NEGATIVE LICHNEROWICZ MODES ON EINSTEIN WARPED PRODUCTS AND SPHERES

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Abstract. Every nonhomogeneous compact Böhm metric on a sphere or a product of spheres in dimensions five through nine has a negative eigenvalue of the full Lichnerowicz Laplacian on transverse-traceless tensors. More generally, for a nonround doubly warped Einstein sphere with factor dimensions $p, q \geq 2$, the inequalities $(p-1)(q-4) \leq 4$ and $(q-1)(p-4) \leq 4$ suffice. They also apply to Wang's $SO(3) \times SO(9)$-invariant Einstein metric on S^{11}. For a positive Einstein warped product of closed connected manifolds, with base dimension $k \geq 3$, fibre dimension $m \geq 2$, and positive nonconstant warping function, the condition $(k-2)(m-4) \leq 4$ gives the same conclusion. Fibre dimensions two through four are therefore covered without a bound on the total dimension. The variational tool is a sharp divergence penalty on smooth trace-free tensors: on a closed connected positive Einstein manifold of dimension $n \geq 3$, subtracting $n/(n-1)$ times the squared divergence from the full Lichnerowicz quadratic form preserves its negative transverse-traceless index. For the sphere problem, this penalty cancels the quadratic radial-derivative term in a smooth interface construction; a weighted-radius rigidity theorem supplies the sign of the remaining interface contribution. Applications include finite quotients and ancient Ricci flows obtained from Kröncke's nonlinear instability theorem. An appendix treats negative spectrum on Ricci-flat asymptotically conical manifolds.

1. Introduction and statements

Böhm constructed inhomogeneous positive Einstein metrics on spheres and products of spheres in dimensions five through nine [1]. Gibbons, Hartnoll and Pope predicted a negative transverse-traceless mode of the full Lichnerowicz Laplacian for every nonhomogeneous member [2, Sections 3.3 and 6]. We prove this compact negative-mode assertion for every member. We also prove a sufficient dimensional criterion for positive Einstein warped products over arbitrary closed bases. These are the conclusions of Theorems 1.1 and 1.2 and Corollary 1.3.

Inverse-power variations changing the relative sizes of the factors already occur in [2, Sections 3.2.2–3.3]. They give analytic negative-mode proofs for the product families on $S^3 \times S^2$ and $S^3 \times S^3$. For sphere metrics, Gibbons, Hartnoll and Pope interpolated numerically between variations involving inverse powers of the two warping functions in selected S^5 examples [2, Section 3.3.3]. Batat, Hall and Murphy extended the relative-factor variations to compact Einstein warped products [7, Definition 4.1 and Section 4]. Here the trial tensors are trace-free but need not be divergence-free. A critical divergence penalty cancels the quadratic term in the radial derivative of the interpolated coefficient. The remaining transition terms have a finite signed limit. A rigidity theorem for the weighted radius ratio determines a point where, under the dimensional inequalities below, the resulting energy is negative. The inverse-power variations and the idea of interpolation are thus antecedents; the argument rests on the critical cancellation and the global sign supplied by rigidity.

Verger proved that the negative index tends to infinity along each compact Böhm sequence [4, Theorem A]. This asymptotic result allows finitely many possible initial exceptions. Our compact argument uses neither convergence to the singular limiting metric nor a choice of a sufficiently late member of a sequence; it asserts one negative mode for every nonhomogeneous member, not divergence of the indices. Batat, Hall and Murphy proved Ricci-flow instability in total dimension at most six and further results for selected higher-dimensional families [7, Theorems A–C]. Our

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warped-product criterion concerns the stronger conclusion of a negative eigenvalue of the full Lichnerowicz operator and includes unbounded total dimension.

All manifolds and metrics are smooth. Closed means compact without boundary. Integrals use the Riemannian density; orientability is not assumed. For an Einstein metric $\mathrm{Ric}_g = \Lambda g$, our conventions are

$$\Delta = -\operatorname{div}\nabla, \qquad (\delta T)_j = -\nabla^i T_{ij}, \qquad L_g = \nabla^*\nabla + 2\Lambda - 2\mathring{R}_g. \tag{1.1}$$

We take $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, with sectional curvature $\langle R(X,Y)Y, X\rangle$ on an orthonormal pair, and

$$(\mathring{R}_g T)(X,Y) = \sum_i T(R(e_i, X)Y, e_i).$$

Thus $\mathring{R}_g(g) = \mathrm{Ric}_g$. The Einstein operator is $L_g - 2\Lambda$. Put

$$Q_g(T) = \langle L_g T, T\rangle_{L^2} = \int_M \left(|\nabla T|^2 + 2\Lambda|T|^2 - 2\langle \mathring{R}_g T, T\rangle\right) dv_g. \tag{1.2}$$

A symmetric tensor is transverse-traceless, or TT, if $\delta T = 0$ and $\mathrm{tr}_g\, T = 0$. Write

$$\lambda_{\mathrm{TT}}(g) = \inf_{\substack{0\neq T\in C^\infty(S^2T^*M)\\ \delta T=0,\ \mathrm{tr}\, T=0}} \frac{Q_g(T)}{\|T\|_{L^2}^2}.$$

The infimum of the empty set is $+\infty$. The closed L^2 TT space is defined by the distributional equations $\delta T = 0$ and $\mathrm{tr}\, T = 0$; its spectral realization is given in Lemma 2.3.

Let $p, q \geq 2$ be integers and put $d = p+q$, $n = d+1$. Denote the unit-round metrics by γ_p, γ_q. On $(0,\ell)\times S^p \times S^q$ consider

$$g = dt^2 + a(t)^2\gamma_p + b(t)^2\gamma_q, \qquad a, b > 0, \qquad \mathrm{Ric}_g = d\, g. \tag{1.3}$$

At the left endpoint assume

$$a(t) = t\alpha_-(t^2), \quad b(t) = \beta_-(t^2), \qquad \alpha_-(0) = 1, \quad \beta_-(0) > 0. \tag{1.4}$$

With $s = \ell - t$, impose one of the following right endpoint conditions:

$$\text{sphere case:}\quad a(\ell - s) = \alpha_+(s^2), \quad b(\ell-s) = s\beta_+(s^2), \qquad \alpha_+(0) > 0, \quad \beta_+(0) = 1; \tag{1.5}$$

$$\text{product case:}\quad a(\ell - s) = s\alpha_+(s^2), \quad b(\ell-s) = \beta_+(s^2), \qquad \alpha_+(0) = 1, \quad \beta_+(0) > 0. \tag{1.6}$$

The coefficient functions are smooth near zero. With the standard collapses, these give smooth closed metrics on S^n and $S^{p+1}\times S^q$, respectively.

Theorem 1.1. *Suppose* (1.3) *satisfies* (1.4) *and* (1.5). *If g is nonround and*

$$(p-1)(q-4) \leq 4, \qquad (q-1)(p-4) \leq 4, \tag{1.7}$$

then $\lambda_{\mathrm{TT}}(g) < 0$. There exist a smooth nonzero TT tensor H and $\mu < 0$ such that $L_g H = \mu H$.

Theorem 1.2. *Let (B^k, g_B) and (F^m, g_F) be closed connected Riemannian manifolds, with $k \geq 3$ and $m \geq 2$. Let $f \in C^\infty(B)$ be positive and nonconstant, and suppose*

$$g = g_B + f^2 g_F \quad \text{on } B\times F, \qquad \mathrm{Ric}_g = \Lambda g, \quad \Lambda > 0.$$

If

$$(k-2)(m-4) \leq 4, \tag{1.8}$$

then $\lambda_{\mathrm{TT}}(g) < 0$.

Corollary 1.3. *Every nonhomogeneous compact Böhm metric on S^{p+q+1} or $S^{p+1}\times S^q$, with $p, q \geq 2$ and $p+q \leq 8$, has a negative TT eigenvalue of the full Lichnerowicz Laplacian.*

The sphere criterion covers every pair with $p+q+1 \leq 10$, and the additional pairs $(2,8), (8,2), (5,5)$ in dimension eleven. For product boundary conditions, Theorem 1.2 applies with $k = p+1$ and $m = q$. Neither theorem asserts existence of a metric for every permitted pair. The existence inputs to the named examples are [1, 5, 6].

The tool for both geometric theorems is the quadratic form

$$\mathcal{J}(T) = Q(T) - \frac{n}{n-1}\|\delta T\|_{L^2}^2$$

on smooth trace-free tensors. The orthogonal decomposition into transverse-traceless, scalar-generated and coclosed-vector-generated tensors, and the corresponding block invariance, are standard; see [10, Section 2] and [3, Section 3.4]. In these blocks, Theorem 2.5 identifies the negative index of $\mathcal{J}$ with that of $L|_{\mathrm{TT}}$ and proves sharpness of the coefficient. In particular, Corollary 2.6 gives

$$\mathcal{J}(T) < 0 \quad \Longrightarrow \quad \lambda_{\mathrm{TT}}(g) \leq \frac{\mathcal{J}(T)}{\|T\|_{L^2}^2} < 0.$$

The divergence constraint can therefore be imposed after the trace-free trial tensor has been chosen.

For the sphere argument, set $w = a^p b^q$ and $\Xi = (\log(a^p/b^q))'$. The dimension-independent Proposition 4.3 proves

$$\Xi \geq 0 \text{ on } (0,\ell) \quad \Longrightarrow \quad a = \sin t, \quad b = \cos t, \quad \ell = \pi/2.$$

Thus nonroundness supplies an interior point c with $\Xi(c) < 0$. For the smooth tensors of Proposition 6.2,

$$\frac{\mathcal{J}(T_\varepsilon)}{C_{p,q}} = c_L I_L(c) + c_R I_R(c) + w(c)\Xi(c) + O(\varepsilon).$$

Here $C_{p,q} = \mathrm{Vol}(S^p, \gamma_p)\,\mathrm{Vol}(S^q, \gamma_q) > 0$, $I_L, I_R \geq 0$, and the optimized coefficients (5.14) satisfy $c_L, c_R \leq 0$ under (1.7). The error tends to zero for each fixed metric and interface point. It is controlled by bounded variation of the interpolated radial coefficient, after the exact cancellation in (5.8). In particular, the quadratic transition term vanishes, but the finite interface contribution does not. For small positive ε the displayed quantity is strictly negative. The warped-product proof uses the same penalty on a globally smooth relative-factor tensor and requires no interface.

The named examples and finite quotients are consequences of the two geometric theorems. In particular, the criteria apply to the three metrics on S^{10} of Nienhaus and Wink [5, Theorem A], to Wang's $SO(3) \times SO(9)$-invariant metric on S^{11}, and to his nonstandard metric on $S^7 \times S^3$ [6, Theorem 1.1]. The ancient-flow conclusion is an application of Kröncke's nonlinear instability theorem [9]. In the appendix, the separated cone form and Hardy threshold of [10, Theorem 1.2 and Section 3.2] lead, by annular transplantation and the min–max principle, to infinite discrete negative spectrum under the stated asymptotically conical hypothesis. The parallel-spinor obstruction uses the local identity of Dai, Wang and Wei [11, Lemma 2.4]. No existence of a Ricci-flat filling is asserted.

