{-1}^{1} z^{2j}(1-z^2)^b \, dz = \frac{1}{2}B(j+\tfrac{1}{2}, b+1). \tag{11.28}$$

For the last bound in (11.25), the difference is

$$140d^4 - m_4 = \frac{4d^3(35d^3 - 145d^2 + 132d - 3)}{(2d-3)(2d-1)}. \tag{11.29}$$

Writing $d = u + 3$ transforms the cubic in the numerator into $35u^3 + 170u^2 + 207u + 33$, which is positive for $u \ge 0$.

On the part of this even spectrum with $s > \lambda$ one has

$$(d+1)(d+2) \le s \le 2d(2d+1). \tag{11.30}$$

The function $s^3/(\lambda^2(s-\lambda))$ has its maximum on this interval at an endpoint: its derivative has the sign of $2s - 3\lambda$. The endpoint values are

$$\frac{(d+2)^3}{2d^2}, \qquad \frac{8(2d+1)^3}{(d+1)^2(3d+1)} < \frac{64}{3}. \tag{11.31}$$

Both are at most d for $d \ge 23$. Indeed, writing $d = u + 8$ gives

$$2d^3 - (d+2)^3 = u^3 + 18u^2 + 84u + 24 > 0 \quad (u \ge 0),$$

and

$$8(d+1)^2(3d+1) - 3(2d+1)^3 = 20d^2 + 22d + 5 > 0.$$

For $s \ne \lambda$,

$$\frac{\lambda}{\lambda - s} = 1 + \frac{s}{\lambda} + \frac{s^2}{\lambda^2} + \frac{s^3}{\lambda^2(\lambda - s)}. \tag{11.32}$$

The last term is nonnegative for $s < \lambda$, and at least $-d$ for $s = s_k > \lambda$. Moreover,

$$\sum_{s_k > \lambda} w_k \le \lambda^{-4} \sum_{s_k > \lambda} w_k s_k^4 \le \frac{m_4}{\lambda^4}. \tag{11.33}$$

Consequently, for odd $d \ge 23$,

$$a_d \ge 1 + \frac{1}{d+1} + \frac{3}{(d+1)^2} - \frac{140d}{(d+1)^4}. \tag{11.34}$$

Now

$$1 + \frac{1}{d+1} - T_d = \frac{3(d+2)}{(d+1)(2d^2+d+3)} > \frac{1}{d^2} \quad (d \ge 3), \tag{11.35}$$

For the strict inequality, multiplication by the positive denominator reduces it to

$$d^3 + 3d^2 - 4d - 3 > 0;$$

with $d = u + 3$ this polynomial is $u^3 + 12u^2 + 41u + 39$. Also

$$\frac{3}{(d+1)^2} - \frac{5}{2d^2} = \frac{d^2 - 10d - 5}{2d^2(d+1)^2} \ge 0 \qquad (d \ge 11).$$

Equation (11.34) therefore gives

$$a_d - T_d > \frac{7}{2d^2} - \frac{140}{d^3} = \frac{7(d-40)}{2d^3} > 0 \qquad (d \ge 41). \tag{11.36}$$

This proves the coefficient gap in every odd degree at least 41.

11.3.2. *Rational coefficients in degrees three through thirty-nine.* Put

$$A_j = \frac{1}{2}\int_{-1}^{1} (1 - z^2)^j \, dz = \prod_{i=1}^{j} \frac{2i}{2i+1}, \quad A_0 = 1, \tag{11.37}$$

and for fixed d set

$$r_0 = 1, \qquad r_{k+1} = -\frac{(2k+1)(d-k)}{(k+1)(2d+2k+3)} r_k, \qquad w_k = (4k+1)\frac{A_d^2}{A_{2d}} r_k^2. \tag{11.38}$$

Indeed, Rodrigues' formula and repeated integration by parts, or expansion followed by the beta integral, give

$$\frac{\langle (1-z^2)^d, P_{2k} \rangle}{A_d} = (-1)^k \frac{(1/2)_k}{k!} \frac{d!}{(d-k)!\,(d+3/2)_k} = r_k. \tag{11.39}$$

The normalization $\langle P_{2k}, P_{2k} \rangle = (4k+1)^{-1}$ proves the formula for w_k. Equations (11.24), (11.14), and (11.38) involve rational numbers only. The remaining values are as follows.

d	Exact value of $b_d - a_d$
3	7/30
5	143/360
7	3111/7700
9	5966/17199
11	324875/1154538
13	689881/3043040
15	1672853/9016920
17	154282156/992759625
19	51054963/383091566
21	64713680975/554798532288
23	177242593/1707481360
25	529413581553/5655539462300
27	2993430197997811/35087339716821000
29	385216681855521817/4912307665746940800
31	19914270803975275/274348642109578704
33	7227731468167576/106943674502944125
35	625974633799006993/9898335309918305250
37	24759114438669087/416608235855735000
39	29541594567462248289 73/52697001033057658777500

Every numerator and denominator is positive. The recurrence (11.38), substituted into (11.13), gives these rational values. For $d = 1$ the same formulas give $a_1 = b_1 = 3/4$, and $b_1 - a_1 = 0$. Together with (11.36), the table completes the proof of (11.8) and Proposition 11.2.

11.4. **The quantitative estimate.**

Proof of Theorem 11.1. Let E_N and O_N be, respectively, the even- and odd-degree subspaces of V_N. Thus $V_N = E_N \oplus O_N$, and O_N contains only degrees $3, 5, \ldots$. Define the finite-dimensional analytic map

$$\mathbf{\Gamma} \,:\, V_N \longrightarrow \mathcal{W}_N := \bigoplus_{\ell=1}^{N+1} \mathrm{Sym}_0(\mathcal{H}_\ell), \qquad \mathbf{\Gamma}(v) = (\Gamma_\ell(v))_{\ell=1}^{N+1}, \tag{11.40}$$

with the product Frobenius norm. Its derivative A vanishes on O_N and is injective on E_N. Indeed $Ae = 0$ implies that $\Pi_{N+1} M_e \Pi_{N+1}$ is scalar. Every harmonic component of $e \in E_N$ has degree at most $N < 2(N+1)$. Equation (10.33) therefore makes e constant, and $e_0 = 0$ gives $e = 0$.

Let $\mathcal{R} = A(E_N)$ and let P_C be orthogonal projection onto its complement in $\mathcal{W}_N$. On O_N define

$$Q_N(o) = \frac{1}{2} P_C D^2 \mathbf{\Gamma}(0)[o, o]. \tag{11.41}$$

For every $o \in O_N$,

$$Q_N(o) \neq 0 \quad \text{for every nonzero } o \in O_N. \tag{11.42}$$

If $Q_N(o) = 0$, there is $h \in E_N$ such that $Ah + \frac{1}{2} D^2 \mathbf{\Gamma}(0)[o, o] = 0$. Thus the curve $v(t) = to + t^2 h$ has scalar second-order coefficients in all selected clusters. Let d be the largest degree of o. Then $3 \le d \le N$, and the selected clusters include d and $d+1$. This contradicts Proposition 11.2, even though that proposition allowed arbitrary smooth h. This proves (11.42).

Suppose first that $N \ge 3$. Compactness of its unit sphere gives $\eta_N > 0$ such that

$$\|Q_N(o)\| \ge \eta_N \|o\|^2 . \tag{11.43}$$

All source norms in this section are $L^2(d\mu)$. Since $A|_{E_N}$ is injective, $\|Ae\| \ge \sigma_N \|e\|$ for some $\sigma_N > 0$. Taylor's formula gives, for small e, o,

$$\begin{aligned} &\mathbf{\Gamma}(e+o) = Ae + B(o,o) + \mathcal{E}(e,o), \\ &\|\mathcal{E}(e,o)\| \le C_N(\|e\|^2 + \|e\|\,\|o\| + \|e+o\|^3), \end{aligned} \tag{11.44}$$

where $B(o,o) = \frac{1}{2}D^2\Gamma(0)[o,o]$ and $P_{\mathcal{C}}B(o,o) = Q_N(o)$.

Write $a = \|e\|$, $b = \|o\|$, and choose $\beta_N \geq 0$ with $\|B(o,o)\| \leq \beta_N b^2$. Choose $\rho_0 > 0$ such that all selected cluster matrices are defined and (11.44) holds whenever $a + b \leq \rho_0$. Increase C_N to at least one and choose

$$K \geq \max\{1, 4\beta_N/\sigma_N\}, \qquad 0 < \rho \leq \min\left\{\rho_0, 1, K^{-1}, \frac{\sigma_N}{40C_N}, \frac{\eta_N}{2C_N(2K+8)}\right\}.$$

Suppose $a + b \leq \rho$. If $a \geq Kb^2$, then

$$\|\mathcal{E}(e,o)\| \leq C_N\left(a^2 + ab + 4a^3 + 4b^3\right) \leq 10C_N\rho a \leq \frac{\sigma_N}{4}a.$$

Here $b^3 \leq \rho a/K$. Together with $\beta_N b^2 \leq \sigma_N a/4$, this gives

$$\|\Gamma(e+o)\| \geq \frac{\sigma_N}{2}a \geq \frac{\sigma_N}{4}(a + b^2).$$

If $a \leq Kb^2$, then $a \leq b$ and

$$\|\mathcal{E}(e,o)\| \leq C_N(K^2b^4 + Kb^3 + 8b^3) \leq C_N(2K+8)b^3 \leq \frac{\eta_N}{2}b^2.$$

Projection onto $\mathcal{C}$ and (11.43) yield

$$\|\Gamma(e+o)\| \geq \frac{\eta_N}{2}b^2 \geq \frac{\eta_N}{2(K+1)}(a + b^2).$$

Thus, with $c'_N = \min\{\sigma_N/4, \eta_N/[2(K+1)]\}$,

$$\|\Gamma(e+o)\| \geq c'_N(\|e\| + \|o\|^2) \geq c'_N\|e+o\|^2, \tag{11.45}$$

since $e \perp o$ and $\|e\| \leq 1$. Set $\epsilon_N = \rho/\sqrt{2}$. The inequality $a + b \leq \sqrt{2}\|e+o\|$ shows that (11.45) holds whenever $\|v\| < \epsilon_N$.

When $N = 2$, $O_N = 0$ and $\sigma_2 := \min_{\|v\|=1}\|Av\| > 0$ by compactness and injectivity. Choose $r_2 > 0$ and $D_2 \geq 1$ such that all selected cluster matrices are defined and

$$\|\Gamma(v) - Av\| \leq D_2\|v\|^2 \qquad (\|v\| \leq r_2).$$

With $\epsilon_2 = \min\{r_2, 1, \sigma_2/(2D_2)\}$, injectivity of A gives

$$\|\Gamma(v)\| \geq \frac{\sigma_2}{2}\|v\| \geq \frac{\sigma_2}{2}\|v\|^2 \qquad (\|v\| < \epsilon_2).$$

Put $c'_2 = \sigma_2/2$.

