NATURAL SPECTRA AND ARITHMETIC OF LENS SPACES

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Abstract. The lens spaces $L(49;1,6,15)$ and $L(49;1,6,20)$ have identical natural spectra precisely for five irreducible SO(5)-types. This proves the finiteness conjecture of Lauret, Miatello and Rossetti. The types admitting bounded or compact self-adjoint corrections are classified, and the least operator norm and essential norm are determined for every one-row type on the bounded-correction locus. The distinction is arithmetic: exact equality requires equality of each Casimir eigenvalue multiplicity, whereas bounded correction requires equality of the total multiplicities in each separated quadratic cluster. Periodic character identities give both conditions and the asymptotic number of unmatched frequencies. For $L(p^2;1,1\pm p,1\pm 3p)$, the bounded-correction classification holds for every prime $p\geq 5$. For flat bundles on $L(q;1^m,a)$, with $q\geq 3$ odd, $m\geq 1$, and a a unit modulo q, equality of connection spectra is characterized in terms of the holonomies. For effective prime-square lens orbifolds, the scalar spectrum determines the gcd classes of the rotation weights and implies isometry of the corresponding proper covers at orders 9 and 25.

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Introduction

For integers $q\geq 2$, $n\geq 1$ and $s_1,\dots,s_n$, let $L(q;s_1,\dots,s_n)$ be the quotient of the unit sphere $S^{2n-1}\subset\mathbb{C}^n$ by the cyclic action generated by

$$(z_1,\dots,z_n)\longmapsto(e^{2\pi i s_1/q}z_1,\dots,e^{2\pi i s_n/q}z_n).$$

We assume $\gcd(q,s_1,\dots,s_n)=1$, so the action is effective. This quotient is a lens orbifold; it is a lens manifold when every s_j is coprime to q. All quotients carry the induced round metric.

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Put $G = \mathrm{SO}(6)$, and let $K \simeq \mathrm{SO}(5)$ be the stabilizer of a unit vector, so that $S^5 = G/K$. Let $\Gamma \subset G$ be finite and act freely on S^5, and put $M = \Gamma\backslash S^5$. An irreducible complex representation (τ, V_τ) of K, with an invariant Hermitian inner product, defines the bundle $E_{\tau,M} = \Gamma\backslash(G \times_K V_\tau)$. Write $\Delta_{\tau,M}$ for the nonnegative self-adjoint G-Casimir on $L^2(E_{\tau,M})$, normalized by the inner product $-\frac{1}{2}\operatorname{Tr}(XY)$ on $\mathfrak{so}(6)$. For exterior powers of the defining representation of K, this is the Hodge Laplacian [18, Section 5.1]. The Hodge spectra therefore involve only finitely many irreducible K-types. All spectral equalities below include multiplicities.

Five exact natural spectra. Consider

$$L = L(49; 1, 6, 15), \qquad L' = L(49; 1, 6, 20).$$

Lauret, Miatello and Rossetti proved that these spaces are nonisometric and isospectral on forms of every degree [17, Theorem 7.1 and Section 8]. They conjectured that only finitely many irreducible K-types give equal natural spectra on L and L' [17, Remark 8.2], [18, Conjecture 5.15]. For integers $u \geq v \geq 0$, let $\tau_{u,v}$ be the irreducible K-representation of integral highest weight (u, v) in orthonormal weight coordinates; these are all the irreducible K-types. Put $\Delta_{u,v,M} = \Delta_{\tau_{u,v},M}$. Theorem 1.1 states that

$$\operatorname{Spec}(\Delta_{u,v,L}) = \operatorname{Spec}(\Delta_{u,v,L'}) \quad \Longleftrightarrow \quad (u, v) \in \{(0,0), (1,0), (2,0), (1,1), (2,1)\}.$$

The three types $(0,0), (1,0), (1,1)$ account for all Hodge degrees by duality. The other two are the trace-free symmetric square $(2,0)$, of dimension 14, and the type $(2,1)$, of dimension 35.

For fixed u, v, a bounded correction is a bounded self-adjoint operator C on $L^2(E_{\tau_{u,v},L})$ such that

$$\operatorname{Spec}(\Delta_{u,v,L} + C) = \operatorname{Spec}(\Delta_{u,v,L'}).$$

The sum has domain $\mathcal{D}(\Delta_{u,v,L})$. Theorem 3.1 specifies a set $\mathscr{Z} \subset (\mathbb{Z}/49\mathbb{Z})^2$ with 289 elements such that a bounded correction exists if and only if $(2u + 3, 2v + 1) \bmod 49 \in \mathscr{Z}$. When it exists, C can be chosen to commute with every spectral projection of $\Delta_{u,v,L}$ and to satisfy

$$\|C\| \leq u(u + 2) - 2v.$$

A compact self-adjoint correction exists precisely for the five types with equal spectra (Corollary 3.8).

Let $\lambda_j(M) = \lambda_j(\Delta_{u,v,M})$, for $M = L, L'$, be the complete eigenvalue lists in nondecreasing order, with multiplicities and $j \geq 1$. Write $\|C\|_{\mathrm{ess}}$ for the distance from C to the compact operators. For each one-row type $\tau_{u,0}$ admitting a bounded correction, Theorem 3.10 determines the common value

$$\min_C \|C\| = \min_C \|C\|_{\mathrm{ess}} = \sup_{j \geq 1} |\lambda_j(L) - \lambda_j(L')|,$$

where both minima range over the bounded corrections just defined. This value is 19 for $u = 9$ and 89 for $u = 48$. This equality concerns the complete eigenvalue sequences, including their initial terms. For general types admitting a bounded correction, the least essential norm is the limit superior of the ordered displacements; the least operator norm is their supremum (Proposition 2.7).

Multiplicity at an offset and multiplicity in a block. For a fixed type $\tau_{u,v}$, the branching rule for $G \supset K$ gives the possible eigenvalues

$$r(r + 4) + w, \qquad r \geq u, \qquad w = k(k + 2) + c^2, \qquad v \leq k \leq u, \quad 0 \leq c \leq v,$$

with integral r, k, c. Let W be the finite set of offsets w. The contributing G-types have highest weights $(r, k, \pm c)$. For $c > 0$, the two signs are combined, as are all contributions with the same offset. Call the resulting multiset for a fixed r the r-block. The initial blocks can overlap. For the order-49 pair, the difference between the two multiplicities at offset w in this block depends only on $a = r + 2 \bmod 49$; denote it by $A_{u,v}(a, w)$. Then

$$\operatorname{Spec}(\Delta_{u,v,L}) = \operatorname{Spec}(\Delta_{u,v,L'}) \quad \Longleftrightarrow \quad A_{u,v}(a, w) = 0 \quad \text{for every } a, w,$$

$$\sup_{j \geq 1} |\lambda_j(L) - \lambda_j(L')| < \infty \quad \Longleftrightarrow \quad \sum_{w \in W} A_{u,v}(a, w) = 0 \quad \text{for every } a,$$

where $a \in \mathbb{Z}/49\mathbb{Z}$ and $w \in W$. Consecutive blocks are eventually separated by gaps tending to infinity. Periodicity therefore makes every nonzero discrepancy recur in infinitely many separated blocks. In particular, every type outside the five listed above has infinitely many unequal eigenvalue multiplicities. When the block totals agree, the partial sums over increasing offsets determine the ordered displacements in every sufficiently late block.

Lauret and Miatello proved rationality and finite determination for multiplicity sequences on finite unions of highest-weight strings, after fixing an integer divisible by the subgroup orders [15, Proposition 2.2 and Theorem 1.2]. They also proved a spectral-to-representation converse for a fixed natural type on odd-dimensional spherical quotients [16, Theorem 4.2]. In the present setting, its hypothesis is injectivity of $(k, c) \mapsto k(k+2) + c^2$ on $v \le k \le u$, $0 \le c \le v$, before equal offsets are combined. For the fixed pair L, L', Theorem 1.1 determines every type for which the spectra agree.

For two frequency lists and $\eta, R \ge 0$, let $E_\eta(R)$ be the minimum total number of unmatched copies after truncation at R, when matched frequencies differ by at most η and each labelled copy occurs in at most one matched pair. For the lists $\{\sqrt{\lambda_j(L)+4}\}_{j\ge1}$ and $\{\sqrt{\lambda_j(L')+4}\}_{j\ge1}$, put $R_N = N + \frac{5}{2}$, with $N \ge u$ integral. For every fixed $\kappa \ge 0$, Theorem 3.7 gives

$$E_{\kappa/R_N}(R_N) = R_N C(\kappa) + O_{u,v}(1) \qquad (N \longrightarrow \infty).$$

The function C is continuous, nonincreasing, and explicitly determined by the periodic multiplicities. Its value at zero vanishes precisely when the spectra are equal; its limit at infinity vanishes precisely when a bounded correction exists. In the latter case, if $C \not\equiv 0$, the least essential norm is twice the largest point of nondifferentiability of the piecewise affine function $\kappa \mapsto \kappa C(\kappa)$. The least essential norm is zero when $C \equiv 0$. Theorems 2.2 and 2.3 also give uniform estimates for tolerances below and at the limiting spacing 1 between consecutive frequency blocks.

Prime-square orders and affine congruences. For a prime $p \ge 5$, put

$$L_p^{\pm} = L(p^2; 1, 1 \pm p, 1 \pm 3p).$$

Their Hodge spectra agree in every degree by DeFord and Doyle [5, Theorem 1 and §8]; for $p \ge 7$, these spaces belong to the construction of Lauret, Miatello and Rossetti [17, Theorem 7.1]. Theorem 4.1 determines every natural type admitting a bounded self-adjoint correction between their Casimir spectra. In the coordinates $(X, Y) = (2u+3, 2v+1) \bmod p^2$, the set has cardinality

$$\begin{cases} 625, & p = 5, \\ 289, & p = 7, \\ 5p^2 - 4p + 40, & p \ge 11. \end{cases}$$

For $p = 5$ the two spaces are isometric. For $p = 7$, orientation-preserving isometries identify L_p^+, L_p^- with L, L', respectively. For $p \ge 11$, the set is the union of

$$\{X = 0\}, \quad \{Y = 0\}, \quad \{X = Y\}, \quad \{X = -Y\}, \quad \{p \mid X,\ p \mid Y\},$$

and the signed-permutation orbits of

$$(3, 1), \quad (5, 1), \quad (5, 3), \quad (7, 1), \quad (7, 3).$$

Here a signed-permutation orbit consists of $(\epsilon X, \delta Y)$ and $(\epsilon Y, \delta X)$ with $\epsilon, \delta \in \{1, -1\}$.

Let t be an indeterminate. By (4.9), bounded correction is equivalent to the vanishing of an explicitly given element of $\mathbb{Z}[t, t^{-1}]/(t^{p^2} - 1)$. This element is obtained by monomial substitutions and an alternating sum from a Laurent polynomial with integer coefficients independent of p. Its coefficient vanishing is a finite Boolean combination of affine congruences between exponents modulo p^2. For fixed integral arrangements, Kamiya, Takemura and Terao proved eventual quasipolynomial counting formulas in the modulus [10, 11]. More generally, fix integers $d, N \ge 1$, an indeterminate T,

polynomial rows $a_i(T) \in \mathbb{Z}[T]^d$ and $b_i(T) \in \mathbb{Z}[T]$ for $1 \le i \le N$, and a Boolean combination of the N conditions

$$a_i(p)x \equiv b_i(p) \pmod{p^k}, \qquad x \in (\mathbb{Z}/p^k\mathbb{Z})^d,$$

for primes p and integers $k \ge 1$. For every fixed k, the analogous count with an integer parameter is eventually quasipolynomial by Bogart, Goodrick and Woods [4, Theorem 1.15]. For each fixed k, Theorem 5.1 expresses the number of solutions of the fixed Boolean combination as a polynomial in p with integer coefficients and degree at most kd, for all primes outside one explicit finite set independent of k.

Flat holonomy and orbifold covers. Let $q \ge 3$ be odd, let a be a unit modulo q, and let $m \ge 1$. In $L(q; 1^m, a)$, the symbol 1^m denotes m entries equal to 1. For a finite-dimensional unitary representation ρ of its fundamental group $C_q = \mathbb{Z}/q\mathbb{Z}$, let Δ_ρ be the connection Laplacian of the associated flat Hermitian bundle. Its spectrum depends only on the real part of $\chi_\rho = \operatorname{tr}\rho$. For line bundles, Lauret's twisted generating functions describe the multiplicities, and his affine-congruence-lattice criterion characterizes isospectrality [12, Theorem 3.6, Proposition 3.7 and Theorem 4.2]. Let ρ, σ be finite-dimensional unitary representations of C_q, and let the bar denote the conjugate representation. If $m \ge 2$, Theorem 6.2 gives

$$\operatorname{Spec}(\Delta_\rho) = \operatorname{Spec}(\Delta_\sigma) \quad \Longleftrightarrow \quad \rho \oplus \overline{\rho} \cong \sigma \oplus \overline{\sigma}.$$

The frequencies of $(\Delta_\rho + m^2)^{1/2}$ belong to $\{\ell + m : \ell \in \mathbb{Z}_{\ge 0}\}$. The same conclusion follows if there are a fixed $\eta \in [0, 1)$ and a sequence $R_\nu \to \infty$ for which the matching deficit of the two frequency lists is $o(R_\nu^m)$.

In dimension three, equality of connection spectra need not determine the real part of the holonomy character. Theorem 7.1 determines the kernel of the spectral multiplicity map on the rational span of the real parts of characters for every odd $q \ge 3$ and every unit a. Its cyclotomic components are explicit scalar multiples of fixed fields in real cyclotomic extensions. Lauret gives examples of isospectral line twists with $a = 2$ [12, Example 4.10].

For a prime p, the scalar spectrum of an effective lens orbifold $L(p^2; s_1, \dots, s_n)$ determines the number of weights in each of the three classes $\gcd(p^2, s_j) = 1, p, p^2$ (Theorem 8.1). Its corresponding order-p cover is $L(p; s_1, \dots, s_n)$, with the weights reduced modulo p. At orders 9 and 25, these covers of scalar-isospectral orbifolds are isometric (Theorem 8.5). Together with the common spherical cover, this proves the corresponding-cover conjecture in [18, Conjecture 7.3] for these two orders in every odd dimension. For lens manifolds, corresponding covers of scalar-isospectral spaces are scalar isospectral [14, Theorem 4.3]. Álzaga and Lauret proved scalar spectral rigidity for spherical orbifolds of group order at most 7 [1, Theorem 4.1]; the assertion at orders 9 and 25 recovers the order-3 or order-5 cover from the scalar spectrum of the larger quotient. In dimension three, scalar isospectrality determines a lens orbifold up to isometry [3, Theorem 3.1].

Finally, for every prime p and every $n \ge 2$, two effective $(2n-1)$-dimensional lens orbifolds of order p^2 with equal Hodge spectra in degrees $0, \dots, n-2$ have equal Hodge spectra in every degree (Corollary 8.6). The gcd invariant supplies the coordinate-axis terms in Lauret's congruence-lattice reduction [14, Remark 4.6]; the required moment identities are those of [13, Corollary 2.3].

Part 1. Ordered spectra and natural operators

1. Exact natural spectra

1.1. **Natural Casimir operators.** Let $e_1, \dots, e_6$ be the standard basis of $\mathbb{R}^6$, and put

$$G = \mathrm{SO}(6), \qquad K = \{g \in G : ge_6 = e_6\} \simeq \mathrm{SO}(5).$$

The inner product $\langle X, Y\rangle = -\frac{1}{2}\operatorname{Tr}(XY)$ on $\mathfrak{so}(6)$ induces the metric of curvature one on $G/K \simeq S^5$. Let R_θ denote a plane rotation through angle θ. On the maximal torus $\operatorname{diag}(R_{\theta_1}, R_{\theta_2}, R_{\theta_3})$, the weight $\sum_{i=1}^3 a_i\varepsilon_i$ denotes the character $e^{i\sum_{i=1}^3 a_i\theta_i}$, where $a_i \in \mathbb{Z}$. The coordinates ε_i are orthonormal for the

induced inner product on weights. The maximal torus of K is given by $\theta_3 = 0$, with weight coordinates $\varepsilon_1, \varepsilon_2$; these are the conventions of [16, Section 4]. Choose the positive roots $\varepsilon_i \pm \varepsilon_j$ with $i < j$ for G, and $\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1, \varepsilon_2$ for K.

Put $s = (1, 6, 15)$ and $s' = (1, 6, 20)$. For $t \in \{s, s'\}$, define

$$g_t = \operatorname{diag}\bigl(R_{2\pi t_1/49}, R_{2\pi t_2/49}, R_{2\pi t_3/49}\bigr), \qquad \Gamma_t = \langle g_t \rangle.$$

Every coordinate of s and s' is a unit modulo 49, so these cyclic groups have order 49 and act freely on S^5. Write

$$\Gamma = \Gamma_s, \qquad \Gamma' = \Gamma_{s'},$$
$$L = \Gamma\backslash S^5 = L(49; 1, 6, 15), \qquad L' = \Gamma'\backslash S^5 = L(49; 1, 6, 20).$$

All finite-dimensional representations are complex and equipped with invariant Hermitian inner products. For integers $u \geq v \geq 0$, let $(\tau_{u,v}, V_{\tau_{u,v}})$ be the irreducible K-representation of highest weight $u\varepsilon_1 + v\varepsilon_2$. These are precisely the irreducible representations of K [16, Section 4]. For $\tau = \tau_{u,v}$, let

$$E_{\tau,L} = \Gamma\backslash(G \times_K V_\tau), \qquad E_{\tau,L'} = \Gamma'\backslash(G \times_K V_\tau).$$

Denote by $\Delta_{\tau,L}$ and $\Delta_{\tau,L'}$ the nonnegative self-adjoint Casimir operators on the respective L^2 spaces of sections, for the inner product fixed above. These elliptic operators have discrete spectra with finite multiplicities [18, Section 4.1]. We call L, L' τ-isospectral when these spectra agree, including multiplicities.

Lauret, Miatello and Rossetti proved that L, L' are nonisometric and isospectral on differential forms of every degree [17, Theorem 7.1 and Section 8]. They conjectured that only finitely many irreducible K-types give equal natural spectra [17, Remark 8.2], [18, Conjecture 5.15].

Theorem 1.1 (Exact natural spectra). *For integers $u \geq v \geq 0$, the lens spaces $L(49; 1, 6, 15)$ and $L(49; 1, 6, 20)$ are $\tau_{u,v}$-isospectral if and only if*

$$(u, v) \in \{(0, 0), (1, 0), (2, 0), (1, 1), (2, 1)\}.$$

The Hodge types are $(0, 0)$, $(1, 0)$ and $(1, 1)$, corresponding to exterior powers of the defining representation and their duals [18, Section 5.1]. The Weyl dimension formula [8, Proposition 28.1], applied to K, gives

$$\dim \tau_{u,v} = \frac{(2u + 3)(2v + 1)(u - v + 1)(u + v + 2)}{6}.$$

The five types in Theorem 1.1 therefore have dimensions $1, 5, 14, 10, 35$, in the order stated there.

1.2. **Branching and Casimir eigenvalues.** Let $\pi_{r,k,c}$ be the irreducible G-representation of highest weight $r\varepsilon_1 + k\varepsilon_2 + c\varepsilon_3$, where

$$r \geq k \geq |c|, \qquad r, k, c \in \mathbb{Z},$$

and denote its character by $\chi_{r,k,c}$. The half-sum of the positive roots of G has coordinates $\rho = (2, 1, 0)$. The Casimir formula [18, (4.3)] gives

$$\lambda(r, k, c) = r(r + 4) + k(k + 2) + c^2. \tag{1.1}$$

Write Hom_K for the space of K-equivariant linear maps. The branching rule [16, (4.1)], with $n = 3$, is

$$\dim \operatorname{Hom}_K(V_{\pi_{r,k,c}}, V_{\tau_{u,v}}) = \begin{cases} 1, & r \geq u \geq k \geq v \geq |c|, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Let $\widehat{G}$ denote the set of equivalence classes of irreducible G-representations, and let V_π^Γ be the subspace fixed by Γ. The decomposition

$$L^2(E_{\tau,L}) \simeq \widehat{\bigoplus_{\pi \in \widehat{G}}} V_\pi^\Gamma \otimes \operatorname{Hom}_K(V_\pi, V_\tau)$$

is a Hilbert direct sum, and the Casimir acts by its scalar on each summand [18, (4.5) and Proposition 4.1]. For $\lambda \geq 0$, write $m_{\tau,L}(\lambda)$ and $m_{\tau,L'}(\lambda)$ for its respective multiplicities, with value zero when λ is absent from the corresponding spectrum. Then

$$m_{\tau_{u,v},L}(\lambda) = \sum_{\substack{r,k,c\in\mathbb{Z},\ r\geq u\geq k\geq v\geq|c| \\ \lambda(r,k,c)=\lambda}} \dim V^{\Gamma}_{\pi_{r,k,c}},$$

and the same formula holds for L' with Γ' in place of Γ. In particular, all eigenvalues are nonnegative integers.

For $r \geq k \geq |c|$, put

$$d(r,k,c) = \dim V^{\Gamma}_{\pi_{r,k,c}} - \dim V^{\Gamma'}_{\pi_{r,k,c}}.$$

The calculation in [17, Lemma 8.1] gives $d(4,3,0) = 8-6 = 2$.

1.3. **Periodic dimension data.** Let $\zeta = e^{2\pi i/49}$. For $t \in \{s, s'\}$ and $x = (a,b,c) \in \mathbb{Z}^3$, define the Laurent polynomials

$$P_t(x;X) = \det\left(X^{x_i t_j} + X^{-x_i t_j}\right)_{1\leq i,j\leq 3}, \qquad D_t(X) = P_t(\rho;X). \tag{1.3}$$

For either t, the six residues $\pm t_1, \pm t_2, \pm t_3$ are distinct modulo 49. At g_t^j, the type-D_3 Weyl denominator is $D_t(\zeta^j)/2$. The Weyl denominator formula [8, Corollary 25.5] therefore gives $D_t(\zeta^j) \neq 0$ for every $j \in (\mathbb{Z}/49\mathbb{Z})^{\times}$. Define

$$T(a,b,c) = \frac{1}{49}\,\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\left(\frac{P_s((a,b,c);\zeta)}{D_s(\zeta)} - \frac{P_{s'}((a,b,c);\zeta)}{D_{s'}(\zeta)}\right). \tag{1.4}$$

For a triple $x = (a,b,c)$, write $T(x) = T(a,b,c)$.

Lemma 1.2. *The function T is integer-valued, even and 49-periodic in each variable. For integers $r \geq k \geq 0$,*

$$d(r,k,0) = T(r+2,k+1,0), \tag{1.5}$$

$$d(r,k,c) + d(r,k,-c) = 2T(r+2,k+1,c) \quad (1 \leq c \leq k). \tag{1.6}$$

Proof. Evenness and periodicity follow from (1.3). For $x \in \mathbb{Z}^3$ and $y = (y_1,y_2,y_3) \in (\mathbb{C}^{\times})^3$, put

$$P_x(y) = \det(y_j^{x_i} + y_j^{-x_i})_{1\leq i,j\leq 3}, \qquad Q_x(y) = \det(y_j^{x_i} - y_j^{-x_i})_{1\leq i,j\leq 3}.$$

The Weyl group of G consists of coordinate permutations and sign changes whose product is 1. Its alternant is $(P_x + Q_x)/2$: in the two determinant expansions the sign vectors with product -1 cancel, and those with product 1 occur twice. Since $Q_\rho = 0$, the denominator alternant is $P_\rho/2$. The Weyl character formula [8, Theorem 25.4] gives, for $t = s, s'$,

$$\chi_{r,k,0}(g_t) = \frac{P_t((r+2,k+1,0);\zeta)}{D_t(\zeta)},$$

$$\frac{\chi_{r,k,c}(g_t) + \chi_{r,k,-c}(g_t)}{2} = \frac{P_t((r+2,k+1,c);\zeta)}{D_t(\zeta)} \quad (1 \leq c \leq k).$$

Indeed, changing c to $-c$ leaves P_x unchanged and reverses the sign of Q_x.

Invariant dimensions are averages of characters over the cyclic group. The identity terms cancel between Γ and Γ'. For $1 \leq j \leq 6$, the elements g_s^{7j} and $g_{s'}^{7j}$ have rotation weights

$$j(1,-1,1) \quad \text{and} \quad j(1,-1,-1) \pmod 7.$$

A reflection in one axis of the third rotation plane conjugates the first element to the second. On regular torus elements this reflection replaces y_3 by y_3^{-1}, leaving P_x unchanged and changing the sign of Q_x. It therefore interchanges $\chi_{r,k,c}$ and $\chi_{r,k,-c}$. The identity holds on the whole torus by continuity. Hence the order-seven terms cancel for $\chi_{r,k,0}$ and for $\chi_{r,k,c} + \chi_{r,k,-c}$.

The remaining powers are indexed by $j \in (\mathbb{Z}/49\mathbb{Z})^\times$. Since the automorphisms of $\mathbb{Q}(\zeta)$ send ζ to these ζ^j, their average difference is

$$\frac{1}{49}\sum_{j\in(\mathbb{Z}/49\mathbb{Z})^\times}\left(\frac{P_s(x;\zeta^j)}{D_s(\zeta^j)}-\frac{P_{s'}(x;\zeta^j)}{D_{s'}(\zeta^j)}\right)=T(x).$$

Taking $x=(r+2,k+1,c)$ proves (1.5)–(1.6).

The longest Weyl element sends (r,k,c) to $(-r,-k,c)$. The dual highest-weight formula [8, Proposition 33.5] gives $V^*_{\pi_{r,k,c}} \simeq V_{\pi_{r,k,-c}}$. Duality preserves the multiplicity of the trivial representation of a finite group; thus $d(r,k,c)=d(r,k,-c)$. It follows that

$$T(r+2,k+1,c)=d(r,k,c)\in\mathbb{Z}\qquad(r\ge k\ge c\ge 0).$$

For any $(a,b,c_0)\in\mathbb{Z}^3$, choose successively

$$\begin{aligned} &c\ge 0, && c\equiv c_0 \pmod{49},\\ &k\ge c, && k+1\equiv b \pmod{49},\\ &r\ge k, && r+2\equiv a \pmod{49}. \end{aligned}$$

Periodicity then gives $T(a,b,c_0)=T(r+2,k+1,c)\in\mathbb{Z}$. □

By periodicity, the first argument of T may be taken in $\mathbb{Z}/49\mathbb{Z}$. Fix integers $u\ge v\ge 0$. Let $\mathbf{1}_{c=0}$ be 1 if $c=0$ and 0 otherwise. For $a\in\mathbb{Z}/49\mathbb{Z}$ and $w\in\mathbb{Z}$, define, with integer summation indices k,c,

$$A_{u,v}(a,w)=\sum_{\substack{v\le k\le u,\ 0\le c\le v\\ k(k+2)+c^2=w}}(2-\mathbf{1}_{c=0})T(a,k+1,c). \tag{1.7}$$

Empty sums are zero. The least and greatest attained offsets are

$$w_-=v(v+2),\qquad w_+=u(u+2)+v^2,\qquad H_{u,v}=w_+-w_-=u(u+2)-2v.$$

For every integer $\lambda\ge 0$, the multiplicity formula and (1.5)–(1.6) give

$$m_{\tau_{u,v},L}(\lambda)-m_{\tau_{u,v},L'}(\lambda)=\sum_{r=u}^{\infty}A_{u,v}\big(r+2 \bmod 49,\ \lambda-r(r+4)\big).$$

The sum is finite, since $A_{u,v}(a,w)=0$ outside $[w_-,w_+]$.

Lemma 1.3. *The pair L,L' is $\tau_{u,v}$-isospectral if and only if $A_{u,v}(a,w)=0$ for every $a\in\mathbb{Z}/49\mathbb{Z}$ and every $w\in\mathbb{Z}$.*

Proof. If all these sums vanish, the displayed multiplicity identity proves isospectrality.

Conversely, suppose that the spectra agree. For an unattained offset w, the defining sum for $A_{u,v}(a,w)$ is empty. For $r\ge u$, put

$$I_r=[r(r+4)+w_-,\ r(r+4)+w_+].$$

For $r>u$ the adjacent gaps are

$$\begin{aligned} \min I_r-\max I_{r-1}&=2r+3-H_{u,v},\\ \min I_{r+1}-\max I_r&=2r+5-H_{u,v}. \end{aligned}$$

Both are positive for $r\ge\max\{u+1,H_{u,v}\}$. Since the endpoints increase with r, such an I_r is disjoint from every $I_{r'}$ with $r'\ge u$ and $r'\ne r$. Given $a\in\mathbb{Z}/49\mathbb{Z}$, choose such an r with $r+2\equiv a \pmod{49}$. At any attained offset w, the multiplicity identity reduces to

$$0=m_{\tau_{u,v},L}\big(r(r+4)+w\big)-m_{\tau_{u,v},L'}\big(r(r+4)+w\big)=A_{u,v}(a,w).$$

This holds for every residue a and every attained offset w. □

1.4. **Finite character identities over the integers.** Let $\mathcal{R} = \mathbb{Z}[X]/(X^{49}-1)$, and denote the class of X by X. Since $X^{49} = 1$, the element X is a unit. Each element of $\mathcal{R}$ has a unique representative of degree at most 48. For $t \in \{s, s'\}$, define $h_t : \mathbb{Z}/49\mathbb{Z} \to \mathbb{Z}$ by the first three columns of Table 1 and by $h_t(-j) = h_t(j)$ for $1 \le j \le 24$. Put

$$H_t(X) = \sum_{j=0}^{48} h_t(j)X^j, \qquad \Phi(X) = \sum_{j=0}^{6} X^{7j}, \qquad C_j = X^j + X^{-j} \quad (j \in \mathbb{Z}).$$

The element C_j depends only on j mod 49. The identities $C_iC_j = C_{i+j} + C_{i-j}$ and $C_{-j} = C_j$ give the images in $\mathcal{R}$ of the denominators in (1.3):

$$D_s = 2(C_3 + C_4 + C_8 - C_{11} - 3C_{13} - C_{17} + C_{18} + C_{20} + C_{22} - C_{24}),$$
$$D_{s'} = 2(-C_3 + C_4 + 3C_8 + C_{10} - C_{11} - C_{13} - C_{15} + C_{17} - C_{18} - C_{22}).$$

Write these expressions as $D_t = 2\sum_{j=1}^{24} d_{t,j}C_j$. For $0 \le \ell \le 48$, the coefficient of X^ℓ in the representative of D_tH_t is

$$2\sum_{j=1}^{24} d_{t,j}\big(h_t(\ell - j) + h_t(\ell + j)\big) = \begin{cases} 12, & \ell = 0, \\ -2, & \ell \in \{7, 14, 21, 28, 35, 42\}, \\ 0, & \text{otherwise}, \end{cases}$$

by the corresponding column of Table 1. Thus, in $\mathcal{R}$,

$$D_s(X)H_s(X) = 14 - 2\Phi(X), \qquad D_{s'}(X)H_{s'}(X) = 14 - 2\Phi(X). \tag{1.8}$$

Evaluation at $X = \zeta$ is defined on $\mathcal{R}$, since $\zeta^{49} = 1$. As $\Phi(\zeta) = 0$, it follows that $D_t(\zeta)^{-1} = H_t(\zeta)/14$ for $t \in \{s, s'\}$. The two columns also give

$$\sum_{j=0}^{6} h_t(a + 7j) = 0 \qquad (t \in \{s, s'\},\ 0 \le a \le 6).$$

For a condition E, let $\mathbf{1}_E$ be 1 if E holds and 0 otherwise. For every $m \in \mathbb{Z}$,

$$\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^m) = \sum_{j=0}^{48} \zeta^{jm} - \sum_{j=0}^{6} \zeta^{7jm} = 49\mathbf{1}_{49|m} - 7\mathbf{1}_{7|m}.$$

Consequently, for $t \in \{s, s'\}$,

$$\frac{1}{49}\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\big(\zeta^m H_t(\zeta)\big) = h_t(-m) - \frac{1}{7}\sum_{\substack{0 \le j \le 48 \\ j \equiv -m \pmod 7}} h_t(j) = h_t(m).$$

Here the sum is zero by the preceding identities, and $h_t(-m) = h_t(m)$. Hence

$$\frac{1}{49}\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\frac{\zeta^m}{D_s(\zeta)} = \frac{h_s(m)}{14}, \qquad \frac{1}{49}\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}\frac{\zeta^m}{D_{s'}(\zeta)} = \frac{h_{s'}(m)}{14}. \tag{1.9}$$

Let S_3 be the permutation group of $\{1, 2, 3\}$, and write $\mathrm{sgn}(\sigma)$ for the sign of $\sigma \in S_3$. For $x = (a, b, c) \in \mathbb{Z}^3$, expansion of the determinants in (1.4) gives

$$14T(a, b, c) = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \sum_{e \in \{\pm 1\}^3} \left[h_s\left(\sum_{i=1}^{3} e_i x_i s_{\sigma(i)}\right) - h_{s'}\left(\sum_{i=1}^{3} e_i x_i s'_{\sigma(i)}\right)\right]. \tag{1.10}$$

The last three columns of Table 1 give the corresponding values of T. Since T is even in its second variable, for every $b \in \mathbb{Z}/49\mathbb{Z}$ there are $a \in \mathbb{Z}/49\mathbb{Z}$ and an integer c with $0 \le c \le 5$ such that $T(a, b, c) \ne 0$.