2. A sharp divergence penalty

Throughout this section $n \geq 3$, (M^n, g) is closed and connected, $\mathrm{Ric}_g = \Lambda g$, and $\Lambda > 0$. Inner products and norms are real $L^2(g)$ inner products and norms. We use the standard Einstein tensor splitting [10, Section 2] and [3, Section 3.4], with the operator conventions in (1.1). Put

$$\theta_n = \frac{n-1}{n}, \qquad P\psi = \nabla^2\psi + \frac{\Delta\psi}{n} g. \tag{2.1}$$

The tensor $P\psi$ is trace-free. The formal adjoint of δ is $(\delta^*\omega)_{ij} = \frac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i)$.

Lemma 2.1. *For smooth functions v, one-forms ω, and symmetric tensors T,*

$$\delta P v = d(\theta_n \Delta v - \Lambda v), \qquad L(Pv) = P\Delta v, \tag{2.2}$$

$$\mathrm{tr}(LT) = \Delta(\mathrm{tr}\, T), \qquad \delta(LT) = (\nabla^*\nabla + \Lambda)\delta T. \tag{2.3}$$

Proof. Commutation of covariant derivatives gives

$$\nabla^*\nabla(dv) = d\Delta v - \Lambda dv, \qquad 2\delta\delta^*\omega = \nabla^*\nabla\omega + d\delta\omega - \Lambda\omega. \tag{2.4}$$

Also $\operatorname{tr}(\delta^*\omega) = -\delta\omega$ and $\delta(fg) = -df$. The first formula in (2.2) follows from $\delta\nabla^2 v = d\Delta v - \Lambda dv$.

The metric variation formula is

$$2(D\,\mathrm{Ric})_g(T) = LT - 2\delta^*\delta T - \nabla^2 \operatorname{tr} T. \tag{2.5}$$

Differentiation of the Levi–Civita connection gives

$$2\dot\Gamma^k_{ij} = \nabla_i T_j{}^k + \nabla_j T_i{}^k - \nabla^k T_{ij}, \qquad (D\,\mathrm{Ric})_g(T)_{ij} = \nabla_k\dot\Gamma^k_{ij} - \nabla_j\dot\Gamma^k_{ik}.$$

Commuting the two covariant derivatives gives

$$\begin{aligned}2(D\,\mathrm{Ric})_g(T)_{ij} = &-\nabla^k\nabla_k T_{ij} + \nabla_i\nabla^k T_{kj} + \nabla_j\nabla^k T_{ki} - \nabla_i\nabla_j(\operatorname{tr} T)\\ &+ \mathrm{Ric}_i{}^k T_{kj} + \mathrm{Ric}_j{}^k T_{ki} - 2(\mathring{R}T)_{ij}.\end{aligned}$$

Substitution of $\mathrm{Ric} = \Lambda g$ and $(\delta T)_j = -\nabla^k T_{kj}$ proves (2.5). These conventions and identities agree with [3, Section 2, equations (2.9)–(2.11)].

Naturality of Ricci curvature, applied to $\delta^*\omega = \frac{1}{2}\mathcal{L}_{\omega^\sharp} g$, gives $(D\,\mathrm{Ric})_g(\delta^*\omega) = \Lambda\delta^*\omega$. Substituting in (2.5) and using (2.4) yields

$$L\delta^*\omega = \delta^*(\nabla^*\nabla\omega + \Lambda\omega). \tag{2.6}$$

Putting $\omega = dv$ gives $L\nabla^2 v = \nabla^2\Delta v$. Since $\mathring{R}(g) = \Lambda g$, one also has $L(vg) = (\Delta v)g$. This proves $L(Pv) = P\Delta v$. Taking formal adjoints of (2.6) and of $L(vg) = (\Delta v)g$ proves (2.3). $\square$

Lemma 2.2. *Let T be a smooth trace-free symmetric tensor with $\delta T = du$. There exists a smooth function ψ such that $H = T - P\psi$ is TT. The tensor $P\psi$ is uniquely determined by T, and*

$$\|T\|^2_{L^2} = \|H\|^2_{L^2} + \|P\psi\|^2_{L^2}, \tag{2.7}$$

$$Q(H) = Q(T) - \frac{n}{n-1}\|du\|^2_{L^2} - \frac{n\Lambda}{n-1}\|P\psi\|^2_{L^2}. \tag{2.8}$$

Proof. Subtract the mean of u. We solve

$$(\theta_n\Delta - \Lambda)\psi = u. \tag{2.9}$$

Integration by parts and (2.4) give

$$\begin{aligned}\|\nabla^2 v\|^2_{L^2} &= \langle\nabla^*\nabla(dv), dv\rangle_{L^2} = \|\Delta v\|^2_{L^2} - \Lambda\|dv\|^2_{L^2},\\ |Pv|^2 &= |\nabla^2 v|^2 - \frac{1}{n}(\Delta v)^2.\end{aligned}$$

Consequently

$$\|Pv\|^2_{L^2} = \theta_n\|\Delta v\|^2_{L^2} - \Lambda\|dv\|^2_{L^2}. \tag{2.10}$$

If v lies in the kernel of $\theta_n\Delta - \Lambda$, then $\Delta v = (\Lambda/\theta_n)v$, and (2.10) implies $Pv = 0$. Integration by parts therefore gives

$$\frac{\Lambda}{\theta_n}\langle u, v\rangle_{L^2} = \langle du, dv\rangle_{L^2} = \langle\delta T, dv\rangle_{L^2} = \langle T, \nabla^2 v\rangle_{L^2} = \langle T, Pv\rangle_{L^2} = 0.$$

The elliptic self-adjoint operator in (2.9) is Fredholm. The displayed orthogonality proves solvability; elliptic regularity gives a smooth solution. Integration of (2.9) shows that ψ has mean zero. Differences of solutions lie in the kernel, on which P vanishes. By (2.2), $H = T - P\psi$ is TT. If $T - P\widetilde\psi$ is TT for any other smooth potential, then $P(\widetilde\psi - \psi)$ is TT. Integration by parts gives

$$\|P(\widetilde\psi - \psi)\|^2_{L^2} = \big\langle\delta P(\widetilde\psi - \psi), d(\widetilde\psi - \psi)\big\rangle_{L^2} = 0.$$

Thus the correction tensor is unique among all TT corrections. For every smooth v,

$$\langle H, Pv\rangle_{L^2} = \langle\delta H, dv\rangle_{L^2} = 0. \tag{2.11}$$

This gives (2.7). It also gives $\langle H, LP\psi\rangle_{L^2} = \langle H, P\Delta\psi\rangle_{L^2} = 0$, hence

$$Q(T) = Q(H) + Q(P\psi). \tag{2.12}$$

Finally,

$$\|P\psi\|_{L^2}^2 = \langle du, d\psi\rangle_{L^2}, \qquad Q(P\psi) = \langle P\psi, P\Delta\psi\rangle_{L^2} = \langle du, d\Delta\psi\rangle_{L^2}.$$

Differentiating (2.9) gives $d\Delta\psi = \theta_n^{-1}(du + \Lambda d\psi)$, so

$$Q(P\psi) = \theta_n^{-1}\|du\|_{L^2}^2 + \Lambda\theta_n^{-1}\|P\psi\|_{L^2}^2.$$

Together with (2.12), this proves (2.8). □

Lemma 2.3. *The full Lichnerowicz operator has a self-adjoint restriction to the closed L^2 space of distributionally TT tensors. This restriction has compact resolvent. Its eigenvectors are smooth, and a smooth TT tensor of negative Q-energy implies a negative TT eigenvalue.*

Proof. Let

$$\mathcal{T} = \{U \in L^2(S^2T^*M) : \operatorname{tr} U = 0,\ \delta U = 0 \text{ in distributions}\}.$$

The maps $\operatorname{tr} : L^2 \to L^2$ and $\delta : L^2 \to H^{-1}$ are continuous, so $\mathcal{T}$ is closed. On the closed manifold, L with domain $H^2(S^2T^*M)$ is self-adjoint, bounded below, and has compact resolvent. Choose $c > 0$ so large that $L + c$ is positive and invertible. Let U be L^2 and $\delta U = 0$, $\operatorname{tr} U = 0$ in distributions, and set $V = (L + c)^{-1}U \in H^2$. Equations (2.3) give

$$(\Delta + c)\operatorname{tr} V = 0, \qquad (\nabla^*\nabla + \Lambda + c)\delta V = 0.$$

Here $\operatorname{tr} V \in H^2$ and $\delta V \in H^1$. Their weak energy identities imply $\operatorname{tr} V = 0$ and $\delta V = 0$. Thus $R_c = (L+c)^{-1}$ preserves $\mathcal{T}$. Since R_c is self-adjoint, it also preserves $\mathcal{T}^\perp$. Its restriction to $\mathcal{T}$ is compact, self-adjoint, positive and injective; its range is dense in $\mathcal{T}$, since the orthogonal complement of that range is its kernel. The spectral theorem gives an orthonormal basis (H_j) of $\mathcal{T}$ with $R_cH_j = \rho_jH_j$, $\rho_j > 0$. Then $H_j \in H^2$ and $LH_j = (\rho_j^{-1} - c)H_j$. Elliptic regularity gives $H_j \in C^\infty$. The restriction of L has domain $H^2 \cap \mathcal{T}$. Expansion of a smooth TT tensor in this basis proves the last assertion. □

Put $c_n = n/(n-1)$ and define the trace-free conformal Killing operator

$$\mathcal{K}\omega = \delta^*\omega + \frac{\delta\omega}{n}g.$$

Then $\mathcal{K}(d\psi) = P\psi$. On one-forms we use $\Delta_H = d\delta + \delta d = \nabla^*\nabla + \Lambda$. For a quadratic form on smooth tensors, ind^- denotes the supremum of dimensions of linear subspaces on which the form is negative definite. For the compact TT operator it is the number of negative eigenvalues, counted with multiplicities.

Lemma 2.4. *Every smooth trace-free symmetric tensor has a decomposition*

$$T = H + P\psi + V, \qquad H \in \mathrm{TT}, \qquad V = \delta^*\eta, \qquad \delta\eta = 0. \tag{2.13}$$

The three tensor summands are unique, mutually orthogonal in L^2 and for Q, and depend linearly on T. Moreover,

$$Q(P\psi) = c_n\|\delta P\psi\|_{L^2}^2 + c_n\Lambda\|P\psi\|_{L^2}^2, \tag{2.14}$$

$$Q(V) = 2\|\delta V\|_{L^2}^2 + 2\Lambda\|V\|_{L^2}^2. \tag{2.15}$$

Proof. On trace-free tensors the formal adjoint of $\mathcal{K}$ is δ. By (2.4),

$$\mathcal{B} := \delta\mathcal{K} = \frac{1}{2}\Delta_H - \Lambda + \left(\frac{1}{2} - \frac{1}{n}\right) d\delta. \tag{2.16}$$

Its principal symbol at $\xi \neq 0$ is

$$\frac{1}{2}|\xi|^2 I + \left(\frac{1}{2} - \frac{1}{n}\right)\xi \otimes \xi,$$

which is positive definite. Also $\mathcal{B} = \mathcal{K}^*\mathcal{K}$, so $\ker \mathcal{B} = \ker \mathcal{K}$. For $\omega \in \ker \mathcal{K}$,

$$\langle \delta T, \omega\rangle_{L^2} = \langle T, \mathcal{K}\omega\rangle_{L^2} = 0.$$

The Fredholm alternative gives a unique smooth solution $\omega \perp \ker \mathcal{B}$ of $\mathcal{B}\omega = \delta T$. Then $H = T - \mathcal{K}\omega$ is TT and orthogonal to $\operatorname{im} \mathcal{K}$.