For a real symmetric matrix, each centered eigenvalue has absolute value at most the full spectral width. Therefore

$$\|\Gamma_\ell(v)\|_{\mathrm{F}} \leq \sqrt{2\ell+1}\, w_\ell(v), \qquad \|\Gamma(v)\| \leq \sqrt{(N+2)^2 - 1}\max_\ell w_\ell(v). \tag{11.46}$$

Together with (11.45) and the bound for $N = 2$, this proves (11.4) with $c_N = c'_N/\sqrt{(N+2)^2-1}$. □

The estimate depends only on the differences between eigenvalues within each selected cluster.

Appendix A. Local inversion and metric calculus

All tensor norms are taken in a fixed finite smooth atlas on the compact manifold.

A.1. **The trace-free exponential.**

Proposition A.1. *Let $h_1, \dots, h_d$ be smooth symmetric two-tensors on a compact Riemannian manifold (M, g). Suppose $\operatorname{tr}_g h_j = 0$ and that their supports have compact union K. For $t \in \mathbb{R}^d$, put*

$$H(t) = \sum_{j=1}^{d} t_j g^{-1} h_j, \qquad g_t(X, Y) = g(e^{H(t)}X, Y). \tag{A.1}$$

Then g_t is a smooth metric, $dv_{g_t} = dv_g$, and $g_t = g$ on $M \setminus K$. Moreover,

$$\partial_{t_j} g_t\big|_{t=0} = h_j, \qquad \|g_t - g\|_{C^k} \le C_k|t| \quad (|t| \le 1), \tag{A.2}$$

for every nonnegative integer k.

Proof. At each point, $g^{-1}h_j$ is self-adjoint: $g(g^{-1}h_jX, Y) = h_j(X, Y) = h_j(Y, X)$. Hence $H(t)$ is self-adjoint. An orthonormal eigenbasis gives

$$H(t) = \operatorname{diag}(\theta_1, \dots, \theta_n), \qquad e^{H(t)} = \operatorname{diag}(e^{\theta_1}, \dots, e^{\theta_n}).$$

Every exponential eigenvalue is positive. This proves symmetry and positive definiteness of g_t. In arbitrary coordinates, $\det g_t = (\det g) \det e^{H(t)}$. Since

$$\det e^{H(t)} = \prod_i e^{\theta_i} = e^{\operatorname{tr} H(t)} = 1,$$

the positive square roots of these determinants agree. If $x \notin K$, then $H(t, x) = 0$, and $g_t(x) = g(x)$.

The matrix exponential is defined by its locally uniformly convergent power series. Its derivative at zero is the identity, so differentiating (A.1) gives the first formula in (A.2). For every fixed k, the Leibniz rule gives

$$\|AB\|_{C^k} \le c_k \|A\|_{C^k} \|B\|_{C^k}$$

for smooth endomorphisms. Multiplying this norm by c_k makes it submultiplicative. In that equivalent norm, $\|H(t)\|_{C^k} \le M_k|t|$ and

$$\left\| e^{H(t)} - I \right\|_{C^k} \le \sum_{j \ge 1} \frac{\|H(t)\|_{C^k}^j}{j!} \le M_k|t| e^{M_k|t|}.$$

Multiplication by the fixed metric and return to the original norm prove the second estimate in (A.2). The same uniformly convergent series, differentiated term by term on bounded parameter sets, proves smooth dependence on t in every C^k space. □

A.2. **A local inverse with parameters.**

Lemma A.2. *Let $f : \overline{B}_r(0) \subset \mathbb{R}^d \to \mathbb{R}^d$ extend to a C^1 map on a neighborhood of the closed ball. Suppose $L = Df(0)$ is invertible and*

$$\sup_{|x| \le r} \left\| I - L^{-1} Df(x) \right\| \le \tfrac{1}{2}. \tag{A.3}$$

If $\left\|L^{-1}(y - f(0))\right\| \le r/2$, there is a unique $x \in \overline{B}_r(0)$ with $f(x) = y$, and

$$|x| \le 2 \left\| L^{-1}(y - f(0)) \right\|. \tag{A.4}$$

The inverse is as smooth as f on the open target neighborhood with strict inequality. If $f = f(x, p)$ is jointly C^j, $1 \le j \le \infty$, in x and additional finite-dimensional parameters p, and the displayed bounds hold locally uniformly in p, the inverse is jointly C^j in (y, p) on the corresponding open target set.

Proof. Set $T_y(x) = x - L^{-1}(f(x) - y)$. The integral mean-value formula and (A.3) give

$$|T_y(x) - T_y(x')| \le \tfrac{1}{2}|x - x'|, \qquad |T_y(0)| = \left\|L^{-1}(y - f(0))\right\| \le r/2.$$

Thus T_y maps $\overline{B}_r$ into itself and contracts distances by at most $1/2$. The iterates $x_{j+1} = T_y(x_j)$, $x_0 = 0$, satisfy

$$|x_{j+1} - x_j| \le 2^{-j}|T_y(0)|.$$

They are Cauchy, and their limit x is a fixed point by continuity. Summing the geometric series proves (A.4). If x' is another fixed point in the ball, the contraction inequality gives $|x - x'| \le |x - x'|/2$, so $x = x'$.

For smoothness, write $Df(x) = L(I + E(x))$, where $\|E(x)\| \le 1/2$. The inverse exists by the matrix series $\sum_{j\ge0}(-E)^j$. For two targets y, y' and their fixed points x, x', subtraction gives

$$|x - x'| \le \tfrac{1}{2}|x - x'| + \left\|L^{-1}\right\| |y - y'|, \qquad |x - x'| \le 2\left\|L^{-1}\right\| |y - y'|.$$

Apply the differentiability remainder of f to

$$f(x(y + h)) - f(x(y)) = h.$$

The Lipschitz bound makes that remainder $o(|h|)$, so

$$D_y x = (Df(x))^{-1}.$$

For a jointly C^1 family $f(x, p)$, its parameter derivative is bounded on a smaller compact product neighborhood. The contraction estimate applied at nearby (y, p) therefore gives a local Lipschitz bound for $x(y, p)$. The differentiability remainder of f then yields

$$D_y x = (D_x f)^{-1}, \qquad D_p x = -(D_x f)^{-1} D_p f. \tag{A.5}$$

Differentiating these identities inductively proves every higher derivative for which f is differentiable. The inverse is therefore smooth in the target and the additional parameters. □

Proposition A.3. *Let $\mathscr{F}$ be a smooth map from a neighborhood of a smooth metric g to a finite-dimensional real vector space W. Suppose*

$$D\mathscr{F}(g) \,:\, C_c^\infty(O; S_0^2 T^*M) \longrightarrow W$$

is onto, where $O \subset M^\circ$ is open. There are a compact set $K \Subset O$, a neighborhood V of $\mathscr{F}(g)$, and a smooth family $y \mapsto g_y$, $y \in V$, such that

$$\mathscr{F}(g_y) = y, \qquad g_{\mathscr{F}(g)} = g, \qquad dv_{g_y} = dv_g, \qquad g_y = g \text{ on } M \setminus K. \tag{A.6}$$

For every fixed k, $\|g_y - g\|_{C^k} \le C_k|y - \mathscr{F}(g)|$ after decreasing V.

Proof. Choose a basis $e_1, \dots, e_d$ of W and tensors h_j with $D\mathscr{F}(g)h_j = e_j$. Their finite support union is compact in O; enlarge it to $K \Subset O$. Define g_t by (A.1) and put $f(t) = \mathscr{F}(g_t)$ in this basis. The chain rule and (A.2) give $Df(0) = I_d$. Continuity of Df gives a ball satisfying (A.3). Lemma A.2 gives a smooth solution $t(y)$ and $|t(y)| \le 2|y - \mathscr{F}(g)|$. Define $g_y = g_{t(y)}$. Proposition A.1 gives (A.6) and its norm bound. □

Let $F : E \supset U \to W$ be smooth, where E is Banach, W is finite-dimensional, $0 \in U$, and $A = DF(0)$ is onto. Choose a linear right inverse $R : W \to E$. It is bounded because W is finite-dimensional. The operator RA is a bounded projection, so

$$E = \ker A \oplus R(W).$$

For $x \in \ker A$ and $z \in R(W)$, the derivative in z of $F(x + z)$ at $(0, 0)$ is the isomorphism $A|_{R(W)}$. The contraction proof of Lemma A.2, with the Banach variable x as a parameter, gives a unique smooth solution $z = z(x, y)$ of $F(x + z) = y$ near $(0, F(0))$. The derivative formulas (A.5) hold with Fréchet derivatives in x. Thus

$$F^{-1}(y) = \{x + z(x, y) \,:\, x \in \ker A \text{ near } 0\}$$

locally, and this level set has codimension $\dim W$.

Lemma A.4. *Let F be a smooth map between finite-dimensional manifolds, with rank constantly r near a point. There are local coordinates (s, z) in the source in which $F(s, z)$ depends only on $s \in \mathbb{R}^r$. The local fibres may be taken connected. Every vector in* $\ker dF$ *is the velocity of a smooth curve in its local level set.*

Proof. Work in coordinate charts. For $r = 0$, each component has zero differential on a connected coordinate ball and is constant there by integration along line segments. For $r > 0$, reorder source and target coordinates so that the first r target components have an invertible derivative in the first r source variables. The map

$$x \longmapsto (F_1(x), \dots, F_r(x), x_{r+1}, \dots, x_N)$$

has invertible derivative. Lemma A.2, after translation and multiplication by that derivative's inverse, supplies local coordinates (s, z). In these coordinates the first r components of F equal s. A nonzero derivative in any z direction of one of the remaining components would add an independent column to the derivative and raise its rank above r. Therefore all those derivatives vanish. Restricting to a product of balls makes each fibre connected and proves that $F(s, z)$ is independent of z. Curves $z(t) = z_0 + tv$, with s fixed, realize all kernel velocities; the inverse coordinate change gives the required curves in the original source. □

A.3. **Quantitative preservation of a strict Ricci inequality**. Let ∇ be the Levi–Civita connection of g, and write $\widetilde{g} = g + h$. The difference of the two connections is the tensor

$$A^k{}_{ij} = \tfrac{1}{2}\widetilde{g}^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}). \tag{A.7}$$

Indeed, the torsion-free identity makes $A^k{}_{ij} = A^k{}_{ji}$. Subtract the two metric-compatibility equations and cycle their three covariant indices; solving for $\widetilde{g}_{k\ell}A^\ell{}_{ij}$ gives (A.7). Expanding the curvature of $\nabla + A$ and contracting its first and third indices gives

$$\mathrm{Ric}_{\widetilde{g},ij} - \mathrm{Ric}_{g,ij} = \nabla_k A^k{}_{ij} - \nabla_j A^k{}_{ik} + A^k{}_{k\ell}A^\ell{}_{ij} - A^k{}_{j\ell}A^\ell{}_{ik}. \tag{A.8}$$

The curvature convention is $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, with sectional curvature $g(R(X, Y)Y, X)$.

Lemma A.5. *For each compact set K_1 and fixed reference metric g, there are constants $C > 0$ and $\epsilon > 0$ such that*

$$\left\|\mathrm{Ric}_{g+h} - \mathrm{Ric}_g\right\|_{C^0(K_1,g)} \le C\,\|h\|_{C^2(K_1,g)} \quad \text{if } \|h\|_{C^2(K_1,g)} < \epsilon. \tag{A.9}$$

Consequently, if $\mathrm{Ric}_g - cg \ge \delta g$ *on K_1, with $\delta > 0$, then*

$$\mathrm{Ric}_{g+h} - c(g + h) \ge \tfrac{\delta}{2}g > 0 \tag{A.10}$$

there whenever $(C + |c|)\,\|h\|_{C^2} < \delta/2$ and $\|h\|_{C^2} < \epsilon$.