TABLE 1. Integer coefficients and nonzero character differences.

j	$h_s(j)$	$h_{s'}(j)$	a	c	$T(a,j,c)$
0	0	0	6	4	-2
1	0	0	6	5	-1
2	-1	0	5	3	1
3	0	-1	5	2	-1
4	1	0	6	0	2
5	1	1	6	1	1
6	0	0	4	0	-2
7	-1	-1	6	4	-1
8	0	0	4	0	2
9	1	1	6	1	-1
10	1	1	6	0	-2
11	-1	0	6	0	2
12	0	-1	6	1	1
13	0	0	4	0	-2
14	1	1	6	4	2
15	0	0	4	0	2
16	-2	0	6	1	-1
17	-1	0	6	0	-2
18	0	-1	6	0	-2
19	1	1	6	1	-1
20	0	0	4	0	2
21	0	0	13	4	2
22	0	0	4	0	-2
23	0	-2	6	1	1
24	0	1	6	0	2

The types $\tau_{0,0}, \tau_{1,0}, \tau_{1,1}$ are the exterior powers of degrees $0, 1, 2$ of the defining representation of K. The Hodge isospectrality of L, L' [17, Theorem 7.1 and Section 8] and Lemma 1.3 therefore give, for every $a \in \mathbb{Z}/49\mathbb{Z}$,

$$\begin{aligned} T(a,1,0) &= A_{0,0}(a,0) = 0, \\ T(a,2,0) &= A_{1,0}(a,3) = 0, \\ 2T(a,2,1) &= A_{1,1}(a,4) = 0. \end{aligned}$$

For $t \in \{s, s'\}$ and $a, b, c \in \mathbb{Z}$, put

$$R_t(a;b,c) = 2 \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \sum_{\epsilon,\delta \in \{\pm 1\}} h_t\big(a t_{\sigma(1)} + \epsilon b\, t_{\sigma(2)} + \delta c\, t_{\sigma(3)}\big).$$

Pairing opposite sign vectors in (1.10) and applying $h_t(-j) = h_t(j)$ gives $14T(a,b,c) = R_s(a;b,c) - R_{s'}(a;b,c)$. Replacing (ϵ, δ) by $(-\epsilon, -\delta)$ gives $R_t(-a;b,c) = R_t(a;b,c)$; these functions are also 49-periodic in a. Table 2 gives the values of the displayed sum for $0 \le a \le 24$ and $(b,c) = (3,0), (3,1)$. The values for $t = s, s'$ agree. Evenness and periodicity extend the equality to every $a \in \mathbb{Z}$, so

$$T(a,1,0) = T(a,2,0) = T(a,3,0) = T(a,2,1) = T(a,3,1) = 0 \qquad (a \in \mathbb{Z}/49\mathbb{Z}). \tag{1.11}$$

1.5. **Proof of the classification.**

TABLE 2. Values of $R_t(a;3,0)/4$ and $R_t(a;3,1)/2$ for both $t = s$ and $t = s'$.

a	$R_t(a;3,0)/4$	$R_t(a;3,1)/2$
0	0	0
1	0	0
2	-6	0
3	0	0
4	3	0
5	0	-3
6	0	-1
7	0	0
8	0	1
9	2	3
10	-3	0
11	-1	0
12	6	1
13	0	5
14	0	0
15	0	-5
16	-4	-1
17	1	0
18	2	0
19	-2	-2
20	0	-3
21	0	0
22	0	3
23	4	2
24	-2	0

Proof of Theorem 1.1. Fix integers $u \geq v \geq 0$, and suppose that L, L' are $\tau_{u,v}$-isospectral. By Lemma 1.3, $A_{u,v}(a, w) = 0$ for every $a \in \mathbb{Z}/49\mathbb{Z}$ and $w \in \mathbb{Z}$. If an integer c_0 satisfies $0 \leq c_0 \leq v$ and $c_0^2 < 2v + 3$, then

$$A_{u,v}\big(a, v(v+2) + c_0^2\big) = (2 - \mathbf{1}_{c_0=0})T(a, v+1, c_0). \tag{1.12}$$

Indeed, $k \geq v + 1$ implies $k(k+2) \geq v(v+2) + 2v + 3 > v(v+2) + c_0^2$. For $k = v$, equality of the offsets gives $c^2 = c_0^2$, hence $c = c_0$, since $c, c_0 \geq 0$. The pair (v, c_0) is admissible because $u \geq v$.

If $v \geq 12$, Table 1 and evenness and periodicity of T in its second variable give $a \in \mathbb{Z}/49\mathbb{Z}$ and $0 \leq c_0 \leq 5$ such that $T(a, v+1, c_0) \neq 0$. Since $c_0 \leq v$ and $c_0^2 \leq 25 < 2v + 3$, equation (1.12) contradicts $A_{u,v} = 0$.

For $2 \leq v \leq 11$, $v \neq 6$, Table 1 gives $T(a, v+1, c_0) \neq 0$ for the choices

v	2	3	4	5	7	8	9	10	11
a	5	6	6	4	4	6	6	6	6
c_0	2	0	1	0	0	1	0	0	1

In each column, $0 \leq c_0 \leq v$ and $c_0^2 < 2v + 3$; equation (1.12) again contradicts $A_{u,v} = 0$.

If $u = v = 6$, the only admissible value of k is 6, and $k(k+2) + c^2 = 64$ forces $c = 4$. Table 1 gives

$$A_{6,6}(6, 64) = 2T(6, 7, 4) = -2.$$

If $v = 6$ and $u \geq 7$, the offset 63 has the unique admissible pair $(k, c) = (7, 0)$: $k = 6$ would require $c^2 = 15$, whereas $k \geq 8$ gives $k(k+2) \geq 80$. Thus

$$A_{u,6}(4, 63) = T(4, 8, 0) = 2$$

by Table 1. Hence $v \le 1$.

If $v \in \{0,1\}$ and $u \ge 3$, then $k(k+2)+c^2 \le 9$ for $k \le 2$, and $k(k+2)+c^2 \ge 24$ for $k \ge 4$. Thus the offset 15 has the unique admissible pair $(3,0)$. By (1.5) and [17, proof of Lemma 8.1],

$$A_{u,v}(6,15) = T(6,4,0) = d(4,3,0) = 2.$$

Consequently $u \le 2$ and $v \le 1$, which gives precisely the five pairs in the statement.

Conversely, for each of these five pairs, every $(k+1,c)$ in (1.7) belongs to

$$\{(1,0),(2,0),(3,0),(2,1),(3,1)\}.$$

Equation (1.11) gives $A_{u,v}(a,w) = 0$ for every a, w. Lemma 1.3 proves isospectrality. □

1.6. **The number of spectral discrepancies.** Fix integers $u \ge v \ge 0$ and put $\tau = \tau_{u,v}$. For a real number $X \ge 0$, let

$$D_\tau(X) = \#\{\lambda \in \mathbb{Z}_{\ge 0} \,:\, \lambda \le X, m_{\tau,L}(\lambda) \ne m_{\tau,L'}(\lambda)\}.$$

Thus $D_\tau(X)$ counts distinct eigenvalues with unequal multiplicities. Let

$$W_{u,v} = \{k(k+2)+c^2 \,:\, k,c \in \mathbb{Z},\ v \le k \le u,\ 0 \le c \le v\}$$

and put

$$\eta_\tau = \frac{1}{49} \sum_{w \in W_{u,v}} \#\{a \in \mathbb{Z}/49\mathbb{Z} \,:\, A_{u,v}(a,w) \ne 0\}. \tag{1.13}$$

Corollary 1.4. *For every irreducible τ,*

$$D_\tau(X) = \eta_\tau \sqrt{X} + O_\tau(1) \qquad (X \longrightarrow \infty).$$

The coefficient η_τ is zero precisely for the five types in Theorem 1.1. For every other type, $\eta_\tau \ge 4/49$, and there exist $a \in \mathbb{Z}$, $w \in W_{u,v}$ and $\delta \in \mathbb{Z} \setminus \{0\}$ such that

$$m_{\tau,L}(r(r+4)+w) - m_{\tau,L'}(r(r+4)+w) = \delta$$

for every sufficiently large integer r with $r+2 \equiv a \pmod{49}$.

Proof. Set $R = \max\{u+1, H_{u,v}\}$ and, for integers $r \ge u$, put $I_r = [r(r+4)+w_-, r(r+4)+w_+]$. The proof of Lemma 1.3 gives $I_r \cap I_{r'} = \emptyset$ whenever $r \ge R$, $r' \ge u$ and $r' \ne r$. Thus $(r,w) \mapsto r(r+4)+w$ is injective on $\{r \in \mathbb{Z} \,:\, r \ge R\} \times W_{u,v}$, and

$$m_{\tau,L}(r(r+4)+w) - m_{\tau,L'}(r(r+4)+w) = A_{u,v}(r+2,w) \qquad (r \ge R,\ w \in W_{u,v}). \tag{1.14}$$

The terms with $u \le r < R$ contribute at most $(R-u)|W_{u,v}|$ distinct eigenvalues.

For $a \in \mathbb{Z}/49\mathbb{Z}$ and $w \in W_{u,v}$, let $N_{a,w}(X)$ count the integers $r \ge R$ with $r+2 \equiv a \pmod{49}$ and $r(r+4)+w \le X$. If $X \ge R(R+4)+w_+$, put $B_w(X) = \sqrt{X+4-w} - 2$. Then $B_w(X) \ge R$ and

$$\left| N_{a,w}(X) - \frac{B_w(X) - R}{49} \right| \le 1.$$

Moreover,

$$B_w(X) - \sqrt{X} = -2 + \frac{4-w}{\sqrt{X+4-w} + \sqrt{X}} = O_\tau(1),$$

uniformly for $w \in W_{u,v}$. Hence $N_{a,w}(X) = \sqrt{X}/49 + O_\tau(1)$, uniformly in a, w. Summing over the pairs with $A_{u,v}(a,w) \ne 0$ proves the asymptotic. All summands in (1.13) are nonnegative, and $A_{u,v}(a,w) = 0$ outside $W_{u,v}$. By Lemma 1.3 and Theorem 1.1, $\eta_\tau = 0$ if and only if τ is one of the five listed types.

The seven-term sums of $h_s, h_{s'}$ in Subsection 1.4 give, for integers α, β, a with $7 \nmid \alpha$,

$$\sum_{j=0}^{6} h_t\big(\alpha(a+7j)+\beta\big) = \sum_{j=0}^{6} h_t(\alpha a + \beta + 7j) = 0 \qquad (t \in \{s, s'\}),$$

since multiplication by α permutes $\mathbb{Z}/7\mathbb{Z}$. Every coordinate of s, s' is a unit modulo 7. Equations (1.10) and (1.7) therefore give

$$\sum_{j=0}^{6} A_{u,v}(a+7j, w) = 0 \qquad (a, w \in \mathbb{Z}).$$

Also, $A_{u,v}(-a, w) = A_{u,v}(a, w)$ by Lemma 1.2 and (1.7). For every type outside the five listed ones, the proof of Theorem 1.1 gives $a_0 \in \{4, 5, 6, 13\}$ and $w \in W_{u,v}$ with $A_{u,v}(a_0, w) \neq 0$. The zero sum forces a second nonzero value in the same residue class modulo 7. Evenness gives two nonzero values in the negative class. The two classes are disjoint, since $7 \nmid 2a_0$. This offset contributes at least four residues to (1.13), so $\eta_\tau \geq 4/49$. Finally, $\delta = A_{u,v}(a_0, w)$ is a nonzero integer by Lemma 1.2 and (1.7). Equation (1.14) gives the asserted multiplicity difference for every $r \geq R$ in that residue class. □

1.7. **A finiteness criterion for natural spectra.**

Proposition 1.5. *Let $\Gamma, \Gamma' \subset \mathrm{SO}(6)$ be finite groups acting freely on S^5, and put $M = \Gamma\backslash S^5$ and $M' = \Gamma'\backslash S^5$. For integers $r \geq k \geq |c|$, put*

$$d(r, k, c) = \dim V^{\Gamma}_{\pi_{r,k,c}} - \dim V^{\Gamma'}_{\pi_{r,k,c}}.$$

Assume that (1.5)–(1.6) *hold for this d and a function $T : \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$ that is periodic with an integer period $q \geq 1$ in its first two variables. Suppose that, for an integer $C \geq 0$,*

(1) for every $b \in \mathbb{Z}/q\mathbb{Z}$ there are $a \in \mathbb{Z}/q\mathbb{Z}$ and an integer c with $0 \leq c \leq C$ and $T(a, b, c) \neq 0$;

(2) there are $a_0, b_0 \in \mathbb{Z}/q\mathbb{Z}$ with $T(a_0, b_0, 0) \neq 0$.

If M, M' are $\tau_{u,v}$-isospectral for the Casimir operators with normalization (1.1)*, then*

$$v \leq \max\left\{C - 1, \left\lfloor \frac{C^2 - 3}{2} \right\rfloor\right\}, \qquad u \leq \left\lceil \frac{v^2}{2} \right\rceil + q - 2.$$

In particular, only finitely many irreducible $\mathrm{SO}(5)$*-types give equal spectra on M, M'.*

Proof. Fix integers $u \geq v \geq 0$. For $a \in \mathbb{Z}/q\mathbb{Z}$ and $w \in \mathbb{Z}$, define $A_{u,v}(a, w)$ by (1.7) with the present function T. The Casimir formula (1.1), the branching rule (1.2) and the two assumed identities give the multiplicity formula preceding Lemma 1.3, with M, M' in place of L, L' and q in place of 49. The disjointness of the intervals in its proof depends only on u, v. For each $a \in \mathbb{Z}/q\mathbb{Z}$, there are arbitrarily large integers r with $r + 2 \equiv a \pmod{q}$. Consequently, if M, M' are $\tau_{u,v}$-isospectral, then

$$A_{u,v}(a, w) = 0 \qquad (a \in \mathbb{Z}/q\mathbb{Z},\ w \in \mathbb{Z}).$$

Suppose that $v \geq C$ and $2v + 3 > C^2$. Condition (1), with $b = v + 1 \bmod q$, gives a, c such that $T(a, v+1, c) \neq 0$, $0 \leq c \leq v$ and $c^2 < 2v + 3$. The identity (1.12), which follows from the defining sum (1.7), gives

$$A_{u,v}\big(a, v(v+2) + c^2\big) = (2 - \mathbf{1}_{c=0})T(a, v+1, c) \neq 0.$$

Thus isospectrality implies $v < C$ or $2v + 3 \leq C^2$, which is the asserted bound on v.

Put $m = \lceil v^2/2 \rceil$. Then $m \geq v$ and $2m + 1 > v^2$. Choose $k \in \{m, m+1, \dots, m+q-1\}$ with $k + 1 \equiv b_0 \pmod{q}$; these q consecutive integers contain exactly one such k. Suppose that $u \geq k$, and let integers k', c satisfy $v \leq k' \leq u$ and $0 \leq c \leq v$. If $k' < k$, then

$$k(k+2) - k'(k'+2) \geq 2k + 1 > v^2 \geq c^2.$$

If $k' > k$, then $k'(k'+2) + c^2 > k(k+2)$. If $k' = k$, equality forces $c = 0$. Hence the offset $k(k+2)$ has the unique admissible pair $(k, 0)$, and

$$A_{u,v}\big(a_0, k(k+2)\big) = T(a_0, k+1, 0) = T(a_0, b_0, 0) \neq 0.$$

It follows that $u < k \leq m + q - 1$, hence $u \leq \lceil v^2/2 \rceil + q - 2$. □

2. Quadratic spectral clusters

Let X, Y be finite multisets in $\mathbb{R}$, and let $\eta \geq 0$. A matching at tolerance η is a collection of pairs (x, y), with one labelled copy from X and one from Y, such that $|x - y| \leq \eta$ and no copy occurs in two pairs. Let $b_\eta(X, Y)$ be the largest possible number of pairs, and put

$$E_\eta(X, Y) = |X| + |Y| - 2b_\eta(X, Y).$$

For multisets X, Y that are bounded below and finite on bounded intervals, define

$$E_\eta(R; X, Y) = E_\eta\big(X \cap (-\infty, R], Y \cap (-\infty, R]\big) \qquad (R \in \mathbb{R}).$$

Intersections retain their multiplicities. For $x \in \mathbb{R}$ and an integer $m \geq 0$, write $x^{[m]}$ for m labelled copies of x.

2.1. Matchings on a finite path.

Lemma 2.1. *Let $s \geq 1$ be an integer, let $x_1 < \cdots < x_s$ be real numbers, and let p_i, q_i be nonnegative integers. Fix $\eta \geq 0$, and put*

$$\delta_i = p_i - q_i, \qquad S = \sum_{i=1}^{s} |\delta_i|.$$

Join i to $i+1$ when $x_{i+1} - x_i \leq \eta$, and let $\mathscr{I}$ be the set of connected components. If $\min(p_i, q_i) \geq S$ for every i, then

$$E_\eta\big(\{x_i^{[p_i]}\}_{i=1}^s, \{x_i^{[q_i]}\}_{i=1}^s\big) = \sum_{C \in \mathscr{I}} \left| \sum_{i \in C} \delta_i \right|. \tag{2.1}$$

More generally, for a component $C = \{a, \dots, b\}$, the minimum number of unmatched copies on C is $|\sum_{i=a}^{b} \delta_i|$ provided that, for some real $L \geq 0$,

$$\left| \sum_{i=a}^{j} \delta_i \right| \leq L \quad (a \leq j < b), \qquad \min(p_i, q_i) \geq L \quad (a \leq i \leq b).$$

Proof. If $x_{j+1} - x_j > \eta$, then $x_b - x_a > \eta$ for $a \leq j < b$. Thus every allowed pair lies in one component, whose total discrepancy is a lower bound for its unmatched count in absolute value. Fix a component and renumber its vertices $1, \dots, \ell$. Put $\Delta = \sum_{i=1}^{\ell} \delta_i$. If $\Delta \geq 0$, remove Δ copies from the first multiset at x_ℓ; if $\Delta < 0$, remove $-\Delta$ copies from the second. These removals are possible: writing $f = \sum_{i<\ell} \delta_i$, one has

$$p_\ell - \Delta = q_\ell - f \geq 0 \quad (\Delta \geq 0), \qquad q_\ell + \Delta = p_\ell + f \geq 0 \quad (\Delta < 0).$$

Let p_i', q_i' be the remaining multiplicities and put

$$f_i = \sum_{j=1}^{i} (p_j' - q_j') \quad (0 \leq i \leq \ell).$$

Then $f_0 = f_\ell = 0$ and $|f_i| \leq L$ for $i < \ell$. For $1 \leq i < \ell$ set

$$m_{i,i+1} = \max(f_i, 0), \qquad m_{i+1,i} = \max(-f_i, 0),$$

and set all other off-diagonal entries to zero. Their row sum is

$$o_i = \max(f_i, 0) + \max(-f_{i-1}, 0).$$

For $i < \ell$, if $f_{i-1} \leq 0 \leq f_i$, then $o_i = f_i - f_{i-1} = p_i - q_i \leq p_i$; if $f_i \leq 0 \leq f_{i-1}$, then $o_i = 0$. In the remaining cases $o_i \leq L \leq p_i$. At the last vertex, $o_\ell = \max(p_\ell' - q_\ell', 0) \leq p_\ell'$. Consequently $m_{ii} = p_i' - o_i$ is a nonnegative integer. The difference between the off-diagonal row and column sums is $f_i - f_{i-1} = p_i' - q_i'$, so the row sums of (m_{ij}) are p_i' and its column sums are q_i'. Its nonzero entries pair copies at equal or adjacent points. Exactly $|\Delta|$ copies remain unmatched. Finally, every proper prefix discrepancy in any component is at most S in absolute value. Taking $L = S$ gives (2.1). □

The lower bound on the common multiplicities cannot be omitted. At $x_1 = 0, x_2 = 1, x_3 = 2$, take $(p_1, p_2, p_3) = (1, 0, 0)$ and $(q_1, q_2, q_3) = (0, 0, 1)$. For $\eta = 1$ the path is connected, but no pair is allowed. Its total discrepancy is zero and its unmatched count is two.

2.2. **The periodic quadratic model.** Fix an integer $s \geq 1$, real numbers $\sigma > 0$, β, c, and real offsets $w_1 < \cdots < w_s$. Choose an integer r_0 such that $\sigma r_0 + \beta > 0$ and $(\sigma r_0 + \beta)^2 + w_1 > 0$. For integers $r \geq r_0$ and $1 \leq i \leq s$, put

$$t_r = \sigma r + \beta, \qquad x_{r,i} = \sqrt{t_r^2 + w_i}, \qquad \lambda_{r,i} = t_r^2 + w_i - c. \tag{2.2}$$

Let P, Q be finite multisets in $[-c, \infty)$ and let $p_{r,i}, q_{r,i}$ be nonnegative integers for these indices. Write $\sqcup$ for multiset union and define the eigenvalue multisets

$$\Lambda = P \sqcup \bigsqcup_{r \geq r_0} \{\lambda_{r,i}^{[p_{r,i}]} \,:\, 1 \leq i \leq s\},$$
$$\Lambda' = Q \sqcup \bigsqcup_{r \geq r_0} \{\lambda_{r,i}^{[q_{r,i}]} \,:\, 1 \leq i \leq s\}.$$

The corresponding frequency multisets are $X = \{\sqrt{\lambda + c} \,:\, \lambda \in \Lambda\}$ and $Y = \{\sqrt{\lambda + c} \,:\, \lambda \in \Lambda'\}$, with multiplicities. Write $E_\eta(R) = E_\eta(R; X, Y)$. All eigenvalue and frequency lists are ordered nondecreasingly, with multiplicities.

Let $q \geq 1$ be an integer and $\delta_i : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}$ for $1 \leq i \leq s$. Assume

$$p_{r,i} - q_{r,i} = \delta_i(r \bmod q), \qquad \min(p_{r,i}, q_{r,i}) \longrightarrow \infty \quad (r \longrightarrow \infty) \tag{2.3}$$

for every i. The multisets Λ, Λ', X, Y are infinite and have finite multiplicity on bounded intervals.

Fix $0 < \theta < 1$ and, for integers $K \geq r_0$, set

$$R_K = \sigma K + \beta + \theta\sigma. \tag{2.4}$$

Since $x_{r,i} = t_r + O(t_r^{-1})$, uniformly in i, for all sufficiently large K one has $x_{K,s} < R_K < x_{K+1,1}$, and all frequencies from P, Q are below R_K. The frequencies at most R_K are then precisely those from P, Q and from the indices $r_0 \leq r \leq K$. Coincident frequencies retain the sum of their multiplicities.

Put

$$g_i = w_{i+1} - w_i \quad (1 \leq i < s), \qquad H = w_s - w_1, \qquad B_a = \sum_{i=1}^{s} \delta_i(a), \qquad G = \{g_i \,:\, 1 \leq i < s\}.$$

All sums in a below run over $\mathbb{Z}/q\mathbb{Z}$. For $t \geq 0$, let $\mathscr{P}(t)$ be the partition of $\{1, \ldots, s\}$ into the components of the path with edge $\{i, i+1\}$ whenever $g_i \leq t$. Define

$$F_a(t) = \sum_{C \in \mathscr{P}(t)} \left| \sum_{i \in C} \delta_i(a) \right|. \tag{2.5}$$

The triangle inequality shows that F_a is nonincreasing. It is right-continuous, and its only possible jumps belong to G. For $a \in \mathbb{Z}/q\mathbb{Z}$ and $g \in G$, put

$$D_a(g) = F_a(g-) - F_a(g), \qquad d_g = \sum_a D_a(g), \qquad c_0 = \frac{1}{q} \sum_a \sum_{i=1}^{s} |\delta_i(a)|, \qquad c_+ = \frac{1}{q} \sum_a |B_a|.$$

Then $D_a(g) \geq 0$ and

$$F_a(t) = |B_a| + \sum_{g \in G} D_a(g) \mathbf{1}_{\{t < g\}} \qquad (t \geq 0). \tag{2.6}$$

This formula includes simultaneous mergers at repeated gaps. For $s = 1$, the set G and all sums indexed by G are empty.

Theorem 2.2 (Resolution of quadratic clusters). *Under* (2.2)–(2.3)*, define*

$$\begin{aligned} C(0) &= c_0, \\ C(\kappa) &= \frac{1}{q}\sum_a \int_0^1 F_a(2\kappa x)\,dx \\ &= c_+ + \frac{1}{q}\sum_{g\in G} d_g \min\left(1, \frac{g}{2\kappa}\right) \qquad (\kappa > 0). \end{aligned} \tag{2.7}$$

For every fixed $0 \le \eta_* < \sigma$*, uniformly for* $0 \le \eta \le \eta_*$ *and* $K \longrightarrow \infty$*,*

$$E_\eta(R_K) = \frac{R_K}{\sigma} C(\eta R_K) + O(1). \tag{2.8}$$

The error constant depends only on the fixed model, θ *and* η_**.*

Proof. Set $M = \max_i |w_i|$ and $S = \max_a \sum_i |\delta_i(a)|$. The identity

$$x_{r,i} - t_r = \frac{w_i}{\sqrt{t_r^2 + w_i} + t_r}$$

gives $|x_{r,i} - t_r| \le M/t_r$. Hence $x_{r+1,1} - x_{r,s} \to \sigma$. Choose an integer $r_1 \ge r_0$ such that

$$x_{r+1,1} - x_{r,s} > \eta_*, \qquad \min(p_{r,i}, q_{r,i}) \ge S \qquad (r \ge r_1,\ 1 \le i \le s).$$

Deleting one labelled copy changes a maximum matching cardinality by zero or one and changes E_η by at most one. Thus deleting P, Q and the finitely many indices $r < r_1$ changes $E_\eta(R_K)$ by $O(1)$, uniformly in η, K. No allowed pair joins different retained indices r. Every proper prefix discrepancy of a component at a fixed r has absolute value at most S, so Lemma 2.1 applies.

The adjacent frequency gap is

$$x_{r,i+1} - x_{r,i} = \frac{g_i}{\sqrt{t_r^2 + w_i} + \sqrt{t_r^2 + w_{i+1}}}. \tag{2.9}$$

The preceding bound gives

$$\left|\frac{x_{r,i} + x_{r,i+1}}{2} - t_r\right| \le \frac{M}{t_r}.$$

For $\eta > 0$, if the conditions $x_{r,i+1} - x_{r,i} \le \eta$ and $g_i \le 2\eta t_r$ differ, then

$$\left|t_r - \frac{g_i}{2\eta}\right| \le \frac{M}{t_r} \le \frac{M}{t_{r_1}}.$$

The number of such indices r for a fixed i is at most $1 + \lfloor 2M/(\sigma t_{r_1})\rfloor$. At $\eta = 0$, both graphs have no edges. Their component-deficit sums lie in $[0, S]$. Consequently,

$$E_\eta(R_K) = \sum_{r=r_1}^{K} F_{r \bmod q}(2\eta t_r) + O(1) \tag{2.10}$$

uniformly for $0 \le \eta \le \eta_*$.

For $a \in \mathbb{Z}/q\mathbb{Z}$, let r_a be the least integer at least r_1 in that residue class. For $K \ge r_1$ and $L > 0$, put

$$N_a(K, L) = \#\{r \in \mathbb{Z} : r_1 \le r \le K,\ r \equiv a \pmod q,\ t_r < L\}.$$

Since $r \le K$ is equivalent to $t_r < t_{K+1}$,

$$\begin{aligned} N_a(K, L) &= \max\left\{0, \left\lceil \frac{\min(t_{K+1}, L) - t_{r_a}}{q\sigma} \right\rceil\right\} \\ &= \frac{\min(R_K, L)}{q\sigma} + O(1). \end{aligned} \tag{2.11}$$

The error is uniform in a, K, L: the difference from $\min(t_{K+1}, L)/(q\sigma)$ is at most $1 + t_{r_a}/(q\sigma)$, and $t_{K+1} - R_K = (1-\theta)\sigma$. Multiplying this count by $D_a(g)$ or $|B_a|$ and summing over a gives

$$\sum_{r=r_1}^{K} D_{r \bmod q}(g)\mathbf{1}_{\{t_r<L\}} = \frac{d_g}{q\sigma}\min(R_K, L) + O(1),$$

$$\sum_{r=r_1}^{K} |B_{r \bmod q}| = \frac{R_K c_+}{\sigma} + O(1),$$

where the second equality follows by taking $L > t_{K+1}$. For $\eta > 0$, substitute $L = g/(2\eta)$ in (2.6) and (2.10). This gives

$$E_\eta(R_K) = \frac{R_K c_+}{\sigma} + \frac{1}{q\sigma}\sum_{g\in G} d_g \min\left(R_K, \frac{g}{2\eta}\right) + O(1).$$

For $\eta = 0$, (2.10) and (2.11) give $E_0(R_K) = R_K c_0/\sigma + O(1)$. Integrating (2.6) gives (2.7) and proves (2.8). □

The function $\mathcal{C}$ is continuous and nonincreasing on $[0,\infty)$, is piecewise a linear combination of 1 and $1/\kappa$ with constant coefficients on $(0,\infty)$, and has limit c_+ at infinity. If $G \neq \emptyset$, then $\mathcal{C}(\kappa) = c_0$ for $0 \le 2\kappa \le \min G$. If $G = \emptyset$, then $\mathcal{C} \equiv c_0 = c_+$.

Write $R = R_K$. For fixed $\kappa \ge 0$, Theorem 2.2 gives

$$E_{\kappa/R}(R) = \frac{R}{\sigma}\mathcal{C}(\kappa) + O(1).$$

Fix $0 \le \eta_* < \sigma$, and let $0 \le \eta_R \le \eta_*$. If $\eta_R R \to \kappa < \infty$, then

$$\frac{E_{\eta_R}(R)}{R} \longrightarrow \frac{\mathcal{C}(\kappa)}{\sigma}.$$

Since $\mathcal{C}$ is constant near zero, $\eta_R R \to 0$ gives $E_{\eta_R}(R) = Rc_0/\sigma + O(1)$. For $\eta_R > 0$ and $2\eta_R R \ge \max(G \cup \{0\})$, one has

$$E_{\eta_R}(R) = \frac{Rc_+}{\sigma} + \frac{1}{2q\sigma\eta_R}\sum_{g\in G} d_g g + O(1). \tag{2.12}$$

Thus $\eta_R R \to \infty$ implies $E_{\eta_R}(R)/R \to c_+/\sigma$, and the second term in (2.12) is $o(R)$. For fixed $0 < \eta < \sigma$ it is constant. If every $d_g = 0$, then $E_\eta(R) = Rc_+/\sigma + O(1)$ throughout the stated uniform range.

Let δ_g denote the unit point mass at g. Define the finite nonnegative measure on $(0,\infty)$

$$\nu = \sum_{g\in G}(d_g/q)\,\delta_g.$$

The identity

$$\kappa\mathcal{C}(\kappa) = \kappa c_+ + \frac{1}{q}\sum_{g\in G} d_g \min(\kappa, g/2) \qquad (\kappa \ge 0)$$

gives, with $d^+/d\kappa$ denoting the right derivative,

$$\frac{d^+}{d\kappa}\big[\kappa\mathcal{C}(\kappa)\big] = \frac{1}{q}\sum_a F_a(2\kappa) \qquad (\kappa \ge 0). \tag{2.13}$$

The right derivative has downward jump d_g/q at $\kappa = g/2$. Thus $\mathcal{C}$ determines the residue-averaged component deficit and the measure ν. It need not determine the individual $\delta_i(a)$: the arrays $(\delta_i(a))$ and $(-\delta_i(a))$ have the same functions F_a, and they differ whenever the array is nonzero.

2.3. **The boundary at the cluster spacing.** Let $R = R_K$ be the cutoffs in (2.4), and put $h_r = x_{r+1,1} - x_{r,s}$ for $r \geq r_0$.

Theorem 2.3. *Assume the periodic quadratic model, $H = w_s - w_1 > 0$, and*

$$\sum_{a=0}^{q-1} B_a = 0. \tag{2.14}$$

For each fixed $0 < \eta_0 < \sigma$, as $K \to \infty$, uniformly for $\eta_0 \leq \eta < \sigma$,

$$E_\eta(R) = \frac{c_+}{\sigma} \max\left(0, R - \frac{H}{2(\sigma - \eta)}\right) + O(1). \tag{2.15}$$

Uniformly for $\eta \geq \sigma$, one has $E_\eta(R) = O(1)$. The error constants depend only on the fixed model, θ and η_0. In particular, for each fixed $\kappa > 0$,

$$E_{\sigma-\kappa/R}(R) = \frac{Rc_+}{\sigma} \max\left(0, 1 - \frac{H}{2\kappa}\right) + O(1). \tag{2.16}$$

Proof. Put $L = \sum_a \sum_{i=1}^{s} |\delta_i(a)|$. For $r \in \mathbb{Z}$ and $1 \leq i \leq s$, define $e_{sr+i} = \delta_i(r \bmod q)$. Then $e_{n+qs} = e_n$ and $\sum_{n=1}^{qs} e_n = 0$. Every finite interval of integers is a disjoint union of intervals of length qs and one interval of length less than qs. The former have sum zero; the latter contains each residue modulo qs at most once. Consequently,

$$\left|\sum_{n=u}^{v} e_n\right| \leq L \qquad (u, v \in \mathbb{Z},\ u \leq v).$$

Uniformly in i, one has

$$x_{r,i} - t_r - \frac{w_i}{2t_r} = -\frac{w_i^2}{2t_r(x_{r,i} + t_r)^2} = O(t_r^{-3}).$$

Consequently, for some fixed $C \geq 0$ and all sufficiently large r,

$$\left|h_r - \sigma + \frac{H}{2t_r}\right| \leq \frac{C}{t_r^2}.$$

Choose an integer $r_1 \geq r_0$, independent of η and K, such that

$$h_r > 0, \qquad x_{r,s} - x_{r,1} < \eta_0, \qquad \min(p_{r,i}, q_{r,i}) \geq L, \qquad \frac{C}{t_r} \leq \frac{H}{4} \quad (r \geq r_1,\ 1 \leq i \leq s).$$

The deletion estimate in the proof of Theorem 2.2 shows that removing P, Q and the indices $r < r_1$ changes $E_\eta(R)$ by $O(1)$, uniformly in η and K.