A harmonic one-form ζ satisfies $0 = \|\nabla\zeta\|_{L^2}^2 + \Lambda\|\zeta\|_{L^2}^2$, so it vanishes. The Hodge decomposition gives $\omega = d\psi + \eta$, with $\delta\eta = 0$. Equation (2.16) preserves exact and coclosed one-forms. Consequently

$$\langle P\psi, \delta^*\eta\rangle_{L^2} = \langle d\psi, \mathcal{B}\eta\rangle_{L^2} = 0, \qquad \langle \delta P\psi, \delta\delta^*\eta\rangle_{L^2} = 0.$$

The commutations in Lemma 2.1, together with $\delta\Delta_H = \Delta\delta$, imply

$$L\mathcal{K}\omega = \mathcal{K}\Delta_H\omega.$$

Thus L preserves each summand, and the decomposition is also Q-orthogonal. Orthogonality implies uniqueness of the tensor summands; the functions and one-forms may differ by elements annihilated by P and δ^*.

Set $u = \theta_n\Delta\psi - \Lambda\psi$. By integration by parts,

$$\|P\psi\|_{L^2}^2 = \langle du, d\psi\rangle_{L^2}, \qquad Q(P\psi) = \langle du, d\Delta\psi\rangle_{L^2}.$$

Since $d\Delta\psi = c_n(du + \Lambda d\psi)$, (2.14) follows. On coclosed one-forms, $\mathcal{B}\eta = \frac{1}{2}\Delta_H\eta - \Lambda\eta$. Using $\delta V = \mathcal{B}\eta$ and $\|V\|_{L^2}^2 = \langle \eta, \mathcal{B}\eta\rangle_{L^2}$ gives

$$Q(V) = \langle \Delta_H\eta, \mathcal{B}\eta\rangle_{L^2} = 2\|\mathcal{B}\eta\|_{L^2}^2 + 2\Lambda\langle \eta, \mathcal{B}\eta\rangle_{L^2},$$

which is (2.15). □

Theorem 2.5. *Let $n \geq 3$ and $\operatorname{Ric}_g = \Lambda g$, $\Lambda > 0$, on a closed connected manifold. For $\gamma \in \mathbb{R}$ define, on smooth trace-free tensors,*

$$\mathcal{J}_\gamma(T) = Q(T) - \gamma\|\delta T\|_{L^2}^2.$$

For every $\gamma \leq c_n$,

$$\operatorname{ind}^- \mathcal{J}_\gamma = \operatorname{ind}^-(L|_{\mathrm{TT}}). \tag{2.17}$$

For $\gamma > c_n$, the form $\mathcal{J}_\gamma$ has infinite negative index on $\operatorname{im} P$.

Proof. For (2.13), orthogonality and (2.14)–(2.15) give

$$\begin{aligned} \mathcal{J}_\gamma(T) = Q(H) &+ (c_n - \gamma)\|\delta P\psi\|_{L^2}^2 + c_n\Lambda\|P\psi\|_{L^2}^2 \\ &+ (2-\gamma)\|\delta V\|_{L^2}^2 + 2\Lambda\|V\|_{L^2}^2. \end{aligned} \tag{2.18}$$

If $\gamma \leq c_n$, then $c_n - \gamma \geq 0$ and $2 - \gamma \geq 2 - c_n > 0$. The four terms after $Q(H)$ are nonnegative, and their sum is zero only if $P\psi = V = 0$. The map $T \mapsto H$ is injective on each $\mathcal{J}_\gamma$-negative subspace, and its image is Q-negative. This gives one inequality in (2.17). The other follows from $\mathcal{J}_\gamma = Q$ on TT.

Let (φ_j) be an orthonormal scalar eigenbasis, $\Delta\varphi_j = \lambda_j\varphi_j$, with $\lambda_j \to \infty$. Equation (2.10) gives

$$\|P\varphi_j\|_{L^2}^2 = \lambda_j(\theta_n\lambda_j - \Lambda), \qquad \|\delta P\varphi_j\|_{L^2}^2 = (\theta_n\lambda_j - \Lambda)\|P\varphi_j\|_{L^2}^2.$$

For all sufficiently large j, these tensors are nonzero, and

$$\mathcal{J}_\gamma(P\varphi_j) = \big((1 - \gamma\theta_n)\lambda_j + \gamma\Lambda\big)\|P\varphi_j\|_{L^2}^2. \tag{2.19}$$

They are mutually orthogonal for Q and for the divergence pairing, by polarization of the same scalar identities. If $\gamma > c_n$, the coefficient in (2.19) is negative for all large j. Their finite spans have arbitrarily large negative dimension. □

Define

$$\mathcal{J}(T) = \mathcal{J}_{c_n}(T) = Q(T) - \frac{n}{n-1}\|\delta T\|_{L^2}^2. \tag{2.20}$$

Corollary 2.6. *For every smooth trace-free tensor,*

$$\mathcal{J}(T) < 0 \quad \Longrightarrow \quad \lambda_{\mathrm{TT}}(g) \le \frac{\mathcal{J}(T)}{\|T\|^2_{L^2}} < 0. \tag{2.21}$$

If $\mathcal{J}(T) = 0$ and $\delta T \neq 0$, then $\lambda_{\mathrm{TT}}(g) < 0$ as well.

Proof. In (2.13), $Q(H) \le \mathcal{J}(T)$ and $\|H\|^2_{L^2} \le \|T\|^2_{L^2}$. If $\mathcal{J}(T) < 0$, then $H \neq 0$ and $Q(H)/\|H\|^2_{L^2} \le \mathcal{J}(T)/\|T\|^2_{L^2}$. Apply Lemma 2.3. If $\mathcal{J}(T) = 0$ and $\delta T \neq 0$, at least one of $P\psi, V$ is nonzero. The positive zero-order terms in (2.18) give $Q(H) < 0$. □

Corollary 2.7. *For every $\tau \le c_n\Lambda$,*

$$\mathrm{ind}^-\big(\mathcal{J} - \tau\|\cdot\|^2_{L^2}\big) = \#\{\lambda \in \sigma(L|_{\mathrm{TT}}) : \lambda < \tau\}, \tag{2.22}$$

with multiplicities.

Proof. Subtract $\tau\|T\|^2_{L^2}$ in (2.18) at $\gamma = c_n$. The scalar term is $(c_n\Lambda - \tau)\|P\psi\|^2_{L^2}$, and the vector terms are

$$(2 - c_n)\|\delta V\|^2_{L^2} + (2\Lambda - \tau)\|V\|^2_{L^2}.$$

They are nonnegative. TT projection is injective on every negative subspace and maps it into a negative subspace of $Q - \tau\|\cdot\|^2_{L^2}$. Conversely the two forms agree on TT. The compact spectral theorem gives (2.22). □

The indices in (2.17) and (2.22) are quadratic-form indices on smooth trace-free tensors. No ellipticity or compact-resolvent assertion is made for the critically penalized operator on that whole space.

3. Compact Einstein warped products

Proof of Theorem 1.2. Put $n = k + m$, $v = \log f$, and $g_V = f^2 g_F$. For $\phi \in C^\infty(B)$, with pullbacks to $B \times F$ understood, set

$$T_\phi = \phi\left(\frac{1}{k}g_B - \frac{1}{m}g_V\right). \tag{3.1}$$

Its horizontal and vertical eigenvalues are ϕ/k and $-\phi/m$, so $\mathrm{tr}\, T_\phi = 0$. In a g-orthonormal frame adapted to the two factors, the warped-product connection gives

$$\delta T_\phi = -\frac{1}{k}(d\phi + n\phi\, dv), \tag{3.2}$$

$$|\nabla T_\phi|^2 = \frac{n}{km}|d\phi|^2 + \frac{2n^2}{k^2 m}\phi^2|dv|^2. \tag{3.3}$$

Indeed the mixed derivatives are

$$(\nabla_{e_\alpha} T_\phi)(e_i, e_\beta) = \frac{n}{km}\phi\, dv(e_i)\delta_{\alpha\beta}.$$

The two symmetric positions give the factor 2 in (3.3); their contraction gives the second term of (3.2). The remaining derivatives differentiate ϕ horizontally.

For any diagonal tensor with eigenvalues s_i, the Einstein equation implies

$$2\Lambda|T|^2 - 2\langle \mathring{R}T, T\rangle = 2\sum_{i<j} K_{ij}(s_i - s_j)^2. \tag{3.4}$$

In fact $\langle \mathring{R}T, T\rangle = 2\sum_{i<j} K_{ij}s_i s_j$ and $\sum_{j\neq i} K_{ij} = \Lambda$. Only mixed pairs contribute for (3.1). For horizontal X and vertical U of unit length,

$$K(X, U) = -f^{-1}\nabla^2_B f(X, X).$$

Writing $D_B = \mathrm{div}_B \nabla_B$, we obtain

$$\langle (2\Lambda - 2\mathring{R})T_\phi, T_\phi\rangle = -\frac{2n^2}{k^2 m}\phi^2 f^{-1} D_B f.$$

Since $f^{-1}D_B f = D_B v + |dv|^2$, equation (3.3) gives

$$Q(T_\phi) = \frac{n}{km}\int_M |d\phi|^2\, dv_g - \frac{2n^2}{k^2 m}\int_M \phi^2 D_B v\, dv_g. \tag{3.5}$$

For $\beta > 0$ take $\phi = f^{-\beta}$ and put

$$I_\beta = \int_M f^{-2\beta}|dv|^2\, dv_g > 0.$$

The strict inequality follows from connectedness and nonconstancy of f. The volume density is $dv_g = f^m dv_{g_B} dv_{g_F}$. Integration by parts on the closed base gives

$$\begin{aligned}\int_M \phi^2 D_B v\, dv_g &= \mathrm{Vol}(F, g_F)\int_B f^{m-2\beta} D_B v\, dv_{g_B}\\ &= -\mathrm{Vol}(F, g_F)\int_B \langle d(f^{m-2\beta}), dv\rangle\, dv_{g_B} = (2\beta - m)I_\beta.\end{aligned}$$

Therefore

$$\|\delta T_\phi\|^2_{L^2} = \frac{(n-\beta)^2}{k^2} I_\beta, \tag{3.6}$$

$$\mathcal{J}(T_\phi) = C_{k,m}(\beta) I_\beta, \tag{3.7}$$

where

$$\begin{aligned}C_{k,m}(\beta) &= \frac{n}{km}\beta^2 - \frac{4n^2}{k^2 m}\beta + \frac{2n^2}{k^2} - \frac{n(n-\beta)^2}{k^2(n-1)}\\ &= \frac{n^2(k-1)}{k^2 m(n-1)}(\beta-\beta_*)^2 + \frac{n^2((k-2)(m-4)-4)}{k^2 m(k-1)},\end{aligned} \tag{3.8}$$

$$\beta_* = \frac{2k+m-2}{k-1}. \tag{3.9}$$

If (1.8) is strict, then $\mathcal{J}(T_{f^{-\beta_*}}) < 0$. If equality holds, this value is zero, but

$$n - \beta_* = \frac{(k-2)(n-1)}{k-1} > 0,$$

so (3.6) is nonzero. In either case Corollary 2.6 gives $\lambda_{\mathrm{TT}}(g) < 0$. □

Corollary 3.1. *The conclusion of Theorem 1.2 holds whenever* $2 \le m \le 4$*, independently of* k*. It also holds whenever* $5 \le k + m \le 10$*.*

Proof. For $m \le 4$, $(k-2)(m-4) \le 0$. For $m \ge 5$ put $x = k-2 \ge 1$, $y = m-4 \ge 1$. If $k+m \le 10$, then $x + y \le 4$, so $xy \le (x+y)^2/4 \le 4$. □

Only mixed sectional curvatures were used in (3.5). Neither factor is assumed to have constant sectional curvature. The conditions that B is closed and $f > 0$ are retained in Theorem 1.2.