Proof. Assume $\|h\|_{C^0(g)} \le 1/2$. Then $\frac{1}{2}g \le \widetilde{g} \le \frac{3}{2}g$, and the inverse metric is uniformly bounded in the g norm. Differentiating $\widetilde{g}^{-1}\widetilde{g} = I$ gives

$$\nabla\widetilde{g}^{-1} = -\widetilde{g}^{-1}(\nabla h)\widetilde{g}^{-1}.$$

Equation (A.7) therefore implies

$$|A|_g \le C_n|\nabla h|_g, \qquad |\nabla A|_g \le C_n(|\nabla^2 h|_g + |\nabla h|_g^2).$$

Substitution into (A.8) gives

$$|\,\mathrm{Ric}_{g+h} - \mathrm{Ric}_g\,|_g \le C_n(|\nabla^2 h|_g + |\nabla h|_g^2).$$

For $\|h\|_{C^2} \le 1$, the quadratic term is bounded by $\|h\|_{C^2}$. This proves (A.9). For a g-unit vector X, the difference between $(\mathrm{Ric}_{g+h} - c(g + h))(X, X)$ and $(\mathrm{Ric}_g - cg)(X, X)$ has absolute value at most $(C + |c|)\,\|h\|_{C^2}$. The assumed bound proves (A.10). Positivity as a quadratic form does not depend on the choice between the two equivalent metric norms. □

If $\mathrm{Ric}_g > cg$ on a compact neighborhood K_1 of the support of a perturbation, the continuous function

$$(x, X) \longmapsto (\mathrm{Ric}_g - cg)_x(X, X), \qquad x \in K_1, \quad |X|_g = 1,$$

has a positive minimum δ. The unit-sphere bundle over K_1 is compact, so the minimum is attained. Lemma A.5 and the C^2 estimate of Proposition A.3 give one parameter neighborhood on which the strict inequality is retained throughout K_1. Outside the support the two metrics and their Ricci tensors are equal.

For a real-analytic finite-dimensional family, complexify the source and parameter variables near the reference point. Continuity of the holomorphic derivative gives a complex ball on which the contraction constant is less than one; decreasing the target and parameter neighborhoods makes that ball invariant. The holomorphic iterates converge locally uniformly, so their limit is holomorphic. Restriction to the real variables gives a real-analytic inverse.

Appendix B. Flat tori and Fourier expansions

The perturbations are trigonometric polynomials.

B.1. **All first-order obstructions on a planar frequency shell.** Let $T^2 = \mathbb{R}^2/\Lambda$ be flat and let $\Lambda^* = \{k : k \cdot \gamma \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Lambda\}$. For a positive eigenvalue, write its complete shell as

$$\{k \in \Lambda^* : |k|^2 = \lambda\} = \{\pm k_1, \dots, \pm k_r\}, \qquad m = 2r. \tag{B.1}$$

The real orthonormal basis consists of $\sqrt{2/\mathrm{Vol}(T^2)}\cos(k_j \cdot x)$ and the corresponding sine. Put $C_a = \mathrm{diag}(a_1 I_2, \dots, a_r I_2)$. If $s = k + \ell \neq 0$ and $|k| = |\ell| = \sqrt{\lambda}$, then $k \cdot s = |s|^2/2$. This line meets the circle in at most two points, exchanged by $k \leftrightarrow \ell$. Therefore s determines the unordered pair. Expanding a real quadratic relation in the complex exponentials shows that all coefficients with nonzero sum frequency vanish separately. The zero-sum pairs are $k_j, -k_j$. In the real basis they give exactly $a_j I_2$ blocks, and their constant terms cancel precisely when $\sum_j a_j = 0$. Thus

$$\begin{aligned} \mathcal{C}_{\mathrm{conf}} &= \{C_a : \sum_j a_j = 0\}, \\ \mathcal{C}_{\mathrm{all}} &= \{C_a : \sum_j a_j k_j \otimes k_j = 0\}. \end{aligned} \tag{B.2}$$

The trace of the second relation implies the first because $|k_j|^2 = \lambda > 0$. The span of the matrices $k_j \otimes k_j$ has dimension $\min\{r, 3\}$: up to a fixed invertible change of coordinates their entries are $(1, \cos 2\theta_j, \sin 2\theta_j)$, with distinct θ_j modulo π. Any three distinct points of a circle are not collinear. Hence

$$\dim \mathcal{C}_{\mathrm{conf}} = r - 1, \qquad \dim \mathcal{C}_{\mathrm{all}} = \max\{r - 3, 0\}. \tag{B.3}$$

Together with Theorem 6.6, this gives smooth full-metric multiplicity loci for $m = 2, 4, 6$ and non-smooth loci for $m \geq 8$; within the conformal class, only $m = 2$ is strongly stable.

B.2. **Differential order.** Let $\mathscr{D}_{2s}$ be the real formally self-adjoint scalar differential operators of order at most $2s$ on T^2, where $s \geq 0$. Consider

$$\mathscr{D}_{2s} \longrightarrow \mathrm{Sym}(E_\lambda), \qquad D \longmapsto P_\lambda D P_\lambda. \tag{B.4}$$

Proposition B.1. *The cokernel dimension of* (B.4) *is*

$$\max\{r - (2s + 1), 0\}. \tag{B.5}$$

The least even differential order for which the compression map is surjective is $2\lfloor r/2 \rfloor$.

Proof. Multiplication by every real smooth function belongs to $\mathscr{D}_{2s}$. Equation (B.2) makes every annihilator a matrix C_a. Testing real differential operators is equivalent to testing their self-adjoint parts: for symmetric real C_a, transposition gives $\operatorname{tr}(C_aPDP) = \operatorname{tr}(C_aP(D+D^*)P/2)$. For a multi-index α, direct trigonometric differentiation gives

$$\cos(k\cdot x)\,\partial^\alpha \cos(k\cdot x) + \sin(k\cdot x)\,\partial^\alpha \sin(k\cdot x) = \begin{cases} 0, & |\alpha| \text{ odd}, \\ (-1)^{|\alpha|/2}k^\alpha, & |\alpha| \text{ even}. \end{cases} \tag{B.6}$$

Testing each coefficient of a differential operator therefore gives precisely $\sum_j a_j k_j^\alpha = 0$ for even $|\alpha| \le 2s$. Contracting two indices in $k_j^{\otimes 2s}$ gives $\lambda k_j^{\otimes(2s-2)}$. Since $\lambda > 0$, all lower moment equations follow from

$$\sum_j a_j k_j^{\otimes 2s} = 0. \tag{B.7}$$

Conversely, the order-$2s$ equation is among the moment equations derived from (B.6). The space of homogeneous polynomials of degree $2s$ in two variables has dimension $2s+1$. Any at most $2s+1$ distinct projective directions give independent evaluation functionals. To isolate a fixed direction in such a collection, choose one linear factor vanishing on each other direction but not on the fixed direction. Their product has degree at most $2s$. Multiply it by a power of another linear form nonzero on the fixed direction to reach degree $2s$. Its evaluations vanish at the other directions and not at the fixed one. Dualizing proves that the map in (B.7) has rank $\min\{r, 2s+1\}$. This gives (B.5). The smallest s satisfying $r \le 2s+1$ is $\lfloor r/2 \rfloor$. □

On the square torus the shell $|k|^2 = 5^s$ has $4(s+1)$ vectors. Indeed, in the Gaussian integers, $5 = (2+i)(2-i)$ with two nonassociate primes; a Gaussian integer of norm 5^s is a unit times $(2+i)^a(2-i)^{s-a}$, $0 \le a \le s$. For $z, w \in \mathbb{Z}[i]$, $w \neq 0$, choose a Gaussian integer q by rounding the two coordinates of z/w. Then $|z - qw|^2 \le |w|^2/2 < |w|^2$. The Euclidean algorithm therefore gives, for any $a, b \in \mathbb{Z}[i]$, a greatest common divisor d and Gaussian integers u, v with $d = ua + vb$. If an irreducible π does not divide a, this identity for π, a gives $1 = u\pi + va$ after multiplication by a unit. Consequently $\pi \mid ab$ implies $\pi \mid b$; every irreducible is prime. Induction on the positive integer norm factors every nonzero nonunit into irreducibles, since a proper nonunit factor has strictly smaller norm. Primality then proves uniqueness by cancellation, up to units and order. The two factors $2+i$ and $2-i$ have prime norm five and are not associates.

If $z\overline{z} = 5^s$, every prime factor of z divides $(2+i)^s(2-i)^s$ and is therefore associated to one of these two primes. Taking norms in $z = u(2+i)^a(2-i)^b$, $u \in \{1, -1, i, -i\}$, gives $a + b = s$. Hence the choices are exactly $u \in \{1, -1, i, -i\}$ and $0 \le a \le s$, with $b = s - a$. Unique factorization makes the displayed $4(s+1)$ choices distinct. Thus $r = 2s+2$, and (B.5) is one. Consequently, for every $s \ge 0$ there is a positive shell on which compression of $\mathscr{D}_{2s}$ has nonzero cokernel. For $s \ge 1$ and $D \in \mathscr{D}_{2s}$, consider the cluster of Δ^s at λ^s. The function $x \mapsto x^s$ is strictly increasing on $[0, \infty)$, so its eigenspace is E_λ. The principal symbol of $\Delta^s + tD$ is $|\xi|^{2s} + t\sigma_{2s}(D)(x, \xi)$. Boundedness of $\sigma_{2s}(D)$ on the compact unit cosphere makes it positive for sufficiently small $|t|$. This formally self-adjoint elliptic family has fixed domain H^{2s}, and its cluster derivative is $P_\lambda D P_\lambda$.