For $\eta \geq \eta_0$, let $\mathscr{J}_\eta(K)$ be the components of the path on $\{r_1, \dots, K\}$ with edge $\{r, r+1\}$ when $h_r \leq \eta$. All points in a fixed block are connected. The interval bound above and Lemma 2.1 give

$$E_\eta(R) = \sum_{J \in \mathscr{J}_\eta(K)} \left|\sum_{r \in J} B_{r \bmod q}\right| + O(1).$$

Every inner sum has absolute value at most L.

Suppose $\eta < \sigma$, and put $\varepsilon = \sigma - \eta$ and $A_\varepsilon = H/(2\varepsilon)$. Replace the condition $h_r \leq \eta$ by $t_r \leq A_\varepsilon$. If these conditions differ, then

$$|\varepsilon t_r - H/2| \leq C/t_r, \qquad \varepsilon t_r \geq H/4, \qquad |t_r - A_\varepsilon| \leq \frac{4C}{H}.$$

Since $t_{r+1} - t_r = \sigma$, at most $1 + \lfloor 8C/(H\sigma) \rfloor$ edges differ. Adding an edge merges two intervals with discrepancies u, v, and changes the sum of absolute discrepancies by

$$|u| + |v| - |u + v| \leq 2\min(|u|, |v|) \leq 2L.$$

Deleting an edge gives the same bound. Thus this replacement changes the preceding sum by $O(1)$, uniformly in η and K.

For the new path, every component except the first is a single block. Apart from at most one index, these single-block components are precisely the indices with $t_r > A_\varepsilon$. The first component has discrepancy at most L in absolute value. Hence

$$E_\eta(R) = \sum_{r=r_1}^{K} |B_{r \bmod q}| \mathbf{1}_{\{t_r > A_\varepsilon\}} + O(1).$$

Equation (2.11) applies with threshold A_ε; replacing a strict inequality by a non-strict one affects at most one index. Subtracting the weighted count below the threshold from the total weighted count gives

$$\sum_{r=r_1}^{K} |B_{r \bmod q}| \mathbf{1}_{\{t_r > A_\varepsilon\}} = \frac{c_+}{\sigma}\big(R - \min(R, A_\varepsilon)\big) + O(1).$$

This proves (2.15), also when the threshold lies outside the retained range.

The choice of r_1 gives $h_r \le \sigma - H/(4t_r) < \sigma$. At $\eta = \sigma$ the retained path is connected, so its unmatched count is at most L. Since $E_\eta(R)$ is nonincreasing in η, the bound is uniform for $\eta \ge \sigma$. □

For $s = 1$ and $w_1 = w$, the derivative of $t \mapsto \sqrt{t^2 + w}$ is $t/\sqrt{t^2 + w}$ on the relevant interval. Thus $h_r < \sigma$ if $w > 0$, $h_r = \sigma$ if $w = 0$, and $h_r > \sigma$ if $w < 0$. If $w > 0$, then

$$h_r = \sigma - \frac{\sigma w}{2t_r^2} + O(t_r^{-3}), \qquad R_K^2(\sigma - h_K) \longrightarrow \frac{\sigma w}{2}.$$

For example, take $\sigma = 1$, $\beta = c = 0$, $w_1 = -1$, $q = 2$, $r_0 = 2$, $P = Q = \emptyset$, and

$$p_{r,1} = r + \max\{(-1)^r, 0\}, \qquad q_{r,1} = r + \max\{-(-1)^r, 0\}.$$

Then $B_0 + B_1 = 0$ and $h_r > 1$. No pair at tolerance 1 joins distinct support points, so $E_1(R_K) = K - 1$ for all sufficiently large K. If every $\delta_i(a)$ vanishes, matching equal copies at each support point gives $E_\eta(R) \le |P| + |Q|$ for every $\eta \ge 0$ and every real R.

2.4. **Complete ordered spectra and the initial prefix.** Write $\lambda_1 \le \lambda_2 \le \cdots$ and $\lambda'_1 \le \lambda'_2 \le \cdots$ for the complete ordered lists of Λ and Λ', respectively, whenever these multisets are infinite, and put $d_{\mathrm{pre}} = |P| - |Q|$.

Theorem 2.4. *Retain the quadratic support and finite prefixes of subsection 2.2. Assume $p_{r,i} - q_{r,i} = \delta_i(r \bmod q)$ and that Λ, Λ' are infinite. The following conditions are equivalent:*

(1) $\sup_j |\lambda_j - \lambda'_j| < \infty$*;*

(2) $B_a = 0$ *for every* $a \in \mathbb{Z}/q\mathbb{Z}$ *and* $d_{\mathrm{pre}} = 0$*;*

(3) $\sqrt{\lambda_j + c} - \sqrt{\lambda'_j + c} \longrightarrow 0$.

Under condition 2, put $m = |P| = |Q|$ and write $P_1 \le \cdots \le P_m$ and $Q_1 \le \cdots \le Q_m$ for the ordered prefixes. Then

$$\sup_j |\lambda_j - \lambda'_j| \le \max\left\{H, \max_{1 \le j \le m} |P_j - Q_j|\right\},$$

where the second maximum is zero if $m = 0$.

Proof. Both ordered lists tend to infinity, since their multisets are infinite and finite on bounded intervals. Thus condition 1 implies condition 3 by the identity

$$\sqrt{\lambda_j + c} - \sqrt{\lambda'_j + c} = \frac{\lambda_j - \lambda'_j}{\sqrt{\lambda_j + c} + \sqrt{\lambda'_j + c}},$$

valid for all sufficiently large j.

Suppose condition 3 holds. For all sufficiently large r, every prefix value lies below the r-block, the blocks are ordered, and

$$h_r = x_{r+1,1} - x_{r,s} \longrightarrow \sigma > 0.$$

Let N_r, N_r' be the numbers of frequencies through block r in the two complete lists. If $N_r \neq N_r'$, put $j_r = \min(N_r, N_r') + 1$. One member of the ordered pair of rank j_r is at most $x_{r,s}$ and the other is at least $x_{r+1,1}$. Hence

$$\left|\sqrt{\lambda_{j_r} + c} - \sqrt{\lambda'_{j_r} + c}\right| \geq h_r.$$

Since $N_r, N_r' \longrightarrow \infty$, condition 3 forces $N_r = N_r'$ for every sufficiently large r. Now

$$N_r - N_r' = d_{\mathrm{pre}} + \sum_{h=r_0}^{r} B_{h \bmod q}.$$

Subtracting the equalities at r and $r-1$ gives $B_{r \bmod q} = 0$ eventually. Periodicity gives $B_a = 0$ for every a, and the displayed identity then gives $d_{\mathrm{pre}} = 0$.

Suppose condition 2 holds, and denote the asserted bound by C. Match the ordered prefixes and, in each block, match the two equal-cardinality multisets. Every paired eigenvalue difference has absolute value at most C. Restrict this matching to the prefixes and the complete blocks $r_0 \leq r \leq K$. The paired entries form two self-adjoint diagonal matrices A_K, A_K' of the same size, with $\|A_K - A_K'\| \leq C$. By [2, Theorem 1.1, p. 419], their ordered eigenvalues differ by at most C. For every fixed rank j, the jth ordered eigenvalues of these matrices equal λ_j, λ_j' for all sufficiently large K, since the omitted eigenvalues tend to infinity. Therefore $|\lambda_j - \lambda_j'| \leq C$ for every j. □

The equality $c_+ = 0$ is equivalent to $B_a = 0$ for every a and imposes no condition on d_{pre}. For example, take $\sigma = 1$, $\beta = c = 0$, $s = 1$, $w_1 = 0$, $r_0 = 1$, $p_{r,1} = q_{r,1} = r^4$, $P = \{0\}$ and $Q = \emptyset$. For every $R \geq 0$ the truncated frequency multisets differ by one copy at zero. Hence $E_\eta(R) = 1$ for every $\eta \geq 0$. Put $n_r = \sum_{h=1}^{r} h^4$. At rank $n_r + 1$ the eigenvalues are r^2 and $(r+1)^2$, respectively. Their absolute difference is $2r + 1$.

Theorem 2.5. *Assume common multiplicity growth, $B_a = 0$ for every a, and $d_{\mathrm{pre}} = 0$. Define*

$$s_i(a) = \sum_{j=1}^{i} \delta_j(a) \quad (1 \leq i < s), \qquad D_* = \max\{g_i \,:\, s_i(a) \neq 0 \text{ for some } a\},$$

with an empty maximum equal to zero. Then

$$\limsup_{j \to \infty} |\lambda_j - \lambda_j'| = D_*. \tag{2.17}$$

All sufficiently late ordered pairs have displacement at most D_*. In every sufficiently late r-block, exactly $|s_i(r \bmod q)|$ ordered pairs join the offsets w_i and w_{i+1}, for each $1 \leq i < s$; every other pair joins equal offsets. Thus each such boundary contributes $|s_i(r \bmod q)|$ displacements equal to g_i. Contributions from equal gaps are added.

Proof. For all sufficiently large r, the blocks are separated and the prefixes precede them. Equality of the prefix and block cardinalities then aligns the complete ordered indices within each block. Fix such a block and put

$$U_i = \sum_{j=1}^{i} p_{r,j}, \qquad V_i = \sum_{j=1}^{i} q_{r,j} \quad (0 \leq i \leq s).$$

Here $U_0 = V_0 = 0$, $U_s = V_s$, and $U_i - V_i = s_i(r \bmod q)$ for $1 \leq i < s$. Set

$$M = \max_{a,\, 1 \leq i < s} |s_i(a)|,$$

with $M = 0$ if $s = 1$. Choose r large enough that $\min(p_{r,i}, q_{r,i}) > M$ for every i. For $0 \le i < s$,

$$\min(U_{i+1}, V_{i+1}) \ge \min(U_i, V_i) + \min(p_{r,i+1}, q_{r,i+1}) > \max(U_i, V_i).$$

The integer intervals

$$I_i = (\min(U_i, V_i), \max(U_i, V_i)] \cap \mathbb{Z} \qquad (1 \le i < s)$$

are therefore disjoint, and $|I_i| = |s_i(r \bmod q)|$. If $U_i > V_i$ and $n \in I_i$, the preceding inequalities give

$$U_{i-1} < n \le U_i, \qquad V_i < n \le V_{i+1}.$$

The ordered pair at rank n is consequently $(\lambda_{r,i}, \lambda_{r,i+1})$. If $U_i < V_i$, the pair is $(\lambda_{r,i+1}, \lambda_{r,i})$ by the same inequalities with U, V interchanged. At any rank $1 \le n \le U_s$ outside the I_i, one has $n \le U_i$ if and only if $n \le V_i$ for every i; both lists therefore have the same offset at rank n. This proves the stated multiplicities. If $D_* = 0$, all sufficiently late displacements vanish. If $D_* > 0$, a boundary with gap D_* and nonzero partial sum occurs in some residue class, hence in infinitely many blocks. This proves (2.17). □

Proposition 2.6. *Assume common multiplicity growth and* (2.14)*. Then*

$$\limsup_{j\to\infty} \left| \sqrt{\lambda_j + c} - \sqrt{\lambda'_j + c} \right| = \begin{cases} 0, & B_a = 0 \text{ for every } a \text{ and } d_{\mathrm{pre}} = 0, \\ \sigma, & \text{otherwise.} \end{cases}$$

Proof. For all sufficiently large r, let N_r, N'_r be the complete cumulative counts through block r, and put

$$D_r = N_r - N'_r = d_{\mathrm{pre}} + \sum_{h=r_0}^{r} B_{h \bmod q}.$$

Condition (2.14) gives $D_{r+q} = D_r$. Choose $L \ge 0$ with $|D_r| \le L$. For all sufficiently large r, both cardinalities of block $r+1$ exceed L, and hence

$$\min(N_{r+1}, N'_{r+1}) > \min(N_r, N'_r) + L \ge \max(N_r, N'_r).$$

The intervals

$$J_r = (\min(N_r, N'_r), \max(N_r, N'_r)] \cap \mathbb{Z}$$

are disjoint. If $N_r > N'_r$ and $n \in J_r$, then

$$N_{r-1} < n \le N_r, \qquad N'_r < n \le N'_{r+1};$$

thus the ordered pair joins blocks r and $r+1$. The reverse inequality interchanges the lists. Outside these intervals, every sufficiently late ordered pair belongs to one block, since its rank lies below the same cumulative boundaries in both lists. Uniformly in i, j,

$$|x_{r,i} - x_{r,j}| = O(r^{-1}), \qquad x_{r+1,j} - x_{r,i} = \sigma + O(r^{-1}).$$

Hence the limsup is at most σ. If $B_a = 0$ for every a and $d_{\mathrm{pre}} = 0$, Theorem 2.4 gives convergence to zero. Otherwise the periodic sequence D_r is not identically zero: $D_r = 0$ for every sufficiently large r would give $B_{r \bmod q} = D_r - D_{r-1} = 0$ for every residue and then $d_{\mathrm{pre}} = 0$. Thus J_r is nonempty for infinitely many r. The corresponding frequency displacements tend to σ. □

2.5. **Bounded and compact spectral corrections.** For self-adjoint matrices of the same size, the operator-norm distance between unitary orbits is the maximum ordered eigenvalue displacement; commuting representatives attain this distance [2, Theorems 1.1–1.2, p. 419].

For bounded operators, $\|\cdot\|$ denotes the operator norm. For a bounded operator T on a Hilbert space H, write

$$\|T\|_{\mathrm{ess}} = \inf\{\|T - K\| : K \text{ is compact on } H\}.$$

Proposition 2.7. *Let A be self-adjoint, bounded below, and have compact resolvent on an infinite-dimensional separable complex Hilbert space H. Write $\mathcal{D}(A)$ for its domain and I for the identity on H. Let $(a_j)_{j\geq 1}$ be its nondecreasing eigenvalue sequence, with multiplicities, and let $(b_j)_{j\geq 1}$ be a nondecreasing real sequence tending to $+\infty$. Put*

$$D = \sup_{j\geq 1} |a_j - b_j|.$$

There exists a bounded self-adjoint operator C on H such that $A + C$, with domain $\mathcal{D}(A)$, has eigenvalue sequence (b_j) if and only if $D < \infty$. In that case,

$$\min_C \|C\| = D, \qquad \min_C \|C\|_{\mathrm{ess}} = \limsup_{j\to\infty} |a_j - b_j|,$$

where both minima range over these operators C. Both are attained by the same operator diagonal in an eigenbasis of A. A compact self-adjoint operator C with the prescribed eigenvalue sequence exists if and only if $a_j - b_j \to 0$.

Proof. For a bounded self-adjoint operator B and $u \in \mathcal{D}(A)$, $\|Bu\| \leq \|B\|\|u\|$, so its relative A-bound is zero. The Kato–Rellich theorem gives self-adjointness of $A + B$ on $\mathcal{D}(A)$ [26, Theorem 6.4, p. 159]. Moreover, $A + B \geq (a_1 - \|B\|)I$, and its resolvent is compact by the second resolvent identity and the ideal property of compact operators [26, Lemmas 5.6 and 6.5, pp. 151 and 159]. For $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(A + B - z)^{-1} = (A - z)^{-1} - (A - z)^{-1}B(A + B - z)^{-1}.$$

Write $\lambda_j(A + B)$ for its ordered eigenvalues, with multiplicities. For bounded self-adjoint operators B_1, B_2, the common domain $\mathcal{D}(A)$ and the variational inequality [26, Corollary 4.13, p. 141] give

$$\left|\lambda_j(A + B_1) - \lambda_j(A + B_2)\right| \leq \|B_1 - B_2\|. \tag{2.18}$$

In particular, $D \leq \|C\|$ for every operator C in the proposition.

The compact positive injective operator $(A - a_1 + 1)^{-1}$ has an orthonormal eigenbasis (e_j) by [26, Theorem 6.6, p. 160]. Choose its ordering so that $Ae_j = a_j e_j$. For $N \geq 1$, let P_N be the orthogonal projection onto $\operatorname{span}\{e_1, \dots, e_N\}$. For compact self-adjoint K, put $K_N = P_N K P_N$. Since $P_N \to I$ strongly, [26, Lemma 6.9(ii), p. 162] gives

$$\|K - K_N\| \leq 2\|(I - P_N)K\| \longrightarrow 0.$$

The operator $A+K_N$ is the orthogonal sum of an operator on $P_N H$ and the restriction of A to $(I-P_N)H$. Let $\beta_{N,1} \leq \cdots \leq \beta_{N,N}$ be the eigenvalues of the first summand. For $t \geq \max(a_N, \beta_{N,N})$,

$$\#\{j \,:\, \lambda_j(A + K_N) \leq t\} = N + \#\{j > N \,:\, a_j \leq t\} = \#\{j \,:\, a_j \leq t\}.$$

Thus, for each N, $\lambda_j(A + K_N) = a_j$ for all sufficiently large j. It follows from (2.18) that

$$\limsup_{j\to\infty} |\lambda_j(A + K) - a_j| \leq \|K - K_N\|.$$

Letting $N \to \infty$ gives

$$\lambda_j(A + K) - a_j \longrightarrow 0. \tag{2.19}$$

For any operator C in the proposition and any compact self-adjoint K, equations (2.18) and (2.19) give

$$\limsup_{j\to\infty} |b_j - a_j| \leq \|C - K\|.$$

For arbitrary compact K, its self-adjoint part $K_0 = (K+K^*)/2$ is compact and satisfies $\|C-K_0\| \leq \|C-K\|$. Consequently,

$$\limsup_{j\to\infty} |b_j - a_j| \leq \|C\|_{\mathrm{ess}}. \tag{2.20}$$

Suppose $D < \infty$. Define C by $Ce_j = (b_j - a_j)e_j$. It is bounded and self-adjoint, $\|C\| = D$, and $(A + C)e_j = b_j e_j$. Since (e_j) is complete, $A + C$ has the prescribed eigenvalue sequence. Also,

$$\|C - P_N C\| = \sup_{j>N} |b_j - a_j|.$$

The operators $P_N C$ have finite rank. Letting $N \to \infty$ and applying (2.20) gives the second minimum. If $a_j - b_j \to 0$, the same formula makes C a norm limit of finite-rank operators, hence compact. The converse follows from (2.19). □

Under the hypotheses of Theorem 2.5, the least essential norm of a bounded self-adjoint correction is therefore D_*.

2.6. **A common-multiplicity estimate on odd spherical quotients.**

Lemma 2.8. *Let $\Gamma \subset \mathrm{SO}(6)$ be a finite group of odd order q acting freely on S^5. Fix integers $k \geq |c|$, and let $\pi_{r,k,c}$ be the complex irreducible representation of highest weight (r, k, c), where $r \in \mathbb{Z}$ and $r \geq k$. For every $\gamma \in \Gamma \setminus \{1\}$,*

$$|\chi_{r,k,c}(\gamma)| = O_{\gamma,k,c}((r+1)^2). \tag{2.21}$$

Moreover,

$$\dim V^{\Gamma}_{\pi_{r,k,c}} = \frac{1}{q} \dim \pi_{r,k,c} + O_{\Gamma,k,c}((r+1)^2), \tag{2.22}$$

$$\dim \pi_{r,k,c} = \frac{\big((r+2)^2 - (k+1)^2\big)\big((r+2)^2 - c^2\big)\big((k+1)^2 - c^2\big)}{12}. \tag{2.23}$$

In particular,

$$\dim V^{\Gamma}_{\pi_{r,k,c}} = \frac{(k+1)^2 - c^2}{12q}\, r^4 + O_{\Gamma,k,c}(r^3),$$

and the leading coefficient is positive. For $\gamma \neq 1$, the function $r \mapsto \chi_{r,k,c}(\gamma)$ is an exact quasipolynomial of period dividing q and degree at most two: on each residue class modulo q, it agrees with a polynomial in r with complex coefficients.

Proof. Fix $\gamma \neq 1$. Choose $\theta = (\theta_1, \theta_2, \theta_3)$ so that γ is conjugate in $\mathrm{SO}(6)$ to $\mathrm{diag}(R_{\theta_1}, R_{\theta_2}, R_{\theta_3})$, where R_ϑ is rotation through angle ϑ in an oriented plane. Since $\gamma^q = 1$, one has $e^{iq\theta_j} = 1$. Freeness excludes $e^{i\theta_j} = 1$, and odd order excludes $e^{i\theta_j} = -1$.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the standard basis of $\mathbb{R}^3$, and let $\langle\ ,\ \rangle$ denote its Euclidean inner product. Put

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \,:\, 1 \leq i < j \leq 3\}, \qquad \Phi^+_\gamma = \{\alpha \in \Phi^+ \,:\, e^{i\langle \alpha, \theta\rangle} = 1\}.$$

Let W be the group of coordinate permutations and sign changes with an even number of sign changes. The degenerate Weyl character formula [15, (2.4)], applied with $\ell = (r+2, k+1, c)$, gives constants $b_{\gamma,s} \in \mathbb{C}$, independent of r, such that

$$\chi_{r,k,c}(\gamma) = \sum_{s \in W} b_{\gamma,s} e^{i\langle s\ell, \theta\rangle} \prod_{\alpha \in \Phi^+_\gamma} \langle \alpha, s\ell\rangle. \tag{2.24}$$

Here Φ^+_γ is the positive root system of the identity component of the centralizer of the chosen torus element; an empty product is 1.

For $i < j$, the two roots $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$ cannot both belong to Φ^+_γ: otherwise $e^{2i\theta_i} = e^{2i\theta_j} = 1$. For each $s \in W$ there are $a \in \{1, 2, 3\}$ and $\eta \in \{1, -1\}$ with $s\varepsilon_1 = \eta\varepsilon_a$. A factor $\langle \alpha, s\ell\rangle$ depends on r only when α involves ε_a. There are two other indices, and at most one root for each such pair. Hence the product in (2.24) is a polynomial in r of degree at most two. Its exponential factor has modulus 1 and is q-periodic in r, since

$$e^{i\langle s(\ell + q\varepsilon_1), \theta\rangle} = e^{i\langle s\ell, \theta\rangle} e^{i\eta q\theta_a} = e^{i\langle s\ell, \theta\rangle}.$$

This proves both (2.21) and exact quasipolynomiality. If Φ_γ^+ is empty, the character is q-periodic.

The character average

$$\dim V^\Gamma_{\pi_{r,k,c}} = \frac{1}{q}\sum_{\gamma\in\Gamma}\chi_{r,k,c}(\gamma)$$

gives (2.22). The Weyl dimension formula [8, Proposition 28.1], with $\rho = (2,1,0)$, gives

$$\dim \pi_{r,k,c} = \prod_{1\le i<j\le 3}\frac{\ell_i^2-\ell_j^2}{\rho_i^2-\rho_j^2}.$$

Its denominator is $(4-1)(4-0)(1-0) = 12$. Thus (2.23) holds, with leading coefficient $((k+1)^2 - c^2)/12 > 0$. □

For fixed q, k, c, these estimates are uniform over all subgroups of order q acting freely on S^5. Indeed, their elements are conjugate in SO(6) to elements of the finite set of q-torsion points of a fixed maximal torus. The constant may depend on q, k, c.

Fix a natural type $\tau_{u,v}$, where $u \ge v \ge 0$ are integers. For an attained offset w, put

$$\mathcal{I}_{u,v}(w) = \{(k,c)\in\mathbb{Z}^2 \,:\, v\le k\le u,\ |c|\le v,\ k(k+2)+c^2 = w\},$$

so that $\mathcal{I}_{u,v}(w)$ is finite and nonempty. Put

$$C_{u,v}(w) := \frac{1}{12}\sum_{(k,c)\in\mathcal{I}_{u,v}(w)}((k+1)^2-c^2) > 0.$$

By (1.2), the multiplicity contributed by the r-block at this offset, for $r \ge u$, is

$$m_{\Gamma,w}(r) := \sum_{(k,c)\in\mathcal{I}_{u,v}(w)}\dim V^\Gamma_{\pi_{r,k,c}} = \frac{C_{u,v}(w)}{|\Gamma|}\,r^4 + O_{\Gamma,u,v}(r^3).$$

Consequently, for any two fixed subgroups $\Gamma, \Gamma' \subset \mathrm{SO}(6)$ of odd order acting freely on S^5,

$$\lim_{r\to\infty} r^{-4}\min\{m_{\Gamma,w}(r), m_{\Gamma',w}(r)\} = \frac{C_{u,v}(w)}{\max\{|\Gamma|,|\Gamma'|\}} > 0.$$

This proves the condition $\min(p_{r,i}, q_{r,i}) \to \infty$ in (2.3) at every attained offset, including an offset at which the two multiplicities agree.

2.7. **Quasipolynomial discrepancies.** Retain the quadratic support, finite prefixes, and cutoffs of (2.2) and (2.4), and suppose that both complete ordered lists are infinite. Let $q, D \ge 1$ be integers. For $a \in \mathbb{Z}/q\mathbb{Z}$ and $1 \le i \le s$, let $P_{a,i}$ be a real polynomial of degree at most D, with integer values at every $r \ge r_0$ in its residue class. Assume

$$p_{r,i} - q_{r,i} = P_{a,i}(r), \qquad a = r \bmod q, \qquad r \ge r_0. \tag{2.25}$$

Put

$$S_r = \sum_{i=1}^{s}|P_{r \bmod q,i}(r)|, \qquad A_a = \sum_{i=1}^{s}P_{a,i}, \qquad T_{a,i} = \sum_{j=1}^{i}P_{a,j} \quad (1\le i<s).$$

Write $b_i(a)$ for the coefficient of r^D in $P_{a,i}$, and define

$$F_a^{(D)}(t) = \sum_{C\in\mathscr{P}(t)}\left|\sum_{i\in C}b_i(a)\right| \quad (t\ge 0), \qquad C_D(\kappa) = \frac{1}{q}\sum_{a\in\mathbb{Z}/q\mathbb{Z}}\int_0^1 x^D F_a^{(D)}(2\kappa x)\,dx \quad (\kappa\ge 0).$$

Theorem 2.9. *Under* (2.25),

$$\sup_j|\lambda_j-\lambda_j'| < \infty \quad\Longleftrightarrow\quad d_{\mathrm{pre}} = 0 \text{ and } A_a \equiv 0 \text{ for every } a\in\mathbb{Z}/q\mathbb{Z}.$$

Suppose in addition that, for every sufficiently large r,

$$\min(p_{r,i}, q_{r,i}) \geq S_r \qquad (1 \leq i \leq s). \tag{2.26}$$

For every fixed $0 \leq \eta_* < \sigma$, uniformly for $0 \leq \eta \leq \eta_*$ and $K \longrightarrow \infty$,

$$E_\eta(R_K) = \frac{R_K^{D+1}}{\sigma^{D+1}} C_D(\eta R_K) + O(R_K^D). \tag{2.27}$$

The error constant depends only on the fixed data, θ, and η_*. If $d_{\mathrm{pre}} = 0$ and $A_a \equiv 0$ for every a, put

$$D_*^{\mathrm{pol}} = \max\{g_i \,:\, 1 \leq i < s,\ T_{a,i} \not\equiv 0 \text{ for some } a \in \mathbb{Z}/q\mathbb{Z}\},$$

with an empty maximum equal to zero. Then

$$\limsup_{j\to\infty} |\lambda_j - \lambda'_j| = D_*^{\mathrm{pol}}, \qquad |\lambda_j - \lambda'_j| \leq D_*^{\mathrm{pol}} \quad \text{for all sufficiently large } j.$$

In every sufficiently late r-block, exactly $|T_{r \bmod q,i}(r)|$ ordered pairs join w_i and w_{i+1}; every other pair joins equal offsets.

Proof. Suppose first that the ordered eigenvalue displacement is bounded. The two ordered frequency lists then have difference tending to zero, by rationalization as in Theorem 2.4. The same theorem's block-gap argument applies, since both lists are infinite and $x_{r+1,1} - x_{r,s} \longrightarrow \sigma$. Hence their cumulative cardinalities N_r, N'_r through block r agree for every sufficiently large r. For such r,

$$0 = N_r - N'_r = d_{\mathrm{pre}} + \sum_{h=r_0}^{r} A_{h \bmod q}(h), \qquad A_{r \bmod q}(r) = 0.$$

Each A_a therefore vanishes at infinitely many integers and is zero. The displayed identity then gives $d_{\mathrm{pre}} = 0$. Conversely, these equalities give equal cardinalities in the prefixes and in every block. Let P_j, Q_j be the ordered prefixes. The finite matching and ordered-eigenvalue inequality in the proof of Theorem 2.4 give

$$\sup_j |\lambda_j - \lambda'_j| \leq \max\left\{H, \max_{1\leq j\leq |P|} |P_j - Q_j|\right\}.$$

The inner maximum is zero when the prefixes are empty.

Now assume (2.26). Fix $r_1 \geq \max\{r_0, 1\}$ so that this inequality holds for $r \geq r_1$ and consecutive retained frequency blocks are separated by more than η_*. Put $R = R_K$. Removing the prefixes and the blocks with $r < r_1$ changes $E_\eta(R)$ by $O(1)$, uniformly in η. Lemma 2.1 applies to each remaining block. The comparison of adjacent edge conditions in the proof of Theorem 2.2 shows that the true partition and $\mathscr{P}(2\eta t_r)$ differ at only $O(1)$ indices r, uniformly for $0 \leq \eta \leq \eta_*$. For either partition the sum of the absolute component discrepancies is at most $S_r = O(r^D)$. Thus

$$E_\eta(R) = \sum_{r=r_1}^{K} \sum_{C \in \mathscr{P}(2\eta t_r)} \left| \sum_{i\in C} P_{r \bmod q,i}(r) \right| + O(R^D).$$

For every residue a and every component C,

$$\left| \left| \sum_{i\in C} P_{a,i}(r) \right| - r^D \left| \sum_{i\in C} b_i(a) \right| \right| \leq \left| \sum_{i\in C} \big(P_{a,i}(r) - b_i(a) r^D\big) \right| = O(r^{D-1}).$$

There are finitely many residues and components. Consequently

$$E_\eta(R) = \sum_{r=r_1}^{K} r^D F^{(D)}_{r \bmod q}(2\eta t_r) + O(R^D).$$

Let r_a be the least integer at least r_1 congruent to a modulo q, and put $N = N_a(K, L)$, with the strict inequality $t_r < L$ as in (2.11). For $L > 0$, that formula and summation of powers give, uniformly in L,

$$\begin{aligned}\sum_{\substack{r_1 \le r \le K \\ r \equiv a \ (q)}} r^D \mathbf{1}_{\{t_r < L\}} &= \sum_{n=0}^{N-1} (r_a + qn)^D \\ &= \frac{q^D N^{D+1}}{D+1} + O((N+1)^D) \\ &= \frac{\min(R, L)^{D+1}}{q\sigma^{D+1}(D+1)} + O(R^D).\end{aligned}$$

The same formula holds for $L = \infty$, with $\min(R, \infty) = R$. Each $F_a^{(D)}$ is a linear combination of the constant function and the functions $\mathbf{1}_{\{t<g\}}$, $g \in G$, by (2.6), with $b_i(a)$ in place of $\delta_i(a)$. For $\eta > 0$ apply the last estimate with $L = g/(2\eta)$, and use

$$\int_0^1 x^D \mathbf{1}_{\{2\eta R x < g\}}\, dx = \frac{1}{D+1} \min\left(1, \frac{g}{2\eta R}\right)^{D+1}.$$

For $\eta = 0$ every such indicator equals one. This proves (2.27) with the asserted uniformity.

Finally suppose $d_{\mathrm{pre}} = 0$ and $A_a \equiv 0$ for every a. If $S_r = 0$, the two block lists agree. If $S_r > 0$, their total discrepancy is zero, so

$$|T_{r \bmod q, i}(r)| \le \frac{S_r}{2} < \min(p_{r,j}, q_{r,j}) \qquad (1 \le i < s,\ 1 \le j \le s).$$

The rank-interval argument in Theorem 2.5, applied to this block, gives the stated number of adjacent pairs and leaves only equal-offset pairs. A nonzero polynomial $T_{a,i}$ has only finitely many zeros in its residue class. Hence ordered pairs join w_i and w_{i+1} in arbitrarily late blocks if and only if $T_{a,i} \not\equiv 0$ for some a. This proves both assertions concerning D_*^{pol}. □

One has

$$C_D(0) = \frac{1}{q(D+1)} \sum_{a \in \mathbb{Z}/q\mathbb{Z}} \sum_{i=1}^{s} |b_i(a)|.$$

Thus $C_D \equiv 0$ if and only if every $P_{a,i}$ has degree less than D. If the largest degree is $d \ge 1$, formula (2.27) applies with d in place of D. For constant discrepancies with common multiplicities tending to infinity, Theorem 2.2 gives the remainder $O(1)$.

Corollary 2.10. *Let $\Gamma, \Gamma' \subset \mathrm{SO}(6)$ be finite subgroups of the same odd order q, acting freely on S^5, and fix a natural type $\tau_{u,v}$. Their multiplicity difference at each quadratic offset is an exact quasipolynomial in r, of period dividing q and degree at most two. The hypotheses of Theorem 2.9 hold after a fixed initial region, with $D = 2$, or with $D = 1$ if all discrepancies have degree at most one.*

For either group Y *and each $\gamma \in \mathrm{Y}$, choose angles $\theta_j(\gamma)$ so that its eigenvalues are $e^{\pm i\theta_j(\gamma)}$, $1 \le j \le 3$, and put*

$$\mathrm{Y}_{\mathrm{nr}} = \{\gamma \in \mathrm{Y} \setminus \{1\} : \cos\theta_i(\gamma) = \cos\theta_j(\gamma) \text{ for some } i < j\}.$$

If, for every attained offset w and every sufficiently large r,

$$\sum_{(k,c) \in \boldsymbol{\mathcal{I}}_{u,v}(w)} \sum_{\gamma \in \Gamma_{\mathrm{nr}}} \chi_{r,k,c}(\gamma) = \sum_{(k,c) \in \boldsymbol{\mathcal{I}}_{u,v}(w)} \sum_{\gamma \in \Gamma'_{\mathrm{nr}}} \chi_{r,k,c}(\gamma), \tag{2.28}$$

then the multiplicity differences are periodic and Theorem 2.2 applies after a fixed initial region.