4. A rigidity theorem for the weighted radius ratio

Return to (1.3). Put

$$A = \frac{a'}{a}, \qquad B = \frac{b'}{b}, \qquad \mathcal{H} = pA + qB, \qquad \Xi = pA - qB, \qquad w = a^p b^q. \tag{4.1}$$

In particular $w' = \mathcal{H}w$ and $\Xi = (\log(a^p/b^q))'$.

Lemma 4.1. *The sectional curvatures of radial, factor-tangential, and mixed planes are, respectively,*

$$K_{0P} = -a''/a, \qquad K_{0Q} = -b''/b,$$

$$K_P = (1 - a'^2)/a^2, \qquad K_Q = (1 - b'^2)/b^2, \qquad K_{PQ} = -AB. \tag{4.2}$$

They have finite limits at both smooth endpoints. Moreover,

$$\begin{aligned} pK_{0P} + qK_{0Q} &= d, \\ K_{0P} + (p-1)K_P + qK_{PQ} &= d, \\ K_{0Q} + (q-1)K_Q + pK_{PQ} &= d. \end{aligned} \tag{4.3}$$

At an a-collapse one has $K_P = K_{0P}$ in the endpoint limit.

Proof. The shape operator of an orbit has eigenvalues A and B on its two factors. The intrinsic sectional curvatures of the orbit are a^{-2}, b^{-2} and zero. The Gauss equation gives the last three formulas in (4.2). The radial curvature equation gives $K_{0P} = -A' - A^2 = -a''/a$ and the analogous formula for b. Taking Ricci traces gives (4.3).

At the left endpoint write $a = t + a_3t^3 + O(t^5)$ and $b = b_0 + b_2t^2 + O(t^4)$. Then

$$\begin{aligned} K_P &= -6a_3 + O(t^2), & K_{0P} &= -6a_3 + O(t^2), \\ K_Q &= b_0^{-2} + O(t^2), & K_{0Q} &= -2b_2/b_0 + O(t^2), \\ K_{PQ} &= -2b_2/b_0 + O(t^2). \end{aligned}$$

The right endpoint is identical after replacing t by $\ell - t$ and, in the sphere case, interchanging a, p with b, q. □

Define the curvature defects

$$X = K_P - 1, \qquad Y = K_Q - 1, \qquad Z = K_{PQ} - 1, \qquad u = p(p-1)X, \quad v = q(q-1)Y. \tag{4.4}$$

The letters u, v in this section denote these scalar defects, not the potential in Section 2.

Lemma 4.2. *On* $(0, \ell)$*,*

$$K_{0P} - 1 = -(p-1)X - qZ, \qquad\qquad K_{0Q} - 1 = -(q-1)Y - pZ,$$

$$u + v + 2pqZ = 0, \tag{4.5}$$

$$u' = -A\big((p+1)u - (p-1)v\big), \qquad\qquad v' = -B\big((q+1)v - (q-1)u\big). \tag{4.6}$$

At the left endpoint,

$$u(0) = \frac{p-1}{p+1}v(0), \qquad v(0) = q(q-1)(b_0^{-2} - 1). \tag{4.7}$$

Proof. The first line follows from the last two equations of (4.3). Multiply them by p, q, respectively, and subtract the first equation of (4.3). This gives (4.5). Differentiating (4.2) gives

$$K_P' = -2A(K_P - K_{0P}), \qquad K_Q' = -2B(K_Q - K_{0Q}),$$

hence $X' = -2A(pX + qZ)$ and $Y' = -2B(qY + pZ)$. Eliminating $Z = -(u+v)/(2pq)$ proves (4.6). At $t = 0$, $K_P = K_{0P}$ implies $pX + qZ = 0$. Combining this with (4.5) gives $p(p+1)X = q(q-1)Y$, which is the first formula in (4.7). The second follows from $b'(0) = 0$. □

Proposition 4.3. *Assume the sphere endpoint conditions. If $\Xi \geq 0$ on $(0, \ell)$, then $a(t) = \sin t$, $b(t) = \cos t$, and $\ell = \pi/2$. In particular every nonround solution has $\Xi(c) < 0$ at some $c \in (0, \ell)$. No upper bound on n is required.*

Proof. Put $r_0 = (p-1)/(p+1)$ and $b_0 = b(0)$. Suppose first that $b_0 = 1$. Then $u(0) = v(0) = 0$. Multiplying the first equation of (4.6) by a^{p+1} gives

$$u(t) = \frac{p-1}{p+1}\, a(t)^{-(p+1)} \int_0^t v(s)(a(s)^{p+1})'\, ds. \tag{4.8}$$

The lower endpoint term vanishes because u is bounded and $a(0) = 0$. For small positive t, $a' > 0$. With $M(t) = \sup_{0\leq s\leq t} |v(s)|$, this gives $|u(t)| \leq r_0 M(t)$. Also $|B(t)| \leq C_B t$ near zero. Integration of the second equation in (4.6) therefore yields

$$M(t) \leq \frac{C_B}{2}\big((q+1) + (q-1)r_0\big)t^2 M(t).$$

For sufficiently small t the coefficient is less than one. Thus $M(t) = 0$, and (4.8) gives $u = 0$ there. Uniqueness for the regular linear system (4.6) on compact interior intervals gives $u = v = 0$ throughout $(0, \ell)$. Equations (4.4)–(4.5) imply $X = Y = Z = 0$ and $K_{0P} = K_{0Q} = 1$. Therefore $a'' = -a$ and $b'' = -b$. The initial conditions give $a = \sin t$ and $b = \cos t$; positivity in the open interval and the right collapse force $\ell = \pi/2$.

Now suppose $b_0 \neq 1$ and $\Xi \geq 0$ on the whole interval. Multiply the pair (u, v) by a common sign so that $v(0) > 0$. This changes neither the homogeneous system nor its initial ratio. Where $v \neq 0$, the ratio $R = u/v$ satisfies

$$R' = F(t, R) := -A\big((p+1)R - (p-1)\big) + BR\big((q+1) - (q-1)R\big). \tag{4.9}$$

Put

$$r_* = \frac{q(p-1)}{p(q-1)}. \tag{4.10}$$

Direct substitution gives

$$F(t, -1) = 2\Xi(t) \geq 0, \qquad F(t, r_*) = -\frac{(p-1)d}{p^2(q-1)}\Xi(t) \leq 0. \tag{4.11}$$

Moreover,

$$-1 < r_0 < r_*, \qquad r_* - r_0 = \frac{(p-1)d}{p(p+1)(q-1)} > 0.$$

Choose $t_0 > 0$ sufficiently small that $v(t_0) > 0$ and $u(t_0)/v(t_0) \in (-1, r_*)$. Let R be the maximal solution of (4.9) with initial value $R(t_0) = u(t_0)/v(t_0)$.

Fix $t_2 \in (t_0, \ell)$. On $[t_0, t_2] \times [-2, r_* + 1]$ let C be a Lipschitz constant for F in its second argument. Up to the first exit from this rectangle, the absolutely continuous functions $z_+ = (R - r_*)_+$ and $z_- = (-1 - R)_+$ satisfy

$$z_+' \leq Cz_+, \qquad z_-' \leq Cz_- \quad \text{almost everywhere}, \qquad z_+(t_0) = z_-(t_0) = 0.$$

These inequalities follow from (4.11) and the chain rule for positive parts. Gronwall's inequality gives $z_+ = z_- = 0$. Hence the solution remains in $[-1, r_*]$ and extends to t_2. Since $t_2 < \ell$ was arbitrary, it exists on $[t_0, \ell)$ with values in that interval.

Define

$$\widetilde{v}(t) = v(t_0) \exp\left(-\int_{t_0}^{t} B(s)\big((q+1) - (q-1)R(s)\big)\, ds\right), \qquad \widetilde{u} = R\widetilde{v}.$$

Then $(\widetilde{u}, \widetilde{v})$ solves (4.6) with the original data at t_0. Uniqueness gives $(\widetilde{u}, \widetilde{v}) = (u, v)$, and hence

$$v(t) > 0, \qquad -1 \leq u(t)/v(t) \leq r_* \qquad (t_0 \leq t < \ell). \tag{4.12}$$

Near the right endpoint, $B < 0$. On this neighbourhood,

$$(q+1) - (q-1)R \geq (q+1) - (q-1)r_* = \frac{d}{p}.$$

The second equation of (4.6) gives

$$\frac{v'}{v} \geq -\frac{d}{p}B.$$

For a fixed $t_1 < \ell$ sufficiently close to ℓ, integration yields

$$v(t) \geq v(t_1)\left(\frac{b(t_1)}{b(t)}\right)^{d/p} \longrightarrow +\infty \qquad (t \uparrow \ell). \tag{4.13}$$

This contradicts the bounded curvature defects at the smooth right endpoint. Thus $\Xi \geq 0$ is impossible when $b_0 \neq 1$, proving the proposition. $\square$

5. The invariant quadratic form

Put $g_P = a^2\gamma_p$, $g_Q = b^2\gamma_q$, and

$$C_{p,q} = \mathrm{Vol}(S^p,\gamma_p)\,\mathrm{Vol}(S^q,\gamma_q).$$

On the regular set, an invariant diagonal trace-free tensor has the form

$$T = x\,dt^2 + y\,g_P + z\,g_Q, \qquad x + py + qz = 0. \tag{5.1}$$

Write $\eta = y - z$. Then

$$y = \frac{-x + q\eta}{d}, \qquad z = \frac{-x - p\eta}{d}, \qquad |T|^2 = \frac{n}{d}x^2 + \frac{pq}{d}\eta^2. \tag{5.2}$$