B.3. **Second variations at the eigenvalue five.** Let $\ell \in \Lambda^*$ satisfy $|\ell| > 2\sqrt{\lambda}$, and put $f_\ell(x) = \cos(\ell \cdot x)$. This function is well-defined on $\mathbb{R}^2/\Lambda$. If $|k|^2 = \lambda$, then $|k \pm \ell| \ge |\ell| - |k| > \sqrt{\lambda}$. Thus multiplication by f_ℓ sends each shell exponential into the orthogonal complement of E_λ, and $PM_{f_\ell}P = 0$. For the conformal curve $(1 + tf_\ell)g$, put $q_a(\ell) = \langle C_a, D^2\mathsf{L}(0)[f_\ell, f_\ell]\rangle$. The dimension-two second variation gives

$$q_a(\ell) = -\lambda^2 \sum_{j=1}^{r} a_j \left(\frac{1}{|\ell|^2 + 2k_j\cdot\ell} + \frac{1}{|\ell|^2 - 2k_j\cdot\ell} \right). \tag{B.8}$$

For the normalized exponential $e_k = \operatorname{Vol}(T^2)^{-1/2} e^{ik\cdot x}$,

$$M_{f_\ell} e_k = \tfrac{1}{2}(e_{k+\ell} + e_{k-\ell}), \qquad \langle M_{f_\ell} R M_{f_\ell} e_k, e_k\rangle = \tfrac{1}{4} \sum_{\epsilon=\pm 1} \frac{1}{|k + \epsilon\ell|^2 - \lambda}.$$

The same sum is obtained for $-k$. The scalar block $a_j I_2$ is unchanged on complexifying the corresponding sine–cosine plane. Its trace contribution is therefore twice this diagonal entry times a_j. In dimension two, $A_0'' = B_0'' = 0$. Since $PM_{f_\ell}P = 0$, equation (2.50) gives the uncontracted identity

$$D^2\mathsf{L}(0)[f_\ell, f_\ell] = -2\lambda^2 U^* M_{f_\ell} R M_{f_\ell} U.$$

Its contraction gives (B.8) for every $a \in \mathbb{R}^r$. All denominators are positive because $|\ell| > 2\sqrt{\lambda}$. For the shell in Example 6.3, with $a = (1, 1, -1, -1)$, the second moment is zero and the fourth is

$$\sum_{j=1}^{4} a_j (k_j \cdot \xi)^4 = -48\xi_1\xi_2(\xi_1^2 - \xi_2^2). \tag{B.9}$$

Equation (B.8) gives

$$q_a((5,1)) = \frac{845}{352}, \qquad q_a((5,-1)) = -\frac{845}{352}. \tag{B.10}$$

For example the bracketed sum for $(5, 1)$ is

$$\frac{1}{40} + \frac{1}{12} + \frac{1}{44} + \frac{1}{8} - \frac{1}{32} - \frac{1}{20} - \frac{1}{48} - \frac{1}{4},$$

and multiplication by -25 gives the first value.

B.4. **A regular zero of the quadratic map.** Let $T^2 = \mathbb{R}^2/\Lambda$ and let its complete positive frequency shell be $\mathscr{S}_\lambda = \{\pm k_1, \dots, \pm k_r\}$. Denote by Π its orthogonal projection and by U the normalized sine–cosine frame. In this appendix put

$$\Delta_0\Phi = (1+v)\Phi\mathsf{L}(v), \qquad \boldsymbol{\mathcal{K}} = \{f \,:\, \Pi M_f \Pi = 0\}, \qquad Q(f) = \tfrac{1}{2} P_C D^2\mathsf{L}(0)[f, f]. \tag{B.11}$$

By (B.2), the conformal cokernel is identified with

$$\mathbb{R}^r_0 = \{a \in \mathbb{R}^r \,:\, \sum_j a_j = 0\}, \qquad a \longmapsto C_a = \operatorname{diag}(a_1 I_2, \dots, a_r I_2). \tag{B.12}$$

For $r = 1$ the cokernel is zero. Suppose $r \geq 2$.

B.5. **Separated Fourier shifts.** Let

$$R_\lambda = (\Delta_0 - \lambda)^{-1}_{E_\lambda^\perp}, \qquad R_\lambda|_{E_\lambda} = 0.$$

For $f \in \boldsymbol{\mathcal{K}}$, differentiating (B.11) first gives

$$(I - \Pi)\Phi'(0) = \lambda R_\lambda M_f U.$$

The second derivative, projected onto E_λ, gives

$$D^2\mathsf{L}(0)[f, f] = -2\lambda^2 U^* M_f R_\lambda M_f U. \tag{B.13}$$

Indeed, $\mathsf{L}'(0) = 0$, the pencil has no direct second derivative, and the possible E_λ component of Φ' disappears because $\Pi M_f \Pi = 0$. Polarization gives

$$D^2\mathsf{L}(0)[f, h] = -\lambda^2 U^*(M_f R_\lambda M_h + M_h R_\lambda M_f)U \quad (f, h \in \boldsymbol{\mathcal{K}}). \tag{B.14}$$

For $\eta \in \Lambda^*$ with $|\eta| > 2\sqrt{\lambda}$, set $f_\eta(x) = \cos(\eta \cdot x)$. The identity

$$M_{f_\eta} e_k = \tfrac{1}{2}(e_{k+\eta} + e_{k-\eta})$$

shows that $f_\eta \in \boldsymbol{\mathcal{K}}$. Returning to the original shell after two shifts by η is possible only with total shift zero, since $|2\eta| > 2\sqrt{\lambda}$. The Hessian is consequently scalar on each sine–cosine pair. Define

$$b_j(\eta) = \frac{1}{|\eta|^2 + 2k_j \cdot \eta} + \frac{1}{|\eta|^2 - 2k_j \cdot \eta}, \qquad \bar{b}(\eta) = \frac{1}{r}\sum_j b_j(\eta). \tag{B.15}$$

Every denominator is positive. Direct substitution in (B.13) gives

$$D^2\mathsf{L}(0)[f_\eta, f_\eta] = \operatorname{diag}\left(-\tfrac{\lambda^2}{2}b_1(\eta)I_2, \dots, -\tfrac{\lambda^2}{2}b_r(\eta)I_2\right), \tag{B.16}$$

$$q(\eta) := -\tfrac{\lambda^2}{4}\big(b(\eta) - \bar{b}(\eta)\mathbf{1}\big) \in \mathbb{R}^r_0. \tag{B.17}$$

Here $q(\eta)$ represents $Q(f_\eta)$ under (B.12). Equivalently,

$$H_a(\eta) := \langle C_a, D^2\mathsf{L}(0)[f_\eta, f_\eta]\rangle = -\lambda^2 \sum_j a_j b_j(\eta) = 4a \cdot q(\eta). \tag{B.18}$$

Thus (B.18) is (B.8) in the coordinates (B.12).

Lemma B.2. *If $|\eta|, |\mu| > 2\sqrt{\lambda}$ and*

$$|\eta - \mu| > 2\sqrt{\lambda}, \qquad |\eta + \mu| > 2\sqrt{\lambda}, \tag{B.19}$$

then $D^2\mathsf{L}(0)[f_\eta, f_\mu] = 0$. Hence for a finite family satisfying these pairwise conditions,

$$Q\left(\sum_{\nu=1}^N x_\nu f_{\eta_\nu}\right) = \sum_{\nu=1}^N x_\nu^2 q(\eta_\nu). \tag{B.20}$$

Proof. The Fourier supports of $M_{f_\eta}E_\lambda$ and $M_{f_\mu}E_\lambda$ lie in $\mathscr{S}_\lambda \pm \eta$ and $\mathscr{S}_\lambda \pm \mu$. A common frequency would make $\eta - \mu$ or $\eta + \mu$ a difference of two shell frequencies, whose length is at most $2\sqrt{\lambda}$. Thus the supports are disjoint. The reduced inverse is diagonal in Fourier variables, so both terms in (B.14) vanish entrywise. Polarization proves (B.20). □

B.6. **Harmonic moment polynomials.** Fix $a \in \mathbb{R}^r_0 \setminus \{0\}$ and define

$$M_p^a(\xi) = \sum_{j=1}^r a_j (k_j \cdot \xi)^{2p}, \qquad p \geq 0. \tag{B.21}$$

Put $\Delta_\xi^{\mathrm{E}} = \partial_{\xi_1}^2 + \partial_{\xi_2}^2$; this Euclidean trace operator has the opposite sign to the nonnegative Laplacian.

Lemma B.3. *There is a least $p \in \{1, \dots, r-1\}$ with $M_p^a \not\equiv 0$. This first nonzero moment is homogeneous harmonic of degree $2p$ and has both signs on the unit circle.*

Proof. The zeroth moment vanishes. Choose ξ_0 outside the finitely many lines $(k_i - k_j)\cdot\xi = 0$ and $(k_i + k_j)\cdot\xi = 0$, $i \neq j$. Then the real numbers $(k_j \cdot \xi_0)^2$ are pairwise distinct. If the moments through $r-1$ all vanished, the resulting Vandermonde system would imply $a = 0$.

Direct differentiation gives

$$\Delta_\xi^{\mathrm{E}} M_p^a = 2p(2p-1)\lambda M_{p-1}^a. \tag{B.22}$$

For the first nonzero moment the right side is zero. A homogeneous harmonic polynomial of positive degree has zero average on the unit circle: the divergence theorem shows the radial derivative of its spherical average is zero, and the average at radius zero is zero. For a nonzero continuous function of zero average, its minimum is negative and its maximum is positive. Homogeneity excludes its vanishing on the entire unit circle. Both signs therefore occur. □

For a fixed nonzero $\nu \in \Lambda^*$, expand (B.15) along $\eta = N\nu$. When N is sufficiently large, the geometric series converges and gives

$$H_a(N\nu) = -2\lambda^2 \sum_{j\geq 0} 4^j N^{-2j-2} \frac{M_j^a(\nu)}{|\nu|^{4j+2}}. \tag{B.23}$$

For the least nonzero p this becomes

$$H_a(N\nu) = -2^{2p+1}\lambda^2 N^{-2p-2}\frac{M_p^a(\nu)}{|\nu|^{4p+2}} + O(N^{-2p-4}). \tag{B.24}$$

The remainder estimate follows from a geometric-series tail, since there are only finitely many k_j and ν is fixed.

The directions of a full-rank lattice are dense in the circle. For example, a closest lattice point to $t\omega$ stays at bounded distance from that ray point as $t \to \infty$, by the bounded diameter of a fundamental parallelogram. Its direction tends to ω. Thus one may choose lattice vectors v_+ and v_- where M_p^a has opposite signs. Equation (B.24) shows that $a \cdot q(\eta)$ has both signs at arbitrarily high lattice frequencies.

Remark B.4. For the multiplicity-eight shell in (B.9), the weights $(1, 1, -1, -1)$ satisfy $M_0 = M_1 = 0$ and

$$M_2(\xi) = -48\xi_1\xi_2(\xi_1^2 - \xi_2^2) \neq 0.$$

B.7. **A regular quadratic zero from finite convexity.** Assume $r \geq 2$. Construct a sequence of pairwise separated frequencies as follows. Enumerate every nonzero dual-lattice vector infinitely many times. At each step, replace the vector by a positive integer multiple whose length exceeds the preceding selected length by more than $4\sqrt{\lambda}$, and start with length greater than $4\sqrt{\lambda}$. Denote the selected frequencies by $\eta_1, \eta_2, \ldots$. Every fixed lattice ray occurs at unbounded lengths, and (B.19) holds for every pair.

By Section B.6, the open subsets

$$\{a \in \mathbb{R}_0^r \,:\, |a| = 1,\ a \cdot q(\eta_\nu) < 0\}, \qquad \nu \geq 1,$$

cover the unit sphere in $\mathbb{R}_0^r$. Compactness gives a finite subcover. Relabel its frequencies $\eta_1, \ldots, \eta_N$ and put $q_\nu = q(\eta_\nu)$.

Lemma B.5. *The origin belongs to the interior of* $\operatorname{conv}\{q_1, \ldots, q_N\}$ *in* $\mathbb{R}_0^r$. *There are strictly positive numbers* $\tau_1, \ldots, \tau_N$ *such that*

$$\sum_\nu \tau_\nu q_\nu = 0, \qquad \operatorname{span}\{q_1, \ldots, q_N\} = \mathbb{R}_0^r. \tag{B.25}$$

Proof. If the origin were not an interior point, finite-dimensional separation would provide $a \neq 0$ with $a \cdot q_\nu \geq 0$ for every ν, contrary to the covering property. Interior containment also gives $\operatorname{span}\{q_\nu\} = \mathbb{R}_0^r$.

