DIRICHLET DEFORMATIONS, MORSE INDEX, AND NONLINEAR SYMMETRY OF RICCI-FLAT BÖHM METRICS

ANASS NIFA

Abstract. Let g be a complete Ricci-flat Böhm metric on $\mathbb{R}^{p+1} \times S^q$, where $p, q \geq 2$ and $p + q \leq 8$. We classify the geometric Dirichlet kernel on every compact truncation and prove that the reduced Morse index counts the preceding umbilic orbits. Every negative L^2 Lichnerowicz eigentensor is $O(p+1) \times O(q+1)$-invariant, and the quadratic form is coercive in homogeneous energy on the complementary subspace. These results imply symmetry of sufficiently small ancient Ricci–DeTurck solutions whose non-invariant component is bounded in L^2, without an assumption of backward convergence. They also give stationary rigidity in a fixed background gauge. Each degeneracy of the smooth induced-metric map is a fold. Arbitrarily large finite collections of pairwise nonisometric Ricci-flat fillings, with arbitrarily large reduced index, persist under nonsymmetric boundary perturbations. The reduced negative spectrum converges to the complete negative spectrum; a transverse Dirichlet-to-Neumann form has a positive-measure representation. All scalar coefficient identities are given explicitly, with their exact integer data.

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Corresponding author: Anass Nifa.

Contents

1. Main results

The existence of negative Lichnerowicz eigenvalues does not determine whether an unstable Ricci-flat metric has symmetry-breaking negative modes. If a compact group acts isometrically, the Lichnerowicz operator commutes with its action. Each isotypic subspace is therefore preserved, but a negative eigenvalue may still occur in a nontrivial representation. The question considered here is whether this occurs for the complete Ricci-flat metrics constructed by Böhm on $\mathbb{R}^{p+1} \times S^q$. Angenent–Knopf proved that these metrics have infinitely many negative eigenvalues in the sign convention used below, and proved simplicity within the invariant sector [6, Main Theorem II]. We prove that there are no negative eigentensors outside that sector.

The proof uses compact truncations with geometric Dirichlet data. The boundary condition fixes the induced metric; it does not set every component of a symmetric tensor equal to zero. Infinitesimal coordinate changes and conformal variations must therefore be removed before the corresponding Hessian can be compared with the complete Lichnerowicz operator. We construct this reduced form, identify its kernel with the geometric Dirichlet kernel, and prove convergence of its counts below each fixed negative threshold. The resulting absolute index is the number of umbilic principal orbits preceding the boundary.

There are three principal conclusions. Theorem A classifies all geometric Dirichlet degeneracies and their folds. Theorem D determines the full negative spectrum and the reduced index. Theorem F gives coercivity on the non-invariant homogeneous-energy space. Theorems B and C supply the operator realization and crossing form used in this classification. The filling and nonlinear consequences are Theorems E, G, and H; Theorem I concerns the associated full-tensor boundary problem. The proof dependencies, including the order of the representation and radius quantifiers, are explained in Section 1.6.

The full spectral symmetry theorem concerns a complete metric with specified behavior at infinity. The filling theorem concerns compact manifolds with perturbed boundary metrics. Thus the absence of non-invariant negative modes on the complete space is compatible with families of nonsymmetric compact Ricci-flat fillings. The finite collections in the latter theorem vary over neighborhoods depending on their cardinality and index bounds. No fixed neighborhood carrying infinitely many simultaneous filling branches is asserted.

1.1. The metric and geometric boundary deformations.

Fix integers $p, q \geq 2$, put $m = p + q$ and $d = m + 1$, and assume $m \leq 8$. The metrics γ_p, γ_q have sectional curvature one. Let

$$g = dt^2 + a(t)^2\gamma_p + b(t)^2\gamma_q, \qquad \mathrm{Ric}_g = 0, \tag{1.1}$$

be the complete noncompact Böhm metric on $X = \mathbb{R}^{p+1} \times S^q$. We fix its scale by prescribing $b(0) > 0$. At the singular orbit,

$$a(t) = t\alpha(t^2), \quad b(t) = \beta(t^2), \quad \alpha(0) = 1, \quad \beta(0) > 0, \tag{1.2}$$

where α, β are smooth. For $R > 0$, the set $M_R = \{t \leq R\}$ includes the singular orbit. It is a smooth compact manifold with boundary $\Sigma_R = \{t = R\}$ and is diffeomorphic to $D^{p+1} \times S^q$.