Lemma 5.1. *For a globally smooth tensor* (5.1),

$$\delta T = -D\,dt, \qquad D = x' + pA(x-y) + qB(x-z) = x' + \frac{n}{d}\mathcal{H}x - \frac{pq}{d}(A-B)\eta, \tag{5.3}$$

and

$$\begin{aligned}\frac{Q(T)}{C_{p,q}} = \int_0^\ell w\big[&x'^2 + py'^2 + qz'^2 - 2pA'(x-y)^2\\ &- 2qB'(x-z)^2 - 2pqAB\eta^2\big]\,dt.\end{aligned} \tag{5.4}$$

Proof. Use $e_0 = \partial_t$ and orthonormal factor frames $e_i = a^{-1}\bar e_i$ and $e_\alpha = b^{-1}\bar e_\alpha$, chosen geodesic within each sphere at the point under consideration. The nonzero radial connection terms are

$$\nabla_{e_i}e_0 = Ae_i, \quad (\nabla_{e_i}e_j)^\perp = -A\delta_{ij}e_0, \qquad \nabla_{e_\alpha}e_0 = Be_\alpha, \quad (\nabla_{e_\alpha}e_\beta)^\perp = -B\delta_{\alpha\beta}e_0.$$

The frames are parallel in the radial direction; the mixed connection terms between the two sphere factors vanish. In addition to the radial derivatives x', y', z', the nonzero tensor derivatives at this point are

$$(\nabla_{e_i}T)(e_0,e_j) = A(x-y)\delta_{ij}, \qquad (\nabla_{e_\alpha}T)(e_0,e_\beta) = B(x-z)\delta_{\alpha\beta},$$

and their symmetric counterparts. Contraction gives (5.3) and

$$|\nabla T|^2 = x'^2 + py'^2 + qz'^2 + 2pA^2(x-y)^2 + 2qB^2(x-z)^2. \tag{5.5}$$

If a symmetric tensor is diagonal with eigenvalues s_i, then $\langle \mathring{R}T, T\rangle = 2\sum_{i<j}K_{ij}s_is_j$. Since $\sum_{j\neq i}K_{ij} = \Lambda$, it follows that

$$2\Lambda|T|^2 - 2\langle \mathring{R}T,T\rangle = 2\sum_{i<j}K_{ij}(s_i - s_j)^2. \tag{5.6}$$

For (5.1) only radial–factor and mixed-factor pairs contribute. Apply (4.2), then use $A^2 - a''/a = -A'$ and $B^2 - b''/b = -B'$. Together with (5.5) and $dv_g = w\,dt\,dv_{\gamma_p}\,dv_{\gamma_q}$, this proves (5.4). $\square$

Let E_J be the integrand in (5.4), before multiplication by w, minus $(n/d)D^2$. Then

$$\frac{\mathcal{J}(T)}{C_{p,q}} = \int_0^\ell wE_J\,dt. \tag{5.7}$$

Since

$$x'^2 + py'^2 + qz'^2 = \frac{n}{d}x'^2 + \frac{pq}{d}\eta'^2,$$

the x'^2 terms cancel identically. More explicitly,

$$E_J = -2\left(\frac{n}{d}\right)^2\mathcal{H}xx' + \frac{2npq}{d^2}(A-B)\eta x' + \mathcal{R}(t,x,\eta,\eta'), \tag{5.8}$$

$$\begin{aligned}\mathcal{R} = {}&\frac{pq}{d}\eta'^2 - \frac{n}{d}\left(\frac{n}{d}\mathcal{H}x - \frac{pq}{d}(A-B)\eta\right)^2\\ &- \frac{2p}{d^2}A'(nx - q\eta)^2 - \frac{2q}{d^2}B'(nx + p\eta)^2 - 2pqAB\eta^2.\end{aligned} \tag{5.9}$$

In particular, there is no $x'\eta'$ term. The coefficients of $\mathcal{R}$ are smooth on every compact interior interval.

Lemma 5.2. *Let $\beta > 0$. For*

$$x = y = \frac{q}{n}\eta, \qquad z = -\frac{p+1}{n}\eta, \qquad \eta' = -\beta B\eta,$$

one has

$$E_J = c_{p,q}(\beta)B^2\eta^2 - \frac{2q}{w}(wB\eta^2)', \tag{5.10}$$

where

$$\begin{aligned} c_{p,q}(\beta) &= \frac{q(p+1)}{n}\beta^2 - \frac{q^2}{nd}(n-\beta)^2 + 2q(q-2\beta) \\ &= \frac{pq}{d}\left(\beta - 2 - \frac{q}{p}\right)^2 + \frac{q}{p}\big((p-1)(q-4)-4\big). \end{aligned} \tag{5.11}$$

For $x = z = -p\eta/n$, $y = (q+1)\eta/n$, and $\eta' = -\beta A\eta$,

$$E_J = c_{q,p}(\beta)A^2\eta^2 - \frac{2p}{w}(wA\eta^2)'. \tag{5.12}$$

Proof. For the first tensor $x - y = 0$, $x - z = y - z = \eta$, and $D = q(\eta' + nB\eta)/n$. Substitution gives

$$E_J = \frac{q(p+1)}{n}\eta'^2 - 2qB'\eta^2 - 2pqAB\eta^2 - \frac{q^2}{nd}(\eta' + nB\eta)^2.$$

Use $\eta' = -\beta B\eta$ and

$$\frac{(wB\eta^2)'}{w} = (B' + pAB + (q-2\beta)B^2)\eta^2.$$

The remaining coefficient is the first expression in (5.11). Its quadratic coefficient is pq/d, its vertex is $2+q/p$, and its value there is $q((p-1)(q-4)-4)/p$. This proves the second expression. Interchanging the factors and replacing η by $-\eta$ gives (5.12). $\square$

Set

$$\beta_L = 2 + q/p, \qquad \beta_R = 2 + p/q, \tag{5.13}$$

$$c_L = \frac{q}{p}\big((p-1)(q-4)-4\big), \qquad c_R = \frac{p}{q}\big((q-1)(p-4)-4\big). \tag{5.14}$$

These minimize the two half-density coefficients independently. At $\beta = 4$, (5.11) gives $c_{p,q}(4) = q^2(n-10)/d$.

6. A smooth interface on the sphere

Assume the sphere endpoint conditions and fix $c \in (0, \ell)$. Put

$$\begin{aligned} \eta_L(t) &= \left(\frac{b(c)}{b(t)}\right)^{\beta_L} && (0 \le t \le c), \\ \eta_R(t) &= \left(\frac{a(c)}{a(t)}\right)^{\beta_R} && (c \le t \le \ell). \end{aligned} \tag{6.1}$$

Both functions are also defined in a fixed open neighbourhood of c, and $\eta_L(c) = \eta_R(c) = 1$. Define

$$T_L = \frac{q}{n}\eta_L(dt^2 + g_P) - \frac{p+1}{n}\eta_L g_Q, \tag{6.2}$$

$$T_R = -\frac{p}{n}\eta_R(dt^2 + g_Q) + \frac{q+1}{n}\eta_R g_P. \tag{6.3}$$

Only the noncollapsing radius is inverted on each respective side.

Lemma 6.1. *The tensor T_L extends smoothly across the left singular orbit, and T_R extends smoothly across the right singular orbit.*

Proof. At the left endpoint, let $X_1, \dots, X_{p+1}$ be Cartesian coordinates on the normal disc, $t^2 = \sum_i X_i^2$, and write $a = t\alpha_-(t^2)$. Then

$$dt^2 + a^2 \gamma_p = \alpha_-(t^2)^2 \sum_i dX_i^2 + \frac{1 - \alpha_-(t^2)^2}{t^2} \left(\sum_i X_i \, dX_i \right)^2. \tag{6.4}$$

The quotient in this formula is a smooth function of t^2, because $\alpha_-(0) = 1$. Thus the normal-disc metric is smooth. Both b^2 and $\eta_L = \exp(\beta_L(\log b(c) - \log b))$ are smooth functions of t^2, with $b > 0$. Formula (6.2) is therefore smooth. At the right endpoint, use $s = \ell - t$, interchange the factors, and apply the same calculation to (6.3). □

Set

$$I_L(c) = \int_0^c w\eta_L^2 B^2 \, dt, \qquad I_R(c) = \int_c^\ell w\eta_R^2 A^2 \, dt, \tag{6.5}$$

$$\mathcal{F}(c) = c_L I_L(c) + c_R I_R(c) + w(c)\Xi(c), \tag{6.6}$$

$$\mathcal{N}(c) = \frac{1}{n} \left(q(p+1) \int_0^c w\eta_L^2 \, dt + p(q+1) \int_c^\ell w\eta_R^2 \, dt \right). \tag{6.7}$$

All these integrals are finite, $I_L, I_R \geq 0$, and $\mathcal{N}(c) > 0$. Indeed $w = O(t^p)$ and $B = O(t)$ at the left endpoint, whereas $w = O((\ell - t)^q)$ and $A = O(\ell - t)$ at the right; the relevant η is bounded at each endpoint.

Proposition 6.2. *For every interior c there are globally smooth invariant trace-free tensors T_ε, $\varepsilon > 0$ sufficiently small, with*

$$\frac{\mathcal{J}(T_\varepsilon)}{C_{p,q}} = \mathcal{F}(c) + O(\varepsilon), \tag{6.8}$$

$$\frac{\|T_\varepsilon\|_{L^2}^2}{C_{p,q}} = \mathcal{N}(c) + O(\varepsilon). \tag{6.9}$$

The constants may depend on g, c and a fixed smoothing function. If $\mathcal{F}(c) < 0$, then

$$\lambda_{\mathrm{TT}}(g) \leq \frac{\mathcal{F}(c)}{\mathcal{N}(c)} < 0. \tag{6.10}$$

Proof. Choose a smooth nondecreasing function $\chi : \mathbb{R} \to [0,1]$ with $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[1, \infty)$. For small $\varepsilon > 0$ put $I_\varepsilon = (c - \varepsilon, c + \varepsilon)$ and $\chi_\varepsilon(t) = \chi((t-c)/\varepsilon)$. On I_ε set

$$\eta_\varepsilon = (1 - \chi_\varepsilon)\eta_L + \chi_\varepsilon \eta_R,$$
$$x_\varepsilon = (1 - \chi_\varepsilon)\frac{q}{n}\eta_L - \chi_\varepsilon \frac{p}{n}\eta_R. \tag{6.11}$$

Outside the band use the corresponding left or right expressions, and define $y_\varepsilon, z_\varepsilon$ by (5.2). The cutoff is smooth and constant to all orders at the edges of the band. Thus the formulas give a smooth tensor on the regular set. Lemma 6.1 gives smoothness at the singular orbits. The tensor is invariant and trace-free by (5.2). For each fixed $\varepsilon > 0$ it lies in $C^\infty(S^2 T^*M)$ and hence in the quadratic-form domain of L.