Proof. For $r \ge u$, the branching rule (1.2) and (1.1) give the offsets $w = k(k+2) + c^2$ and the eigenvalues $(r+2)^2 + w - 4$. The r-block contribution at w is $m_{\Gamma,w}(r)$, and similarly for Γ'. The identity terms $q^{-1} \dim \pi_{r,k,c}$ in the two character averages are equal. By Lemma 2.8, their difference

is a quasipolynomial of period dividing q and degree at most two. On each residue class its imaginary part vanishes at infinitely many integers; its polynomial coefficients are therefore real. Its values are integers, being differences of multiplicities.

For every attained offset w, the preceding multiplicity estimate gives

$$\min\{m_{\Gamma,w}(r), m_{\Gamma',w}(r)\} = \frac{C_{u,v}(w)}{q}\, r^4 + O(r^3), \qquad C_{u,v}(w) > 0.$$

Since $S_r = O(r^2)$, (2.26) holds eventually, and both ordered spectra are infinite. The quadratic model has $\sigma = 1$, $\beta = 2$, and constant $c = 4$ in (2.2). For an element outside $\mathrm{Y}_{\mathrm{nr}} \cup \{1\}$, Lemma 2.8 gives a periodic character. Under (2.28), only these periodic terms remain in each multiplicity difference. The common multiplicities tend to infinity, so Theorem 2.2 applies. □

3. Bounded and compact perturbations of natural spectra

Retain the lens spaces and the Casimir normalization of Section 1. For integers $u \geq v \geq 0$ and $M \in \{L, L'\}$, write

$$\mathcal{H}_{u,v,M} = L^2(E_{\tau_{u,v},M}), \qquad \Delta_{u,v,M} = \Delta_{\tau_{u,v},M}, \qquad \lambda_j(M) = \lambda_j(\Delta_{u,v,M}) \quad (j \geq 1).$$

The eigenvalues are listed in nondecreasing order, with multiplicities; equalities of spectra include multiplicities. For a bounded self-adjoint operator C on $\mathcal{H}_{u,v,M}$, the sum $\Delta_{u,v,M} + C$ has domain $\mathcal{D}(\Delta_{u,v,M})$, as in Proposition 2.7.

Let $\mathscr{Z} \subset (\mathbb{Z}/49\mathbb{Z})^2$ consist of the pairs (X, Y) satisfying at least one of the following conditions:

(1) $X = 0$, $Y = 0$, $X = Y$ or $X = -Y$;
(2) $7 \mid X$ and $7 \mid Y$;
(3) one coordinate is divisible by 7, and the other belongs to $\{\pm 1, \pm 3\}$;
(4) both coordinates belong to $\{\pm 1, \pm 3, \pm 5\}$.

Here $7 \mid X$ means $X \in 7(\mathbb{Z}/49\mathbb{Z})$, and similarly for Y.

Theorem 3.1. *For integers $u \geq v \geq 0$, the following conditions are equivalent.*

(1) $(2u + 3, 2v + 1) \bmod 49 \in \mathscr{Z}$.
(2) $\sup_{j \geq 1} |\lambda_j(L) - \lambda_j(L')| < \infty$.
(3) $|\sqrt{\lambda_j(L) + 4} - \sqrt{\lambda_j(L') + 4}| \longrightarrow 0$ *as* $j \longrightarrow \infty$.
(4) There is a bounded self-adjoint operator C on $\mathcal{H}_{u,v,L}$ such that $\mathrm{Spec}(\Delta_{u,v,L} + C) = \mathrm{Spec}(\Delta_{u,v,L'})$.
(5) The operator C in condition 4 can be chosen to commute with every spectral projection of $\Delta_{u,v,L}$.

If these conditions hold, then

$$|\lambda_j(L) - \lambda_j(L')| \leq H_{u,v} = u(u + 2) - 2v \qquad (j \geq 1),$$

and $\|C\| \leq H_{u,v}$ is possible. They are also equivalent to the existence of bounded self-adjoint operators C, C' on $\mathcal{H}_{u,v,L}, \mathcal{H}_{u,v,L'}$, respectively, such that

$$\mathrm{Spec}(\Delta_{u,v,L} + C) = \mathrm{Spec}(\Delta_{u,v,L'} + C').$$

The set $\mathscr{Z}$ has 289 elements. The uncorrected spectra are equal precisely for

$$(u, v) = (0, 0), (1, 0), (2, 0), (1, 1), (2, 1).$$

Every other dominant type satisfying condition 1 has unequal spectra and admits a bounded correction commuting with every spectral projection.

Lemma 3.2. *The function T is alternating in its three variables, and its sum over a complete period in any one variable is zero.*

Proof. Interchanging two variables interchanges two rows in (1.3), hence negates T. For $t \in \{s, s'\}$ and $1 \le j \le 3$, the residue t_j is a unit modulo 49, so

$$\sum_{b=0}^{48} \left(\zeta^{bt_j} + \zeta^{-bt_j}\right) = 0.$$

The determinant is linear in each row. Summing one variable in (1.4) therefore gives zero. □

3.1. **The alternating rectangle.** Fix integers $u \ge v \ge 0$. Let

$$W = \{k(k+2) + c^2 \,:\, k, c \in \mathbb{Z},\ v \le k \le u,\ 0 \le c \le v\},$$

and put

$$A(a, w) = A_{u,v}(a, w), \qquad B_{u,v}(a) = \sum_{w \in W} A(a, w), \qquad B(a) = B_{u,v}(a),$$

where $A_{u,v}$ is defined in (1.7). For every $r \ge u$, the representations $\pi_{r,k,c}$ in (1.2) have eigenvalues in

$$I_r = [r(r+4) + v(v+2),\ r(r+4) + u(u+2) + v^2].$$

This interval has length $H_{u,v}$. By (1.5)–(1.6), the sum of $d(r, k, c)$ over $v \le k \le u$ and $|c| \le v$ is $B_{u,v}(r+2)$.

Lemma 3.3. *The functions $B_{u,v}$ extend from $u \ge v \ge 0$ to $u, v \ge 0$ by the formula*

$$2B_{u,v}(a) = \sum_{b=-u-1}^{u+1} \sum_{c=-v}^{v} T(a, b, c). \tag{3.1}$$

This extension is 49-periodic in each parameter. Write $\mathcal{B}_a(X, Y) = B_{u,v}(a)$ when $X = 2u+3$ and $Y = 2v+1$ in $\mathbb{Z}/49\mathbb{Z}$. Then

$$\mathcal{B}_a(-X, Y) = \mathcal{B}_a(X, -Y) = \mathcal{B}_a(Y, X) = -\mathcal{B}_a(X, Y).$$

In particular, $\mathcal{B}_a(X, Y) = 0$ whenever $X = 0$, $Y = 0$, $X = Y$ or $X = -Y$.

Proof. By the evenness of T in its third variable,

$$B_{u,v}(a) = \sum_{b=v+1}^{u+1} \sum_{c=-v}^{v} T(a, b, c).$$

The sum over $-v \le b, c \le v$ is zero by alternation. Evenness in the second variable now gives (3.1).

Let $(e_b)_{b \in \mathbb{Z}/49\mathbb{Z}}$ be the standard basis of $\mathbb{Q}[\mathbb{Z}/49\mathbb{Z}]$. For each a, define the bilinear form β_a by $\beta_a(e_b, e_c) = T(a, b, c)$. It is alternating. Put

$$J_N = \sum_{j=-N}^{N} e_{j \bmod 49} \quad (N \ge 0), \qquad \mathbf{e} = \sum_{b \in \mathbb{Z}/49\mathbb{Z}} e_b.$$

Lemma 3.2 gives $\beta_a(\mathbf{e}, z) = \beta_a(z, \mathbf{e}) = 0$ for every $z \in \mathbb{Q}[\mathbb{Z}/49\mathbb{Z}]$, and

$$2B_{u,v}(a) = \beta_a(J_{u+1}, J_v).$$

The identities

$$J_{N+49} = J_N + 2\mathbf{e} \quad (N \ge 0), \qquad J_{48-N} = 2\mathbf{e} - J_N \quad (0 \le N \le 48)$$

prove periodicity and the two sign identities. Indeed, $2(48-N)+1 \equiv -(2N+1) \pmod{49}$. Alternation of β_a gives the interchange identity. Finally $J_{24} = \mathbf{e}$, so $X = 0$ or $Y = 0$ gives zero; if $X = Y$, the two arguments of the alternating form agree modulo $\mathbb{Q}\mathbf{e}$. The identity for $-Y$ then gives zero also when $X = -Y$. □

Lemma 3.4. *Outside the four lines $X = 0$, $Y = 0$, $X = Y$, $X = -Y$, the zeros of $\big(\mathcal{B}_a(X, Y)\big)_{a \bmod 49}$ are exactly the signed-permutation orbits of the following unordered pairs of least positive absolute representatives of X, Y:*

$$\{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 7\}, \{1, 14\}, \{1, 21\}, \{3, 7\}, \{3, 14\}, \{3, 21\}, \{7, 14\}, \{7, 21\}, \{14, 21\}.$$

Proof. Each orbit in the complement of the four lines has a unique representative with $1 \le Y < X \le 24$. For $Z \in \mathbb{Z}/49\mathbb{Z}$, let N_Z be the integer determined by

$$0 \le N_Z \le 48, \qquad N_Z \equiv 25(Z - 1) \pmod{49},$$

and, for $b \in \mathbb{Z}/49\mathbb{Z}$, put

$$j_Z(b) = \#\{j \in \mathbb{Z} : -N_Z \le j \le N_Z,\ j \equiv b \pmod{49}\}.$$

Since $2 \cdot 25 \equiv 1 \pmod{49}$, (3.1) gives

$$\mathcal{B}_a(X, Y) = \frac{1}{2} \sum_{b,c \in \mathbb{Z}/49\mathbb{Z}} j_X(b) j_Y(c) T(a, b, c).$$

Define

$$Q(X, Y) = \frac{1}{4} \sum_{a=0}^{24} \mathcal{B}_a(X, Y)^2.$$

The values of $\mathcal{B}_a$ are real, and they are even and 49-periodic in a. Thus $Q(X, Y) = 0$ if and only if $\mathcal{B}_a(X, Y) = 0$ for every $a \in \mathbb{Z}/49\mathbb{Z}$. Substitution of (1.10), with the coefficients in Table 1, gives Table 3. Its zero entries are precisely the twelve pairs in the statement.

TABLE 3. Values of $Q(X, Y)$ for $1 \le Y < X \le 24$.

X	$Q(X, 1), \dots, Q(X, X-1)$
2	12
3	0, 12
4	12, 24, 48
5	0, 12, 0, 36
6	24, 83, 72, 59, 72
7	0, 48, 0, 84, 6, 84
8	24, 89, 72, 71, 72, 12, 84
9	6, 12, 6, 36, 6, 81, 6, 79
10	12, 30, 48, 24, 36, 71, 84, 59, 34
11	6, 24, 24, 48, 18, 75, 28, 71, 12, 40
12	12, 6, 12, 30, 12, 89, 34, 83, 10, 24, 22
13	12, 51, 36, 45, 36, 24, 28, 20, 39, 41, 27, 49
14	0, 24, 0, 28, 10, 28, 0, 28, 10, 28, 28, 10, 28
15	12, 49, 36, 41, 36, 20, 28, 24, 49, 45, 47, 51, 20, 28
16	10, 6, 10, 30, 10, 88, 34, 92, 12, 34, 24, 10, 51, 10, 45
17	6, 22, 24, 40, 18, 71, 28, 75, 22, 48, 40, 24, 47, 28, 27, 18
18	10, 26, 40, 24, 30, 72, 56, 80, 32, 40, 48, 30, 47, 28, 35, 20, 24
19	6, 10, 6, 34, 6, 79, 20, 81, 10, 36, 22, 12, 49, 10, 39, 6, 12, 26
20	20, 78, 60, 62, 60, 12, 56, 20, 71, 70, 67, 82, 24, 28, 12, 64, 55, 44, 65
21	0, 40, 0, 56, 12, 56, 0, 56, 12, 56, 28, 26, 28, 0, 28, 26, 28, 56, 12, 56
22	20, 82, 60, 70, 60, 20, 56, 12, 65, 62, 55, 78, 12, 28, 24, 76, 67, 68, 71, 24, 56
23	10, 10, 10, 34, 10, 92, 20, 88, 6, 30, 18, 6, 45, 10, 51, 12, 24, 32, 12, 76, 12, 64
24	10, 30, 40, 40, 30, 80, 56, 72, 26, 24, 24, 26, 35, 28, 47, 32, 48, 48, 32, 68, 56, 44, 20

□

Each of the four lines has 49 points. Their pairwise intersections are the origin, since 2 is a unit modulo 49; their union therefore has $1 + 4 \cdot 48 = 193$ points. A pair outside this union has no nonidentity stabilizer under independent sign changes and interchange of the coordinates, so its orbit has eight points. The number of zeros is consequently $193 + 12 \cdot 8 = 289$. The twelve pairs in Lemma 3.4 consist of two distinct members of $\{1, 3, 5\}$, one member of $\{1, 3\}$ and one of $\{7, 14, 21\}$, or two distinct members of $\{7, 14, 21\}$. Together with the four lines, their orbits give exactly $\mathscr{Z}$.

3.2. **Complete ordered eigenvalue lists.**

Lemma 3.5. *For fixed integers $u \geq v \geq 0$, the following conditions are equivalent:*

$$B(a) = 0 \quad \text{for every } a \in \mathbb{Z}/49\mathbb{Z},$$
$$\sup_{j \geq 1} |\lambda_j(L) - \lambda_j(L')| < \infty,$$
$$|\sqrt{\lambda_j(L) + 4} - \sqrt{\lambda_j(L') + 4}| \longrightarrow 0 \quad (j \longrightarrow \infty).$$

If they hold, then $|\lambda_j(L) - \lambda_j(L')| \leq H_{u,v}$ for every $j \geq 1$.

Proof. Write $W = \{w_1 < \cdots < w_s\}$. With the notation following Lemma 2.8, put

$$p_{r,i} = m_{\Gamma, w_i}(r), \qquad q_{r,i} = m_{\Gamma', w_i}(r) \qquad (r \geq u,\ 1 \leq i \leq s).$$

The branching rule (1.2) and the multiplicity formula in subsection 1.2 give the complete eigenvalue multisets of the quadratic model with

$$\sigma = 1, \quad \beta = 2, \quad c = 4, \quad r_0 = u, \quad P = Q = \emptyset.$$

Here $t_u = u + 2 > 0$ and $t_u^2 + w_1 > 0$, since $u \geq 0$ and $w_1 = v(v + 2) \geq 0$. Moreover, $t_r^2 + w_i - 4 = r(r + 4) + w_i$, and there are no summands with $r < u$. The estimates following Lemma 2.8 give $\min(p_{r,i}, q_{r,i}) \to \infty$ for every i; in particular, both spectra are infinite. For $a \in \mathbb{Z}/49\mathbb{Z}$, put

$$\delta_i(a) := A(a + 2, w_i).$$

The functions δ_i are integer-valued on $\mathbb{Z}/49\mathbb{Z}$ by Lemma 1.2, and (1.5)–(1.6) give

$$p_{r,i} - q_{r,i} = A(r + 2, w_i) = \delta_i(r \bmod 49).$$

Their sum is $B(a + 2)$, and $w_s - w_1 = H_{u,v}$. Theorem 2.4 now gives all three equivalences and the bound at every index. Its multiset formulation includes coincident eigenvalues and overlapping initial blocks. □

Proof of Theorem 3.1. Lemmas 3.3 and 3.4 identify condition 1 with $B = 0$. Lemma 3.5 identifies this condition with conditions 2 and 3. The operators $\Delta_{u,v,L}$ and $\Delta_{u,v,L'}$ are nonnegative and have compact resolvent on their infinite-dimensional separable complex Hilbert spaces. Proposition 2.7 therefore identifies conditions 2 and 4 and gives a correction of norm at most $H_{u,v}$, diagonal in an eigenbasis of $\Delta_{u,v,L}$. Each spectral projection is diagonal in that basis, so the correction commutes with it. This proves condition 5, which implies condition 4.

If bounded self-adjoint corrections C, C' give a common ordered eigenvalue sequence $(\mu_j)_{j \geq 1}$, then (2.18) gives

$$|\lambda_j(L) - \lambda_j(L')| \leq |\lambda_j(L) - \mu_j| + |\mu_j - \lambda_j(L')| \leq \|C\| + \|C'\| \qquad (j \geq 1).$$

Conversely, condition 4 gives such a pair with $C' = 0$. The count of $\mathscr{Z}$ follows from Lemma 3.4 and the count immediately following its proof. Theorem 1.1 gives the five equality types and the assertion for all other types. □

3.3. **The eventual displacement.**

Proposition 3.6. *Suppose $B(a) = 0$ for every $a \in \mathbb{Z}/49\mathbb{Z}$, and write $W = \{w_1 < \cdots < w_s\}$. Put*

$$D_* = \max\left\{ w_{i+1} - w_i \,:\, 1 \le i < s, \quad \sum_{j=1}^{i} A(a, w_j) \neq 0 \text{ for some } a \in \mathbb{Z}/49\mathbb{Z} \right\}, \tag{3.2}$$

where an empty maximum is zero. Then

$$\limsup_{j\to\infty} |\lambda_j(L) - \lambda_j(L')| = D_*,$$

and $|\lambda_j(L) - \lambda_j(L')| \le D_*$ for all sufficiently large j.

Proof. The multiplicities in the proof of Lemma 3.5 satisfy common multiplicity growth, have empty prefixes, and have equal block cardinalities because $B = 0$. Thus Theorem 2.5 applies. Since $a \mapsto a+2$ permutes $\mathbb{Z}/49\mathbb{Z}$, its maximum is exactly (3.2), and it gives both assertions. □

3.4. **Resolution at every tolerance.** Fix integers $u \ge v \ge 0$ and retain the notation A, B, W of subsection 3.1. Write $W = \{w_1 < \cdots < w_s\}$ and put

$$G_W = \{w_{i+1} - w_i \,:\, 1 \le i < s\}.$$

For $t \ge 0$, let $\mathscr{P}_W(t)$ be the partition of W into the components of the path joining w_i to w_{i+1} whenever $w_{i+1} - w_i \le t$. For $a \in \mathbb{Z}/49\mathbb{Z}$, put

$$F_a(t) = \sum_{C \in \mathscr{P}_W(t)} \left| \sum_{w \in C} A(a, w) \right|.$$

For $g \in G_W$, write $F_a(g-) = \lim_{t\uparrow g} F_a(t)$ and define

$$d_g = \sum_{a=0}^{48} \big(F_a(g-) - F_a(g) \big).$$

Thus edges with the same gap are joined simultaneously. Put

$$c_0 = \frac{1}{49} \sum_{a=0}^{48} \sum_{w \in W} |A(a, w)|, \qquad c_+ = \frac{1}{49} \sum_{a=0}^{48} |B(a)|.$$

When $s = 1$, the set G_W and all sums over it are empty. Let

$$X = \{\sqrt{\lambda_j(L) + 4} \,:\, j \ge 1\}, \qquad Y = \{\sqrt{\lambda_j(L') + 4} \,:\, j \ge 1\},$$

with multiplicities, and write $E_\eta(R) = E_\eta(R; X, Y)$. For integers $K \ge u$, put $R_K = K + 5/2$.

Theorem 3.7. *Define*

$$C(0) = c_0, \qquad C(\kappa) = c_+ + \frac{1}{49} \sum_{g \in G_W} d_g \min\left\{1, \frac{g}{2\kappa}\right\} \quad (\kappa > 0).$$

As $K \to \infty$, the following estimates hold uniformly in the indicated ranges of η:

$$E_\eta(R_K) = R_K C(\eta R_K) + O_{u,v}(1) \qquad (0 \le \eta \le \tfrac{1}{2}), \tag{3.3}$$

$$E_\eta(R_K) = c_+ \max\left\{0, R_K - \frac{H_{u,v}}{2(1-\eta)}\right\} + O_{u,v}(1) \qquad (\tfrac{1}{2} \le \eta < 1), \tag{3.4}$$

$$E_\eta(R_K) = O_{u,v}(1) \qquad (\eta \ge 1). \tag{3.5}$$

The error constants depend only on u, v. For $(u, v) = (0, 0)$, $E_\eta(R) = 0$ for every $\eta \ge 0$ and $R \in \mathbb{R}$. For every fixed $\kappa \ge 0$, as $K \to \infty$,

$$E_{\kappa/R_K}(R_K) = R_K C(\kappa) + O_{u,v}(1),$$

and for every fixed $\kappa > 0$, as $K \to \infty$,

$$E_{1-\kappa/R_K}(R_K) = R_K c_+ \max\left\{0, 1 - \frac{H_{u,v}}{2\kappa}\right\} + O_{u,v}(1).$$

Moreover,

$$\limsup_{j\to\infty} \left|\sqrt{\lambda_j(L)+4} - \sqrt{\lambda_j(L')+4}\right| = \begin{cases} 0, & (2u+3, 2v+1) \bmod 49 \in \mathscr{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The proof of Lemma 3.5 gives the complete quadratic model with

$$\sigma = 1, \quad \beta = 2, \quad c = 4, \quad r_0 = u, \quad q = 49, \qquad P = Q = \emptyset, \qquad \delta_i(a) = A(a+2, w_i).$$

Translation by 2 permutes $\mathbb{Z}/49\mathbb{Z}$, so the quantities c_0, c_+, d_g and the function C are those of Theorem 2.2. By (1.7) and Lemma 3.2,

$$\sum_{a=0}^{48} B(a) = 0.$$

Taking $\theta = 1/2$ in (2.4) gives $R_K = K + 5/2$. Theorem 2.2, with $\eta_* = 1/2$, proves (3.3).

If $(u, v) \neq (0, 0)$, then $H_{u,v} = u(u+2) - 2v \geq u^2 > 0$. Theorem 2.3, with $\eta_0 = 1/2$, proves (3.4) and (3.5). For $(u, v) = (0, 0)$, Theorem 1.1 and Lemma 1.3 give equality of the spectra and $A = 0$; hence $E_\eta = 0$ and $C \equiv 0$. For each fixed κ, the two substitutions $\eta = \kappa/R_K$ and $\eta = 1 - \kappa/R_K$ belong to the respective ranges for all sufficiently large K, with $\kappa > 0$ in the second substitution. This gives the two further estimates.

Since $P = Q = \emptyset$, Proposition 2.6 gives limsup 0 if $B = 0$ and 1 otherwise. Lemma 3.5 and Theorem 3.1 identify $B = 0$ with $(2u+3, 2v+1) \bmod 49 \in \mathscr{Z}$. □

Corollary 3.8. *A compact self-adjoint operator C on $\mathcal{H}_{u,v,L}$ with $\mathrm{Spec}(\Delta_{u,v,L} + C) = \mathrm{Spec}(\Delta_{u,v,L'})$ exists precisely for*

$$(u, v) \in \{(0,0), (1,0), (2,0), (1,1), (2,1)\}.$$

Suppose $B = 0$, and put $D_{u,v} = D_*$, with D_* defined by (3.2). *Let C range over the bounded self-adjoint operators on $\mathcal{H}_{u,v,L}$ such that $\mathrm{Spec}(\Delta_{u,v,L} + C) = \mathrm{Spec}(\Delta_{u,v,L'})$. Then*

$$\min_C \|C\|_{\mathrm{ess}} = \limsup_{j\to\infty} |\lambda_j(L) - \lambda_j(L')| = D_{u,v},$$

$$\min_C \|C\| = \sup_{j\geq 1} |\lambda_j(L) - \lambda_j(L')| \leq H_{u,v}.$$

Both minima are attained by the same operator, diagonal in an eigenbasis of $\Delta_{u,v,L}$. For every sufficiently large r and every $1 \leq i < s$, exactly

$$\left|\sum_{j=1}^{i} A(r+2, w_j)\right|$$

ordered pairs in the r-block join the offsets w_i and w_{i+1}; all other pairs join equal offsets.

Proof. The Laplacians are nonnegative with compact resolvent on infinite-dimensional separable complex Hilbert spaces. Proposition 2.7, Lemma 3.5, and Proposition 3.6 give the two minima and their common diagonal minimizer. Theorem 2.5, with $\delta_i(a) = A(a+2, w_i)$ as in the proof of Lemma 3.5, gives the stated pair counts.

If a compact correction exists, Proposition 2.7 and Lemma 3.5 give $B = 0$ and $\lambda_j(L) - \lambda_j(L') \to 0$. Proposition 3.6 then gives $D_{u,v} = 0$. Since every adjacent gap is positive, (3.2) implies

$$\sum_{j=1}^{i} A(a, w_j) = 0 \qquad (a \in \mathbb{Z}/49\mathbb{Z},\ 1 \leq i < s).$$

Together with $B(a) = 0$, these identities give $A(a, w_i) = 0$ for every a and i. Lemma 1.3 and Theorem 1.1 give the five displayed types. Conversely, their spectra are equal, so $C = 0$ suffices. □

Corollary 3.9. *A bounded self-adjoint correction giving equality of the two natural spectra exists if and only if* $\lim_{\kappa\to\infty} C(\kappa) = 0$. *A compact self-adjoint correction exists if and only if* $C(0) = 0$, *equivalently, if the uncorrected spectra are equal. Suppose* $B = 0$, *and let* C *range over the bounded self-adjoint corrections specified in Corollary 3.8. Then*

$$\min_{C} \|C\|_{\mathrm{ess}} = D_{u,v} = \max\{g \in G_W \,:\, d_g > 0\},$$

where an empty maximum is zero. If $\mathcal{C} \not\equiv 0$, *then* $D_{u,v}/2$ *is the greatest point in* $(0,\infty)$ *at which the piecewise affine function* $\kappa \mapsto \kappa C(\kappa)$ *is not differentiable. If* $\mathcal{C} \equiv 0$, *then* $D_{u,v} = 0$.

Proof. The definitions give

$$\lim_{\kappa\to\infty} C(\kappa) = c_+, \qquad c_+ = 0 \iff B = 0, \qquad C(0) = c_0, \qquad c_0 = 0 \iff A = 0.$$

Lemma 3.5 and Proposition 2.7 give the bounded correction criterion. Lemma 1.3 and Corollary 3.8 give the compact correction criterion and the equivalence with equality of spectra.

Suppose $B = 0$. For $a \in \mathbb{Z}/49\mathbb{Z}$, put

$$S_0(a) = 0, \qquad S_i(a) = \sum_{j=1}^{i} A(a, w_j) \quad (1 \le i \le s).$$

Then $S_s(a) = 0$. At threshold $t \ge 0$, each prefix ending at a missing edge is a union of components. Each component $\{w_{\ell+1}, \dots, w_m\}$ has sum $S_m(a) - S_\ell(a)$. Since $S_0(a) = S_s(a) = 0$ and $F_a(t)$ is a sum of absolute values,

$$F_a(t) = 0 \quad \iff \quad S_i(a) = 0 \quad \text{whenever } 1 \le i < s \text{ and } w_{i+1} - w_i > t.$$

Thus

$$\sum_{a=0}^{48} F_a(t) = 0 \quad \iff \quad t \ge D_{u,v}.$$

If $D_{u,v} > 0$, the finite step function on the left is positive for $t < D_{u,v}$ and zero for $t \ge D_{u,v}$. Hence $d_{D_{u,v}} > 0$ and $d_g = 0$ for $g > D_{u,v}$. If $D_{u,v} = 0$, every d_g is zero. This proves the formula for the maximum. By (2.13), the nondifferentiability points of $\kappa \mapsto \kappa C(\kappa)$ in $(0,\infty)$ are exactly $g/2$ with $g \in G_W$ and $d_g > 0$. The last assertions follow. □

For each fixed $0 < \eta < 1$, equations (3.3)–(3.5) give

$$E_\eta(R_K) = c_+ R_K + O_{u,v,\eta}(1), \qquad E_1(R_K) = O_{u,v}(1).$$

If $B = 0$ and (u, v) is not one of the five types in Corollary 3.8, then $D_{u,v} > 0$ and $d_{D_{u,v}} > 0$. Consequently $C(\kappa) > 0$ for every finite $\kappa \ge 0$, whereas $E_\eta(R_K) = O_{u,v,\eta}(1)$ for every fixed $\eta > 0$.

3.5. **Optimal correction norms for one-row types.** Fix an integer $u \ge 0$, and put $A_u = \Delta_{u,0,L}$ and $A'_u = \Delta_{u,0,L'}$, with the groups, bundles, and Casimir normalization of Section 1. Write $(\lambda_j(A_u))_{j\ge1}$ and $(\lambda_j(A'_u))_{j\ge1}$ for their eigenvalues in nondecreasing order, with multiplicities. For integers $r \ge k \ge 0$, let $V_{r,k,0}$ be the irreducible G-representation of highest weight $(r, k, 0)$, and put

$$p_{r,k} = \dim V_{r,k,0}^{\Gamma}, \qquad q_{r,k} = \dim V_{r,k,0}^{\Gamma'}.$$

For $v = 0$, the branching rule (1.2) has $c = 0$ and $r \ge u \ge k \ge 0$, with multiplicity one. Together with (1.1), it gives the complete multiset decompositions

$$\begin{aligned} \mathrm{Spec}(A_u) &= \coprod_{r=u}^{\infty} \coprod_{k=0}^{u} \{r(r+4) + k(k+2)\}^{[p_{r,k}]}, \\ \mathrm{Spec}(A'_u) &= \coprod_{r=u}^{\infty} \coprod_{k=0}^{u} \{r(r+4) + k(k+2)\}^{[q_{r,k}]}. \end{aligned} \tag{3.6}$$

These representations form the family in [19, Remark 3.17]; their weight multiplicities are given by [19, Theorem 3.3].

Let $\mathscr{C}_u$ be the set of bounded self-adjoint operators C on $\mathcal{H}_{u,0,L}$ such that $\mathrm{Spec}(A_u + C) = \mathrm{Spec}(A'_u)$, with multiplicities. The domain of $A_u + C$ is $\mathcal{D}(A_u)$. Put

$$\mathcal{B} = \{0, 1, 2, 9, 16, 23, 30, 37, 44, 45, 46, 47, 48\}.$$

Theorem 3.10 (Optimal norms for one-row types). *The set $\mathscr{C}_u$ is nonempty if and only if $u \bmod 49 \in \mathcal{B}$. For $u = 0, 1, 2$, the spectra are equal and $0 \in \mathscr{C}_u$. For every other u with residue in $\mathcal{B}$, write $u = 49t + b$, where $t \in \mathbb{Z}_{\geq 0}$ and $0 \leq b < 49$, and put*

$$d_u = \begin{cases} 98t - 9, & b \in \{0, 1, 2\}, \quad t \geq 1, \\ 98t + 2b + 1, & b \in \{9, 16, 23, 30, 37\}, \\ 98t + 89, & b \in \{44, 45, 46, 47, 48\}. \end{cases} \tag{3.7}$$

Then

$$\min_{C \in \mathscr{C}_u} \|C\| = \min_{C \in \mathscr{C}_u} \|C\|_{\mathrm{ess}} = d_u,$$
$$\sup_{j \geq 1} |\lambda_j(A_u) - \lambda_j(A'_u)| = \limsup_{j \to \infty} |\lambda_j(A_u) - \lambda_j(A'_u)| = d_u.$$

Both minima are attained by the same operator, diagonal in an eigenbasis of A_u. The set $\mathscr{C}_u$ contains a compact operator if and only if $u \in \{0, 1, 2\}$.

3.5.1. *Two invariant tensors.* For a dominant integral weight λ of G, let V_λ denote the irreducible representation of highest weight λ.

Lemma 3.11. *Let λ, μ be dominant integral weights of G, and let $\Lambda \in \{\Gamma, \Gamma'\}$. For every nonzero $\eta \in V_\mu^\Lambda$, Cartan multiplication by η induces an injection*

$$V_\lambda^\Lambda \longrightarrow V_{\lambda+\mu}^\Lambda.$$

In particular, for integers $r \geq k \geq 0$,

$$\dim V_{r+2,k,0}^\Lambda \geq \dim V_{r,k,0}^\Lambda, \qquad \dim V_{r+1,k+1,0}^\Lambda \geq \dim V_{r,k,0}^\Lambda. \tag{3.8}$$

Proof. Put $G_{\mathbb{C}} = \mathrm{SO}(6, \mathbb{C})$, and let U be a maximal unipotent subgroup. The Cartan algebra $\mathbb{C}[G_{\mathbb{C}}/U]$ contains each V_λ with multiplicity one, and its multiplication sends $V_\lambda V_\mu$ into $V_{\lambda+\mu}$ [22, p. 173, (3.20)–(3.21)]. Since $G_{\mathbb{C}}$ is connected, $\mathbb{C}[G_{\mathbb{C}}]$ is an integral domain. Its subalgebra $\mathbb{C}[G_{\mathbb{C}}/U]$ is therefore an integral domain, and its multiplication is $G_{\mathbb{C}}$-equivariant. Multiplication by η is therefore injective and preserves Λ-invariant vectors.