For strict positivity, put $b = \sum_\nu q_\nu$. Choose $\epsilon > 0$ small enough that $-\epsilon b$ lies in the convex hull, and write $-\epsilon b = \sum_\nu \beta_\nu q_\nu$, with $\beta_\nu \geq 0$ and $\sum \beta_\nu = 1$. Then $\tau_\nu = \beta_\nu + \epsilon > 0$ proves the displayed relation. If $b = 0$, taking every $\tau_\nu = 1$ is also possible. □

Set

$$z_0 = \sum_\nu \sqrt{\tau_\nu}\, f_{\eta_\nu}. \tag{B.26}$$

Distinct frequencies up to sign make these cosine functions linearly independent. Thus $z_0 \neq 0$. Lemma B.2 and (B.25) imply

$$Q(z_0) = 0, \qquad DQ(z_0)\left[\sum_\nu h_\nu f_{\eta_\nu}\right] = 2\sum_\nu \sqrt{\tau_\nu} h_\nu q_\nu.$$

The derivative is onto $\mathcal{C}$. Put $d = r - 1$ and $D = m(m+1)/2$. Choose a d-dimensional subspace Z_1 of the span of these cosines on which it is an isomorphism. Since $DQ(z_0)z_0 = 2Q(z_0) = 0$, the vector z_0 does not belong to Z_1. Therefore

$$Z = \mathbb{R}z_0 \oplus Z_1 \subset \mathcal{K}, \qquad \dim Z = d + 1.$$

Choose a trigonometric space X of dimension $a = D - d$ such that $D\mathsf{L}(0)|_X : X \to \mathcal{R}$ is an isomorphism. Indeed the first derivative depends only on the Fourier coefficients at the finite set $\mathcal{S}_\lambda + \mathcal{S}_\lambda$. Restriction to trigonometric polynomials therefore has the same derivative range as restriction to smooth functions, and (B.2) gives its dimension $D - d$. Then

$$W = X \oplus Z, \qquad \dim W = D + 1. \tag{B.27}$$

Theorem B.6. *Every positive eigenvalue of a flat two-torus has a local* $1/2$*-Hölder conformal cluster section and satisfies the weak Arnold hypothesis. At a conformally unstable reference,* $D+1$ *trigonometric perturbations suffice, where* $D = m(m+1)/2$*, and the exact scalar level contains a nonconstant analytic curve of regular points approaching the reference.*

Proof. The preceding convexity construction gives a nonzero regular zero of Q and a derivative complement of dimension $r-1$. Their span has dimension r, since the regular zero is in the kernel of its own derivative. Add $D-(r-1)$ trigonometric range perturbations; their span has dimension $D+1$. The spectral map is analytic on this finite space by its resolvent series. Theorem 4.5 gives the section and exact-level curve. Corollary 4.6 gives the essential-value property. If $r = 1$, the analytic inverse function theorem applies on D range perturbations. □

B.8. **The full conformal second derivative on a surface.** Let Δ be the nonnegative Laplacian of a closed surface, U a real orthonormal basis of an isolated λ-eigenspace, $\Pi = UU^*$, and $R = (\Delta-\lambda)^{-1}_{E_\lambda^\perp}$, extended by zero. For arbitrary smooth real functions f, h, put $M_f u = fu$ and $A_f = U^* M_f U$. In the fixed-Hilbert-space Riesz–Gram frame for $(1+v)g$,

$$D^2\mathsf{L}(0)[f,h] = \lambda(A_f A_h + A_h A_f) - \lambda^2 U^*(M_f R M_h + M_h R M_f)U. \tag{B.28}$$

To prove (B.28), write $T(v) = (1+v)^{-1/2}\Delta(1+v)^{-1/2}$ and let $U(v)$ be its Riesz–Gram frame, with $U(0) = U$. Differentiation gives

$$\begin{aligned} T_f' &= -\tfrac{1}{2}(M_f\Delta + \Delta M_f),\\ T_{f,h}'' &= \tfrac{3}{4}(M_{fh}\Delta + \Delta M_{fh}) + \tfrac{1}{4}(M_f\Delta M_h + M_h\Delta M_f). \end{aligned}$$

At the reference, $U^*DU = 0$ and $DU[f] = -RT_f'U$. Twice differentiating $\mathsf{L} = U^*TU$ and substituting $D^2(U^*U) = 0$ gives

$$D^2\mathsf{L}[f,h] = U^*T_{f,h}''U - U^*(T_f'RT_h' + T_h'RT_f')U.$$

Insert an orthonormal eigenbasis of Δ between the multiplication operators. On an intermediate eigenvalue $\nu \neq \lambda$, the coefficient of each ordered product is

$$\frac{3\lambda+\nu}{4} - \frac{(\lambda+\nu)^2}{4(\nu-\lambda)} = -\frac{\lambda^2}{\nu-\lambda}.$$

On $\nu = \lambda$, the resolvent is zero and the coefficient is λ. These are the two terms in (B.28).

On a flat two-torus, take the normalized complex exponentials e_k and write

$$f(x) = \sum_{\alpha\in\Lambda^*} f_\alpha e^{i\alpha\cdot x}, \qquad h(x) = \sum_{\alpha\in\Lambda^*} h_\alpha e^{i\alpha\cdot x}.$$

For shell vectors p, q, define

$$c_z = f_{p-z}h_{z-q} + h_{p-z}f_{z-q}.$$

Equation (B.28) becomes

$$D^2\mathsf{L}(0)[f,h]_{pq} = \lambda\sum_{|z|^2=\lambda} c_z - \lambda^2\sum_{|z|^2\neq\lambda}\frac{c_z}{|z|^2-\lambda}. \tag{B.29}$$

For smooth f, h, the bounds $|f_\alpha|+|h_\alpha| \le C_M(1+|\alpha|)^{-M}$, valid for every M, imply absolute convergence. For trigonometric polynomials both sums are finite. A unitary change from the complex exponentials to the real cosine–sine basis gives the real symmetric matrix. In particular,

$$D^2\mathsf{L}(0)[1,1] = 2\lambda I_m.$$

B.9. **A regular quadratic zero at multiplicity eight.** Take $T = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $\lambda = 5$, and order the shell representatives as

$$k_1 = (1,2), \quad k_2 = (2,-1), \quad k_3 = (1,-2), \quad k_4 = (2,1).$$

The real multiplicity is eight, the conformal cokernel dimension is three, and $D = 36$. Equation (B.3) gives these dimensions.

Choose

$$\eta_1 = (9,3), \quad \eta_2 = (-3,9), \quad \eta_3 = (9,-3), \quad \eta_4 = (3,9), \qquad f_\nu = \cos(\eta_\nu \cdot x). \tag{B.30}$$

Every $|\eta_\nu|^2 = 90$, while

$$\min_{\nu\neq\mu,\ \epsilon=\pm1} |\eta_\nu + \epsilon\eta_\mu|^2 = 36 > 4\lambda = 20.$$

Thus all mixed Hessians vanish. Formula (B.17) gives the following exact matrix of quadratic vectors:

$$[q(\eta_1)\ q(\eta_2)\ q(\eta_3)\ q(\eta_4)] = \frac{15}{39424}\begin{pmatrix} 3 & 3 & 47 & -53 \\ 3 & 3 & -53 & 47 \\ 47 & -53 & 3 & 3 \\ -53 & 47 & 3 & 3 \end{pmatrix}. \tag{B.31}$$

Every column has zero coordinate sum. The four columns add to zero, and the determinant of the first three rows and first three columns of the displayed integer matrix is -30000. In particular the rank is three. Therefore

$$z_0 = f_1 + f_2 + f_3 + f_4, \qquad Q(z_0) = 0, \qquad DQ(z_0)|_{\mathrm{span}\{f_1,f_2,f_3\}} \text{ is an isomorphism onto } \mathcal{C}. \tag{B.32}$$

Define

$$\begin{aligned}\mathscr{A} = \{&(0,2),(0,4),(1,-3),(1,-1),(1,1),(1,3),\\ &(2,-4),(2,0),(2,4),(3,-3),(3,-1),(3,1),\\ &(3,3),(4,-2),(4,0),(4,2)\}.\end{aligned}$$

These are representatives up to sign of all nonzero frequencies in $\mathscr{S}_5 + \mathscr{S}_5$. Put

$$X = \mathrm{span}\big(\{1\} \cup \{\cos(\sigma\cdot x), \sin(\sigma\cdot x) \,:\, \sigma \in \mathscr{A}\}\big). \tag{B.33}$$

The real span of the products $u_i u_j$ equals this 33-dimensional space: it is contained in it, and Equation (B.2) gives product-span dimension $36-3 = 33$. Hence $D\mathsf{L}(0)|_X$ is injective, by its integral pairing with the product span, and maps isomorphically onto $\mathcal{R}$. Its entries are $-5\int f u_i u_j\, dv_g$, and its rank is 33.

The corresponding finite-dimensional space is

$$W = X \oplus \mathrm{span}\{f_1,f_2,f_3,f_4\}, \qquad \dim W = 37. \tag{B.34}$$

Section 4 produces an analytic curve

$$v(s) = s\sum_{\nu=1}^{4}\cos(\eta_\nu\cdot x) + O(s^2), \qquad \mathsf{L}(v(s)) = 5I_8, \tag{B.35}$$

whose conformal cluster derivative on W is onto for all sufficiently small $s \neq 0$. Here $v(s) \in W$ for every sufficiently small s.

Summing the four uncontracted Hessians gives

$$D^2\mathsf{L}(0)[z_0,z_0] = -\frac{6205}{4928}I_8.$$

Since $D\mathsf{L}(0)[1] = -5I_8$, comparison of the coefficients of s^2 gives

$$v(s) = sz_0 + s^2\left(-\frac{1241}{9856}\,1 + z_2\right) + O(s^3), \qquad z_2 \in \mathrm{span}\{f_1,f_2,f_3\}.$$

Every f_ν has mean zero. Therefore

$$\mathrm{Area}\big((1+v(s))g\big) = 4\pi^2\left(1 - \frac{1241}{9856}s^2 + O(s^3)\right).$$

B.10. **Invariant perturbations at multiplicity eight.** On the square torus at $\lambda = 5$, let the involution act by $(x, y) \mapsto (-x, -y)$. In the frame of four cosines followed by four sines, its representation is $\mathrm{diag}(I_4, -I_4)$. The target is

$$\mathrm{Sym}_4(\mathbb{R}) \oplus \mathrm{Sym}_4(\mathbb{R}), \qquad D = 20.$$

Let X_+ be the span of the constant function and the sixteen cosine functions at the frequencies in (B.33). The derivative on X_+ has rank 17: every nonzero sum frequency in a shell determines its unordered pair, and the even product relations are exactly the three trace-free repeated sine–cosine diagonal relations of (B.2). The four high-frequency functions in (B.30) are even. Their quadratic matrix is still (B.31); its column sum is zero and the first three columns are independent in the cokernel. Theorem 4.5 therefore gives a local section from the 20-dimensional invariant target to the 21-dimensional space $X_+ \oplus \mathrm{span}\{f_1, f_2, f_3, f_4\}$.