On a fixed compact neighbourhood of c, the functions η_L, η_R and their first derivatives are bounded. Their difference vanishes at c, so it is $O(\varepsilon)$ on I_ε. Differentiating (6.11) therefore gives uniform bounds

$$\sup_{I_\varepsilon}\big(|x_\varepsilon| + |\eta_\varepsilon| + |\eta_\varepsilon'|\big) \leq C, \qquad \sup_{I_\varepsilon} |\eta_\varepsilon - 1| \leq C\varepsilon, \qquad \int_{I_\varepsilon} |x_\varepsilon'| \, dt \leq C. \tag{6.12}$$

For the last estimate, the terms containing derivatives of η_L and η_R integrate to $O(\varepsilon)$, and the remaining terms are bounded by a constant times $\int_{I_\varepsilon} |\chi_\varepsilon'| dt = 1$.

Choose $\delta > 0$ with $[c-\delta, c+\delta] \subset (0,\ell)$ and fix $0 < \varepsilon < \delta$. The coefficients in (5.9) are bounded on this fixed interval. Equations (5.9) and (6.12) imply

$$\int_{I_\varepsilon} w\mathcal{R}(t, x_\varepsilon, \eta_\varepsilon, \eta_\varepsilon')\,dt = O(\varepsilon).$$

For the other two terms in (5.8), replace $w\mathcal{H}$ and $w(A-B)$ by their values at c, and replace η_ε in the second term by one. The errors are $O(\varepsilon)$ by (6.12). Therefore

$$\begin{aligned}\int_{I_\varepsilon} wE_J\,dt = &-\left(\frac{n}{d}\right)^2 w(c)\mathcal{H}(c)[x_\varepsilon^2]_{c-\varepsilon}^{c+\varepsilon}\\ &+\frac{2npq}{d^2}w(c)(A(c)-B(c))[x_\varepsilon]_{c-\varepsilon}^{c+\varepsilon} + O(\varepsilon).\end{aligned} \tag{6.13}$$

The endpoint values are

$$x_\varepsilon(c-\varepsilon) = q/n + O(\varepsilon), \qquad x_\varepsilon(c+\varepsilon) = -p/n + O(\varepsilon).$$

Consequently,

$$\begin{aligned}[x_\varepsilon]_{c-\varepsilon}^{c+\varepsilon} &= -\frac{d}{n} + O(\varepsilon),\\ [x_\varepsilon^2]_{c-\varepsilon}^{c+\varepsilon} &= \frac{p^2-q^2}{n^2} + O(\varepsilon).\end{aligned}$$

Substitution in (6.13) gives the limiting value

$$\begin{aligned}&-\frac{p^2-q^2}{d^2}w(c)\mathcal{H}(c) - \frac{2pq}{d}w(c)(A(c)-B(c))\\ &\qquad = -w(c)(pA(c)-qB(c)) = -w(c)\Xi(c).\end{aligned} \tag{6.14}$$

Here $p^2 - q^2 = d(p-q)$.

On the left, integrate (5.10); on the right, integrate (5.12). The physical endpoint terms vanish, since

$$wB\eta_L^2 = O(t^{p+1}), \qquad wA\eta_R^2 = O((\ell-t)^{q+1})$$

at the respective endpoints. Replacing the truncated upper and lower limits by c changes the integrals and their boundary values by $O(\varepsilon)$. The left contribution is

$$c_L I_L(c) - 2qw(c)B(c) + O(\varepsilon),$$

and the right contribution is

$$c_R I_R(c) + 2pw(c)A(c) + O(\varepsilon).$$

Adding (6.14) leaves the boundary sum $w(c)\Xi(c)$. This proves (6.8).

The pointwise squared norms of the pure tensors are

$$|T_L|^2 = \frac{q(p+1)}{n}\eta_L^2, \qquad |T_R|^2 = \frac{p(q+1)}{n}\eta_R^2.$$

The band tensor has uniformly bounded norm by (5.2) and (6.12); its contribution is $O(\varepsilon)$. This proves (6.9).

If $\mathcal{F}(c) < 0$, let $C_0 \geq 0$ and $\varepsilon_0 > 0$ satisfy

$$\left|\frac{\mathcal{J}(T_\varepsilon)}{C_{p,q}} - \mathcal{F}(c)\right| \leq C_0\varepsilon \qquad (0 < \varepsilon < \varepsilon_0).$$

For $0 < \varepsilon < \min\{\varepsilon_0, -\mathcal{F}(c)/(2(C_0+1))\}$, one has $\mathcal{J}(T_\varepsilon)/C_{p,q} < \mathcal{F}(c)/2 < 0$. By Corollary 2.6, (2.21) gives

$$\lambda_{\mathrm{TT}}(g) \leq \frac{\mathcal{J}(T_\varepsilon)}{\|T_\varepsilon\|_{L^2}^2}.$$

Taking the limit and using $\mathcal{N}(c) > 0$ proves (6.10). For each fixed ε, the TT projection is smooth by Lemma 2.4. No convergence in H^1 of the tensors T_ε, or of their TT projections, is asserted or required. $\square$

Proof of Theorem 1.1. If the metric is nonround, Proposition 4.3 supplies an interior point c with $\Xi(c) < 0$. By (1.7), $c_L, c_R \leq 0$. Thus (6.6) is strictly negative, since $I_L, I_R \geq 0$ and $w(c)\Xi(c) < 0$. Hence Proposition 6.2 gives $\lambda_{\mathrm{TT}}(g) < 0$. Lemma 2.3 supplies a smooth nonzero TT eigentensor. □

Corollary 6.3. *Up to exchange of the factors, the pairs satisfying* (1.7) *are exactly*

p	q
2	$2 \leq q \leq 8$
3	$3 \leq q \leq 6$
4	$4 \leq q \leq 5$
5	$q = 5.$

They include all pairs of total dimension at most ten and, in dimension eleven, $(2, 8)$ *and* $(5, 5)$.

Proof. Assume $p \leq q$. If $p \geq 6$, then $(p-1)(q-4) \geq 10 > 4$. For $p = 2, 3, 4, 5$, substitution in (1.7) gives the four displayed rows. If $p + q + 1 \leq 10$, then the possibilities with $p \leq q$ are $p = 2, q \leq 7$; $p = 3, q \leq 6$; and $p = 4, q \leq 5$. The only additional pairs in the table have $p + q + 1 = 11$. □

7. Compact consequences

Corollary 7.1. *Suppose* (1.3) *satisfies* (1.4) *and* (1.6). *If* b *is nonconstant and* $(p-1)(q-4) \leq 4$, *then* $\lambda_{\mathrm{TT}}(g) < 0$.

Proof. The metric $dt^2 + a^2\gamma_p$ extends smoothly to S^{p+1}. The positive function b is smooth on this base by its endpoint parities. Apply Theorem 1.2 with $k = p+1$, $m = q$, and $f = b$. □

Proof of Corollary 1.3. The metrics of [1, Theorem 3.6 and Section 7], and those of [1, Theorem 3.4] with $Q = S^q$, have the form (1.3), with $p, q \geq 2$ and $p + q \leq 8$. In the notation of [1, Sections 1–3],

$$d_S = p, \qquad d_Q = q, \qquad d_P = p + q, \qquad f = a, \qquad h = b, \qquad \lambda = d_P.$$

The submersion integrability tensor vanishes and $\mathrm{Ric}^Q = q - 1$. The smooth extensions in [1, Theorem 2.3], together with the terminal conditions in [1, Section 3, p. 151, and Section 7], give (1.4) and, respectively, (1.5) or (1.6). The sphere metrics are nonhomogeneous, hence nonround; Theorem 1.1 and Corollary 6.3 apply. For the product metrics put $k = p + 1$, $m = q$. Then $k + m \leq 9$. If b is constant, the Einstein equations give

$$a'' = -\frac{d}{p}a, \qquad b^2 = \frac{q-1}{d}.$$

The initial data and positivity up to the first positive zero imply

$$a(t) = \sqrt{p/d}\,\sin(\sqrt{d/p}\,t), \qquad \ell = \pi\sqrt{p/d}.$$

This is the homogeneous Einstein product. Thus b is nonconstant for every nonhomogeneous product member. Corollaries 3.1 and 7.1 give the conclusion. □

Corollary 7.2. *The three nonround Einstein metrics on* S^{10} *constructed by Nienhaus and Wink* [5, Theorem A], *Wang's nonround* $SO(3) \times SO(9)$*-invariant metric on* S^{11}, *and Wang's nonstandard metric on* $S^7 \times S^3$ [6, Theorem 1.1] *have* $\lambda_{\mathrm{TT}}(g) < 0$.

Proof. The three ten-sphere metrics have factor dimensions $(p, q) = (2, 7), (3, 6), (4, 5)$ [5, Introduction]. The two left sides in (1.7) are, respectively,

$$(3, -12), \qquad (4, -5), \qquad (3, 0).$$

They are at most four. The endpoint conditions are (1.4) and (1.5); hence Theorem 1.1 applies.

For Wang's S^{11} metric the pair is $(2, 8)$, after reversing the orbit interval if necessary. Equations (5.13)–(5.14) give

$$\beta_L = 6, \qquad \beta_R = 9/4, \qquad c_L = 0, \qquad c_R = -9/2.$$

The conclusion again follows from Theorem 1.1.

For the product in [6, Theorem 1.1], the factor dimensions in that paper are $(d_1, d_2) = (7, 2)$, and the S^2 factor collapses at both endpoints. Thus, in (1.3), $(p, q) = (2, 7)$ and (1.6) holds. The base has dimension $k = 3$ and the fibre dimension is $m = 7$. If b were constant, the computation in the proof of Corollary 1.3 would give the homogeneous product, contrary to nonstandardness. Hence b is nonconstant and

$$(k-2)(m-4) = 3 < 4.$$

Apply Theorem 1.2. □

The higher-dimensional sphere pairs $(3, 7)$, $(2, 9)$ and $(2, 10)$ in [6, Theorem 1.1] do not satisfy (1.7). No existence assertion for a nonround $(5, 5)$ sphere is used. The sufficient criterion is not claimed to be a sharp stability boundary among all tensor variations.