Let $e_1, \dots, e_6$ be the standard basis of $\mathbb{C}^6$, let $x_1, \dots, x_6$ be its coordinate functions, and let $\mathcal{H}_2$ be the space of homogeneous quadratic polynomials annihilated by $\sum_{i=1}^6 \partial_{x_i}^2$. The standard identifications [8, Exercises 32.8 and 32.11(iii)] are

$$\bigwedge^2 \mathbb{C}^6 \simeq V_{1,1,0}, \qquad \mathcal{H}_2 \simeq V_{2,0,0}.$$

Both groups preserve each oriented coordinate two-plane. Thus

$$e_1 \wedge e_2 \in V_{1,1,0}^\Lambda, \qquad x_1^2 + x_2^2 - x_3^2 - x_4^2 \in V_{2,0,0}^\Lambda$$

are nonzero; the Laplacian of the second is $2 + 2 - 2 - 2 = 0$. The choices $\mu = (2, 0, 0)$ and $\mu = (1, 1, 0)$ give (3.8). □

Lemma 3.12. *For integers r, k with $r \geq 9$ and $0 \leq k \leq r$,*

$$p_{r,k} \geq 28, \qquad q_{r,k} \geq 28.$$

Proof. Under the exterior-square covering $\mathrm{SU}(4) \longrightarrow \mathrm{SO}(6)$ [8, Example 32.10], choose diagonal torus coordinates z_1, z_2, z_3, z_4, with $z_1z_2z_3z_4 = 1$, so that the characters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ pull back to z_1z_2, z_1z_3, z_1z_4, respectively. Put

$$t = (11, 39, 44, 4), \qquad t' = (38, 12, 17, 31).$$

Both tuples have sum zero modulo 49, and

$$(t_1 + t_2, t_1 + t_3, t_1 + t_4) \equiv (1, 6, 15), \qquad (t'_1 + t'_2, t'_1 + t'_3, t'_1 + t'_4) \equiv (1, 6, 20) \pmod{49}.$$

The complementary pair sums are their negatives. Put $\zeta = e^{2\pi i/49}$. The diagonal matrices with entries ζ^{t_i} and $\zeta^{t'_i}$ have order 49 and map to the generators of Γ and Γ', respectively.

Fix $\Lambda \in \{\Gamma, \Gamma'\}$, and let $\mathbf{v} = (v_1, v_2, v_3, v_4)$ be t or t', respectively. The highest weight of the pullback of $V_{r,k,0}$ is $z_1^{r+k}z_2^r z_3^k$; its A_3 Dynkin labels are $(k, r-k, k)$. Thus its character is the Schur polynomial associated with the partition $\lambda = (r+k, r, k, 0)$ [8, §29.3].

For $a \in \mathbb{Z}/49\mathbb{Z}$, let $[Z^a]$ denote the coefficient of the corresponding basis element in $\mathbb{Z}[Z]/(Z^{49}-1)$. With X a formal variable, define

$$\sum_{n\geq 0} h_n^{(\mathbf{v})} X^n = \prod_{i=1}^{4} (1 - XZ^{v_i})^{-1}, \qquad h_n^{(\mathbf{v})} = 0 \quad (n < 0). \tag{3.9}$$

The Jacobi–Trudi identity [20, Chapter I, §3, (3.4)] is a polynomial identity, so it remains valid after the substitution $z_i = Z^{v_i}$. Since $\lambda_4 = 0$, it gives

$$\dim V_{r,k,0}^{\Lambda} = [Z^0] \det\big(h_{\lambda_i - i + j}^{(\mathbf{v})}\big)_{i,j=1}^{3}. \tag{3.10}$$

Indeed, the coefficient of Z^a in the restricted character is the multiplicity of the character sending the generator to ζ^a.

For $a \in \mathbb{Z}/49\mathbb{Z}$, define integers $H_n^{(j)}(a)$ by

$$\begin{aligned} H_n^{(0)}(a) &= \mathbf{1}_{n=0}\mathbf{1}_{a=0} \quad (n \geq 0), \qquad H_{-1}^{(j)}(a) = 0 \quad (0 \leq j \leq 4), \\ H_n^{(j)}(a) &= H_n^{(j-1)}(a) + H_{n-1}^{(j)}(a - v_j) \quad (n \geq 0,\ 1 \leq j \leq 4). \end{aligned} \tag{3.11}$$

Then $h_n^{(\mathbf{v})} = \sum_a H_n^{(4)}(a) Z^a$ for $n \geq 0$. Equations (3.10) and (3.11) give

(r,k)	$p_{r,k}$	$q_{r,k}$	(r,k)	$p_{r,k}$	$q_{r,k}$
$(9,0)$	28	28	$(9,6)$	660	660
$(9,1)$	91	91	$(9,7)$	844	842
$(9,2)$	226	226	$(9,8)$	594	594
$(9,3)$	334	334	$(9,9)$	527	527
$(9,4)$	472	472	$(10,0)$	30	30
$(9,5)$	655	657			

Put $d = r - k$. If $d \geq 9$, choose $d_0 \in \{9, 10\}$ with $d \equiv d_0 \pmod 2$. Starting at $(d_0, 0)$, apply the first injection in (3.8) $(d - d_0)/2$ times and the second k times. The resulting pair is $(d + k, k) = (r, k)$. If $0 \leq d \leq 8$, start at $(9, 9 - d)$ and apply the second injection $r - 9$ times. The resulting pair is $(r, r - d) = (r, k)$. All intermediate pairs are dominant, and every initial dimension in the table is at least 28. □

3.5.2. *The cumulative multiplicity differences.* Let $T(a, b, c)$ be the integer-valued function in (1.10), and put $T_0(a, b) = T(a, b, 0)$. By (1.5),

$$p_{r,k} - q_{r,k} = T_0(r + 2, k + 1). \tag{3.12}$$

The function T_0 is even and 49-periodic in each variable. Moreover, Lemma 3.2 gives $\sum_{b=0}^{48} T_0(a, b) = 0$.

For $a \in \mathbb{Z}/49\mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, put

$$S_a(k) = \sum_{j=0}^{k} T_0(a, j+1), \qquad S_a(-1) = 0.$$

The zero period sum gives $S_a(k+49) = S_a(k)$; extend S_a to $\mathbb{Z}$ by this identity. For $0 \leq j \leq 5$, put

$$I_j = \{7j+3, \dots, 7j+8\}, \qquad J_j = \{7j+5, 7j+6\},$$

and write $I = (I_j)_{j=0}^5$, $J = (J_j)_{j=0}^5$. Define

$$c^{(1)} = (1,1,-1,1,-1,-1), \quad c^{(2)} = (1,0,2,-2,0,-1), \quad c^{(3)} = (1,-2,1,-1,2,-1).$$

For $k \in \mathbb{Z}$, let $\bar{k}$ be its representative in $\{0, \dots, 48\}$. For $K = I$ or J, write $\mathbf{1}_{K_j}$ for the indicator function of K_j. For $c = (c_j)_{j=0}^5 \in \mathbb{Z}^6$, define

$$F_{K,c}(k) = 2 \sum_{j=0}^{5} c_j \mathbf{1}_{K_j}(\bar{k}) \qquad (k \in \mathbb{Z}).$$

In the following table, a row (a_+, a_-, K, c) means $S_{a_+} = S_{-a_+} = F_{K,c}$ and $S_{a_-} = S_{-a_-} = -F_{K,c}$. Substitution of Table 1 into (1.10), followed by summation over $b = 1, \dots, k+1$, gives

a_+	a_-	K	c
6	8	I	$c^{(1)}$
10	4	J	$c^{(1)}$
13	15	I	$c^{(2)}$
17	11	J	$c^{(2)}$
22	20	I	$c^{(3)}$
18	24	J	$c^{(3)}$

(3.13)

For every remaining residue a, $S_a = 0$. In particular,

$$|S_a(k)| \leq 4, \qquad S_a(k)S_a(k-1) \geq 0 \qquad (a \in \mathbb{Z}/49\mathbb{Z},\ k \in \mathbb{Z}). \tag{3.14}$$

Indeed, $|c_j^{(\ell)}| \leq 2$, and successive intervals in either family are separated by an integer outside all its intervals. The union of the supports, reduced modulo 49, is $\bigcup_{j=0}^5 I_j$, and S_6 is nonzero throughout this union. Hence

$$S_a(k) = 0 \text{ for every } a \in \mathbb{Z}/49\mathbb{Z} \iff k \bmod 49 \in \mathcal{B}. \tag{3.15}$$

3.5.3. *A matching in every block.* Suppose $u \geq 9$ and $u \bmod 49 \in \mathcal{B}$. Put $w_k = k(k+2)$ for $0 \leq k \leq u$, and define

$$D(u) = \max\{2k+3 : 0 \leq k < u,\ S_a(k) \neq 0 \text{ for some } a \in \mathbb{Z}/49\mathbb{Z}\}, \tag{3.16}$$

where an empty maximum is zero. Fix $r \geq u$, and set $a = r+2 \pmod{49}$. Apply Lemma 2.1 to the offsets $w_0 < \cdots < w_u$ with tolerance $D(u)$. If $w_{k+1} - w_k = 2k+3 > D(u)$, then $S_a(k) = 0$ by (3.16). Also $S_a(-1) = S_a(u) = 0$. Thus, for every component $\{\ell, \dots, m\}$ of this path,

$$\sum_{k=\ell}^{m} (p_{r,k} - q_{r,k}) = S_a(m) - S_a(\ell - 1) = 0,$$

and, for $\ell \leq j < m$,

$$\left| \sum_{k=\ell}^{j} (p_{r,k} - q_{r,k}) \right| = |S_a(j)| \leq 4.$$

Lemma 3.12 gives $\min(p_{r,k}, q_{r,k}) \geq 28$. The component form of Lemma 2.1, with $L = 4$, therefore gives

$$E_{D(u)}\left(\{w_k^{[p_{r,k}]}\}_{k=0}^u, \{w_k^{[q_{r,k}]}\}_{k=0}^u \right) = 0 \qquad (r \geq u). \tag{3.17}$$

Translate each such matching by $r(r+4)$ and take their disjoint union. By (3.6), this gives a bijection between the labelled copies of the complete spectra with displacement at most $D(u)$. For $E \in \mathbb{R}$, define

$$N_u(E) = \#\{j \geq 1 \,:\, \lambda_j(A_u) \leq E\}, \qquad N'_u(E) = \#\{j \geq 1 \,:\, \lambda_j(A'_u) \leq E\}.$$

These numbers are finite, and the bijection gives

$$N_u(E) \leq N'_u(E + D(u)), \qquad N'_u(E) \leq N_u(E + D(u)).$$

Taking $E = \lambda_j(A_u)$ in the first inequality and $E = \lambda_j(A'_u)$ in the second gives

$$|\lambda_j(A_u) - \lambda_j(A'_u)| \leq D(u) \qquad (j \geq 1). \tag{3.18}$$

In the notation of Proposition 3.6,

$$W = \{w_0, \dots, w_u\}, \qquad A(a, w_k) = T_0(a, k+1), \qquad B(a) = S_a(u) = 0.$$

Its cumulative sums are $S_a(k)$, so $D_* = D(u)$. That proposition and (3.18) yield

$$\sup_{j \geq 1} |\lambda_j(A_u) - \lambda_j(A'_u)| = \limsup_{j \to \infty} |\lambda_j(A_u) - \lambda_j(A'_u)| = D(u). \tag{3.19}$$

Put $H = u(u+2)$. The r-block is contained in $[r(r+4), r(r+4)+H]$, and its lower endpoint exceeds the upper endpoint of the preceding block by $2r + 3 - H$. This is positive for

$$r \geq R_u := \max\left\{u, \left\lceil \frac{H}{2} \right\rceil - 1\right\}. \tag{3.20}$$

At these indices the blocks are separated, all earlier blocks lie below them, and the sums of the multiplicities of all preceding blocks are equal. Since $\min(p_{r,k}, q_{r,k}) \geq 28 > 4 \geq |S_a(k)|$, the proof of Theorem 2.5 applies to every $r \geq R_u$: exactly $|S_{r+2}(k)|$ ordered pairs join the offsets w_k, w_{k+1} for each $0 \leq k < u$, and every other pair joins equal offsets.

3.5.4. *The complete formula.*

Proof of Theorem 3.10. For $\tau_{u,0}$, the block total in Lemma 3.5 is $B(a) = S_a(u)$. That lemma and Proposition 2.7 imply that a bounded self-adjoint correction exists if and only if $S_a(u) = 0$ for every a. By (3.15), this is equivalent to $u \bmod 49 \in \mathcal{B}$.

Theorem 1.1 gives equality of the spectra for $u = 0, 1, 2$. Every other u with residue in $\mathcal{B}$ is at least 9, so (3.19) applies. Write $u = 49t + b$, with $t \geq 0$ and $0 \leq b < 49$. By (3.13), the integers k for which $S_a(k) \neq 0$ for some a have residues

$$\bigcup_{j=0}^{5} \{7j + 3, \dots, 7j + 8\} \pmod{49}.$$

For $b \in \{0, 1, 2\}$ and $t \geq 1$, the largest such integer $k < u$ is $49t - 6$. For $b \in \{9, 16, 23, 30, 37\}$, it is $u - 1$. For $44 \leq b \leq 48$, it is $49t + 43$. Substitution in $2k + 3$ gives (3.7). Proposition 2.7 and (3.19) give both norm minima and their simultaneous attainment by a correction diagonal in an eigenbasis of A_u. Since $d_u > 0$ for every such $u \geq 9$, the same proposition excludes compact corrections in these cases. □

Corollary 3.13. *For $\tau_{9,0}$, the least operator and essential correction norms are 19. For $\tau_{48,0}$, both norms are 89.*

Proof. Substitute $u = 9$ and $u = 48$ in (3.7). For $u = 48$, the largest $k < 48$ with $S_a(k) \neq 0$ for some a is 43, between $w_{43} = 1935$ and $w_{44} = 2024$, and $S_6(43) = -2$. Equation (3.20) gives $R_{48} = 1199$. □

3.5.5. *Two resolution profiles.* For each integer $u \geq 0$, put $w_k = k(k+2)$ for $0 \leq k \leq u$. Let C_u be the function C of Theorem 3.7 for the type $\tau_{u,0}$, and denote its coefficients d_g by $d_g^{(u)}$. The gaps $w_{k+1} - w_k = 2k+3$ increase strictly with k. At the gap $2k+3$, the component $\{w_0, \dots, w_k\}$ joins the singleton $\{w_{k+1}\}$. Consequently, for $0 \leq k < u$,

$$\begin{aligned} d_{2k+3}^{(u)} &= \sum_{a=0}^{48} \big(|S_a(k)| + |S_a(k+1) - S_a(k)| - |S_a(k+1)| \big) \\ &= 2 \sum_{a=0}^{48} \max\{|S_a(k)| - |S_a(k+1)|, 0\}, \end{aligned}$$

where the second equality follows from (3.14).

For $u = 9$, the offsets are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99$. The only nonzero coefficients are $d_{15}^{(9)} = d_{19}^{(9)} = 48$. Since $B(a) = S_a(9) = 0$, Theorem 3.7 gives

$$C_9(0) = \frac{96}{49}, \qquad C_9(\kappa) = \frac{48}{49} \left(\min\left\{1, \frac{15}{2\kappa}\right\} + \min\left\{1, \frac{19}{2\kappa}\right\} \right) \quad (\kappa > 0).$$

For $u = 48$, the dimension formula in subsection 1.1 gives $\dim \tau_{48,0} = 99 \cdot 49 \cdot 50/6 = 40425$. The only decreases from $|S_a(k)|$ to $|S_a(k+1)|$ occur at $k = 7j+6$ or $7j+8$, with $0 \leq j \leq 5$. Formula (3.13) gives

$$d_{14j+15}^{(48)} = d_{14j+19}^{(48)} = 16 \sum_{\ell=1}^{3} |c_j^{(\ell)}|.$$

Thus the nonzero coefficients are

j	0	1	2	3	4	5
$14j+15$	15	29	43	57	71	85
$14j+19$	19	33	47	61	75	89
$d_{14j+15}^{(48)} = d_{14j+19}^{(48)}$	48	48	64	64	48	48

Put

$$\mathcal{G} = \{15, 19, 29, 33, 43, 47, 57, 61, 71, 75, 85, 89\}.$$

Since $B(a) = S_a(48) = 0$, the same theorem gives

$$C_{48}(0) = \frac{640}{49}, \qquad C_{48}(\kappa) = \frac{1}{49} \sum_{g \in \mathcal{G}} d_g^{(48)} \min\left\{1, \frac{g}{2\kappa}\right\} \quad (\kappa > 0). \tag{3.21}$$

Finally, $\sum_{g \in \mathcal{G}} g\, d_g^{(48)} = 33280$, whence

$$C_{48}(\kappa) = \frac{16640}{49\kappa} \qquad (\kappa \geq 89/2).$$

Part 2. Prime-square orders and congruences

4. Natural spectra at prime-square orders

Let $p \geq 5$ be prime, put $q = p^2$, and set

$$L_p^{\pm} = L(q; 1, 1 \pm p, 1 \pm 3p).$$

For $p \geq 7$, these are the lens spaces of [17, Theorem 7.1], with parameters $r = p$ and $t = 1$. For every $p \geq 5$, their Hodge spectra agree in every degree by [5, Theorem 1 and Sections 4, 8–9]. In that theorem, take $r = p$, $t = 1$, and $a = (0, 1, 3)$. The only divisors of p are 1 and p; the triple a is reversible modulo 1 and has distinct entries modulo p, as required there.

For integers $u \geq v \geq 0$, let $\tau_{u,v}$ be the irreducible SO(5)-representation with highest weight $u\varepsilon_1 + v\varepsilon_2$, and let $\Delta^{\pm}_{u,v}$ be the nonnegative SO(6) Casimir operators on the associated natural bundles over $L^{\pm}_p$, with the curvature-one normalization of Section 1. Put

$$X = 2u + 3, \qquad Y = 2v + 1, \qquad H = u(u+2) - 2v.$$

For $(x, y) \in (\mathbb{Z}/q\mathbb{Z})^2$, its *signed-permutation orbit* consists of $(\epsilon x, \delta y)$ and $(\epsilon y, \delta x)$, where $\epsilon, \delta \in \{1, -1\}$. Define subsets of $(\mathbb{Z}/q\mathbb{Z})^2$ by

$$\mathcal{W}_q = \{x = 0\} \cup \{y = 0\} \cup \{x = y\} \cup \{x = -y\}, \qquad \mathcal{G}_p = \{(x, y) : p \mid x,\ p \mid y\},$$

and let $\mathcal{O}_q$ be the union of the five signed-permutation orbits represented by

$$(3, 1), \quad (5, 1), \quad (5, 3), \quad (7, 1), \quad (7, 3).$$

Theorem 4.1 (Bounded corrections at prime-square orders). *There is a bounded self-adjoint operator K on $L^2(E_{\tau_{u,v}, L^+_p})$ such that*

$$\operatorname{Spec}(\Delta^+_{u,v} + K) = \operatorname{Spec}(\Delta^-_{u,v}),$$

including multiplicities, if and only if one of the following holds.

(1) $p = 5$.

(2) $p = 7$, and the reduction of (X, Y) modulo 49 belongs to $\mathcal{W}_{49} \cup \mathcal{G}_7 \cup \mathcal{O}_{49}$ or to one of the four signed-permutation orbits represented by

$$(14, 1), \quad (21, 1), \quad (14, 3), \quad (21, 3).$$

(3) $p \geq 11$, and the reduction of (X, Y) modulo p^2 belongs to $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$.

Whenever K exists, it can be chosen to commute with $\Delta^+_{u,v}$ and to satisfy $\|K\| \leq H$. For $p = 5$, an isometry identifies the two natural operators, so $K = 0$ is possible. Allowing bounded self-adjoint corrections on both sides gives the same classification.

The numbers of residue pairs in these three cases are

$$625 \quad (p = 5), \qquad 289 \quad (p = 7), \qquad 5p^2 - 4p + 40 \quad (p \geq 11).$$

4.1. **The fixed Laurent polynomial.** Let t, w, x, y be independent indeterminates. For $d \in \mathbb{Z}$, put

$$E_d = 1 - t^2 w^{2d}, \qquad F_d = 1 - t^4 w^{2d}, \qquad A(w) = (w^2 - 1)(w^4 - 1)(w^6 - 1),$$

and define

$$\Lambda_t = E_3 E_{-3} \prod_{1 \leq |j| \leq 5} F_j, \qquad \mathcal{P} = \{(1, 3), (-1, -3), (-1, 2), (1, -2), (-3, -2), (3, 2)\}.$$

For $(d, e) \in \mathcal{P}$, set

$$Q_{d,e} = \frac{\Lambda_t}{F_d F_e F_{d+e} E_d E_e}.$$

These quotients belong to $\mathbb{Z}[t, w, w^{-1}]$. Indeed, the three indices $d, e, d + e$ are distinct, nonzero, and of absolute value at most 5. After removing the corresponding F-factors from Λ_t, cancel E_f directly when $|f| = 3$. When $f \in \{d, e\}$ and $|f| \leq 2$, the index $2f$ is not one of $d, e, d + e$, and

$$F_{2f} = E_f(1 + t^2 w^{2f}).$$

The factors needed for the two cancellations are distinct. Since $\deg_t \Lambda_t = 44$, it follows that $\deg_t Q_{d,e} = 28$; only even powers of t occur.

Define the signed monomials $m_{d,e}(w)$ by

(d, e)	$(1, 3)$	$(-1, -3)$	$(-1, 2)$	$(1, -2)$	$(-3, -2)$	$(3, 2)$
$m_{d,e}$	w^{18}	w^{-6}	$-w^9$	$-w^3$	w^{-9}	w^{21}.

The fixed Laurent polynomial is

$$J(t,w,x,y)=\sum_{(d,e)\in\mathcal{P}} m_{d,e}(w)Q_{d,e}(t,w)(x^dy^e-x^ey^d)\in\mathbb{Z}[t,w^{\pm1},x^{\pm1},y^{\pm1}]. \tag{4.1}$$

Index its nonzero monomials by a finite set I, and write

$$J=\sum_{\nu\in I} j_\nu t^{a_\nu}w^{b_\nu}x^{c_\nu}y^{d_\nu}.$$

Multiplication of the factors gives 136, 192, and 126 nonzero monomials in $Q_{1,3}$, $Q_{-1,2}$, and $Q_{-3,-2}$, respectively, with coefficients of absolute value at most 4, 6, and 5. Moreover, $Q_{-d,-e}(t,w)=Q_{d,e}(t,w^{-1})$. The twelve ordered exponent pairs $(d,e),(e,d)$ occurring in (4.1) are distinct. Thus J has $4(136+192+126)=1816$ nonzero monomials, with $a_\nu\in\{0,2,\dots,28\}$ and $|j_\nu|\le 6$. If N_a denotes their number at t-degree a, the same products give

a	0	2	4	6	8	10	12	14
N_a	12	32	84	136	152	188	196	216

$$N_{28-a}=N_a.$$

In particular, $N_a\le 216$.

For integers X, Y and $s\in\mathbb{Z}/q\mathbb{Z}$, define

$$C_s(p;X,Y)=\sum_{\substack{\nu\in I,\ \epsilon,\delta\in\{\pm1\}\\ a_\nu+pb_\nu+\epsilon(1+pc_\nu)X+\delta(1+pd_\nu)Y\equiv s \pmod q}} \epsilon\delta j_\nu. \tag{4.2}$$

It depends only on the residue classes of X and Y modulo q. The full sum over s has $4\cdot 1816=7264$ indexed summands before equal residues are combined.

Theorem 4.2. *For p,u,v as above, write $\lambda_j(\Delta^{\pm}_{u,v})$ for the complete eigenvalue sequences in nondecreasing order, with multiplicities. The following conditions are equivalent:*

(1) For every $0\le r<p$, the p integers $C_r(p;X,Y),C_{r+p}(p;X,Y),\dots,C_{r+(p-1)p}(p;X,Y)$ are equal.

(2) $\sup_{j\ge1}|\lambda_j(\Delta^+_{u,v})-\lambda_j(\Delta^-_{u,v})|<\infty$.

(3) There is a bounded self-adjoint operator K on the space $L^2(E_{\tau_{u,v}},L^+_p)$ such that

$$\operatorname{Spec}(\Delta^+_{u,v}+K)=\operatorname{Spec}(\Delta^-_{u,v}),$$

including multiplicities; the sum has domain $\mathcal{D}(\Delta^+_{u,v})$.

(4) The operator K in condition 3 can be chosen to commute with every spectral projection of $\Delta^+_{u,v}$.

If these conditions hold, then

$$|\lambda_j(\Delta^+_{u,v})-\lambda_j(\Delta^-_{u,v})|\le H\qquad(j\ge1),$$

and K can be chosen with $\|K\|\le H$. The same conditions characterize the existence of bounded self-adjoint corrections on both spaces of sections that make the corrected spectra equal, including multiplicities.

4.2. **Reduction from spectra to one Fourier coefficient.** Let $\Gamma^{\pm}_p$ be the cyclic groups defining $L^{\pm}_p$. Their rotation weights $1\pm pa_j$, where $(a_1,a_2,a_3)=(0,1,3)$, are units modulo $q=p^2$. Thus both groups have order q and act freely on S^5. Fix a primitive q-th root ξ, put $\eta=\xi^p$, and write

$$z^{\pm}_{h,j}=\xi^{2h(1\pm pa_j)},\qquad z^{\pm}_h=(z^{\pm}_{h,1},z^{\pm}_{h,2},z^{\pm}_{h,3})\quad(h\in\mathbb{Z}/q\mathbb{Z}).$$

Since 2 is a unit modulo q, these are the rotation coordinates of all powers of a generator of $\Gamma^{\pm}_p$.

For $\ell=(\ell_1,\ell_2,\ell_3)\in\mathbb{Z}^3$ and $z\in(\mathbb{C}^\times)^3$, put

$$C_\ell(z)=\det(z_j^{\ell_i}+z_j^{-\ell_i})_{i,j=1}^3,\qquad U_\ell(z)=\det(z_j^{\ell_i}-z_j^{-\ell_i})_{i,j=1}^3,\qquad D(z)=C_\rho(z),\quad \rho=(2,1,0).$$

For the representations $\pi_{r,k,c}$ of subsection 1.2, equations (1.1) and (1.2) give the eigenvalue

$$r(r+4)+k(k+2)+c^2$$

and the branching condition $r \geq u \geq k \geq v \geq |c|$. The type-$D_3$ character formula in the proof of Lemma 1.2 is

$$\chi_{r,k,c}(z) = \frac{C_{(r+2,k+1,c)}(z) + U_{(r+2,k+1,c)}(z)}{D(z)} \qquad (D(z) \neq 0).$$

Changing c to $-c$ fixes the cosine determinant and negates the sine determinant.

If $p \mid h$, then $z_h^+ = z_h^-$, so these terms cancel in the two character averages, including $h = 0$. If h is a unit modulo q, the coordinates are regular. Indeed, the differences between distinct a_j are $\pm 1, \pm 2, \pm 3$, and

$$1 \pm pa_i \not\equiv 1 \pm pa_j \pmod{q} \quad (i \neq j), \qquad 1 \pm pa_i \not\equiv -(1 \pm pa_j) \pmod{q}.$$

The first assertion follows since $p \geq 5$, and the second follows by reduction modulo p. Multiplication by $2h$ preserves these inequalities. Thus $D(z_h^\pm) \neq 0$ for every unit h.

For integers a, b, c, define

$$T_p(a,b,c) = \frac{1}{q} \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^\times} \left(\frac{C_{(a,b,c)}(z_h^+)}{D(z_h^+)} - \frac{C_{(a,b,c)}(z_h^-)}{D(z_h^-)} \right).$$

Write

$$\delta_p(r,k,c) = \dim V_{\pi_{r,k,c}}^{\Gamma_p^+} - \dim V_{\pi_{r,k,c}}^{\Gamma_p^-}.$$

The character formula and the cancellation of the nonunit powers give

$$\delta_p(r,k,0) = T_p(r+2,k+1,0), \qquad \delta_p(r,k,c) + \delta_p(r,k,-c) = 2T_p(r+2,k+1,c) \quad (1 \leq c \leq k).$$

The function T_p is q-periodic and even in each variable, and alternating under interchange of two variables. Its sum over a full period in any variable is zero: each row entry has frequency $\pm 2h(1 \pm pa_j)$, which is a unit modulo q, and the determinant is linear in each row.

The total multiplicity discrepancy in the block with first shifted coordinate $a = r + 2$ is therefore

$$B_{u,v}(a) = \sum_{b=v+1}^{u+1} \sum_{c=-v}^{v} T_p(a,b,c).$$

The square $-v \leq b, c \leq v$ contributes zero by alternation, and the two remaining strips in the b-variable agree by evenness. Hence

$$2B_{u,v}(a) = \sum_{b=-u-1}^{u+1} \sum_{c=-v}^{v} T_p(a,b,c). \tag{4.3}$$

Define $B_{u,v}$ for all $u, v \geq 0$ by (4.3). Increasing either parameter by q adds two strips of width q, whose sums vanish. Thus this extension is q-periodic in u, v.

For independent variables t, w and integers d, X, put

$$S_d = tw^d - t^{-1}w^{-d}, \qquad z_d = t^2w^{2d}, \qquad c_d = z_d + z_d^{-1}, \qquad R_X(d) = \frac{t^X w^{dX} - t^{-X} w^{-dX}}{S_d}.$$

For every positive odd integer X, the Dirichlet identity is

$$\sum_{b=-(X-1)/2}^{(X-1)/2} \left(t^{2b} w^{2db} + t^{-2b} w^{-2db} \right) = 2R_X(d).$$

At $(t, w) = (\xi, \eta)$, the denominator S_d is nonzero, since $1 + pd$ is a unit modulo q. At these roots,

$$R_{X+q}(d) = R_X(d), \qquad R_{-X}(d) = -R_X(d).$$

The quotients for the rotation coordinates $z_h^\pm$ are obtained by taking $d = \pm a_j$ and replacing (ξ, η) by (ξ^h, η^h). Apply the Dirichlet identity to the last two rows in (4.3), with $X = 2u + 3$ and $Y = 2v + 1$.

The resulting expression depends only on X, Y modulo q; changing the sign of either variable negates it, and interchanging X, Y interchanges two determinant rows. Consequently the condition $B_{u,v} = 0$ is invariant under signed permutations of (X, Y).

For $(d, e) \in \mathcal{P}$ and integers X, Y, put

$$f_{d,e}(X, Y) = \frac{R_X(d)R_Y(e) - R_X(e)R_Y(d)}{2(c_0 - c_d)(c_0 - c_e)(c_d - c_e)}.$$

Define the rational function

$$\mathcal{F}_{X,Y}(t, w) = \sum_{(d,e)\in\{(1,3),(-1,2),(-3,-2)\}} \big(f_{d,e}(X, Y) - f_{-d,-e}(X, Y)\big). \tag{4.4}$$

For each of these six pairs, d, e, $d - e$ are nonzero modulo every prime $p \geq 5$. The triples with rotation weights $1, 1 + pd, 1 + pe$ are therefore regular. It follows that every denominator in (4.4) is nonzero at $(t, w) = (\xi, \eta)$.

Lemma 4.3. *The vector $(B_{u,v}(a))_{a \bmod q}$ is zero if and only if $\mathcal{F}_{X,Y}(\xi, \xi^p) = 0$.*

Proof. Write the finite Fourier expansion in the form

$$B_{u,v}(a) = \sum_{h \bmod q} \widehat{B}(h)\xi^{2ha}, \qquad \widehat{B}(h) = \frac{1}{q} \sum_{a \bmod q} B_{u,v}(a)\xi^{-2ha}.$$

Expansion of the first row gives $\widehat{B}(h) = 0$ for $p \mid h$.

To compute $\widehat{B}(1)$, expand the first row after summing the last two rows in (4.3). The two Dirichlet sums contribute a factor 4, and the factor 2 on the left leaves $2/q$ in the character average. For the jth rotation coordinate in the plus family, the required power of the generator is $h = \pm(1 + pa_j)^{-1}$. Simultaneous inversion of the rotation coordinates fixes the cosine determinant and the Dirichlet quotients, so these two powers contribute equally.

For the positive choice of h one has

$$(1 + pa_k)(1 + pa_j)^{-1} \equiv 1 + p(a_k - a_j) \pmod{q}.$$

Move the jth column to the first position, preserving the relative order of the other two columns. The signs of this permutation cancel between numerator and denominator. For $a_j = 0, 1, 3$, the other two normalized weights give, respectively,

$$(d, e) = (1, 3), \quad (-1, 2), \quad (-3, -2).$$

The minus family gives $(-d, -e)$. If $\gamma_j = z_j + z_j^{-1}$, then

$$D(z) = 2\det(\gamma_j^{3-i})_{i,j=1}^{3} = 2\prod_{i<j}(\gamma_i - \gamma_j).$$

Thus the normalized cofactor divided by its denominator is exactly $f_{d,e}(X, Y)$. Including both signs of h gives

$$\widehat{B}(1) = \frac{4}{q}\mathcal{F}_{X,Y}(\xi, \eta).$$

For any unit h, the same calculation with ξ^h in place of ξ gives

$$\widehat{B}(h) = \frac{4}{q}\mathcal{F}_{X,Y}(\xi^h, \eta^h) = \sigma_h\big(\widehat{B}(1)\big), \qquad \sigma_h(\xi) = \xi^h.$$

Here σ_h is the automorphism of $\mathbb{Q}(\xi)$ given by [21, Lemma 5.9 and Theorem 5.10]; it satisfies $\sigma_h(\eta) = \eta^h$ since $\eta = \xi^p$. All other Fourier coefficients vanish. Fourier inversion now proves the equivalence. □

4.3. **Clearing denominators and the cyclotomic criterion.** For $(d, e) \in \mathcal{P}$, put

$$A_{d,e} = (1 - w^{2d})(1 - w^{2e})(w^{2d} - w^{2e}).$$

For integers a, b, the identities

$$c_a - c_b = -t^{-2} w^{-2(a+b)} (w^{2a} - w^{2b}) F_{a+b}, \qquad S_d = -t^{-1} w^{-d} E_d$$

give

$$2(c_0 - c_d)(c_0 - c_e)(c_d - c_e) S_d S_e = -2t^{-8} w^{-5(d+e)} A_{d,e} F_d F_e F_{d+e} E_d E_e. \tag{4.5}$$

For $(d, e) = (1, 3), (-1, 2), (-3, -2)$, respectively,

$$\frac{A(w)}{A_{d,e}} = -w^{-2}, \quad w^4, \quad -w^{16}, \qquad A_{-d,-e} = -w^{-4(d+e)} A_{d,e}.$$

Hence, for these three pairs,

$$m_{d,e} = -w^{5(d+e)} \frac{A}{A_{d,e}}, \qquad m_{-d,-e} = w^{-5(d+e)} \frac{A}{A_{-d,-e}}.$$

The second formula includes the subtraction of the minus family in (4.4).