Appendix C. Multiplication of spherical harmonics

The following real multiplication isomorphism gives strong conformal stability of the round two-sphere [18, p. 186]; see also [28, Proposition 4.6]. Let γ be the unit-round metric on S^2. Denote by $\mathcal{H}_d$ the real homogeneous harmonic polynomials of degree d in $\mathbb{R}^3$, restricted to S^2. Their eigenvalue is $d(d+1)$ and their dimension is $2d+1$.

Lemma C.1 (Round-sphere multiplication; [18, p. 186], [28, Proposition 4.6]). *For every $\ell \geq 0$, multiplication induces an isomorphism*

$$\mathrm{Sym}^2 \mathcal{H}_\ell \longrightarrow \bigoplus_{j=0}^{\ell} \mathcal{H}_{2j}. \tag{C.1}$$

Here multiplication sends $u \odot v$ to uv, where $u \odot v = (u \otimes v + v \otimes u)/2$. Consequently, a symmetric matrix C in a basis $u_1, \dots, u_{2\ell+1}$ of $\mathcal{H}_\ell$ for which $\sum C_{ij} u_i u_j$ vanishes on a nonempty open subset of S^2 is zero.

Proof. Write $\mathscr{D} = \partial_{X_1}^2 + \partial_{X_2}^2 + \partial_{X_3}^2$ for the Euclidean trace operator and $\rho^2 = X_1^2 + X_2^2 + X_3^2$. Every homogeneous polynomial of degree d has the unique decomposition

$$p = \sum_{a=0}^{\lfloor d/2 \rfloor} \rho^{2a} h_{d-2a}, \qquad \mathscr{D} h_{d-2a} = 0.$$

Existence follows from the polynomial projection formula (7.6), specialized to $n = 2$, and induction on d. For uniqueness, suppose a sum of the displayed form is zero, and choose the largest index a with nonzero h_{d-2a}. Applying $\mathscr{D}^a$ annihilates all terms of smaller index and gives a nonzero scalar multiple of h_{d-2a}, a contradiction. The scalar is nonzero because

$$\mathscr{D}(\rho^{2a} h_e) = 2a(2e + 2a + 1)\rho^{2a-2} h_e, \qquad a > 0.$$

It follows that the image in (C.1) is contained in the stated direct sum.

The decomposition also determines dimensions. Let $\mathcal{P}_d$ be the homogeneous polynomials of degree d in three variables, with $\mathcal{P}_d = 0$ for $d < 0$. Uniqueness gives

$$\mathcal{P}_d = \mathcal{H}_d^{\mathrm{hom}} \oplus \rho^2 \mathcal{P}_{d-2}.$$

Restriction to the sphere is injective on homogeneous harmonic polynomials by homogeneity. Therefore

$$\dim \mathcal{H}_d = \binom{d+2}{2} - \binom{d}{2} = 2d + 1,$$

where the second binomial term is zero for $d = 0, 1$.

Put $z = X_1 + iX_2$. Both $\mathrm{Re}\, z^\ell$ and $\mathrm{Im}\, z^\ell$ are harmonic. Their sum of squares is

$$p_\ell = (X_1^2 + X_2^2)^\ell,$$

so $p_\ell|_{S^2}$ belongs to the real multiplication image. Write its harmonic decomposition as

$$p_\ell = \sum_{j=0}^{\ell} \rho^{2(\ell-j)} h_{2j}.$$

For each j,

$$\mathscr{D}^{\ell-j} p_\ell = 4^{\ell-j} \left(\frac{\ell!}{j!} \right)^2 (X_1^2 + X_2^2)^j.$$

The top homogeneous harmonic component of the left side is a nonzero scalar multiple of h_{2j}: terms indexed above j are annihilated, and terms indexed below j retain a positive power of ρ^2. The top harmonic component of $(X_1^2 + X_2^2)^j$ is nonzero. Otherwise that polynomial would be divisible by ρ^2; evaluation at $(X_1, X_2, X_3) = (1, 0, i)$ over $\mathbb{C}$ gives the contradiction $1 = 0$. Thus every h_{2j} is nonzero.

Let $\mathcal{I}$ be the multiplication image on S^2. It is invariant under rotations. For $L_{ij} = X_i \partial_{X_j} - X_j \partial_{X_i}$, differentiation of the rotation action preserves $\mathcal{I}$ because this finite-dimensional subspace is closed. Direct differentiation gives, with $\mathcal{E} = \sum_i X_i \partial_{X_i}$,

$$\sum_{i<j} L_{ij}^2 = \rho^2 \mathscr{D} - \mathcal{E}^2 - \mathcal{E}.$$

On the restriction of a degree-d homogeneous harmonic polynomial, the right-hand side equals $-d(d+1)$. The sphere decomposition thus identifies $-\sum_{i<j} L_{ij}^2$ with Δ_γ on the finite sum containing $\mathcal{I}$. Hence $\mathcal{I}$ is invariant under Δ_γ. Put $\mu_j = 2j(2j+1)$. The projection onto $\mathcal{H}_{2j}$ on the displayed direct sum is

$$\mathcal{P}_j = \prod_{\substack{0 \le k \le \ell \\ k \ne j}} \frac{\Delta_\gamma - \mu_k I}{\mu_j - \mu_k}.$$

The denominators are nonzero. Since $\mathcal{I}$ is invariant under Δ_γ and contains $p_\ell|_{S^2}$, it contains $\mathcal{P}_j(p_\ell|_{S^2}) = h_{2j}|_{S^2}$.

Each h_{2j} is invariant under the rotations fixing the X_3-axis, by uniqueness of harmonic decomposition. In the variables $z = X_1 + iX_2$, $\bar{z} = X_1 - iX_2$, and X_3, invariance under rotations about the X_3-axis forces equal powers of z and $\bar{z}$. An invariant homogeneous polynomial therefore has the form

$$h = \sum_{a=0}^{\lfloor d/2 \rfloor} c_a (X_1^2 + X_2^2)^a X_3^{d-2a}.$$

The coefficients of $\mathscr{D}h = 0$ satisfy

$$4(a+1)^2 c_{a+1} + (d-2a)(d-2a-1)c_a = 0 \qquad \left(0 \le a < \left\lfloor \frac{d}{2} \right\rfloor \right).$$

At the largest index the remaining X_3-second derivative is zero because its exponent is 0 or 1. Every coefficient is consequently determined by c_0, and every choice of c_0 produces a harmonic polynomial. Thus this invariant subspace is exactly one-dimensional. For an orthonormal basis $Y_1, \dots, Y_{2d+1}$ of $\mathcal{H}_d$, the evaluation kernel

$$K_d(\omega, \nu) = \sum_i Y_i(\omega) Y_i(\nu)$$

is rotation invariant, reproduces every member of $\mathcal{H}_d$, and is nonzero at (ν, ν). The diagonal is positive because $K_d(\nu, \nu)$ is rotation invariant and its integral in ν equals $2d+1$. Thus a nonzero axis-invariant harmonic is a scalar multiple of $K_d(\cdot, e_3)$. Its rotations span $\mathcal{H}_d$: a function orthogonal to all those rotations has zero evaluation at every point. Since $\mathcal{I}$ is rotation invariant and contains h_{2j}, it contains $\mathcal{H}_{2j}$ for every j. This proves surjectivity. Finally,

$$\dim \operatorname{Sym}^2 \mathcal{H}_\ell = (\ell+1)(2\ell+1) = \sum_{j=0}^{\ell} (4j+1),$$

so the map is an isomorphism. The restriction of a polynomial to S^2 is real analytic. Its vanishing on a nonempty open set implies its vanishing on the connected sphere. Injectivity then gives $C = 0$. □

C.1. **The tensor-product degree bound.** The tensor-product decomposition is

$$\mathcal{H}_j^{\mathbb{C}} \otimes \mathcal{H}_k^{\mathbb{C}} \cong \bigoplus_{a=|j-k|}^{j+k} \mathcal{H}_a^{\mathbb{C}}.$$

The weight decomposition follows from the infinitesimal rotation operators. Put $z = X_1 + iX_2$, $\bar{z} = X_1 - iX_2$ and

$$H = -iL_{12}, \qquad E_+ = L_{13} + iL_{23}, \qquad E_- = -L_{13} + iL_{23}.$$

They satisfy $[H, E_\pm] = \pm E_\pm$ and $[E_+, E_-] = 2H$. On any finite-dimensional representation of $SO(3)$, averaging a Hermitian form makes rotations unitary. Thus H is diagonalizable, its weights are integers, and the representation is completely reducible. Let $0 \neq v$ have largest weight a in an irreducible summand. Then $E_+ v = 0$ and

$$E_+ E_-^s v = s(2a - s + 1) E_-^{s-1} v.$$

If r is the largest integer with $E_-^r v \neq 0$, the identity with $s = r + 1$ gives $r = 2a$. The span of $v, E_- v, \dots, E_-^{2a} v$ is invariant under H, E_+, E_- and hence is the whole irreducible summand. Its weights are $a, a-1, \dots, -a$, each once. In $\mathcal{H}_j^{\mathbb{C}}$, z^j has highest weight j. Its successive images under E_- span a subspace of dimension $2j + 1 = \dim \mathcal{H}_j^{\mathbb{C}}$. Therefore $\mathcal{H}_j^{\mathbb{C}}$ is irreducible with weights $-j, -j+1, \dots, j$.

Assume $j \geq k$. The multiplicity of weight $n \in \mathbb{Z}$ in the tensor product is

$$c_n = \#\{(a, b) \in \mathbb{Z}^2 \,:\, |a| \leq j,\ |b| \leq k,\ a + b = n\} = \begin{cases} 2k + 1, & |n| \leq j - k, \\ j + k - |n| + 1, & j - k \leq |n| \leq j + k, \\ 0, & |n| > j + k. \end{cases}$$

For $n \geq 0$, subtraction gives $c_n - c_{n+1} = 1$ exactly when $j - k \leq n \leq j + k$, and zero otherwise. Complete reducibility of orthogonal representations and the weight list of $\mathcal{H}_a^{\mathbb{C}}$ therefore give one copy of each $\mathcal{H}_a^{\mathbb{C}}$ in that interval. This weight decomposition is that of the spherical-harmonic product formulas [21, §34.3]. For the symmetric square, the multiplication isomorphism (C.1) already exhibits exactly the even summands $0, 2, \dots, 2j$. In particular, the top degree $j + k$ occurs once and no greater degree occurs.

For the harmonic-degree pairs in the three-cluster calculation, the degree-two projection of multiplication is nonzero: its zonal Gaunt integral is the nonzero denominator in (10.31). Hence any equivariant bilinear map from the indicated pair to $\mathcal{H}_2$ is a scalar multiple of that projection. Its scalar is determined by one nonzero zonal coefficient. These statements also follow from the explicit product formulas in [21, §34.3].