Corollary 7.3. *Let g be a sphere metric covered by Theorem 1.1, or a product metric covered by Corollary 7.1. If a finite subgroup $\Gamma \subset O(p+1) \times O(q+1)$ acts freely by the standard factorwise action on the smooth completion, the quotient metric has a negative TT Lichnerowicz eigenvalue. In the sphere case this includes the simultaneous antipodal quotient $\mathbb{RP}^n$.*

Proof. The sphere tensors T_ε and the product tensor $T_{b^{-\beta_*}}$ of (3.1) are invariant under the factorwise orthogonal group. Isometric pullback preserves the L^2 inner product and the distributional TT subspace. It therefore commutes with orthogonal TT projection. Naturality of the Levi–Civita connection and curvature gives $\sigma^* L = L\sigma^*$ for every isometry σ. The negative TT test tensor supplied by Corollary 2.6 is Γ-invariant. At the equality case in (1.8), the same conclusion follows from (2.18). Let H be this invariant TT test tensor with $Q(H) < 0$. Its orthogonal expansion into eigenspaces of $L|_{\mathrm{TT}}$ has a nonzero component in a negative eigenspace. The spectral projections commute with Γ, so that component is invariant. It descends to a smooth TT eigentensor on M/Γ. Pullback commutes with L, trace and divergence because the covering is a local isometry. The eigenvalue is unchanged. The element $(-I, -I)$ has no fixed point on a principal orbit; at either singular orbit it acts antipodally on the surviving sphere. On the sphere completion the map

$$(t, x, y) \longmapsto \big(\sin(\pi t/(2\ell))x,\ \cos(\pi t/(2\ell))y\big) \in S^n \subset \mathbb{R}^{p+1} \oplus \mathbb{R}^{q+1}$$

extends over both collapsed orbits. The quotient definition gives a continuous bijection from the compact completion to S^n, hence a homeomorphism. Near either singular orbit its normal coordinate map has radial multiplier $\sin(cr)/r$, where $c = \pi/(2\ell) > 0$. This multiplier is smooth and even, with value c at zero; the inverse has multiplier $\arcsin(r)/(cr)$ near zero. The map is therefore a diffeomorphism and intertwines $(-I, -I)$ with the antipodal map. □

Remark 7.4. For $g = c\widehat{g}$, $c > 0$, one has $L_g = c^{-1} L_{\widehat{g}}$ on covariant two-tensors. Trace-freeness and vanishing divergence are unchanged. Thus every sign conclusion is invariant under constant rescaling.

Corollary 7.5. *Let (M^n, g) be a nonhomogeneous compact Böhm metric covered by Corollary 1.3, normalized by $\mathrm{Ric}_g = (n-1)g$. For every $r_0 > 0$, the zero-cosmological-constant generalized Schwarzschild–Tangherlini spacetime*

$$\widehat{g} = -V(r)\, d\tau^2 + V(r)^{-1} dr^2 + r^2 g, \qquad V(r) = 1 - \left(\frac{r_0}{r}\right)^{n-1}, \qquad r > r_0, \tag{7.1}$$

is linearly unstable in the tensor sector.

Proof. For (7.1), the sufficient condition of [2, Section 2.2, equations (11)–(15)] is

$$\lambda_{\mathrm{TT}}(g) < 4 - \frac{(5-n)^2}{4} = \frac{(n-1)(9-n)}{4}.$$

By Corollary 1.3, $\lambda_{\mathrm{TT}}(g) < 0$. Since $5 \le n \le 9$, the right-hand side is nonnegative. □

In dimensions ten and eleven, bare negativity does not imply this physical criterion: the thresholds are $-9/4$ and -5, respectively. The compact negative-mode assertion is distinct from the Einstein Dirichlet problem and from the conjecture on connected components of Einstein moduli spaces in [1, pp. 146 and 153].

8. Ancient Ricci flows

For a closed connected manifold define Perelman's shrinker entropy by

$$\mathcal{W}(g,f,\tau) = \int_M \big[\tau(R_g + |df|^2) + f - n\big](4\pi\tau)^{-n/2} e^{-f}\, dv_g,$$
$$\nu(g) = \inf_{\tau>0} \inf_{\substack{f\in C^\infty(M)\\ \int_M (4\pi\tau)^{-n/2}e^{-f}\,dv_g=1}} \mathcal{W}(g,f,\tau).$$

Near a positive Einstein metric, ν is analytic in the $C^{2,\alpha}$ topology, $0<\alpha<1$ [9, Lemma 4.1]. The specialization of [9, Theorem 1.3, proof of Theorem 5.4, and Remark 5.5] used below is the following. Let (M, g_E) be closed and connected, with $\mathrm{Ric}_{g_E} = \Lambda g_E$, $\Lambda > 0$. Suppose that smooth metrics g_j satisfy

$$g_j \longrightarrow g_E \quad \text{in every fixed } C^k \text{ topology}, \qquad \nu(g_j) > \nu(g_E). \tag{8.1}$$

For each integer $r \geq 3$, there are a smooth ancient volume-normalized Ricci flow $g(s)$, $s \leq 0$, and diffeomorphisms Ψ_s such that

$$\Psi_s^* g(s) \longrightarrow g_E \quad \text{in } C^r \quad (s \to -\infty), \qquad \nu(g(0)) > \nu(g_E). \tag{8.2}$$

The construction with parameter k gives C^{k-6} convergence; choose $k > r+6$. The strict entropy inequality follows from [9, equation (5.12)]; the volume-normalized formulation is [9, Remark 5.5]. The time origin can be chosen in the interior of the existence interval. No integrability hypothesis on infinitesimal Einstein deformations occurs in this implication.

Corollary 8.1. *Let (M^n, g_E) be closed and connected, with $n \geq 3$, $\mathrm{Ric}_{g_E} = \Lambda g_E$ and $\Lambda > 0$. Suppose $\lambda_{\mathrm{TT}}(g_E) < 0$. For every prescribed integer $r \geq 3$ there is a smooth ancient volume-normalized Ricci flow $g(s)$, $s \in (-\infty, 0]$, nonstationary modulo diffeomorphisms, converging to g_E in C^r modulo diffeomorphisms as $s \to -\infty$. Its volume is $\mathrm{Vol}(M, g_E)$. There are a non-self-similar ancient unnormalized flow $\widehat{g}(t)$, $t \in (-\infty, 0]$, and diffeomorphisms Φ_t such that*

$$\Phi_t^*\big((-t)^{-1}\widehat{g}(t)\big) \longrightarrow 2\Lambda g_E \quad \text{in } C^r \quad (t \to -\infty), \tag{8.3}$$

and

$$\sup_{t<0} |t| \sup_M |\mathrm{Rm}_{\widehat{g}(t)}| < \infty. \tag{8.4}$$

In particular these conclusions hold for the metrics of Theorems 1.1 and 1.2 and Corollaries 1.3, 7.2 and 7.3.

Proof. Choose $0 \neq H \in \mathrm{TT}$ with $LH = \lambda H$, $\lambda < 0$. The Einstein second-variation formula [8, Proposition 8.2], with Einstein operator $L - 2\Lambda$, gives

$$D^2\nu_{g_E}(H,H) = \frac{2\Lambda - \lambda}{4\Lambda\,\mathrm{Vol}(M, g_E)} \|H\|_{L^2}^2 > 0. \tag{8.5}$$

Since g_E is a critical point, Taylor's formula implies $\nu(g_E + \epsilon H) > \nu(g_E)$ for all sufficiently small nonzero ϵ. For $|\epsilon|\|H\|_{g_E, L^\infty} < 1$, these tensors are metrics. To prescribe the volume, set

$$g_\epsilon^0 = \left(\frac{\mathrm{Vol}(M, g_E)}{\mathrm{Vol}(M, g_E + \epsilon H)}\right)^{2/n} (g_E + \epsilon H).$$

Then $g_\epsilon^0 \to g_E$ in every fixed C^k topology and $\nu(g_\epsilon^0) = \nu(g_E + \epsilon H) > \nu(g_E)$. Choose nonzero $\epsilon_j \to 0$ in this range and apply (8.1)–(8.2) to $g_j = g_{\epsilon_j}^0$. The resulting ancient flow is smooth up to the chosen regular final time. Its volume is constant, and backward convergence gives $\mathrm{Vol}(M, g(s)) = \mathrm{Vol}(M, g_E)$. On the fixed compact manifold, C^r convergence with $r \geq 3$ implies $C^{2,\alpha}$ convergence for each

$0 < \alpha < 1$: in a finite atlas, $[D^2u]_\alpha \le C\|D^3u\|_\infty$ for differences u tending to zero in C^3. Continuity of ν in this neighbourhood and its invariance under diffeomorphisms give

$$\lim_{s\to-\infty} \nu(g(s)) = \nu(g_E) < \nu(g(0)).$$

Thus the entropy is not constant, and the flow is not stationary modulo diffeomorphisms.

Set

$$\rho(s) = \frac{1}{n\,\mathrm{Vol}(M, g(s))}\int_M R_{g(s)}\, dv_{g(s)}.$$

Then $\rho(s) \to \Lambda$ and $\partial_s g = -2\,\mathrm{Ric}_{g(s)} + 2\rho(s)g(s)$. Define

$$c(s) = \exp\left(-2\int_0^s \rho(u)\,du\right), \qquad t(s) = -\int_s^0 c(u)\,du, \qquad \widehat{g}(t(s)) = c(s)g(s). \tag{8.6}$$

We have $dt/ds = c(s) > 0$ and $c' = -2\rho c$. Since covariant Ricci curvature is invariant under spatially constant scaling,

$$\partial_t \widehat{g} = c^{-1}(c'g + c\partial_s g) = -2\,\mathrm{Ric}_{\widehat{g}}\,.$$

For sufficiently negative s, $\rho(s) \ge \Lambda/2$. It follows that both $c(s)$ and $-t(s)$ tend to infinity as $s \to -\infty$. The quotient of their derivatives is $2\rho(s)$; l'Hôpital's rule gives

$$\frac{c(s)}{-t(s)} \longrightarrow 2\Lambda. \tag{8.7}$$

Combining the normalized convergence with (8.7) proves (8.3).

The C^r convergence, $r \ge 3$, bounds the curvature of $g(s)$ for sufficiently negative s; smoothness bounds it on the remaining compact time interval. Since

$$|\,\mathrm{Rm}_{\widehat{g}(t(s))}|_{\widehat{g}(t(s))} = c(s)^{-1}|\,\mathrm{Rm}_{g(s)}|_{g(s)},$$

and $(-t(s))/c(s)$ is bounded, (8.4) follows. The entropy is invariant under scaling and diffeomorphisms. A self-similar unnormalized flow would therefore have constant ν, contrary to its construction. □

The estimate (8.4) is a backward Type-I curvature bound. The chosen endpoint $t = 0$ is a regular time; no maximal finite-time singularity is asserted there. The nonlinear implication in Corollary 8.1 uses [9]; it does not require integrability of infinitesimal Einstein deformations.

Appendix A. Ricci-flat asymptotically conical metrics

Let (Z^n, h) be closed and connected, with $\mathrm{Ric}_h = (n-1)h$, $n \ge 3$. Put

$$\bar{G} = dr^2 + r^2 h \quad \text{on } (0,\infty)\times Z.$$

This metric is Ricci-flat. On a Ricci-flat manifold use $\mathcal{E}_G = \nabla^*\nabla - 2\mathring{R}_G$ and its quadratic form Q_G. In this appendix, linear stability means $Q_G(T) \ge 0$ for all $T \in C_c^\infty(S^2T^*X)$. The vertex of the cone is excluded from every test support. The separated TT form below is the one in [10, Section 3.2]; see also [3, Lemma 3.13]. Its spectral parameter is expressed here in terms of the full Lichnerowicz operator on the link.