For integers X, Y, expansion of the numerators gives

$$\begin{aligned} &S_d S_e \big(R_X(d) R_Y(e) - R_X(e) R_Y(d) \big) \\ &\quad = \sum_{\epsilon, \delta = \pm 1} \epsilon \delta \, t^{\epsilon X + \delta Y} \big(w^{\epsilon d X + \delta e Y} - w^{\epsilon e X + \delta d Y} \big). \end{aligned}$$

Define

$$\mathcal{P}_{X,Y}(t, w) = \sum_{\epsilon, \delta = \pm 1} \epsilon \delta \, t^{\epsilon X + \delta Y} J(t, w, w^{\epsilon X}, w^{\delta Y}).$$

Substitution into (4.4) now yields the identity in $\mathbb{Q}(t, w)$

$$\mathcal{F}_{X,Y}(t, w) = \frac{t^8}{2A(w)\Lambda_t(t, w)} \mathcal{P}_{X,Y}(t, w). \tag{4.6}$$

Let ξ be a primitive q-th root of unity, with $q = p^2$. At $(t, w) = (\xi, \xi^p)$, the exponents in the factors E_d and F_d are congruent to 2 and 4 modulo p, respectively. These factors are nonzero. Since ξ^p has order $p \geq 5$, $A(\xi^p) \neq 0$ as well. Thus (4.6) gives

$$\mathcal{F}_{X,Y}(\xi, \xi^p) = 0 \quad \Longleftrightarrow \quad \mathcal{P}_{X,Y}(\xi, \xi^p) = 0.$$

In the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$, reduction modulo $t^q - 1$ gives

$$\mathcal{P}_{X,Y}(t, t^p) \equiv C(t) := \sum_{s=0}^{q-1} C_s(p; X, Y) t^s \pmod{t^q - 1}.$$

Indeed, the term indexed by (ν, ϵ, δ) has coefficient $\epsilon \delta j_\nu$ and exponent

$$a_\nu + p b_\nu + \epsilon(1 + p c_\nu) X + \delta(1 + p d_\nu) Y.$$

Every Laurent polynomial has a unique representative of degree less than q modulo $t^q - 1$, and evaluation at ξ is unchanged by this reduction.

By the irreducibility of cyclotomic polynomials [21, Theorem 5.10, p. 66], the minimal polynomial over $\mathbb{Q}$ of ξ is

$$\Phi_{p^2}(t) = \frac{t^{p^2} - 1}{t^p - 1} = 1 + t^p + \cdots + t^{(p-1)p}.$$

Since $\deg C < p^2$, one has $C(\xi) = 0$ if and only if $C = U \Phi_{p^2}$ for a polynomial $U(t) = \sum_{r=0}^{p-1} b_r t^r \in \mathbb{Q}[t]$. The exponents $r + jp$, with $0 \leq r, j < p$, are distinct and range from 0 to $p^2 - 1$. In this

factorization, comparison of coefficients gives $C_{r+jp} = b_r = C_r$. Conversely, these equalities give $C(t) = (\sum_{r=0}^{p-1} C_r(p; X, Y)t^r)\Phi_{p^2}(t)$. It follows that $\mathcal{F}_{X,Y}(\xi, \xi^p) = 0$ is equivalent to condition (1) of Theorem 4.2. Let

$$W = \{k(k+2) + c^2 \,:\, k, c \in \mathbb{Z},\ v \le k \le u,\ |c| \le v\} = \{w_1 < \cdots < w_s\}.$$

For $r \ge u$, put

$$p_{r,i} = \sum_{(k,c)\in \mathcal{I}_{u,v}(w_i)} \dim V_{\pi_{r,k,c}}^{\Gamma_p^+}, \qquad q_{r,i} = \sum_{(k,c)\in \mathcal{I}_{u,v}(w_i)} \dim V_{\pi_{r,k,c}}^{\Gamma_p^-},$$

where $\mathcal{I}_{u,v}(w_i)$ is defined after Lemma 2.8. The branching and multiplicity formulas of subsection 1.2 give the complete eigenvalue multisets, with parameters

$$\sigma = 1, \qquad \beta = 2, \qquad c = 4, \qquad r_0 = u, \qquad P = Q = \emptyset.$$

Indeed, $t_r^2 + w_i - 4 = r(r+4) + w_i$, and no representation occurs with $r < u$. The least and greatest offsets are

$$w_1 = v(v+2), \qquad w_s = u(u+2) + v^2, \qquad w_s - w_1 = H.$$

In particular, $t_u = u + 2 > 0$ and $t_u^2 + w_1 > 0$.

For $a \in \mathbb{Z}/q\mathbb{Z}$, define

$$\delta_i(a) = \sum_{(k,c)\in \mathcal{I}_{u,v}(w_i)} T_p(a+2, k+1, c).$$

The set $\mathcal{I}_{u,v}(w_i)$ is invariant under $c \mapsto -c$. The cosine-character identities therefore give

$$p_{r,i} - q_{r,i} = \delta_i(r \bmod q), \qquad \sum_{i=1}^{s} \delta_i(a) = B_{u,v}(a+2).$$

Each δ_i is integer-valued: every residue class contains an integer $r \ge u$, where its value is a difference of multiplicities. The period sums of T_p also give $\sum_{a \bmod q} \delta_i(a) = 0$. Both groups have odd order q and act freely, so the estimate following Lemma 2.8 applies at every attained offset:

$$\min(p_{r,i}, q_{r,i}) = \frac{C_{u,v}(w_i)}{q}\, r^4 + O(r^3), \qquad C_{u,v}(w_i) > 0.$$

Thus both complete lists are infinite. Since the prefixes are empty, Theorem 2.4 gives

$$\sup_{j\ge 1} |\lambda_j(\Delta_{u,v}^+) - \lambda_j(\Delta_{u,v}^-)| < \infty \quad \Longleftrightarrow \quad B_{u,v} = 0.$$

When these conditions hold, the absolute difference is at most H for every index. Lemma 4.3 and the preceding coefficient equalities give the equivalence of conditions 1 and 2 in Theorem 4.2.

The Casimirs $\Delta_{u,v}^{\pm}$ are nonnegative self-adjoint operators with compact resolvent on their infinite-dimensional separable complex L^2 spaces of sections. Proposition 2.7 applies to $A = \Delta_{u,v}^+$ and the complete ordered eigenvalue sequence of $\Delta_{u,v}^-$. It gives conditions 3 and 4 and a correction of norm at most H, diagonal in an eigenbasis of $\Delta_{u,v}^+$. Such a correction commutes with every spectral projection, and the perturbed operator has domain $\mathcal{D}(\Delta_{u,v}^+)$.

Finally, if bounded self-adjoint corrections K_+, K_- give a common ordered eigenvalue sequence (μ_j), then (2.18) gives

$$|\lambda_j(\Delta_{u,v}^+) - \lambda_j(\Delta_{u,v}^-)| \le |\lambda_j(\Delta_{u,v}^+) - \mu_j| + |\mu_j - \lambda_j(\Delta_{u,v}^-)| \le \|K_+\| + \|K_-\|.$$

Conversely, a correction on the plus side gives such a pair with $K_- = 0$. This proves the last assertion and completes the proof of Theorem 4.2.

At roots of unity, write $t = \xi$ and $w = \xi^p$.

4.4. **Universal families and two uniform obstructions.**

Proposition 4.4. *For every prime $p \geq 7$, bounded correction is possible on $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$. This union has $5p^2 - 4p + 40$ elements. For every prime $p \geq 7$, the types $\tau_{3,0}$ and $\tau_{2,2}$ admit no bounded self-adjoint correction. At $p = 5$, an orientation-preserving isometry identifies the two natural operators for every type, so the zero correction is possible.*

Proof. At $(t, w) = (\xi, \xi^p)$, the Dirichlet quotients satisfy $R_0(d) = 0$, $R_{-X}(d) = -R_X(d)$, and $R_{X+p^2}(d) = R_X(d)$. Hence the numerator of every $f_{d,e}(X, Y)$ vanishes when $X = 0$, $Y = 0$, or $X = \pm Y$ modulo p^2. Lemma 4.3 and Theorem 4.2 give bounded correction on the four walls. If $p \mid X, Y$, then $w^X = w^Y = 1$. Each skew difference in (4.1) therefore vanishes after every sign substitution, so $\mathcal{P}_{X,Y}(\xi, \xi^p) = 0$ on $\mathcal{G}_p$.

For the five remaining pairs, expansion in the two independent Laurent variables t, w gives

$$\mathcal{P}_{3,1} = \mathcal{P}_{5,1} = \mathcal{P}_{5,3} = \mathcal{P}_{7,1} = \mathcal{P}_{7,3} = 0.$$

These are identities in $\mathbb{Z}[t^{\pm 1}, w^{\pm 1}]$. Indeed, a monomial $jt^a w^b x^c y^d$ of J contributes $\epsilon\delta j$ at the exponent

$$(a + \epsilon X + \delta Y,\ b + \epsilon c X + \delta d Y) \qquad (\epsilon, \delta \in \{1, -1\});$$

the sums at equal exponents vanish at each displayed pair. The signed-permutation invariance proved in subsection 4.2 gives the assertion on $\mathcal{O}_{p^2}$.

The same substitutions for $(X, Y) = (9, 1)$ and $(7, 5)$ give

$$\mathcal{F}_{9,1}(t, w) = \frac{(t^4 - 1)(w^2 - 1)(w^8 - 1)(w^{10} - 1)}{2t^2 w^{10}}, \tag{4.7}$$

$$\mathcal{F}_{7,5}(t, w) = -\frac{(t^8 - 1)(w^2 - 1)(w^8 - 1)(w^{10} - 1)}{2t^4 w^{10}}. \tag{4.8}$$

If t has order p^2 and $w = t^p$ has order $p \geq 7$, all the numerator factors are nonzero. The two pairs correspond to $\tau_{3,0}$ and $\tau_{2,2}$, respectively. Their obstructions follow from Lemma 4.3 and Theorem 4.2.

The four walls meet pairwise only at the origin, since 2 is a unit modulo p^2. Their union has $4p^2 - 3$ elements. Their intersection with the grid has $4p - 3$ elements, so the grid adds $(p - 1)(p - 3)$ elements. For $p \geq 7$, the five small orbits are disjoint from the walls and grid and each has eight elements. The union thus has

$$4p^2 - 3 + (p - 1)(p - 3) + 40 = 5p^2 - 4p + 40$$

elements.

Finally, modulo 25,

$$21(1, 6, 16) = (21, 1, 11).$$

Multiplication by 21 changes the generator of the same cyclic group. Interchanging the first two complex coordinates gives the minus triple $(1, 21, 11)$. This permutation has real determinant 1, so the two groups are conjugate in SO(6). The induced unitary maps on their natural bundles intertwine the Casimir operators. □

4.5. **Vanishing in the cyclic group algebra.** Let

$$C(t) = \sum_{s=0}^{p^2-1} C_s(p; X, Y) t^s.$$

In $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$, the substitution $w = t^p$ gives $w = w^{\epsilon X} = w^{\delta Y} = 1$. Since $J(t, 1, 1, 1) = 0$, the definition of $\mathcal{P}_{X,Y}$ gives

$$C(t) \equiv \mathcal{P}_{X,Y}(t, t^p) \equiv 0 \pmod{t^p - 1}.$$

Thus, for every $0 \le r < p$,

$$\sum_{j=0}^{p-1} C_{r+jp}(p; X, Y) = 0.$$

By Theorem 4.2, bounded correction is equivalent to equality of the p coefficients in each such sum. Their common integer value must then be zero. Hence

$$\sup_j |\lambda_j(\Delta^+_{u,v}) - \lambda_j(\Delta^-_{u,v})| < \infty \quad \Longleftrightarrow \quad C_s(p; X, Y) = 0 \quad (s \bmod p^2). \tag{4.9}$$

These are equalities of integers.

4.6. **Affine cancellations over the prime field.** Write $J_0(w, x, y)$ for the coefficient of t^0 in J. Each $Q_{d,e}$ has constant term 1 in t. Expansion of the six skew differences in (4.1) gives

$$\begin{aligned} J_0(w, x, y) &= w^6 D(w^3 x, w^3 y), \\ D(z, h) &= \frac{(z-1)(z-h)(h-1)(zh-1)(zh^2-1)(z^2h-1)}{z^3h^3}. \end{aligned} \tag{4.10}$$

Here D is the Weyl denominator of G_2, with the convention $e^{(a,b)} = z^a h^b$; see [8, §19.2 and Corollary 25.5]. The simple roots are $(1, -1)$ and $(0, 1)$ for the inner product with matrix $\left(\begin{smallmatrix} 1 & -1/2 \\ -1/2 & 1 \end{smallmatrix}\right)$. The positive roots are

$$(1, -1), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1),$$

their half-sum is $(3, 2)$, and $D = e^{-(3,2)} \prod_{\alpha>0}(e^\alpha - 1)$.

Suppose that $C_s(p; X, Y) = 0$ for every $s \bmod p^2$, and set $x = X \bmod p$, $y = Y \bmod p$. For $p > 28$, two distinct affine functions

$$a + \epsilon x + \delta y, \qquad a \in \{0, 2, \dots, 28\}, \quad \epsilon, \delta \in \{1, -1\},$$

can have the same value only on one of the 116 lines

$$x = k, \quad y = k, \quad x + y = k, \quad x - y = k, \qquad -14 \le k \le 14. \tag{4.11}$$

Assume that (x, y) lies on none of these lines. The four first-residue groups with $a = 0$ are then separate from one another and from every other group. Evaluating each zero group at a primitive pth root w gives

$$J_0(w, w^{\epsilon x}, w^{\delta y}) = 0 \qquad (\epsilon, \delta \in \{1, -1\}).$$

For $(\epsilon, \delta) = (1, 1)$ and $(-1, -1)$, the first four factors of D are nonzero: their vanishing would put (x, y) on $x = \pm 3$, $y = \pm 3$, $x = y$, or $x + y = \pm 6$. The last two factors therefore give

$$\big(x + 2y = -9 \text{ or } 2x + y = -9\big), \qquad \big(x + 2y = 9 \text{ or } 2x + y = 9\big).$$

Since $18 \neq 0$ in $\mathbb{F}_p$, the two chosen equations involve different linear forms. Adding them gives $3(x + y) = 0$, contrary to the assumption $x + y \neq 0$. Thus (x, y) lies on (4.11).

The minors of orders one and two of the augmented coefficient matrix of (4.11) have absolute values at most 14 and 28, respectively. For any three normals from $(1, 0), (0, 1), (1, 1), (1, -1)$, the sum of the absolute values of their three pairwise determinants is at most 4. Expansion in the constant column therefore bounds each minor of order three by $4 \cdot 14 = 56$. For $p > 56$, reduction preserves the ranks of all coefficient and augmented submatrices. It therefore preserves the nonempty intersections of the lines and all their incidences.

Over $\mathbb{Q}$, the arrangement has 2101 intersection points. Indeed, for an integer $N \ge 1$ and the range $-N \le k \le N$, the coordinate lines give $(2N + 1)^2$ points in the square $[-N, N]^2$. On $x = k$, there are $2|k|$ diagonal intersections outside the square. The vertical–diagonal intersections therefore give $2 \sum_{k=-N}^{N} |k| = 2N(N + 1)$ further integral points. The horizontal–diagonal intersections give the same number, in a disjoint region. Diagonal lines with constants of opposite parity give $2N(N + 1)$ points

in $(\mathbb{Z}+\frac{1}{2})^2$; their remaining intersections are integral points of the square. The total is $10N(N+1)+1$, which is 2101 for $N = 14$. There is one signed-permutation orbit of size 1. The orbits of size 4 have 14 representatives on $y = 0 < x$ and 21 on $x = y > 0$: the latter have $x \in \{1, \dots, 14\}$ or $x \in \{1/2, 3/2, \dots, 13/2\}$. All remaining orbits have size 8. Hence there are $1+35+(2101-1-4\cdot 35)/8 = 281$ representatives, chosen with $x \geq y \geq 0$. The open line strata have the 30 representatives

$$x = k \quad \text{and} \quad x + y = k, \qquad 0 \leq k \leq 14,$$

with all intersection points removed.

On an open representative line, write

$$(x, y) = (a_x z + b_x, a_y z + b_y), \qquad z \in \mathbb{F}_p,$$

where $(a_x, b_x, a_y, b_y) = (0, k, 1, 0)$ or $(1, 0, -1, k)$. Choose a lift $\widehat{z} \in \mathbb{Z}/p^2\mathbb{Z}$ of z and write, with $U, V \in \mathbb{F}_p$,

$$X \equiv a_x \widehat{z} + b_x + pU, \qquad Y \equiv a_y \widehat{z} + b_y + pV \pmod{p^2}.$$

For a monomial $jt^a w^b x^c y^d$ of J and a sign pair (ϵ, δ), put

$$L(z) = (\epsilon a_x + \delta a_y)z + a + \epsilon b_x + \delta b_y.$$

Its exponent modulo p^2 is $L(\widehat{z}) + pM(z, U, V)$, where

$$M(z, U, V) = (\epsilon c a_x + \delta d a_y)z + \epsilon U + \delta V + b + \epsilon c b_x + \delta d b_y. \tag{4.12}$$

Within one first-residue group, the functions L agree identically. Their carries are therefore common and do not affect equality of the second residues. Distinct functions L do not meet on the open stratum. Changing $\widehat{z}$ merely changes U, V in the same fiber.

At a rational intersection point (x, y), both coordinate denominators divide 2. Regard x, y as elements modulo p^2 and write $X \equiv x + pU$, $Y \equiv y + pV \pmod{p^2}$, with $U, V \in \mathbb{F}_p$. Group the terms by the rational value $a + \epsilon x + \delta y$. The second-residue forms, after removing the common carry, are

$$\epsilon U + \delta V + b + \epsilon c x + \delta d y. \tag{4.13}$$

Multiplication by 2 makes their coefficients integral. For $p > 56$, distinct first-residue groups remain distinct at each point stratum, since any new equality would give a new incidence with a line of (4.11).

For a field K, a finite family G of affine forms on K^n with integer weights n_f has the zero set

$$Z_K(G) = \left\{ v \in K^n \;:\; \sum_{\substack{f \in G \\ f(v) = c}} n_f = 0 \text{ in } \mathbb{Z} \text{ for every } c \in K \right\}.$$

Here $n = 3$ on a line stratum and $n = 2$ on a point stratum. The forms are those of (4.12) or (4.13), and their integer weights are the corresponding sums of $\epsilon\delta j$. For linear congruences with a parametric prime, extended quantifier elimination gives solvability conditions and rational sample solutions; see [7, Sections 4–6, pp. 153–160].

The following recursion is over $\mathbb{Q}$. Represent an affine subspace by reduced row echelon equations, with variable order (z, U, V) on a line and (U, V) at a point. Write an affine form as its coefficient tuple, with the constant coefficient last, and order these tuples lexicographically over $\mathbb{Q}$. Restriction to a subspace means elimination of its pivot variables. For each initial stratum, identify subspaces with the same reduced row echelon equations.

Start with $S = \mathbb{Q}^n$. For each first-residue group G and each affine subspace S, restrict the forms to S, combine equal forms, and add their integer weights. If a group has nonzero total weight, discard S: the group cannot vanish at any point of S. If all weights in all groups are zero, retain S. Otherwise, for each nonzero group G, write $F_+(G, S)$ and $F_-(G, S)$ for the ordered lists of forms whose weights

are positive and negative. Both lists are nonempty, since the total weight is zero. If $|F_-| \le |F_+|$, set $f = \min F_+$ and $\mathcal{H} = F_-$; otherwise set $f = \min F_-$ and $\mathcal{H} = F_+$. Choose a nonzero group for which $(|\mathcal{H}|, f, \mathcal{H})$ is lexicographically least; ties are resolved by the original coefficient tuple of L on a line stratum, or by the original value $a + \epsilon x + \delta y$ at a point stratum. At a zero of the group, the weight of f must be cancelled by an opposite-sign weight. Hence the zero belongs to

$$S \cap \{f = g\} \qquad \text{for some } g \in \mathcal{H}.$$

Apply the same rule to these intersections, omitting the inconsistent equations. Each nonempty intersection has smaller dimension, since f and g have distinct restrictions to S. Induction on the dimension shows that the union of the retained subspaces is $\bigcap_G Z_{\mathbb{Q}}(G)$. On an open line, remove the points whose parameter belongs to an intersection of the original arrangement.

For (4.12) and (4.13), these rules give 871 distinct affine subspaces, counted separately within each initial stratum, and 5754 inconsistent intersections. Exactly 73 subspaces have all weight sums zero. On $x = 0$, their union before removal of intersection parameters is $\{U = 0\} \cup \{z = 0\}$; on $x + y = 0$, it is $\{U + V = 0\} \cup \{z = 0\}$. The other representative lines have no retained subspace. At the representative intersection points, the retained subspaces give Table 4.

Every denominator in the restricted forms and equations has its prime divisors in $\{2, 3, 5, 11\}$. The numerator and denominator of every nonzero pivot have their prime divisors in $\{2, 3, 5, 7, 11\}$. The prime divisors of the nonzero constant equations are exactly

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,$$
$$67, 71, 73, 79, 89, 97, 101, 103, 107, 109, 113, 127, 131, 157, 173\}.$$

In particular, put

$$\mathcal{S} = \{\ell : \ell \text{ is prime and } \ell \le 173\}. \tag{4.14}$$

For $p \notin \mathcal{S}$, all these divisions remain valid and all the discarded constant equations remain inconsistent over $\mathbb{F}_p$.

TABLE 4. Solutions on the representative strata and their reductions for $p > 173$.

First-residue stratum over $\mathbb{Q}$	Second-residue solutions
Open line $x = 0$	$U = 0$
Open line $x + y = 0$	$U + V = 0$
Other open representative lines	$\emptyset$
Point $(0, 0)$	All (U, V)
Points $(x, 0)$ with $x > 0$	$V = 0$
Points (x, x) with $x > 0$	$U = V$
$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3)$	$U = V = 0$
Other representative intersection points	$\emptyset$

Let B be the product of the primes in $\mathcal{S}$ and let $p > 173$. The preceding arrangement calculation shows that the original first-residue groups obtained from J are unchanged on the corresponding strata. All the affine identities above hold over $\mathbb{Z}[1/B]$; their reductions therefore hold over $\mathbb{F}_p$. Every retained subspace still has zero integer weight sums. Moreover, each intersection in the recursion reduces to the intersection of its two defining affine sets. Equal reduced row echelon systems over $\mathbb{Q}$ are equal over $\mathbb{Z}[1/B]$, so their identifications remain valid after reduction.

A group of nonzero total integer weight has no zero after reduction. In every other subspace S that has not been retained, consider a common zero of the original groups on its reduction modulo p. The vanishing conditions are equalities in $\mathbb{Z}$; only the affine forms take values in $\mathbb{F}_p$. The nonzero weight of the chosen form f thus requires an opposite-sign form $g \in \mathcal{H}$ with the same value. This remains true if further restricted forms become equal modulo p. The solution lies on the reduction of

$S \cap \{f = g\}$, which cannot be one of the discarded intersections. Induction on the dimension proves that it lies on a retained subspace.

Each retained subspace lies either in the space displayed for its stratum or, on a line stratum, in $z = z_0$ for an excluded parameter z_0. Conversely, each nonempty displayed subspace occurs among the retained subspaces. Substitution of the reduced row echelon equations into the defining equations of these containing spaces gives identities over $\mathbb{Z}[1/B]$. The containments therefore hold after reduction. Removing the excluded parameters gives Table 4 over every $\mathbb{F}_p$ with $p > 173$. After signed permutations and the lift formulas, the solution set is $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$.

4.7. **The exceptional primes.** The signed-permutation invariance proved in subsection 4.2 and the criterion (4.9) give this invariance for the common zero set of the C_s. Write $X \equiv x + pU$, $Y \equiv y + pV \pmod{p^2}$, with $0 \le x, y, U, V < p$. The base pairs have the representatives

$$0 \le y \le x \le (p-1)/2.$$

Their orbit sizes are 1 at $(0,0)$, 4 when $y = 0 < x$ or $x = y > 0$, and 8 otherwise. The induced maps of full fibers are bijections. More precisely, a sign $\eta \in \{1, -1\}$ sends

$$x \longmapsto x' \equiv \eta x \pmod{p}, \quad 0 \le x' < p, \qquad U \longmapsto \eta U + \frac{\eta x - x'}{p} \pmod{p};$$

a coordinate interchange also interchanges the two lift coordinates.

For a monomial $jt^a w^b x^c y^d$ of J and signs $\epsilon, \delta \in \{1, -1\}$, put

$$r \equiv a + \epsilon x + \delta y \pmod{p}, \quad 0 \le r < p, \qquad h = \frac{a + \epsilon x + \delta y - r}{p}.$$

Expanding its exponent modulo p^2 gives the second-residue form

$$h + b + \epsilon c x + \delta d y + \epsilon U + \delta V \pmod{p}. \tag{4.15}$$

For each representative base pair, group the forms by r and combine equal forms with their integer weights $\epsilon\delta j$. Apply the cancellation rule of subsection 4.6 over $\mathbb{F}_p$. Retain the plane if all weights vanish, and discard it if a group has nonzero total weight. Otherwise choose a form of nonzero weight in one group and equate it to the forms of opposite sign in that group. Discard nonzero constant equations; every remaining equality defines an affine line.

On each line, combine equal restricted forms. Retain the line if all weights vanish. Otherwise the same rule gives finitely many candidate parameters; retain a candidate precisely when every original group vanishes there. The completeness argument of subsection 4.6 applies over $\mathbb{F}_p$, since the weights remain integers.

For the primes $5 \le p \le 173$, there are

$$\sum_{\substack{5 \le p \le 173 \\ p \text{ prime}}} \frac{(p+1)(p+3)}{8} = 45069$$

representative fibers. The second-residue equations give the following complete zero sets. At $p = 5$ every fiber is the whole plane. For primes $17 \le p \le 173$, the nonempty representative fibers are

(x, y)	solutions in (U, V)
$(0,0)$	$\mathbb{F}_p^2$
$(x,0),\ x > 0$	$V = 0$
$(x,x),\ x > 0$	$U = V$
$(3,1), (5,1), (5,3), (7,1), (7,3)$	$\{(0,0)\}$.

At $p = 11$ or 13, replace the last two base points $(7,1), (7,3)$ by $(p-7,1), (p-7,3)$, and replace their lift $(0,0)$ by $(-1,0)$. At $p = 7$, the fibers at $(1,0)$ and $(3,0)$ are $\{U = 0\} \cup \{V = 0\}$; the fiber at $(2,0)$ is $\{V = 0\}$; the fibers at (x,x), $1 \le x \le 3$, are $\{U = V\}$; and the remaining nonzero base fibers are

$$(2,1) : \{(-1,0)\}, \qquad (3,1) : \{(0,0)\}, \qquad (3,2) : \{(0,-1)\}.$$

The fiber at $(0,0)$ is again the whole plane. It follows, after signed permutations and the lift formulas, that for every prime $11 \le p \le 173$ the zero set is $\mathcal{W}_{p^2} \cup \mathcal{G}_p \cup \mathcal{O}_{p^2}$. At $p = 7$, it consists of this union and the four signed-permutation orbits of $(14,1), (14,3), (21,1), (21,3)$.

TABLE 5. Cardinalities for primes $5 \le p \le 173$.

p	Number of solutions modulo p^2
5	625
7	289
$11 \le p \le 173$	$5p^2 - 4p + 40$

Since $\mathcal{S}$ has no prime greater than 173, the two finite eliminations prove Theorem 4.1 for every prime $p \ge 5$. The fiber orbit sizes give Table 5. For $p \ge 11$, the count is $5p^2 - 4p + 40$, as in Proposition 4.4. At $p = 7$, the four additional eight-point orbits give $257 + 32 = 289$. At $p = 5$ there are $25^2 = 625$ residue pairs.

For $p = 7$, multiplication of the generator weights by 6 sends $(1, 8, 22)$ to $(6, -1, -15)$ modulo 49. Interchanging the first two complex coordinates and conjugating the first and third then gives $(1, 6, 15)$. Conjugating the last two coordinates sends $(1, -6, -20)$ to $(1, 6, 20)$. A permutation of complex coordinates preserves real orientation, and each of the two coordinate transformations contains two complex conjugations. They therefore belong to SO(6). These identifications intertwine the natural SO(6) operators and give the pair of [17, Section 8], treated in Section 1.

5. Polynomial counting for prime-dependent affine congruences

For fixed integral arrangements, the central and affine counting formulas are given by Kamiya, Takemura and Terao [10, Lemma 2.1 and Theorem 2.4] and [11, Lemma 2.3 and Theorem 3.2]. Dolzmann and Sturm give solubility conditions for prime-power congruences and rational solutions integral at p under explicit parameter conditions [7, Sections 5–6].

Let T be an indeterminate, let $d, N \ge 1$ be integers, and fix rows

$$(a_i(T), b_i(T)) \in \mathbb{Z}[T]^d \times \mathbb{Z}[T], \qquad 1 \le i \le N.$$

For a prime p and an integer $k \ge 1$, consider the predicates

$$H_i(p, k; x) : \quad a_i(p)x = b_i(p) \pmod{p^k}, \qquad x \in (\mathbb{Z}/p^k\mathbb{Z})^d.$$

Fix a Boolean function $F : \{0,1\}^N \to \{0,1\}$, independent of p and k. A vector x satisfies F if F takes the value 1 on the truth values of these predicates.

For each fixed k, the analogous count for an integer parameter $t \ge 1$ is eventually a quasipolynomial by Bogart, Goodrick and Woods [4, Property 2 and Theorem 1.15]. Indeed, on the box $0 \le x_j < t^k$, each congruence is equivalent to

$$\exists z \in \mathbb{Z} : \qquad a_i(t)x - b_i(t) = t^k z.$$

The Boolean combination defines a parametric Presburger family with finite fibers. Theorem 5.1 gives one integral polynomial on the primes, with an explicit exceptional set independent of k.

For a nonzero polynomial $f(T) \in \mathbb{Z}[T]$, let $v = \operatorname{ord}_T f$ be its least exponent with nonzero coefficient. Thus

$$f(T) = T^v(c(f) + Tg(T)), \qquad c(f) \in \mathbb{Z} \setminus \{0\}, \quad g(T) \in \mathbb{Z}[T].$$

Let $\mathcal{M}(T)$ be the $N \times (d+1)$ matrix with rows $(a_i(T), b_i(T))$. Define the prime set

$$\mathcal{S}(\mathcal{M}) = \{p \text{ prime} : \ p \mid c(f) \text{ for some nonzero square minor } f \text{ of } \mathcal{M}\}. \tag{5.1}$$

Only minors of sizes at most $d+1$ occur, and the set is finite.

Theorem 5.1. *For every $k \geq 1$, there is a unique polynomial $P_k(Z) \in \mathbb{Z}[Z]$, of degree at most kd, such that the number of x satisfying F is $P_k(p)$ for every prime $p \notin S(\mathcal{M})$. The exceptional set is independent of k.*

Proof. Fix a prime $p \notin S(\mathcal{M})$ and an integer $k \geq 1$. Let $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ denote the additive p-adic valuation, normalized by $v_p(p) = 1$. For every nonzero minor f in (5.1),

$$v_p(f(p)) = \operatorname{ord}_T f,$$

since $p \nmid c(f) + pg(p)$. Zero minors remain zero.

Take $I \subseteq \{1, \dots, N\}$, and write the corresponding system as $A(T)x = b(T)$. For $G = A$ or $G = (A, b)$, put $r(G) = \operatorname{rank}_{\mathbb{Q}(T)} G$. The preceding identity gives $\operatorname{rank}_{\mathbb{Q}} G(p) = r(G)$. Put $v_0(G) = 0$, and let $v_j(G)$ be the minimum T-order of its nonzero $j \times j$ minors for $1 \leq j \leq r(G)$. Set

$$\alpha_j(G) = v_j(G) - v_{j-1}(G), \qquad \kappa_k(G) = \sum_{j=1}^{r(G)} \max\{k - \alpha_j(G), 0\},$$

where $1 \leq j \leq r(G)$ in the first definition and an empty sum is zero. Let s_j, $1 \leq j \leq r(G)$, be the nonzero invariant factors of $G(p)$ over $\mathbb{Z}$, ordered by divisibility. Smith normal form and the determinantal-divisor formula [25, Theorems 2.1(4) and 2.4] give

$$v_p(s_1 \cdots s_j) = v_j(G), \qquad \alpha_j(G) = v_p(s_j) \geq 0 \qquad (1 \leq j \leq r(G)).$$

The image of $G(p)$ modulo p^k therefore has cardinality [10, Lemma 2.1]

$$|\operatorname{im}(G(p) \bmod p^k)| = p^{\kappa_k(G)}.$$

Modulo p^k, the image of $A(p)$ is contained in that of $(A(p), b(p))$. The system is soluble precisely when these finite images have equal cardinality, since equality of the images is equivalent to $b(p)$ belonging to the image of $A(p)$. Every nonempty fiber of $A(p)$ is a coset of its kernel. Thus the number of solutions is

$$\begin{cases} p^{kd - \kappa_k(A)}, & \kappa_k(A) = \kappa_k((A, b)), \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Since $0 \leq \kappa_k(A) \leq kr(A) \leq kd$, each nonzero count in (5.2) is a monomial in p of degree at most kd. All exponents and the solubility condition in (5.2) are independent of p outside $S(\mathcal{M})$. For the empty subset of predicates both matrices have rank zero, $\kappa_k = 0$, and the count is p^{kd}.