Appendix D. Relative completion in a prescribed graph form

Assume $d \geq 1$ and (4.3). Choose a range complement X, and let $L_R : R \to X$ be the inverse of $A|_X$. Put

$$K_1 = \{v \in K \,:\, P_C D^2 F(0)[v, L_R y] = 0 \text{ for all } y \in R\}.$$

This subspace has codimension at most ad in K. Corollary 4.4 gives J_r in K_1. On

$$V_r = X \oplus J_r(\mathbb{R}^r \oplus C^r), \qquad \dim V_r = a + r(d + 1),$$

solve the range equation and write the reduced C-valued map as

$$G(t, w, y) = \sum_i t_i w_i + b(y) + E_0(t, w, y), \qquad j_0^2 E_0 = 0. \tag{D.1}$$

For a target $f = y + z$, a *graph lift* means a family

$$h(t) = \zeta(t, w(t), y(t)) + J_r(t, w(t)), \qquad w(0) = 0,$$

where $\zeta \in X$ solves the range equation. The first coordinate of the selected kernel block is required to equal the parameter t.

Lemma D.1 (Exact graph correction with parameters). *Let $H(t,v) = \sum_i t_i v_i + E(t,v)$ be smooth, with $j^2E = 0$. There is a smooth $v(t) = O(|t|^2)$ satisfying $H(t,v(t)) = 0$. If E depends smoothly on a parameter in a compact smooth manifold, v may be chosen smoothly in that parameter on a common t-neighborhood. For an integer $k \geq 1$, the condition $E(t,0) = O(|t|^{k+2})$ implies $v = O(|t|^{k+1})$. A flat $E(t,0)$ gives a flat v.*

Proof. Let S be the symmetric auxiliary variable and $K(t,S)$ the integral coefficient of Lemma 4.7. The equation $S + K(t,S) = 0$ has derivative the identity in S at $t = 0$, uniformly on a compact external parameter set. Its local solutions agree by uniqueness and therefore give one smooth solution on a common smaller neighborhood. Set $v(t) = S(t)t$.

If $E(t,0) = O(|t|^{k+2})$, Taylor's formula gives $D_t^2 E(t,0) = O(|t|^k)$, hence $K(t,0) = O(|t|^k)$. Define

$$M(t,S) = \int_0^1 D_S K(t, sS)\, ds.$$

Then $K(t,S) = K(t,0) + M(t,S)S$ and $M(0,S) = 0$. The equation becomes $(I + M(t,S))S = -K(t,0)$. On a smaller common neighborhood $\|M\| \leq 1/2$, so $\|S\| \leq 2\|K(t,0)\| = O(|t|^k)$ and $v = O(|t|^{k+1})$. Smoothness implies $j_0^k v = 0$. In the flat case the same conclusion holds for every k, so S and v are flat. □

Theorem D.2 (Relative graph completion). *Let P be a compact smooth external parameter manifold, $A_0 \subset P$ closed, and let $f(p,t) = y(p,t) + z(p,t)$ be smooth with $f(p,0) = 0$ and $D_t z(p,0) = 0$. Suppose graph lifts are already given smoothly on a neighborhood of A_0 in P. They extend to smooth graph lifts on all of P, agreeing on a possibly smaller neighborhood of A_0, after shrinking the common t-domain.*

Proof. Let $U \subset P$ be an open neighborhood of A_0 on which the given graph is $w_A(p,t)$. Write

$$q_p(t) = z_2(p,t) - b(y_1(p,t)) = \sum_{i,j} t_i t_j S_{ij}(p), \qquad S_{ij} = S_{ji},$$

where S is the uniquely determined symmetric coefficient array. The derivative $L_A(p)t = D_t w_A(p,0)t$ satisfies $\sum_i t_i (L_A(p)t)_i = q_p(t)$. Consequently $K_A = L_A - S$ is skew in its two parameter indices. Also $w_A(p,t) - L_A(p)t = O(|t|^2)$.

Choose open sets $A_0 \subset U_0 \Subset U_1 \Subset U$ and a smooth function $\chi : P \to [0,1]$ equal to one on U_1 and supported in U. Extend the products with χ by zero outside U, and set

$$\bar{w}(p,t) = S(p)t + \chi(p)K_A(p)t + \chi(p)\big(w_A(p,t) - L_A(p)t\big).$$

Then $\bar{w} = w_A$ on U_1, $\bar{w}(p,0) = 0$, and $\sum_i t_i (D_t \bar{w}(p,0)t)_i = q_p(t)$ for every p. The residual

$$r(p,t) = G(t, \bar{w}(p,t), y(p,t)) - z(p,t)$$

therefore has zero two-jet in t, uniformly on compact P, and vanishes on U_1. Choose a smooth $\eta : P \to [0,1]$ equal to zero on U_0 and equal to one on $P \setminus U_1$. Then $\eta r = r$.

Define

$$H(p,t,v) = r(p,t) + \int_0^1 D_w G\big(t, \bar{w}(p,t) + s\eta(p)v, y(p,t)\big)[v]\, ds.$$

The fundamental theorem of calculus and $\eta r = r$ give the identity

$$G(t, \bar{w} + \eta v, y) - z = \eta H(p,t,v).$$

By (D.1), the degree-two part of H in (t,v) is $\sum_i t_i v_i$: the derivative of $b(y)$ in w is zero, r has zero two-jet, and $D_w E_0$ vanishes to order two. Lemma D.1, with parameter $p \in P$, gives a smooth $v(p,t) = O(|t|^2)$ on a common neighborhood with $H(p,t,v(p,t)) = 0$. Thus $w = \bar{w} + \eta v$ is an exact graph on all of P and equals w_A on U_0. For $A_0 = \emptyset$, take $\chi = 0$, $\eta = 1$; for $A_0 = P$, retain the original graph. □

Each graph lift is an embedding, since the projection onto the first coordinate of J_r is its left inverse. Moreover,

$$D_w G = \big(\xi_i \mapsto \sum_i t_i \xi_i\big) + O(|t|^2)$$

along any such graph. Let $S_t c = (t_i c/|t|^2)_i$. Then $\|S_t\| = |t|^{-1}$ and $D_w G S_t = I + O(|t|)$. Thus $S_t(D_w G S_t)^{-1}$ is a right inverse of $D_w G$ for small nonzero t. Together with the invertible range derivative, it gives a right inverse of $DF|_{V_r}$ of norm $O(|t|^{-1})$.

Corollary D.3 (Finite and infinite graph jets). *For a fixed target f and a polynomial w_* of degree $k \geq 1$ with $w_*(0) = 0$, an exact smooth graph with $j^k w = j^k w_*$ exists if and only if*

$$j_0^{k+1}\big(G(t, w_*(t), y(t)) - z(t)\big) = 0.$$

Every formal graph solution admits an exact smooth graph with the identical entire formal jet.

Proof. Suppose $j_0^k w = j_0^k w_*$. Smoothness gives $w - w_* = O(|t|^{k+1})$. Along the segment between these graphs, $D_w G = O(|t|)$, and hence

$$G(t, w, y) - G(t, w_*, y) = \int_0^1 D_w G(t, w_* + s(w - w_*), y)[w - w_*]ds = O(|t|^{k+2}).$$

If the first graph is exact, its prescribed residual has zero $(k+1)$-jet. Conversely, that jet condition gives a residual $O(|t|^{k+2})$. After setting $w = w_* + v$, the equation is $\sum_i t_i v_i + E(t, v) = 0$, with $j_0^2 E = 0$ and $E(t, 0) = O(|t|^{k+2})$. Lemma D.1 gives $v = O(|t|^{k+1})$, so $j_0^k w = j_0^k w_*$.

For an arbitrary formal graph write its formal series as $\sum_{j\geq 1} p_j(t)$, with each p_j homogeneous of degree j and values in C^r. Choose $\chi \in C_c^\infty(\mathbb{R}^r)$ equal to one near zero. For $|\alpha| \leq j - 1$, homogeneity and the Leibniz rule give

$$\|D^\alpha(\chi(t/\varepsilon)p_j(t))\|_\infty \leq C_{j,\alpha}\varepsilon^{j-|\alpha|}.$$

Choose $\varepsilon_j \downarrow 0$ so that the right side is at most 2^{-j} for every $|\alpha| \leq j-1$. For each fixed k, the derivatives of order at most k of the terms with $j \geq k+1$ converge uniformly. Thus

$$\bar{w}(t) = \sum_{j\geq 1} \chi(t/\varepsilon_j)p_j(t)$$

is smooth. At zero, each cutoff is locally one, and every term of degree greater than k has zero k-jet. Uniform convergence of the kth derivatives therefore gives $j_0^k \bar{w} = j_0^k \sum_{j=1}^k p_j$ for every k. The chain rule expresses each finite jet of $G(t, \bar{w}(t), y(t)) - z(t)$ in terms of finitely many jets of $\bar{w}$; the formal equations make all these jets zero. Its residual is flat. The flat clause of Lemma D.1 gives an exact graph differing from $\bar{w}$ by a flat function. The exact graph has the prescribed formal Taylor series. □

Appendix E. Continuous relative sections of scalar quadratic forms

E.1. **A bounded level with finite-dimensional sphere topology.** Let $v_1 > v_2 > \cdots > 0$ tend to zero and put

$$q(x, y) = \sum_{j\geq 1} v_j(x_j^2 - y_j^2), \qquad (x, y) \in \ell^2 \oplus \ell^2.$$

Fix $a, \rho > 0$ and $N \geq 1$ such that

$$v_{N+1} < a/\rho^2 < v_N.$$

Then $\{q = a\} \cap B_\rho$ deformation retracts onto

$$\left\{ y = 0,\ x_j = 0\ (j > N),\ \sum_{j=1}^N v_j x_j^2 = a \right\} \cong S^{N-1}.$$

To prove this, write $q_+(x) = \sum \nu_j x_j^2$ and first replace (x, y) by $(\alpha_s x, sy)$, where s decreases from 1 to 0 and

$$\alpha_s^2 = \frac{a + s^2 q_+(y)}{a + q_+(y)}.$$

The level is unchanged and the norm does not increase. Now write $x = x_H + x_T$ at the first N coordinates and replace it by $\beta_s x_H + s x_T$, where

$$\beta_s^2 = \frac{a - s^2 q_+(x_T)}{q_+(x_H)}.$$

The denominator is positive: otherwise $a \le \nu_{N+1}\|x\|^2 < \nu_{N+1}\rho^2 < a$. Its squared norm minus the initial squared norm is

$$(1 - s^2)\left(\frac{q_+(x_T)}{q_+(x_H)}\|x_H\|^2 - \|x_T\|^2\right) \le 0,$$

because $q_+(x_T) \le \nu_{N+1}\|x_T\|^2$ and $q_+(x_H) \ge \nu_N\|x_H\|^2$. The final ellipsoid lies inside the ball since $\|x_H\|^2 \le a/\nu_N < \rho^2$. Both deformations fix that ellipsoid. The zero fiber in the same ball contracts by $(x, y) \mapsto (1 - t)(x, y)$. For each N, take $a_N = \rho^2(\nu_N + \nu_{N+1})/2$. Then $a_N \downarrow 0$, whereas the fiber at a_N has the homotopy type of S^{N-1}. Since fibers of a Hurewicz fibration over an interval are homotopy equivalent, the restriction $q : B_\rho \to \mathbb{R}$ is not a Hurewicz fibration over any neighborhood of zero.