Lemma A.1. *Let $L_h k = \lambda k$, where $k \ne 0$ is a smooth TT tensor on Z. For $f \in C_c^\infty((0,\infty))$, set $T = r^2 f(r) k$ and $T(\partial_r, \cdot) = 0$. Then T is TT on the regular cone and*

$$Q_{\bar{G}}(T) = \|k\|^2_{L^2(Z,h)} \int_0^\infty \left[r^n (f')^2 + (\lambda - 2(n-1))r^{n-2} f^2\right] dr. \tag{A.1}$$

Proof. Choose $e_0 = \partial_r$ and $e_i = r^{-1}\bar{e}_i$, with the $\bar{e}_i$ orthonormal for h. The cone connection has $\nabla_{e_i} e_0 = r^{-1}e_i$ and $(\nabla_{e_i} e_j)^\perp = -r^{-1}\delta_{ij}e_0$. Consequently the radial tensor derivative is $f'k$, the tangential ones have components $r^{-1}f\nabla^h k$, and their mixed tensor slots contribute $2r^{-2}f^2|k|_h^2$. Thus

$$|\nabla T|^2_{\bar{G}} = (f')^2|k|_h^2 + r^{-2}f^2\left(|\nabla^h k|_h^2 + 2|k|_h^2\right).$$

The trace is zero. The tangential divergence is a multiple of $\delta_h k = 0$, and the radial divergence is a multiple of $\mathrm{tr}_h\, k = 0$. The tangential cone curvature is r^{-2} times the link curvature minus the unit constant-curvature tensor. Its contraction on a trace-free tensor gives

$$\langle \mathring{R}_{\bar G} T, T\rangle_{\bar G} = r^{-2} f^2 \big(\langle \mathring{R}_h k, k\rangle_h + |k|_h^2\big).$$

Using $dv_{\bar G} = r^n\, dr\, dv_h$ and $\langle(\nabla_h^*\nabla_h - 2\mathring{R}_h)k, k\rangle_{L^2} = (\lambda - 2(n-1))\|k\|_{L^2}^2$ proves (A.1). □

Theorem A.2. *Suppose L_h has a TT eigenvalue satisfying*

$$\lambda < \frac{(n-1)(9-n)}{4}. \tag{A.2}$$

Let (X^{n+1}, G) be complete, connected, smooth, without boundary, and Ricci-flat. Assume that there are a compact smooth domain $K \subset X$, $R_0 > 0$, and a diffeomorphism of manifolds with boundary

$$\Phi : [R_0, \infty) \times Z \longrightarrow X \setminus \operatorname{int} K$$

such that, with $D_R(r, z) = (Rr, z)$ and $\Phi_R = \Phi \circ D_R$,

$$G_R := R^{-2}\Phi_R^* G \longrightarrow \bar G \quad \text{in } C^2 \tag{A.3}$$

on every fixed compact annulus, as $R \to \infty$. Then the Friedrichs realization of $\mathcal{E}_G$ has infinitely many negative L^2 eigenvalues of finite multiplicity, accumulating only at zero. The associated eigentensors are smooth and TT. If $N_G(\epsilon)$ is the number of eigenvalues below $-\epsilon$, counted with multiplicities, then

$$N_G(\epsilon) \ge c_1 \log(1/\epsilon) - c_2 \quad (0 < \epsilon < \epsilon_0) \tag{A.4}$$

for constants $c_1 > 0$, $c_2 < \infty$, $\epsilon_0 > 0$.

Proof. In Lemma A.1 put $f(r) = r^{-(n-1)/2}\chi(\log r)$. With $s = \log r$, integration of the mixed term gives

$$Q_{\bar G}(T) = \|k\|_{L^2}^2 \int_{\mathbb{R}} \big[(\chi')^2 - \kappa\chi^2\big]\, ds, \qquad \kappa = \frac{(n-1)(9-n)}{4} - \lambda > 0. \tag{A.5}$$

For $L_0 > \pi/\sqrt{\kappa}$, the function $\sin(\pi s/L_0)$ on $(0, L_0)$, extended by zero, belongs to $H^1(\mathbb{R})$ and has negative energy in (A.5). Approximation in H^1 gives a smooth compactly supported χ with negative energy. Fix the resulting tensor T, with support in $(a, b) \times Z$, where $0 < a < b < \infty$.

Equation (A.3) implies $Q_{G_R}(T) \to Q_{\bar G}(T) < 0$ and $\|T\|_{L^2(G_R)}^2 \to \|T\|_{L^2(\bar G)}^2 > 0$. Define T_R by $\Phi_R^* T_R = R^2 T$, extended by zero. It is smooth and compactly supported, and

$$\begin{aligned} \|T_R\|_{L^2(G)}^2 &= R^{n+1}\|T\|_{L^2(G_R)}^2, \\ Q_G(T_R) &= R^{n-1} Q_{G_R}(T). \end{aligned} \tag{A.6}$$

For some $c > 0$ and all sufficiently large R,

$$Q_G(T_R) \le -cR^{-2}\|T_R\|_{L^2(G)}^2. \tag{A.7}$$

Choose $B > b/a$ and $R_j = R_* B^j$, with R_* large. Their supports are disjoint, so both the L^2 pairing and the quadratic-form cross terms vanish. The spans of the first j tensors are negative definite for every j in $C_c^\infty(S^2T^*X)$, without imposing the TT constraint.

On $[1, 2] \times Z$, (A.3) bounds the rescaled curvature uniformly. Thus $|\mathrm{Rm}_G|_G = O(r^{-2})$ on the entire end. It follows that $V = -2\mathring{R}_G$ is bounded and tends to zero outside compact sets. Let $A = \nabla^*\nabla \ge 0$ be the Friedrichs realization of the connection Laplacian on $C_c^\infty(S^2T^*X)$. Its closed quadratic form is the closure of $T \mapsto \|\nabla T\|_2^2$. Since V is bounded, adding $\langle VT, T\rangle_{L^2}$ leaves this form domain unchanged. The representation theorem for closed forms identifies the Friedrichs realization of $\mathcal{E}_G$ with $A + V$; its operator domain is $\mathrm{Dom}(A)$, and it is bounded below by $-\|V\|_{L^\infty}$.

The operator $V(A+1)^{-1}$ is compact. Indeed, the resolvent maps bounded L^2 sets into sets bounded in the graph norm of A. Interior elliptic estimates give bounded H^2 norms on any fixed compact set, and Rellich's theorem gives L^2 precompactness there. On its complement the operator norm is bounded by $\sup|V|\,\|(A+1)^{-1}\|$, which tends to zero as the compact set exhausts X. Hence the compactly supported approximations converge in operator norm. Weyl's essential-spectrum theorem yields

$$\sigma_{\mathrm{ess}}(A+V)=\sigma_{\mathrm{ess}}(A)\subset[0,\infty). \tag{A.8}$$

The negative subspaces and the min–max principle give infinitely many discrete negative eigenvalues. They have finite multiplicity, are bounded below, and can accumulate only at zero by (A.8).

Let $(A+V)H=\mu H$, with $\mu<0$. Elliptic regularity makes H smooth. Boundedness of V gives $H\in\mathrm{Dom}(A)$, and therefore $\nabla H\in L^2$. The Ricci-flat trace and divergence commutations imply

$$\Delta\operatorname{tr}H=\mu\operatorname{tr}H,\qquad \nabla^*\nabla\delta H=\mu\delta H.$$

Both sections u belong to L^2 and are smooth. Fix $o\in X$ and choose $\vartheta\in C^\infty([0,\infty),[0,1])$ equal to one on $[0,1]$ and zero on $[2,\infty)$. Set $\chi_R(x)=\vartheta(d_G(o,x)/R)$. Hopf–Rinow makes its support compact; it is Lipschitz and $|d\chi_R|\le\|\vartheta'\|_\infty/R$ almost everywhere. For each fixed R, χ_R^2u is an admissible compactly supported H^1 test section, by smooth approximation on its compact support. Testing either equation gives

$$\|\nabla(\chi_R u)\|_2^2-\mu\|\chi_R u\|_2^2=\int_X|d\chi_R|^2|u|^2\,dv_G.$$

The right side tends to zero. Since $-\mu>0$, dominated convergence gives $u=0$. Thus every negative eigentensor is TT.

Finally, for $cR_*^{-2}B^{-2j}>\epsilon$, the span of the corresponding tensors is negative for $Q_G+\epsilon\|\cdot\|_2^2$, by (A.7). Counting such integers $j\ge0$ and using min–max gives (A.4), with any fixed $c_1<1/(2\log B)$ after adjusting c_2,ϵ_0. □

Corollary A.3. *Let (Z^n,h) be a compact nonhomogeneous Böhm metric with $5\le n\le9$ and $\mathrm{Ric}_h=(n-1)h$. Every Ricci-flat metric satisfying the complete-end hypothesis of Theorem A.2 with link (Z,h) has infinite negative index. No such metric is linearly stable or carries a nonzero parallel spinor.*

Proof. By Corollary 1.3, the link has a TT eigenvalue $\lambda<0$. The right side of (A.2) is nonnegative for $5\le n\le9$, so Theorem A.2 applies.

Suppose that a spin structure on X admits a nonzero parallel spinor σ. Its length is constant because X is connected; normalize $|\sigma|=1$. Let ΣX be the complex spinor bundle and put

$$\mathcal{S}=\Sigma X\otimes_{\mathbb{C}}(T^*X\otimes_{\mathbb{R}}\mathbb{C}),\qquad \nabla^{\mathcal{S}}=\nabla^{\Sigma X}\otimes1+1\otimes\nabla^{T^*X\otimes\mathbb{C}}.$$

Clifford multiplication acts on the first factor. The twisted Dirac operator and its formal $L^2(G)$ adjoint are denoted by $\mathcal{D}$ and $\mathcal{D}^*$, where

$$\mathcal{D}\Psi=\sum_{a=1}^{n+1}e_a\cdot\nabla^{\mathcal{S}}_{e_a}\Psi.$$

Define

$$\Phi(H)=\sum_{i,j}H_{ij}\,e_i\cdot\sigma\otimes e^j.$$

The definition is independent of the orthonormal frame. The Clifford relations imply

$$\mathrm{Re}\langle e_i\cdot\sigma,e_k\cdot\sigma\rangle=\delta_{ik},$$

so $\mathrm{Re}\langle\Phi(H),\Phi(T)\rangle=\langle H,T\rangle$ for real symmetric tensors. Since σ is parallel, Φ is parallel. With the curvature action of (1.1), the local identity [11, Lemma 2.4, equation (2.21)] is

$$\mathcal{D}^*\mathcal{D}\Phi(H)=\Phi(\mathcal{E}_GH).$$

For compactly supported H, integration gives $Q_G(H)=\|\mathcal{D}\Phi(H)\|_2^2\ge0$. This contradicts (A.7). □

The complete complement of a compact set is required in Theorem A.2; an uncontrolled additional end would not justify (A.8). The corollary is conditional on existence of a Ricci-flat filling, and excludes stable and parallel-spinor fillings, not all fillings. For the sphere metric on S^{11}, Corollary 7.2 proves only $\lambda_{\mathrm{TT}}<0$, whereas (A.2) requires $\lambda<-5$.

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