For $J \subseteq \{1, \dots, N\}$, let $\mathbf{1}_J \in \{0, 1\}^N$ be its indicator vector. Put

$$c_I = \sum_{J \subseteq I} (-1)^{|I| - |J|} F(\mathbf{1}_J) \qquad (I \subseteq \{1, \dots, N\}).$$

Möbius inversion on the Boolean lattice [24, Proposition 3.7.1 and Example 3.8.3] gives

$$F(e) = \sum_{I \subseteq \{1, \dots, N\}} c_I \prod_{i \in I} e_i \qquad (e \in \{0, 1\}^N),$$

with empty product equal to 1. Substitution of the predicate indicators expresses the count as an integral linear combination of the intersection counts (5.2). This gives $P_k \in \mathbb{Z}[Z]$ of degree at most kd. Its uniqueness follows from the infinitude of primes outside the finite set $S(\mathcal{M})$. □

For each fixed prime p, the counting series is rational by a theorem of Denef [6, Theorem 4.1].

Corollary 5.2. *Let U be an indeterminate. In the formal power series ring $\mathbb{Z}[Z][[U]]$, the polynomials of Theorem 5.1 satisfy*

$$\sum_{k \geq 1} P_k(Z) U^k \in \mathbb{Z}\big[Z, U, (1 - U)^{-1}, (1 - ZU)^{-1}, \dots, (1 - Z^d U)^{-1}\big].$$

For every prime $p \notin S(\mathcal{M})$, substituting $Z = p$ gives the counting series for all moduli p^k.

Proof. Choose an integer $h \geq 0$ bounding the degrees of the nonzero entries of $\mathcal{M}(T)$, and put $M = (d+1)h$. For a subset of the predicates, write its system as $Ax = b$ and put $A' = (A, b)$. For $G = A$ or $G = A'$ of rank $r > 0$, every nonzero r-minor has degree at most rh. Hence

$$0 \leq \alpha_j(G) \leq \nu_r(G) \leq rh \leq M \qquad (1 \leq j \leq r).$$

It follows that

$$\kappa_k(G) = kr - \nu_r(G) \qquad (k > M);$$

this identity also holds for $r = 0$ by the empty-sum convention. Put $r = r(A)$. If $r(A') = r + 1$, then, for $k > M$,

$$\kappa_k(A') - \kappa_k(A) = k - \nu_{r+1}(A') + \nu_r(A) \geq k - M > 0,$$

so there are no solutions for $k > M$. If $r(A') = r$, equality of the image exponents for $k > M$ is equivalent to $\nu_r(A') = \nu_r(A)$. If these orders differ, the counts again vanish for $k > M$. If they agree, the terms with $k > M$ give

$$\sum_{k>M} Z^{k(d-r)+\nu_r(A)} U^k = \frac{Z^{(M+1)(d-r)+\nu_r(A)} U^{M+1}}{1 - Z^{d-r}U}.$$

The finitely many terms with $1 \leq k \leq M$ are polynomials in Z, U. Taking the integral linear combination with coefficients c_I gives the asserted rational form. □

For the conditions $C_s(p; X, Y) = 0$ for all $s \in \mathbb{Z}/p^2\mathbb{Z}$, with C_s defined by (4.2), Proposition A.1 bounds the exceptional primes in (5.1) by 14400. Corollary A.2 gives the corresponding five-prime interpolation criterion.

Part 3. Flat bundles and lens orbifolds

6. Flat bundles on lens spaces with two weights

Fix an odd integer $q \geq 3$, an integer $m \geq 2$, and an integer a coprime to q. Put $\zeta = e^{2\pi i/q}$ and let $C_q = \langle h \rangle$ act on $S^{2m+1} \subset \mathbb{C}^{m+1}$ by

$$h(z_1, \dots, z_m, w) = (\zeta z_1, \dots, \zeta z_m, \zeta^a w).$$

This action is free: a fixed point of h^s forces $s = 0$ or $as = 0$ modulo q, and a is a unit. Equip $Q = L(q; 1^m, a) = S^{2m+1}/C_q$ with the curvature-one quotient metric. We identify $\pi_1(Q)$ with C_q.

For a finite-dimensional unitary representation ρ of C_q, let E_ρ be the flat Hermitian bundle with holonomy ρ, and let ∇_ρ be its flat unitary connection. Put $\Delta_\rho = \nabla_\rho^* \nabla_\rho$. Write $\chi_\rho(g) = \operatorname{tr} \rho(g)$ and let $\overline{\rho}$ denote the conjugate representation. For $t \in \mathbb{Z}/q\mathbb{Z}$, denote by χ_t the one-dimensional representation $\chi_t(h^s) = \zeta^{ts}$. The eigenvalues of Δ_ρ belong to $\{k(k+2m) : k \geq 0\}$; denote their multiplicities, possibly zero, by $d_\rho(k)$ [12, Theorems 2.3 and 3.1]. For one-dimensional representations, isospectrality is equivalent to equality of the one-norm counting functions of the corresponding affine congruence lattices [12, Theorem 4.2]. Thus $P_\rho = (\Delta_\rho + m^2)^{1/2}$ has frequency $k + m$ with multiplicity $d_\rho(k)$. For two such representations ρ, σ, write $E_\eta(R) = E_\eta(R; P_\rho, P_\sigma)$ for the matching defect of their frequency multisets, as defined in Section 2.

Define

$$\mathcal{C} = \{s \in \mathbb{Z}/q\mathbb{Z} \setminus \{0\} : (a-1)s = 0 \text{ or } (a+1)s = 0\},$$
$$\mathcal{N} = (\mathbb{Z}/q\mathbb{Z} \setminus \{0\}) \setminus \mathcal{C}.$$

Let d be the round distance on S^{2m+1}. An isometry g is a Clifford translation if $x \mapsto d(x, gx)$ is constant. The elements h^s with $s \in \mathcal{C}$ are exactly the nonidentity Clifford translations in C_q. Indeed, for $(z, w) \in S^{2m+1}$,

$$\cos d\big((z, w), h^s(z, w)\big) = \cos(2\pi s/q) \sum_{j=1}^{m} |z_j|^2 + \cos(2\pi as/q)|w|^2.$$

This is independent of (z, w) precisely when $as = \pm s$ modulo q.

Remark 6.1. Put $A_\pm = \ker(a \pm 1 : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z})$. An element of $A_- \cap A_+$ is killed by 2, hence is zero. Writing $g_\pm = \gcd(q, a \pm 1)$ gives $A_\pm = \{jq/g_\pm : 0 \le j < g_\pm\}$. Thus

$$\mathcal{C} = (A_- \cup A_+) \setminus \{0\}, \qquad A_- \cap A_+ = \{0\}, \qquad |\mathcal{C}| = \gcd(q, a-1) + \gcd(q, a+1) - 2.$$

The union of two subgroups is a subgroup if and only if one contains the other. Since $A_- \cap A_+ = \{0\}$, the set $\mathcal{C} \cup \{0\}$ is a subgroup precisely when one of A_- and A_+ is trivial. For $q = 15$ and $a = 4$, these kernels are $\{0, 5, 10\}$ and $\{0, 3, 6, 9, 12\}$; their union omits $3 + 5 = 8$.

Theorem 6.2 (Flat spectral rigidity). *Let ρ, σ be finite-dimensional unitary representations of C_q, and put*

$$c_s = \operatorname{Re}\chi_\rho(h^s) - \operatorname{Re}\chi_\sigma(h^s) \qquad (s \in \mathbb{Z}/q\mathbb{Z}), \qquad c = (c_s)_{s\in\mathbb{Z}/q\mathbb{Z}}.$$

The following conditions are equivalent:

(1) There exist $\eta \in [0, 1)$ and a sequence $R_\nu \to \infty$ such that $E_\eta(R_\nu; P_\rho, P_\sigma) = o(R_\nu^m)$.

(2) $c_s = 0$ for every $s \in \mathbb{Z}/q\mathbb{Z}$.

(3) $\rho \oplus \overline{\rho} \cong \sigma \oplus \overline{\sigma}$.

(4) Δ_ρ and Δ_σ are isospectral.

Suppose that $\dim\rho = \dim\sigma$ *and c is not identically zero. Set*

$$r = \begin{cases} m+1, & c_s \neq 0 \text{ for some } s \in \mathcal{C}, \\ m, & c_s = 0 \text{ on } \mathcal{C}. \end{cases}$$

Then, for every fixed $\eta \in [0, 1)$,

$$E_\eta(R; P_\rho, P_\sigma) = \kappa_r R^r + O(R^{r-1}), \qquad \kappa_r > 0,$$

where κ_r is given by (6.6)*. For representations of equal dimension and each fixed $\eta \in [0, 1)$, $E_\eta(R) = o(R^{m+1})$ if and only if $c_s = 0$ on $\mathcal{C}$. If the dimensions differ, then $E_\eta(R) = \Theta(R^{2m+1})$.*

Proof. For $k \ge 0$, let $\mathcal{P}_k$ be the complex homogeneous polynomials of degree k on $\mathbb{R}^{2m+2}$. In the standard coordinates $x = (x_1, \dots, x_{2m+2})$, put $\mathcal{D} = \sum_{i=1}^{2m+2} \partial_{x_i}^2$, and let $\mathcal{H}_k = \ker(\mathcal{D}|_{\mathcal{P}_k})$. For $\phi \in \mathcal{H}_\ell$,

$$\mathcal{D}(|x|^{2j}\phi) = 4j(\ell + m + j)|x|^{2j-2}\phi \qquad (j \ge 1).$$

The $\mathrm{O}(2m+2)$-equivariant harmonic decomposition is

$$\mathcal{P}_k = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} |x|^{2j}\mathcal{H}_{k-2j};$$

see [9, (1.1)]. For $|z| < 1$, the character-weighted Molien formula [23, Theorem 2.1], applied to this decomposition, gives

$$\sum_{k\ge0} d_\rho(k) z^k = \frac{1-z^2}{q} \sum_{j=0}^{q-1} \frac{\operatorname{Re}\chi_\rho(h^j)}{(1-z\zeta^j)^m(1-z\zeta^{-j})^m(1-z\zeta^{aj})(1-z\zeta^{-aj})}. \tag{6.1}$$

The irreducible representations of C_q are the χ_t. Their multiplicities add under direct sums, and the terms indexed by j and $-j$ have equal denominators and conjugate numerators. Consequently, (2) implies (4).

Distinct frequencies $k + m$ are separated by one. Hence, for $0 \le \eta < 1$,

$$E_\eta(R) = \sum_{\substack{k\ge0 \\ k+m\le R}} |d_\rho(k) - d_\sigma(k)|. \tag{6.2}$$

Put $D(z) = \sum_{k\ge0}(d_\rho(k) - d_\sigma(k))z^k$. Since $c_0 = \dim\rho - \dim\sigma$, the identity contribution to $D(z)$ is

$$\frac{c_0}{q}\frac{1+z}{(1-z)^{2m+1}}.$$

Since q is odd, $\zeta^j \neq \zeta^{-j}$ for $j \neq 0$. Every nonidentity summand has poles of order at most $m+1$ and hence coefficients $O(k^m)$. Therefore

$$d_\rho(k) - d_\sigma(k) = \frac{2c_0}{q(2m)!}k^{2m} + O(k^{2m-1}).$$

If $c_0 \neq 0$, (6.2) gives

$$E_\eta(R) = \frac{2|c_0|}{q(2m+1)!}R^{2m+1} + O(R^{2m}).$$

In particular, (1) implies $c_0 = 0$. Suppose now that $c_0 = 0$ and $c \neq 0$.

The rational function D is proper, since each summand in (6.1) is $O(z^{-2m})$ at infinity. For $s \neq 0$, let q_s be the order of h^s; it is odd and at least three. On the complexification of $\mathbb{R}^{2m+2}$, the eigenvalues of h^s are $\zeta^{\pm s}$ and $\zeta^{\pm as}$, and ζ^s is a primitive q_s-th root of unity. Thus [9, Lemma 2.6] applies with root ζ^s, by cyclotomic conjugation, and local coordinate $1 - z/\zeta^s$. For $s \neq 0$, only the indices $j = \pm s, \pm a^{-1}s$ can contribute a pole at $z = \zeta^s$. If $s \in \mathcal{N}$, these four indices are distinct. The first pair has poles of order m, and the second pair has simple poles. Since $m \geq 2$ and $c_{-s} = c_s$, the coefficient of $(1-z/\zeta^s)^{-m}$ is

$$C_{s,m} = \frac{2c_s}{q(1-\zeta^{2s})^{m-1}(1-\zeta^{(a+1)s})(1-\zeta^{(1-a)s})}. \tag{6.3}$$

The denominator is nonzero by the definition of $\mathcal{N}$ and the oddness of q.

If $s \in C$, the two pairs of indices coincide. Their denominators are $(1-z\zeta^s)^{m+1}(1-z\zeta^{-s})^{m+1}$, so the coefficient of $(1-z/\zeta^s)^{-(m+1)}$ is

$$C_{s,m+1} = \frac{2c_s}{q(1-\zeta^{2s})^m}. \tag{6.4}$$

It is nonzero if and only if $c_s \neq 0$. If $c_s = 0$ on C, all summands indexed by C vanish identically. Thus, when $c \neq 0$, the largest pole order of D is the integer r in the statement.

Put $S_{m+1} = C$ and $S_m = \mathcal{N}$, and define

$$\tau_{s,m+1} = \frac{2\zeta^{sm}}{q\,m!(1-\zeta^{2s})^m} \qquad (s \in C),$$

$$\tau_{s,m} = \frac{2\zeta^{sm}}{q(m-1)!(1-\zeta^{2s})^{m-1}(1-\zeta^{(a+1)s})(1-\zeta^{(1-a)s})} \qquad (s \in \mathcal{N}).$$

$$\gamma_b = \sum_{s \in S_r} \tau_{s,r} c_s \zeta^{-sb} \qquad (b \in \mathbb{Z}/q\mathbb{Z}). \tag{6.5}$$

Write $\mu = k + m$. The coefficient of z^k in $(1-z/\zeta^s)^{-r}$ is $\zeta^{-sk}\binom{k+r-1}{r-1}$. Partial fractions therefore give

$$d_\rho(\mu - m) - d_\sigma(\mu - m) = \gamma_{\mu \bmod q}\mu^{r-1} + O(\mu^{r-2}) \qquad (\mu \in \mathbb{Z},\ \mu \geq m).$$

The set S_r is stable under sign, $\tau_{-s,r} = \overline{\tau_{s,r}}$, and $c_{-s} = c_s$; hence $\gamma_b \in \mathbb{R}$. For every $b \in \mathbb{Z}/q\mathbb{Z}$,

$$\sum_{\substack{m \leq \mu \leq R \\ \mu \equiv b\ (q)}} \mu^{r-1} = \frac{R^r}{rq} + O(R^{r-1}).$$

Since $r \geq 2$, the sum of the $O(\mu^{r-2})$ errors is $O(R^{r-1})$. The inequality $\big||x+y| - |x|\big| \leq |y|$ and (6.2) yield

$$\kappa_r = \frac{1}{rq}\sum_{b=0}^{q-1}|\gamma_b|. \tag{6.6}$$

For $s \in S_r$, finite Fourier inversion [8, Corollary 35.12] gives

$$\tau_{s,r}c_s = \frac{1}{q}\sum_{b=0}^{q-1}\gamma_b\zeta^{sb}.$$

Every $\tau_{s,r}$ is nonzero, and $c_s \neq 0$ for some $s \in S_r$. Thus some γ_b is nonzero, and $\kappa_r > 0$.

Since $r \geq m$, $\liminf_{R\to\infty} E_\eta(R)/R^m > 0$. Hence (1) implies (2). Since $\chi_{\rho\oplus\overline{\rho}} = 2\operatorname{Re}\chi_\rho$, the character criterion for finite-dimensional representations [8, Corollaries 35.6 and 35.12] gives (2)$\Longleftrightarrow$(3). Finally, (4) implies (1), which proves the equivalence. □

6.1. **Two discrepancy orders.** Let $q = 15$, $m = 2$, and $a = 4$. Then $Q = L(15; 1, 1, 4)$, and $C = \{3, 5, 6, 9, 10, 12\}$. For $\rho = \chi_1$ and $\sigma = \chi_2$, the nonzero Fourier coefficients in (6.5) are

$$\tau_{3,3}c_3 = \tau_{12,3}c_{12} = \frac{1-\sqrt{5}}{60}, \qquad \tau_{6,3}c_6 = \tau_{9,3}c_9 = \frac{1+\sqrt{5}}{60}.$$

Thus γ_b is 5-periodic, with

$$(\gamma_0, \dots, \gamma_4) = (1/15, -1/10, 1/15, 1/15, -1/10).$$

It follows that $\sum_{b=0}^{14} |\gamma_b| = 6/5$ and

$$E_\eta(R) = \frac{2}{75}R^3 + O(R^2) \qquad (0 \leq \eta < 1).$$

Let

$$\rho = \chi_0^{\oplus 4} \oplus \chi_2 \oplus \chi_7 \oplus \chi_8 \oplus \chi_{13},$$
$$\sigma = \chi_3^{\oplus 2} \oplus \chi_{12}^{\oplus 2} \oplus \chi_5^{\oplus 2} \oplus \chi_{10}^{\oplus 2}.$$

Both representations have dimension 8, and

$$c_s = (2 - 2\cos(2\pi \cdot 3s/15))(2 - 2\cos(2\pi \cdot 5s/15)).$$

This vanishes on C. For $s \in \mathcal{N}$,

$$\tau_{s,2}c_s = \frac{2}{15}\zeta^{2s}(1-\zeta^{3s})\sum_{j=0}^{4}\zeta^{2js}.$$

The expression on the right vanishes also for $s \in C \cup \{0\}$. Fourier inversion gives $\gamma_b = 2$ for $b = 2, 4, 6, 8, 10$, $\gamma_b = -2$ for $b = 5, 7, 9, 11, 13$, and $\gamma_b = 0$ at the other residues modulo 15. Thus $\sum_{b=0}^{14} |\gamma_b| = 20$, and

$$E_\eta(R) = \frac{2}{3}R^2 + O(R) \qquad (0 \leq \eta < 1).$$

7. The three-dimensional flat-bundle kernel

Let $q \geq 3$ be an integer, let a be a unit modulo q, and put $\zeta = e^{2\pi i/q}$. Let $C_q = \langle h \rangle$ act on $S^3 \subset \mathbb{C}^2$ by

$$h(z, w) = (\zeta z, \zeta^a w).$$

Equip $L(q; 1, a) = S^3/C_q$ with the curvature-one quotient metric. For a finite-dimensional unitary representation ρ of C_q, let Δ_ρ be its nonnegative flat connection Laplacian. Its eigenvalues belong to $\{k(k+2) : k \in \mathbb{Z}_{\geq 0}\}$; write $d_\rho(k)$ for their multiplicities, possibly zero [12, Theorems 2.3 and 3.1]. Thus $P_\rho = (\Delta_\rho + 1)^{1/2}$ has frequency $k+1$ with multiplicity $d_\rho(k)$. Ranks and multiplicities are taken over $\mathbb{C}$. Write $\chi_\rho = \operatorname{tr}\rho$ and $\chi_b(h^s) = \zeta^{bs}$ for $b, s \in \mathbb{Z}/q\mathbb{Z}$. For two such representations ρ, σ, put

$$c = \operatorname{Re}\chi_\rho - \operatorname{Re}\chi_\sigma, \qquad c_s = c(h^s), \qquad t_s = \cos(2\pi s/q),$$
$$b_k = d_\rho(k) - d_\sigma(k) \qquad (k \geq 0).$$

In particular $c_0 = \dim_{\mathbb{C}}\rho - \dim_{\mathbb{C}}\sigma$ and $c_{-s} = c_s$. Write $E_\eta(R) = E_\eta(R; P_\rho, P_\sigma)$ for the matching defect of Section 2. Since distinct frequencies differ by at least one,

$$E_\eta(R) = \sum_{\substack{\mu\in\mathbb{Z}\\ 1\leq\mu\leq R}} |b_{\mu-1}| \qquad (0 \leq \eta < 1).$$

Define

$$\mathcal{C} = \{s \in \mathbb{Z}/q\mathbb{Z} \setminus \{0\} \,:\, (a-1)s = 0 \text{ or } (a+1)s = 0\},$$
$$\mathcal{N} = (\mathbb{Z}/q\mathbb{Z} \setminus \{0\}) \setminus \mathcal{C}.$$

These sets are invariant under multiplication by a and under sign. Moreover, $t_s - t_{as} \neq 0$ precisely for $s \in \mathcal{N}$. Let ι be the involution $X \mapsto X^{-1}$ of $\mathbb{Q}[X]/(X^q - 1)$, and put

$$\mathscr{R}_q = \big(\mathbb{Q}[X]/(X^q-1)\big)^{\iota}, \qquad \mathscr{R}_q^0 = \{f \in \mathscr{R}_q \,:\, f(1) = 0\}.$$

Evaluation gives a $\mathbb{Q}$-algebra isomorphism

$$\mathscr{R}_q \longrightarrow \mathcal{R}_q := \sum_{b \in \mathbb{Z}/q\mathbb{Z}} \mathbb{Q}\,\mathrm{Re}\,\chi_b, \qquad f \longmapsto \big(h^s \longmapsto f(\zeta^s)\big).$$

Indeed, $(X^b + X^{-b})/2$ maps to $\mathrm{Re}\,\chi_b$, and evaluation at all qth roots of unity is injective on $\mathbb{Q}[X]/(X^q - 1)$. Under this identification, $\mathscr{R}_q^0$ corresponds to $\mathcal{R}_q^0 = \{c \in \mathcal{R}_q \,:\, c_0 = 0\}$, and augmentation sends $\mathrm{Re}\,\chi_\rho$ to $\dim_{\mathbb{C}} \rho$.

Put $D_s(z) = 1 - 2t_s z + z^2$. The harmonic decomposition on $\mathbb{R}^4$ [9, (1.1)] and the character-weighted Molien formula [23, Theorem 2.1] give, for $|z| < 1$,

$$B(z) := \sum_{k \geq 0} b_k z^k = \frac{1 - z^2}{q} \sum_{s=0}^{q-1} \frac{c_s}{D_s(z) D_{as}(z)}. \tag{7.1}$$

The terms indexed by s and $-s$ have equal denominators and conjugate character values. This gives the real-character numerator also when q is even, since characters are real at self-inverse elements. For arbitrary $c \in \mathcal{R}_q$, the coefficients on the right define a $\mathbb{Q}$-linear map $S_{q,a} : \mathcal{R}_q \to \mathbb{Q}^{\mathbb{Z}_{\geq 0}}$. They are rational because $S_{q,a}(\mathrm{Re}\,\chi_b) = (d_{\chi_b}(k))_{k \geq 0}$ and the $\mathrm{Re}\,\chi_b$ span $\mathcal{R}_q$.

Theorem 7.1 (The three-dimensional spectral kernel). *Suppose that q is odd. The flat connection Laplacians $\Delta_\rho, \Delta_\sigma$ on $L(q;1,a)$ are isospectral if and only if their ranks agree and*

$$c_s = 0 \quad (s \in \mathcal{C}), \qquad \frac{c_s}{t_s - t_{as}} = \frac{c_{a^{-1}s}}{t_{a^{-1}s} - t_s} \quad (s \in \mathcal{N}).$$

For each divisor $d > 1$ of q, put $\zeta_d = e^{2\pi i/d}$, $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, and let σ_a be the automorphism of K_d given by $\sigma_a(\zeta_d + \zeta_d^{-1}) = \zeta_d^a + \zeta_d^{-a}$. Under the cyclotomic decomposition of $\mathscr{R}_q^0$, the kernel of $S_{q,a}$ is the product, over $d \mid q$, $d > 1$, of

$$\begin{cases} \{0\}, & a \equiv \pm 1 \pmod{d}, \\ \big(\cos(2\pi/d) - \cos(2\pi a/d)\big) K_d^{\langle \sigma_a \rangle}, & a \not\equiv \pm 1 \pmod{d}. \end{cases}$$

For a finite group G and $g \in G$, write $\mathrm{ord}_G(g)$ for the order of g; put $\varphi(d) = |(\mathbb{Z}/d\mathbb{Z})^\times|$. The dimension of the kernel over $\mathbb{Q}$ is

$$\sum_{\substack{d \mid q,\ d > 1 \\ a \not\equiv \pm 1 \pmod{d}}} \frac{\varphi(d)}{2\,\mathrm{ord}_{(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}}(a)}.$$

For every fixed $0 \leq \eta < 1$, the matching defect satisfies exactly one of the following alternatives:

(1) If $c_0 \neq 0$, then $E_\eta(R) = \Theta(R^3)$.

(2) If $c_0 = 0$ and c is nonzero on $\mathcal{C}$, then $E_\eta(R) = \kappa_2 R^2 + O(R)$ with $\kappa_2 > 0$.

(3) If $c_0 = 0$, c vanishes on $\mathcal{C}$, and $c \notin \ker S_{q,a}$, then $E_\eta(R) = \kappa_1 R + O(1)$ with $\kappa_1 > 0$.

(4) If $c \in \ker S_{q,a}$, then $E_\eta(R) = 0$ for every R.

In the third case $b_{k+q} = b_k$ for every $k \geq 0$, and $\kappa_1 = q^{-1} \sum_{k=0}^{q-1} |b_k|$. In the second case,

$$\gamma_b = \sum_{s \in \mathcal{C}} \frac{2\zeta^s c_s}{q(1 - \zeta^{2s})} \zeta^{-sb}, \qquad \kappa_2 = \frac{1}{2q} \sum_{b=0}^{q-1} |\gamma_b|.$$

Here $b \in \mathbb{Z}/q\mathbb{Z}$, and every γ_b is real. Consequently, $E_\eta(R_\nu) = o(R_\nu)$ for a sequence $R_\nu \to +\infty$ if and only if the two Laplacians are isospectral.

Proof. In (7.1), the identity term is $c_0(1+z)/(q(1-z)^3)$, whereas the nonidentity terms are regular at $z = 1$. Thus $B = 0$ implies $c_0 = 0$. Assume $c_0 = 0$. For $s \in \mathcal{C}$, one has $a^{-1}s = \pm s$. Only the terms indexed by s and $-s$ can have a pole at $z = \zeta^s$, and the coefficient of $(1 - z\zeta^{-s})^{-2}$ is

$$C_{s,2} = \frac{2c_s}{q(1-\zeta^{2s})}.$$

Its denominator is nonzero since q is odd. Hence $B = 0$ forces $c_s = 0$ on $\mathcal{C}$. Conversely, these equalities remove all collision summands.

Suppose that c vanishes on $\mathcal{C}$, and put $u_s = c_s/(t_s - t_{as})$ for $s \in \mathcal{N}$. The identity

$$\frac{c_s}{D_s D_{as}} = \frac{u_s}{2z}\left(\frac{1}{D_s} - \frac{1}{D_{as}}\right)$$

and the change of index $s \mapsto as$ give

$$B(z) = \frac{1-z^2}{2qz} \sum_{s \in \mathcal{N}} \frac{u_s - u_{a^{-1}s}}{D_s(z)}. \tag{7.2}$$

Two factors $D_s, D_{s'}$ have a common root if and only if $s' = \pm s$. Moreover $u_{-s} = u_s$, and $\zeta^s \notin \{0, 1, -1\}$ for $s \neq 0$. Taking the coefficient of the simple pole at each ζ^s in (7.2) therefore gives $B = 0$ if and only if $u_s = u_{a^{-1}s}$ on $\mathcal{N}$.

The Chinese remainder theorem, applied to the distinct cyclotomic factors of $X^q - 1$, gives

$$\mathscr{R}_q \cong \mathbb{Q} \times \prod_{\substack{d \mid q \\ d>1}} K_d, \qquad f \longmapsto \big(f(1), (f(\zeta_d))_{d \mid q,\ d>1}\big);$$

inversion acts on each cyclotomic field as complex conjugation [21, Theorems 5.10 and 8.1]. The first factor is augmentation. For $f \in \mathscr{R}_q$ corresponding to c, put $f_d = f(\zeta_d) = c_{q/d}$. If v is a unit modulo d, then $c_{vq/d} = \sigma_v(f_d)$, where $\sigma_v(\zeta_d + \zeta_d^{-1}) = \zeta_d^v + \zeta_d^{-v}$. If $a \equiv \pm 1 \pmod{d}$, the indices of order d lie in $\mathcal{C}$, so $f_d = 0$. Otherwise put $\delta_d = \cos(2\pi/d) - \cos(2\pi a/d) \neq 0$. Then

$$u_{vq/d} = \sigma_v(f_d/\delta_d).$$

The equations $u_s = u_{a^{-1}s}$ on the indices of order d are therefore equivalent to $f_d/\delta_d \in K_d^{\langle \sigma_a \rangle}$. By [21, Lemma 5.9 and Theorem 3.17(d)], $\mathrm{Gal}(K_d/\mathbb{Q}) \cong (\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$. The order of σ_a is the order of a in this quotient. The degree formula [21, Theorem 3.17(b)] therefore gives the stated dimension. The real parts of virtual characters in the kernel are precisely its intersection with $\sum_{b \in \mathbb{Z}/q\mathbb{Z}} \mathbb{Z}\,\mathrm{Re}\,\chi_b$.

Still assuming $c_0 = 0$, the rational function B is $O(z^{-2})$ at infinity and has at most double poles at qth roots of unity. Partial fractions give the exact identity

$$b_{\mu-1} = \mu\gamma_{\mu \bmod q} + \beta_{\mu \bmod q} \qquad (\mu \geq 1),$$

where β is a real q-periodic sequence. The coefficient $C_{s,2}$ gives the displayed formula for γ; pairing s with $-s$ shows that $\gamma_b \in \mathbb{R}$. Finite Fourier inversion [8, Corollary 35.12] shows that $\gamma \neq 0$ precisely when c is nonzero on $\mathcal{C}$. For each residue b modulo q,

$$\sum_{\substack{1 \leq \mu \leq R \\ \mu \equiv b \ (q)}} \mu = \frac{R^2}{2q} + O(R).$$

The inequality $\big|\|\mu\gamma_b + \beta_b| - \mu|\gamma_b\big| \leq |\beta_b|$ therefore proves the second alternative and its leading coefficient. If $\gamma = 0$, then $b_{k+q} = b_k$ for $k \geq 0$. Summing one period proves the third and fourth alternatives. If $c_0 \neq 0$, the identity term has coefficient $c_0(k+1)^2/q$, while the remaining coefficients are $O(k+1)$. Thus

$$E_\eta(R) = \frac{|c_0|}{3q} R^3 + O(R^2).$$

These alternatives also prove the final equivalence. □

7.1. **Prime orders and Gaussian periods.** Let $q = p \geq 5$ be prime and let $a \in \mathbb{F}_p^\times \setminus \{1, -1\}$. Since $a - 1$ and $a + 1$ are nonzero in $\mathbb{F}_p$, the collision set is empty. Theorem 7.1 shows that equal-rank twists are isospectral precisely when $u_s = c_s/(t_s - t_{as})$ is constant on the a-orbits in $\mathbb{F}_p^\times/\{\pm 1\}$. For every equal-rank pair, $b_{k+p} = b_k$ for $k \geq 0$, and

$$E_\eta(R) = \kappa R + O(1), \qquad \kappa = \frac{1}{p}\sum_{k=0}^{p-1} |b_k| \qquad (0 \leq \eta < 1).$$

Put

$$K = \mathbb{Q}(\zeta + \zeta^{-1}), \qquad M = (p-1)/2, \qquad \tau = \operatorname{ord}_{\mathbb{F}_p^\times/\{\pm 1\}}(a), \qquad r = M/\tau.$$

Corollary 7.2. *Evaluation $c \mapsto c_1 = c(h)$ identifies $\mathcal{R}_p^0$ with K. Under this identification the spectral kernel is*

$$(t_1 - t_a)K^{\langle \sigma_a \rangle}, \qquad \sigma_a(\zeta + \zeta^{-1}) = \zeta^a + \zeta^{-a}. \tag{7.3}$$

Its dimension over $\mathbb{Q}$ is r. Let $H = \langle -1, a \rangle \subset \mathbb{F}_p^\times$; then $|H| = 2\tau$. For each multiplicative coset O of H, put

$$V_O = (\chi_1 + \chi_{-1} - \chi_a - \chi_{-a}) \sum_{b \in O} \chi_b.$$

These r real-valued virtual characters have rank zero and form a $\mathbb{Q}$-basis of the spectral kernel. The real parts of virtual characters in this kernel form its intersection with $\sum_{b \in \mathbb{F}_p} \mathbb{Z} \operatorname{Re} \chi_b$.

Proof. For $q = p$, the decomposition in Theorem 7.1 is $\mathscr{R}_p \cong \mathbb{Q} \times K$. Its first coordinate is c_0 and its second is c_1. The theorem therefore gives (7.3) and its dimension $M/\tau = r$.