E.2. **Relative sections over the initial point.** Let $H = \ell^2 \oplus \ell^2$ and

$$q(x, y) = \sum_{j\ge1} \alpha_j x_j^2 - \sum_{j\ge1} \beta_j y_j^2, \tag{E.1}$$

where α_j, β_j are positive nonincreasing sequences converging to zero. A zero eigenspace may be added without changing the arguments.

Definition E.1. A local continuous relative section for q is a continuous map Φ on a product neighborhood of $(0, 0) \in H \times \mathbb{R}$ such that

$$\Phi(p, 0) = p, \qquad q(\Phi(p, s)) = q(p) + s. \tag{E.2}$$

Theorem E.2 (Successive-ratio criterion). *For* (E.1)*, a local correction* (E.2) *exists if and only if*

$$\sup_j \frac{\alpha_j}{\alpha_{j+1}} < \infty, \qquad \sup_j \frac{\beta_j}{\beta_{j+1}} < \infty. \tag{E.3}$$

When (E.3) *holds, a correction exists on all of $H \times \mathbb{R}$. Consequently $q : H \to \mathbb{R}$ is a Hurewicz fibration and every full fiber is contractible.*

Proof. Necessity. Suppose that a local Φ exists and fix $\epsilon > 0$ small. Since $\Phi(0, 0) = 0$, there are $\rho, \eta > 0$ such that

$$\|p\| < \rho, \quad |s| < \eta \quad \Longrightarrow \quad \|\Phi(p, s)\| < \epsilon.$$

If the positive ratios are unbounded, choose $N \ge 2$ and $a > 0$ with

$$\alpha_{N+1}\epsilon^2 < a < \alpha_N\rho^2, \qquad a < \eta. \tag{E.4}$$

Indeed choose N so large that $\alpha_N\rho^2 < \eta$ and $\alpha_N/\alpha_{N+1} > \epsilon^2/\rho^2$. Such indices exist arbitrarily far out because the finite initial segment has bounded ratios. Any a strictly between the two displayed endpoints has all the required properties.

The ellipsoid

$$S_a = \left\{ y = 0,\ x_j = 0\ (j > N),\ \sum_{j=1}^{N} \alpha_j x_j^2 = a \right\}$$

is an $(N-1)$-sphere contained in B_ρ. The formula

$$H_t(p) = \Phi\big((1-t)p,\ a-(1-t)^2 a\big), \qquad 0 \le t \le 1, \tag{E.5}$$

contracts this sphere in $\{q = a\} \cap B_\epsilon$.

However, the first N positive coordinates cannot vanish on that set: otherwise

$$q(x,y) \le \alpha_{N+1}\|x\|^2 < \alpha_{N+1}\epsilon^2 < a.$$

Their normalized projection therefore defines a map to S^{N-1}. On S_a its restriction is a homeomorphism of nonzero degree. Composing with (E.5) would make this map nullhomotopic, a contradiction. The involution $(x,y,s) \mapsto (y,x,-s)$ transforms a relative section for q into one for $-q$, whose positive coefficients are the β_j. The preceding contradiction therefore also excludes unbounded negative-coefficient ratios.

In particular,

$$\alpha_j = \beta_j = 2^{-2^j} \tag{E.6}$$

gives an analytic quadratic map with infinite positive and negative index and with no local continuous relative section. Theorem 4.9 applies to this map.

Sufficiency. First let the form be positive:

$$q_+(x) = \sum_j \lambda_j x_j^2, \qquad \lambda_1 = 1, \qquad \lambda_j/\lambda_{j+1} \le K.$$

Multiplying both the quadratic form and its increment by λ_1^{-1} reduces the general case to $\lambda_1 = 1$. Define

$$T_N e_j = \begin{cases} e_j, & j < N, \\ \sqrt{\lambda_j/\lambda_{j+1}}\, e_{j+1}, & j \ge N. \end{cases}$$

Then $q_+(T_N x) = q_+(x)$, $\|T_N\| \le \sqrt{K}$, and the range misses e_N. Put $E_j = e_j/\sqrt{\lambda_j}$, so that (E_j) is orthonormal for the positive bilinear form associated to q_+.

In the plane spanned by E_N, E_{N+1}, let

$$R_\theta E_N = \cos\theta E_N - \sin\theta E_{N+1}, \qquad R_\theta E_{N+1} = \sin\theta E_N + \cos\theta E_{N+1},$$

with identity on the other coordinates. The operators

$$T_{N,\theta} = R_\theta T_N, \qquad r_{N,\theta} = (-1)^N R_\theta E_N$$

satisfy

$$q_+(T_{N,\theta}x) = q_+(x), \qquad q_+(r_{N,\theta}) = 1, \qquad B_+(T_{N,\theta}x, r_{N,\theta}) = 0. \tag{E.7}$$

At $\theta = \pi/2$, $T_{N,\theta} = T_{N+1}$, and the signed free vector equals $(-1)^{N+1}E_{N+1}$. Thus both objects join continuously to the next stage.

Set $\tau_N = \lambda_N/N$, a strictly decreasing sequence even when eigenvalues repeat. On $[\tau_{N+1}, \tau_N]$, interpolate θ continuously from $\pi/2$ to 0 as τ increases. This defines T_τ, r_τ for $0 < \tau \le 1$ with

$$\|T_\tau\| \le K, \qquad T_\tau \longrightarrow I \text{ strongly as } \tau \downarrow 0, \qquad \|r_\tau\| \le \sqrt{K/\tau}. \tag{E.8}$$

For the last estimate, $\lambda_{N+1} \ge \lambda_N/K \ge N\tau/K \ge \tau/K$ and $\|r_{N,\theta}\| \le \lambda_{N+1}^{-1/2}$. If Π_{N-1} is the orthogonal projection onto the first $N-1$ coordinates, then

$$\|(T_\tau - I)x\| \le (K+1)\|(I - \Pi_{N-1})x\| \qquad (\tau_{N+1} \le \tau \le \tau_N).$$

Since $N \to \infty$ as $\tau \downarrow 0$, this proves strong convergence.

For a positive increment s, normalize the positive weights by α_1, put $\delta = s/\alpha_1$, and set

$$\tau = \frac{\sqrt{\delta}}{\|p\| + \sqrt{\delta}}, \qquad \Phi(p,s) = (T_\tau x + \sqrt{\delta}\, r_\tau,\ y).$$

For $s<0$, let the operators and vectors associated with the normalized coefficients β_j/β_1 be $T_\tau^{(-)}, r_\tau^{(-)}$. Put

$$\delta_- = -s/\beta_1, \qquad \tau_- = \frac{\sqrt{\delta_-}}{\|p\|+\sqrt{\delta_-}}, \qquad \Phi(p,s) = (x, T_{\tau_-}^{(-)}y + \sqrt{\delta_-}\, r_{\tau_-}^{(-)}).$$

The negative quadratic summand increases in magnitude by $\beta_1\delta_- = -s$, so $q(\Phi(p,s)) = q(p)+s$. For $s>0$, (E.7) gives the same identity with the increase in the positive summand. Set $\Phi(p,0)=p$.

The added vector obeys

$$\left\| \sqrt{\delta}\, r_\tau \right\| \le \sqrt{K}\, \delta^{1/4}(\|p\|+\sqrt{\delta})^{1/2}.$$

Suppose $(p_j, s_j)\to(p_0,0)$ with $p_0\neq 0$. On the subsequence $s_j>0$, one has $\tau_j\to 0$ and

$$\|T_{\tau_j}x_j - x_0\| \le K\|x_j-x_0\| + \|(T_{\tau_j}-I)x_0\| \longrightarrow 0.$$

The added vector tends to zero by the displayed bound, and the negative coordinate is unchanged. For $s_j<0$ the same inequality in the negative coordinate, with its own ratio bound, proves convergence; the zero-increment subsequence is the identity. If $p_j\to 0$, uniform boundedness gives $\|T_{\tau_j}x_j\| \le K\|x_j\|\to 0$, and the added-vector bound again tends to zero. This proves continuity at $(0,0)$. At a nonzero increment, τ stays positive locally; only finitely many successive intervals are encountered in a sufficiently small neighborhood. Their endpoint identities and their continuous matrix formulas prove continuity there. Thus Φ is continuous on $H\times\mathbb{R}$.

For any topological parameter space Z, homotopy $a : Z\times[0,1]\to\mathbb{R}$, and initial lift $u_0 : Z\to H$, the formula

$$u(z,t) = \Phi\big(u_0(z), a(z,t)-a(z,0)\big)$$

is a homotopy lift. Hence q is a Hurewicz fibration. On $q^{-1}(a)$, the homotopy

$$H_t(p) = \Phi((1-t)p, a-(1-t)^2 a)$$

satisfies $q(H_t(p)) = a$, $H_0(p)=p$, and $H_1(p)=\Phi(0,a)$. Thus every full fiber is contractible. □

E.3. **Negative-order pseudodifferential quadratic forms.**

Proposition E.3. *Let P be a real self-adjoint classical pseudodifferential operator of order $-s<0$ on a closed n-manifold, $n\ge 1$. Suppose its principal symbol has nonzero positive and negative parts on sets of positive cosphere measure. The scalar quadratic form $u\mapsto\langle Pu,u\rangle_{L^2}$ admits a global continuous relative section on its Hilbert space.*

Proof. The signed Birman–Solomyak asymptotics [50, Theorem 6.1] give positive and negative eigenvalue magnitudes

$$\alpha_j = c_+ j^{-s/n}(1+o(1)), \qquad \beta_j = c_- j^{-s/n}(1+o(1)), \qquad c_+, c_- > 0.$$

Consequently their consecutive ratios tend to one; the supremum of each sequence of ratios is finite. The spectral theorem identifies the positive and negative subspaces with the model of Theorem E.2. On $H_+\oplus H_-\oplus\ker P$, extend the correction by

$$\widehat{\Phi}((x,y,z),s) = (\Phi((x,y),s),z).$$

This map is continuous, is the identity at $s=0$, and increases the quadratic form by s. The homotopy $\widehat{\Phi}((1-t)(x,y,z), a-(1-t)^2 a)$ contracts its fiber at a. □

Let $\lambda>0$ be an isolated Laplace eigenvalue on a closed manifold, and let $0\neq C\in\operatorname{coker} D\mathsf{L}(0)$ for the full conformal parameter space. Set $T=-2\mathcal{A}B_C\mathcal{A}$, with $\mathcal{A}$ and B_C as in (2.39) and (2.47), and write $\sigma=\operatorname{ord}T\le 0$. For $s>0$, the isometry $\Lambda_s=(1+\Delta_g)^{s/2} : H^s\to L^2$ gives the representative

$$A_s = \Lambda_s^{-1}T\Lambda_s^{-1}, \qquad \operatorname{ord}A_s = \sigma-2s<0,$$

whose principal symbol is $|\xi|_g^{-2s}\sigma_\sigma(T)(x,\xi)$. It has both signs by Theorem 3.4. If P_K is the orthogonal projection onto the transformed first-variation kernel, then $P_K - I$ has finite rank and smooth range. Consequently

$$P_K A_s P_K - A_s = (P_K - I)A_s P_K + A_s(P_K - I)$$

has a smooth finite-rank kernel. Proposition E.3 therefore applies to the scalar quadratic form represented by $P_K A_s P_K$ in this fixed Hilbert space.

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