Since H contains -1 and a, the Gaussian periods $\eta_O = \sum_{b \in O} \zeta^b$ lie in $K^{\langle \sigma_a \rangle}$. Represent each nonzero residue by an integer $1 \leq b \leq p - 1$. If $\sum_O \lambda_O \eta_O = 0$ with $\lambda_O \in \mathbb{Q}$, then $P(X) = \sum_O \lambda_O \sum_{b \in O} X^b$ has degree at most $p - 1$, has constant term zero, and vanishes at ζ. By [21, Theorem 5.10], the minimal polynomial is $\Phi_p(X) = 1 + X + \cdots + X^{p-1}$. Hence P is a scalar multiple of Φ_p; its constant term gives $P = 0$ and every $\lambda_O = 0$. There are r cosets, hence these periods form a basis of the fixed field. Each V_O is real-valued, has rank zero, and evaluates at h to $2(t_1 - t_a)\eta_O$. Since $t_1 - t_a \neq 0$, the virtual characters V_O form a basis. □

7.2. **Isospectral twists and real holonomy.** The case $a = 2$ for odd q is contained in [12, Example 4.10 and Theorem 4.2].

Corollary 7.3. *For every integer $q \geq 3$ and every unit $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, the flat line Laplacians on the round lens space $L(q; 1, a)$ twisted by χ_1 and χ_a are isospectral. If $a \not\equiv \pm 1 \pmod{q}$, their realified holonomy representations of C_q are inequivalent.*

Proof. For this pair, $c_s = t_s - t_{as}$. For every $s \in \mathbb{Z}/q\mathbb{Z}$,

$$\frac{t_s - t_{as}}{D_s D_{as}} = \frac{1}{2z}\left(\frac{1}{D_s} - \frac{1}{D_{as}}\right).$$

Multiplication by a permutes $\mathbb{Z}/q\mathbb{Z}$, so summing gives zero. Equation (7.1), which is valid also for even q, proves isospectrality. The realified holonomy representations have characters $2t_s$ and $2t_{as}$; these coincide for every s if and only if $a \equiv \pm 1 \pmod{q}$. □

On $L(13; 1, 3)$, the class of 3 has order 3 modulo sign, so the kernel has dimension $6/3 = 2$. One element is $\operatorname{Re} \chi_1 - \operatorname{Re} \chi_3$. A second is

$$\operatorname{Re} \chi_2 + \operatorname{Re} \chi_3 + \operatorname{Re} \chi_5 - \operatorname{Re} \chi_1 - 2 \operatorname{Re} \chi_6. \tag{7.4}$$

Indeed,

$$2(t_1 - t_3)(t_1 + t_3 + t_4) = t_2 + t_3 + t_5 - t_1 - 2t_6,$$

by $2t_it_j = t_{i+j} + t_{i-j}$, and σ_3 permutes t_1, t_3, t_4. The first element is nonzero; the coefficients of χ_2 in the two character expansions are 0 and $1/2$. Hence the elements are linearly independent. Consequently the complex rank-three bundles

$$\rho = \chi_2 \oplus \chi_3 \oplus \chi_5, \qquad \sigma = \chi_1 \oplus \chi_6 \oplus \chi_6$$

are isospectral and have inequivalent realified holonomy.

7.3. **Three-dimensional discrepancy coefficients.** Fix $0 \le \eta < 1$. On $L(9;1,2)$, the collision indices are $3, 6$. For the ordered pair of twists (χ_0, χ_1), $c_3 = c_6 = 3/2$, so $\gamma_b = 2\sin(2\pi b/3)/(3\sqrt{3})$. Its values modulo 3 are $(0, 1/3, -1/3)$, and $\sum_{b=0}^{8} |\gamma_b| = 2$. Hence

$$E_\eta(R) = \frac{1}{9}R^2 + O(R).$$

For the ordered pair (χ_1, χ_4) on the same space, c vanishes on C. The numbers t_1, t_2, t_4 are the roots of $8T^3 - 6T + 1$, so $t_1 + t_2 + t_4 = 0$ and $t_1^2 + t_2^2 + t_4^2 = 3/2$. Consequently

$$D_1D_2D_4 = 1 + z^3 + z^6,$$
$$2\big((t_1 - t_4)D_4 + (t_2 - t_1)D_1 + (t_4 - t_2)D_2\big) = 9z.$$

Pairing the terms indexed by s and $-s$ in (7.1) therefore gives

$$\sum_{k\ge 0} b_k z^k = \frac{z(1-z^2)}{1+z^3+z^6} = \frac{z - z^3 - z^4 + z^6}{1 - z^9}.$$

The multiplicity difference, indexed by the frequency μ, is

$$(b_{\mu-1})_{\mu \bmod 9} = (0, 0, 1, 0, -1, -1, 0, 1, 0),$$

where the entries are ordered by the residues $0, \dots, 8$. Thus

$$E_\eta(R) = \frac{4}{9}R + O(1).$$

On $L(15;1,4)$, the ordered pair (χ_1, χ_2) satisfies $c_3 = c_{12} = \sqrt{5}/2$, $c_6 = c_9 = -\sqrt{5}/2$, and $c_5 = c_{10} = 0$. The leading sequence is

$$\gamma_b = \frac{\sqrt{5}}{15}\left(\frac{\sin(2\pi b/5)}{\sin(2\pi/5)} - \frac{\sin(4\pi b/5)}{\sin(4\pi/5)}\right).$$

Its values modulo 5 are $(0, 0, 1/3, -1/3, 0)$, so $\sum_{b=0}^{14} |\gamma_b| = 2$ and

$$E_\eta(R) = \frac{1}{15}R^2 + O(R).$$

8. Prime-square lens orbifolds and corresponding covers

Lauret proved that scalar-isospectral lens manifolds have scalar-isospectral corresponding covers [14, Theorem 4.3]. The extension to lens orbifolds was proposed in [14, Remark 4.4] and restated as [18, Conjecture 7.3]. At orders 9 and 25, the corresponding proper covers of scalar-isospectral effective lens orbifolds are isometric (Theorem 8.5). In dimension three, scalar isospectrality already implies isometry for every group order [3, Theorem 3.1]. All metrics in this section are round quotient metrics.

8.1. **The Molien series and the isotropy filtration.** Let p be prime, let $q = p^2$, and let $n \geq 1$. For every positive integer m, put $\zeta_m = e^{2\pi i/m}$. For $w = (w_1, \dots, w_n) \in (\mathbb{Z}/q\mathbb{Z})^n$, let $L(q; w)$ denote the quotient of the unit sphere in $\mathbb{C}^n$ by the cyclic action generated by

$$g_w(z_1, \dots, z_n) = (\zeta_q^{w_1} z_1, \dots, \zeta_q^{w_n} z_n).$$

Assume that the action is effective, equivalently $\gcd(q, w_1, \dots, w_n) = 1$. Weights equal to zero are permitted. For $d \mid q$, the subgroup $\langle g_w^d \rangle$ has order q/d and quotient $L(q/d; w)$, with weights reduced modulo q/d. For a pair of lens orbifolds, the quotients obtained with the same d are their corresponding covers; these are proper when $d > 1$. Write

$$\begin{aligned} b &= \#\{j \,:\, w_j = 0 \pmod{p^2}\}, \\ s &= \#\{j \,:\, p \mid w_j,\ w_j \neq 0 \pmod{p^2}\}, \\ r &= \#\{j \,:\, p \nmid w_j\}. \end{aligned} \tag{8.1}$$

Thus $n = b + s + r$ and $r \geq 1$. Each of b, s, r counts complex coordinate lines; the corresponding real dimensions are $2b, 2s, 2r$. Put $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and let $U = (u_1, \dots, u_r) \in (\mathbb{F}_p^\times)^r$ be the reductions modulo p of the unit weights.

The Hilbert series of invariant complex-valued polynomials on $\mathbb{R}^{2n}$ is

$$H_w(t) = \frac{1}{p^2} \sum_{k=0}^{p^2-1} \prod_{j=1}^{n} \frac{1}{(1 - t\zeta_{p^2}^{kw_j})(1 - t\zeta_{p^2}^{-kw_j})}. \tag{8.2}$$

By the harmonic decomposition and Molien formula [1, Sections 2.1–2.2, (2.1)–(2.3)], the scalar multiplicities have generating series $(1-t^2)H_w(t)$, where harmonic degree k has eigenvalue $k(k+2n-2)$. Weyl's law determines the dimension and volume, hence n and q, since $\operatorname{vol} L(q; w) = q^{-1} \operatorname{vol} S^{2n-1}$ [1, (1.1) and Remark 2.1]. The eigenvalues $k(k + 2n - 2)$ are strictly increasing for $k \geq 0$. Thus scalar isospectrality is equivalent to equality of the Hilbert series.

Define

$$R_U(t) = \sum_{a=1}^{p-1} \prod_{j=1}^{r} \frac{1}{(1 - t\zeta_p^{au_j})(1 - t\zeta_p^{-au_j})}. \tag{8.3}$$

Separate in (8.2) the identity, the elements of order p, and the elements of order p^2. This gives

$$p^2(1-t)^{2b} H_w(t) = (1-t)^{-2(r+s)} + (1-t)^{-2s} R_U(t) + T_w(t), \tag{8.4}$$

where T_w is the sum over k prime to p after deleting the b zero-weight coordinate lines. The function T_w is analytic at $t = 1$: for such k, every nonzero weight w_j satisfies $kw_j \neq 0 \pmod{p^2}$.

Donnelly's equivariant heat expansion [3, Theorem 6.2], together with the multiplicity of invariant linear polynomials, gives the following spectral invariant.

Theorem 8.1. *The scalar spectrum of an effective order-p^2 lens orbifold determines b, s, r. In particular, scalar-isospectral lens orbifolds $L(p^2; w)$ and $L(p^2; w')$ have equal multisets*

$$\{\!\{\gcd(p^2, w_j) \,:\, 1 \leq j \leq n\}\!\} = \{\!\{1^r, p^s, (p^2)^b\}\!\}. \tag{8.5}$$

Proof. The coefficient of t in H_w is the dimension of the invariant linear polynomials, namely $2b$. Thus b is determined. Remove the known identity contribution and set

$$K_w(t) = p^2(1-t)^{2b} H_w(t) - (1-t)^{-2(n-b)}. \tag{8.6}$$

By (8.4),

$$K_w(t) = (1-t)^{-2s} R_U(t) + T_w(t).$$

The function R_U is analytic at 1, and

$$R_U(1) = \sum_{a=1}^{p-1} \prod_{j=1}^{r} \frac{1}{4\sin^2(\pi a u_j/p)} > 0. \tag{8.7}$$

If $s > 0$, K_w therefore has a pole of exact order $2s$ at 1. If $s = 0$, it is analytic there. This determines s, and then $r = n - b - s$. □

8.2. **The reciprocal residual polynomial.** The order-p subgroup is generated by g_w^p. Its weight list consists of U and $b+s$ zero weights, so the Hilbert series of its spherical quotient is

$$H_w^{(p)}(t) = \frac{1}{p}\left[(1-t)^{-2n} + (1-t)^{-2(b+s)} R_U(t)\right]. \tag{8.8}$$

Suppose $H_w = H_{w'}$. Theorem 8.1 gives common values of r, s, b. Write $\Delta R = R_U - R_{U'}$ and $\Delta T = T_w - T_{w'}$. Equation (8.4) implies

$$\Delta R(t) = -(1-t)^{2s} \Delta T(t). \tag{8.9}$$

In particular, if $s > 0$, the function ΔR has a zero of order at least $2s$ at $t = 1$.

Theorem 8.2. *Let p be odd, and let $L(p^2; w)$ and $L(p^2; w')$ be scalar-isospectral effective lens orbifolds with $s > 0$. Put $\Phi_p(t) = 1 + t + \cdots + t^{p-1}$ and*

$$E = r(p-3) - 2s - 4.$$

There is a polynomial $P \in \mathbb{Q}[t]$ such that

$$\Delta R(t) = \frac{t^2(1-t)^{2s} P(t)}{\Phi_p(t)^r}, \qquad P(t) = t^E P(1/t). \tag{8.10}$$

If $E < 0$, then $P = 0$. If $P \neq 0$, then $E \geq 0$ and $\deg P \leq E$. In particular, the corresponding order-p covers are scalar isospectral whenever

$$2s + 4 > r(p-3). \tag{8.11}$$

Their Hilbert-series difference is always

$$H_w^{(p)}(t) - H_{w'}^{(p)}(t) = \frac{t^2 P(t)}{p(1-t)^{2b} \Phi_p(t)^r}. \tag{8.12}$$

Proof. Every root of each denominator in (8.3) is a primitive p-th root. The two roots contributed by one coordinate line are distinct, since p is odd and $u_j \neq 0$. The multiplicity at a fixed primitive root is at most r. Hence

$$N(t) = \Phi_p(t)^r \Delta R(t)$$

is a polynomial. The Galois group of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ permutes the summands of R_U, so $N \in \mathbb{Q}[t]$. Repeated weights are counted with multiplicity in the bound r.

Put $D = r(p-3)$. Each summand in (8.3) is $O(t^{-2r})$ at infinity, so

$$\deg N \leq r(p-1) - 2r = D \qquad (N \neq 0). \tag{8.13}$$

Each paired denominator is reciprocal. Consequently

$$\Delta R(1/t) = t^{2r} \Delta R(t), \qquad N(t) = t^D N(1/t).$$

Moreover,

$$R_U(0) = p - 1, \qquad R_U'(0) = \sum_{j=1}^{r} \sum_{a=1}^{p-1} (\zeta_p^{au_j} + \zeta_p^{-au_j}) = -2r. \tag{8.14}$$

The same identities hold for U', so t^2 divides N.

Since $\Phi_p(1) = p$, equation (8.9) implies that $(1-t)^{2s}$ divides N. The two factors t^2 and $(1-t)^{2s}$ are coprime, and hence $N = t^2(1-t)^{2s} P$ for $P \in \mathbb{Q}[t]$. Substitution into the reciprocity identity for N gives $P(t) = t^{D-2s-4} P(1/t)$. If $P \neq 0$, let m and d be its least and greatest exponents with nonzero coefficients. Comparison of the least exponents gives

$$m + d = D - 2s - 4 = E.$$

Since $0 \leq m \leq d = \deg P$, one has $E \geq 0$ and $\deg P \leq E$. This proves (8.10) and (8.11). Finally, (8.8) gives (8.12). □

Remark 8.3. Let $L(p^2; w)$ and $L(p^2; w')$ be scalar-isospectral effective lens orbifolds, with p odd. For $c \in \mathbb{F}_p^\times/\{\pm 1\}$, put

$$m_c(U) = \#\{j : u_j \in c\}, \qquad M = \max_c\{m_c(U), m_c(U')\}.$$

Thus $m_c(U)$ is the multiplicity of the nontrivial irreducible real representation indexed by c. Multiplication by $a \in \mathbb{F}_p^\times$ permutes these classes, so each summand denominator in (8.3) divides Φ_p^M. Consequently $N_M = \Phi_p^M \Delta R \in \mathbb{Q}[t]$ satisfies

$$\deg N_M \le M(p-1) - 2r \quad (N_M \neq 0), \qquad N_M(t) = t^{M(p-1)-2r} N_M(1/t).$$

Equations (8.9) and (8.14) give $t^2(1-t)^{2s} \mid N_M$. Hence there is $P_M \in \mathbb{Q}[t]$ with

$$\Delta R(t) = \frac{t^2(1-t)^{2s} P_M(t)}{\Phi_p(t)^M}, \qquad P_M(t) = t^{E_M} P_M(1/t),$$

where $E = r(p-3) - 2s - 4$ and

$$E_M = M(p-1) - 2r - 2s - 4 = E - (r-M)(p-1) \le E.$$

The comparison of exponents in the proof of Theorem 8.2 gives $P_M = 0$ when $E_M < 0$, hence $\Delta R = 0$. When $E = 0$ in (8.10), P is constant.

Proposition 8.4. *If $s = 0$, scalar-isospectral effective order-p^2 lens orbifolds have scalar-isospectral corresponding order-p covers.*

Proof. Delete the b zero weights and denote the remaining lists by v, v'. All their entries are units modulo p^2, and

$$H_v(t) = (1-t)^{2b} H_w(t) = (1-t)^{2b} H_{w'}(t) = H_{v'}(t).$$

If $r \ge 2$, the lens manifolds $L(p^2; v)$ and $L(p^2; v')$ are therefore scalar isospectral. By [14, Theorem 4.3], their corresponding order-p covers are scalar isospectral. Reinserting the zero weights gives $H_w^{(p)} = (1-t)^{-2b} H_v^{(p)}$ and the same identity for w', so the original corresponding covers are scalar isospectral. If $r = 1$, multiplication by u_1 permutes $\mathbb{F}_p^\times$, so (8.3) is independent of u_1. Then $R_U = R_{U'}$, and (8.8) gives the conclusion. □

8.3. **Orders nine and twenty-five.**

Theorem 8.5 (Isometric corresponding covers). *Let $q = 9$ or $q = 25$. If two effective lens orbifolds $L(q; w)$ and $L(q; w')$ are scalar isospectral, then their corresponding proper orbifold covers are isometric. In particular, the corresponding-cover conjecture holds for these group orders in every odd dimension.*

Proof. Write $q = p^2$, with $p \in \{3, 5\}$. The trivial subgroup gives the common round sphere. It remains to consider the subgroup of order p. When $p = 3$, the cyclic group has only one nontrivial irreducible real representation. Thus the order-3 representation is determined by its r nontrivial and $b + s$ trivial complex coordinate lines. These numbers coincide by Theorem 8.1. Permute the matching coordinate lines and conjugate any complex coordinate whose nonzero weight has the opposite sign. This real orthogonal map conjugates the generators, and hence induces an isometry of the quotients.

Let $p = 5$. Up to signs, the nonzero weights modulo 5 are represented by 1 and 2. Write their multiplicities in U as x and $r - x$. Set

$$A = 4\sin^2(\pi/5) = \frac{5-\sqrt{5}}{2}, \qquad B = 4\sin^2(2\pi/5) = \frac{5+\sqrt{5}}{2}.$$

The pairs $a = \pm 1$ and $a = \pm 2$ respectively preserve and exchange these two sign classes. Hence, as in [1, (4.3)],

$$\begin{aligned} R_U(1) &= 2\big(A^{-x}B^{-(r-x)} + A^{-(r-x)}B^{-x}\big) \\ &= 4(AB)^{-r/2}\cosh\big((x - r/2)\log(B/A)\big). \end{aligned} \tag{8.15}$$

For fixed r, this determines $|x - r/2|$ and therefore the unordered pair $\{x, r - x\}$, because $B > A > 0$ and cosh is strictly increasing on the nonnegative real axis.

If $s > 0$, equation (8.9) gives $R_U(1) = R_{U'}(1)$. If $s = 0$, Proposition 8.4 gives equality of the entire R-functions. Thus in either case the unordered pairs of character multiplicities agree. Multiplication by 2 exchanges these two classes. Replace one generator by its square if necessary. A permutation and individual complex conjugations then match all coordinate weights, including the common $b + s$ zero weights. The resulting real orthogonal map conjugates the order-5 subgroups and induces an isometry of the corresponding covers. □

8.4. **Recovery of the missing form degree.** The gcd multiset in Theorem 8.1 supplies the axis data in Lauret's reduction [14, Remark 4.6].

Corollary 8.6. *Let $n \geq 2$ and let p be any prime. Suppose two effective $(2n-1)$-dimensional lens orbifolds of order p^2 are isospectral on differential forms in every degree $0, 1, \dots, n-2$. Then they are isospectral on differential forms in every degree.*

Proof. For a weight list w, let

$$\Lambda_w = \left\{ a \in \mathbb{Z}^n \,:\, \sum_{j=1}^{n} a_j w_j = 0 \pmod{p^2} \right\}.$$

For $0 \leq \ell \leq n$, define the formal series

$$Z_w^{(\ell)}(z) = \sum_{\substack{a \in \Lambda_w \\ \#\{j\,:\,a_j=0\}=\ell}} z^{\|a\|_1}, \qquad \|a\|_1 = \sum_{j=1}^{n} |a_j|.$$

The series $Z_w^{(n)}$ equals 1. The vectors counted by $Z_w^{(n-1)}$ are ke_j, with $k \neq 0$ and e_j the jth coordinate vector. Such a vector belongs to Λ_w precisely when $p^2/\gcd(p^2, w_j)$ divides k. Therefore

$$Z_w^{(n-1)}(z) = 2 \sum_{j=1}^{n} \frac{z^{p^2/\gcd(p^2,w_j)}}{1 - z^{p^2/\gcd(p^2,w_j)}}. \tag{8.16}$$

Theorem 8.1, applied to the scalar spectra included in the hypothesis, proves equality of these series for w and w'.

Put $\delta_\ell = Z_w^{(\ell)} - Z_{w'}^{(\ell)}$. Thus $\delta_n = \delta_{n-1} = 0$. By the form-spectrum criterion for lens orbifolds [13, Corollary 2.3], the assumed form-spectrum equalities imply

$$\sum_{\ell=0}^{n} \ell^h \delta_\ell(z) = 0, \qquad 0 \leq h \leq n-2, \tag{8.17}$$

where the $h = 0$ coefficient is 1 for every ℓ. After deleting the two zero terms, this is the Vandermonde system at $0, 1, \dots, n-2$. Its determinant is $\prod_{0 \leq i < j \leq n-2}(j - i) \neq 0$, so every δ_ℓ vanishes. For $n = 2$ this is the one-by-one matrix (1), with empty determinant product equal to 1. The criterion gives equality of all form spectra in degrees $0, \dots, n-1$. Write Δ_j for the Hodge Laplacian on j-forms. The cyclic actions lie in $\mathrm{SO}(2n)$, so the Hodge star $*$ descends to each quotient and satisfies $* \Delta_j = \Delta_{2n-1-j} *$. It identifies the eigenspaces in complementary degrees, giving equality of the remaining spectra. □

Appendix A. Coefficient bounds for prime-square congruences

With the notation of (4.1), write

$$J = \sum_{v \in I} j_v \, t^{a_v} w^{b_v} x^{c_v} y^{d_v}, \qquad j_v \in \mathbb{Z} \setminus \{0\}.$$

For each prime p, let C_s be given by (4.2) and put

$$\mathcal{T}_p = \{(X, Y) \in (\mathbb{Z}/p^2\mathbb{Z})^2 \,:\, C_s(p; X, Y) = 0 \text{ for every } s \in \mathbb{Z}/p^2\mathbb{Z}\}.$$

The binomial factorizations preceding (4.1) give the following bounds for the w-exponents:

(d, e)	w-exponents in $m_{d,e}Q_{d,e}$
$\pm(1, 3)$	$[-18, 30]$
$\pm(-1, 2)$	$[-23, 35]$
$\pm(-3, -2)$	$[-15, 27]$

Each $Q_{d,e}$ has t-degree 28, and the powers of x, y in (4.1) are the coordinates of (d, e) or (e, d). Thus every nonzero monomial of J satisfies

$$0 \le a_\nu \le 28, \qquad -23 \le b_\nu \le 35, \qquad |c_\nu|, |d_\nu| \le 3.$$

Proposition A.1. *There is a polynomial $P(Z) \in \mathbb{Z}[Z]$ of degree at most four such that $|\mathcal{T}_p| = P(p)$ for every prime $p > 14400$.*

Proof. Let T be an indeterminate and put $\Omega = I \times \{\pm 1\}^2$. For $\alpha = (\nu, \epsilon, \delta) \in \Omega$, put

$$\omega_\alpha = \epsilon\delta j_\nu, \qquad L_\alpha(T; X, Y) = a_\nu + Tb_\nu + \epsilon(1 + Tc_\nu)X + \delta(1 + Td_\nu)Y.$$

The set Ω and the integer weights ω_α are independent of p. For distinct $\alpha = (\nu, \epsilon, \delta)$ and $\beta = (\mu, \epsilon', \delta')$, the predicate $L_\alpha(p; X, Y) = L_\beta(p; X, Y) \pmod{p^2}$ has the form $A_{\alpha\beta}(p)X + B_{\alpha\beta}(p)Y = D_{\alpha\beta}(p)$ modulo p^2, where

$$\begin{aligned}(A_{\alpha\beta}, B_{\alpha\beta}, D_{\alpha\beta})(T) = \big(&\epsilon - \epsilon' + T(\epsilon c_\nu - \epsilon' c_\mu),\\ &\delta - \delta' + T(\delta d_\nu - \delta' d_\mu),\\ &a_\mu - a_\nu + T(b_\mu - b_\nu)\big).\end{aligned}$$

Let $\mathcal{E}$ be the set of unordered pairs of distinct elements of Ω. Choose an orientation for each pair, and let $\mathcal{M}(T)$ be the matrix with the corresponding rows. In each of its three columns, the absolute values of the coefficients are bounded coefficientwise by

$$2 + 6T, \qquad 2 + 6T, \qquad 28 + 58T,$$

respectively. The determinant formula therefore bounds the absolute values of the coefficients of each 3×3 minor coefficientwise by

$$6(2 + 6T)^2(28 + 58T) = 672 + 5424T + 14400T^2 + 12528T^3.$$

For a 2×2 minor involving the last column, the bound is

$$2(2 + 6T)(28 + 58T) = 112 + 568T + 696T^2;$$

for one in the first two columns, it is

$$2(2 + 6T)^2 = 8 + 48T + 72T^2.$$

Every coefficient of a 1×1 minor has absolute value at most 58. Hence, for every nonzero square minor $f(T)$, the coefficient $c(f)$ of its least power of T satisfies $0 < |c(f)| \le 14400$. If $p > 14400$, then $p \nmid c(f)$. Thus the exceptional set $S(\mathcal{M})$ of (5.1) contains no prime greater than 14400.

The pairwise equality predicates partition Ω according to the residues of $L_\alpha(p; X, Y)$ modulo p^2. By (4.2), membership in $\mathcal{T}_p$ is equivalent to the vanishing of $\sum_{\alpha\in B} \omega_\alpha$ for every block B of this partition. Define $F : \{0, 1\}^{\mathcal{E}} \to \{0, 1\}$ to have value 1 precisely on the truth assignments arising from partitions of Ω whose blocks have weight sum zero, and value 0 on all other assignments. This function is independent of p. On the truth values of the equality predicates, it is the indicator of $\mathcal{T}_p$. Theorem 5.1, with $d = k = 2$, gives $P \in \mathbb{Z}[Z]$ of degree at most four. □

The same bound applies when the equations defining the walls, grid, and finite orbits of Section 4 are included. Indeed, adjoin to $\mathcal{M}(T)$ the rows

$$(1,0,0), \quad (0,1,0), \quad (1,-1,0), \quad (1,1,0), \quad (T,0,0), \quad (0,T,0).$$

The first four rows represent $X=0$, $Y=0$, $X=Y$, $X=-Y$; their disjunction defines $\mathcal{W}_{p^2}$. At $T=p$, the last two represent $pX=0$, $pY=0$ modulo p^2, or equivalently $p \mid X$, $p \mid Y$; their conjunction defines $\mathcal{G}_p$. Let $\mathcal{R} \subset \mathbb{Z}^2$ consist of $(\epsilon r, \delta s)$ and $(\epsilon s, \delta r)$ for

$$(r,s) \in \{(3,1),(5,1),(5,3),(7,1),(7,3)\}, \qquad \epsilon, \delta \in \{\pm 1\}.$$

For every $(r,s) \in \mathcal{R}$, also adjoin $(1,0,r)$ and $(0,1,s)$. The conjunction of the corresponding equations is $(X,Y)=(r,s)$ modulo p^2, and the disjunction over $\mathcal{R}$ defines $\mathcal{O}_{p^2}$. Since $|r|,|s| \le 7$, every added row satisfies the same three column bounds. Consequently every fixed Boolean combination of the exponent equalities and these additional equations has a counting polynomial in $\mathbb{Z}[Z]$ of degree at most four for all primes $p > 14400$.

Corollary A.2. *For every prime $p > 14400$, let $S_p \subset \mathcal{T}_p$. Suppose that one polynomial $Q(Z) \in \mathbb{R}[Z]$ of degree at most four satisfies $|S_p| = Q(p)$ for every such prime. If $|S_p| = |\mathcal{T}_p|$ at five distinct primes greater than 14400, then $S_p = \mathcal{T}_p$ for every prime greater than 14400.*

Proof. By Proposition A.1, $P - Q$ has degree at most four and vanishes at five distinct primes. Hence $P = Q$. For each prime $p > 14400$, the inclusion $S_p \subset \mathcal{T}_p$ and equality of their finite cardinalities give $S_p = \mathcal{T}_p$. □

References

[1] A. Álzaga and E. A. Lauret, *Isospectral spherical space forms and orbifolds of highest volume*, Revista de la Unión Matemática Argentina 69 (2026), no. 1, 71–91. doi:10.33044/revuma.4943.

[2] E. A. Azoff and C. Davis, *On distances between unitary orbits of self-adjoint operators*, Acta Scientiarum Mathematicarum (Szeged) 47 (1984), 419–439.

[3] N. S. Bari and E. Hunsicker, *Isospectrality for orbifold lens spaces*, Canadian Journal of Mathematics 72 (2020), no. 2, 281–325. doi:10.4153/S0008414X19000178.

[4] T. Bogart, J. Goodrick and K. Woods, *Parametric Presburger arithmetic: logic, combinatorics, and quasi-polynomial behavior*, Discrete Analysis 2017 (2017), Paper No. 4, 34 pp. doi:10.19086/da.1254.

[5] D. DeFord and P. G. Doyle, *Cyclic groups with the same Hodge series*, Revista de la Unión Matemática Argentina 59 (2018), no. 2, 241–254.

[6] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Inventiones Mathematicae 77 (1984), 1–23. doi:10.1007/BF01389133.

[7] A. Dolzmann and T. Sturm, *Parametric systems of linear congruences*, in: Computer Algebra in Scientific Computing CASC 2001, Springer, Berlin, Heidelberg, 2001, 149–166. doi:10.1007/978-3-642-56666-0_12.

[8] P. Etingof, *18.745: Lie Groups and Lie Algebras, I*, lecture notes, Massachusetts Institute of Technology. https://math.mit.edu/~etingof/lnlg.pdf.

[9] A. Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature*, Osaka Journal of Mathematics 17 (1980), no. 1, 75–93. doi:10.18910/5171.

[10] H. Kamiya, A. Takemura and H. Terao, *Periodicity of hyperplane arrangements with integral coefficients modulo positive integers*, Journal of Algebraic Combinatorics 27 (2008), 317–330. doi:10.1007/s10801-007-0091-2.

[11] H. Kamiya, A. Takemura and H. Terao, *Periodicity of non-central integral arrangements modulo positive integers*, Annals of Combinatorics 15 (2011), 449–464. doi:10.1007/s00026-011-0105-6. arXiv:0803.2755.

[12] E. A. Lauret, *Spectra of orbifolds with cyclic fundamental groups*, Annals of Global Analysis and Geometry 50 (2016), 1–28. doi:10.1007/s10455-016-9498-0. arXiv:1510.05948v3.

[13] E. A. Lauret, *The spectrum on p-forms of a lens space*, Geometriae Dedicata 197 (2018), 107–122. doi:10.1007/s10711-018-0322-9. arXiv:1604.02471.

[14] E. A. Lauret, *A computational study on lens spaces isospectral on forms*, Experimental Mathematics 30 (2021), no. 2, 268–282. doi:10.1080/10586458.2018.1538908. arXiv:1703.03077.

[15] E. A. Lauret and R. J. Miatello, *Strong multiplicity one theorems for locally homogeneous spaces of compact type*, Proceedings of the American Mathematical Society 148 (2020), 3163–3173. doi:10.1090/proc/14980. arXiv:1804.08288.

[16] E. A. Lauret and R. J. Miatello, *Strong representation equivalence for compact symmetric spaces of real rank one*, Pacific Journal of Mathematics 314 (2021), no. 2, 333–373. doi:10.2140/pjm.2021.314.333. arXiv:1812.09606.

[17] E. A. Lauret, R. J. Miatello and J. P. Rossetti, *Spectra of lens spaces from 1-norm spectra of congruence lattices*, International Mathematics Research Notices 2016 (2016), no. 4, 1054–1089. doi:10.1093/imrn/rnv159. arXiv:1311.7167.

[18] E. A. Lauret, R. J. Miatello and J. P. Rossetti, *Recent results on the spectra of lens spaces*, São Paulo Journal of Mathematical Sciences 15 (2021), 240–267. doi:10.1007/s40863-019-00154-3. arXiv:1904.01146v2.
[19] E. A. Lauret and F. Rossi Bertone, *Weight multiplicity formulas for bivariate representations of classical Lie algebras*, Journal of Mathematical Physics 59 (2018), no. 8, 081705. doi:10.1063/1.5043305. arXiv:1708.07905.
[20] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1995.
[21] J. S. Milne, *Fields and Galois Theory*, version 5.10, September 2022. https://www.jmilne.org/math/CourseNotes/FT.pdf.
[22] V. L. Popov and E. B. Vinberg, *Invariant Theory*, in: Algebraic Geometry IV, Encyclopaedia of Mathematical Sciences 55, Springer, Berlin, Heidelberg, 1994, 123–284. doi:10.1007/978-3-662-03073-8_2.
[23] R. P. Stanley, *Invariants of finite groups and their applications to combinatorics*, Bulletin of the American Mathematical Society (N.S.) 1 (1979), no. 3, 475–511. doi:10.1090/S0273-0979-1979-14597-X.
[24] R. P. Stanley, *Enumerative Combinatorics*, Volume 1, second edition, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, Cambridge, 2012. doi:10.1017/CBO9781139058520.
[25] R. P. Stanley, *Smith normal form in combinatorics*, Journal of Combinatorial Theory, Series A 144 (2016), 476–495. doi:10.1016/j.jcta.2016.06.013.
[26] G. Teschl, *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators*, second edition, Graduate Studies in Mathematics 157, American Mathematical Society, Providence, RI, 2014.