All boundary metrics in the spectral statements are induced by the unscaled restrictions of g. Write $h^\top$ for the restriction of a symmetric two-tensor to $T\Sigma_R$. Let

$$K_R = \frac{\{h \in C^\infty(S^2T^*M_R) \,:\, \mathrm{Ric}'_g(h) = 0,\ h^\top = 0\}}{\{\mathcal{L}_Y g \,:\, Y \in C^\infty(TM_R),\ Y|_{\Sigma_R} = 0\}}. \tag{1.3}$$

The group $\mathfrak{G} = O(p+1) \times O(q+1)$ acts by pullback.

Theorem A (Geometric Dirichlet deformations). *For every $R > 0$, $K_R = K_R^{\mathfrak{G}}$. More precisely,*

$$\dim K_R = \begin{cases} 0, & a'(R)/a(R) \neq b'(R)/b(R), \\ 1, & a'(R)/a(R) = b'(R)/b(R). \end{cases} \tag{1.4}$$

At a radius in the second case, the induced-metric map on the full smooth Ricci-flat moduli space, modulo diffeomorphisms equal to the identity on the boundary, is a fold. In smooth tame local coordinates it has the form

$$(y, s) \longmapsto (y, s^2). \tag{1.5}$$

The geometric existence and conical-limit result is Böhm's theorem [7, Theorem 6.1 and Sections 10–11]. The deformation and boundary-moduli results used for the fold conclusion are stated in Section 9. The two-factor scalar estimate is proved in Section 8. Its higher-harmonic and exceptional coefficient identities are given in full in Sections F to I. The algebraic inequalities hold for all real $p, q \geq 2$ in their stated spectral ranges. The geometric theorems retain the dimensions fixed above.

1.2. **The reduced Hessian, index, and full negative spectrum.** For a compact Ricci-flat manifold (M, g) of dimension d, define

$$\mathcal{W}(M, g) = \{T \in C^\infty(S^2 T^*M) \,:\, \delta T = 0,\ \operatorname{tr} T = 0,\ (T^\top)_0 = 0\}. \tag{1.6}$$

The subscript 0 denotes the trace-free tangential part. If $T \in \mathcal{W}$, let f_T be the harmonic function determined by

$$\Delta f_T = 0, \qquad f_T|_{\partial M} = \frac{T(\nu, \nu)}{d-1}. \tag{1.7}$$

Thus $T^\top = -f_T\gamma$, where $\gamma = g^\top$ and ν is the outward unit normal. Put

$$q_M(T) = \int_M \langle LT - (d-2)\nabla^2 f_T, T\rangle\, dv_g, \qquad L = \nabla^*\nabla - 2\mathring{R}. \tag{1.8}$$

This is twice the Hessian of the negative Einstein–Hilbert action after eliminating the zero-boundary conformal factor; see Section 10. The Hilbert norm used for this form is the L^2 norm of the TT representative, not the L^2 norm of $T + f_T g$.

Theorem B (The reduced self-adjoint realization). *For every $R > 0$, q_{M_R} is closable and bounded below on*

$$\mathscr{H}_R = \overline{\mathcal{W}(M_R, g)}^{L^2}.$$

Its closure defines a self-adjoint operator A_R with compact resolvent. Its eigentensors are smooth up to the boundary, and

$$\ker A_R \simeq K_R. \tag{1.9}$$

Let L_X be the Friedrichs realization of L on all symmetric tensors on the complete manifold X. For $\varepsilon > 0$ with $-\varepsilon \notin \sigma(L_X)$,

$$\#\{\lambda \in \sigma(A_R) \,:\, \lambda < -\varepsilon\} = \#\{\lambda \in \sigma(L_X) \,:\, \lambda < -\varepsilon\} \tag{1.10}$$

for all sufficiently large R. Multiplicities are included.

Every negative L^2 eigentensor of L_X is TT. The count on the right of (1.10) is finite. The negative eigenvalues of L_X, arranged increasingly with multiplicity, are therefore limits of the correspondingly numbered eigenvalues of A_R. No monotonicity of these latter eigenvalues is asserted.

Theorem C (The geometric crossing form). *Let R_* be an umbilic radius. For the geometric Hessian $B = q/2$, its crossing form under outward displacement of the boundary with normal speed $v_* > 0$ is*

$$\Gamma_B([h], [k]) = -2 \int_{\Sigma_{R_*}} v_* \big(\langle A_g'(h), A_g'(k)\rangle - \operatorname{tr} A_g'(h)\, \operatorname{tr} A_g'(k)\big)\, dA. \tag{1.11}$$

It is negative definite on K_{R_*}. The fold has one-dimensional kernel by Theorem A.

Let $\mathcal{P}$ be normalized group averaging on the tensor spaces. It is an orthogonal projection commuting with the relevant forms and operators.

Theorem D (The full negative spectrum and absolute index). *The closed cap forms have smooth local Hilbert form domains on $(0,\infty)$. Every negative eigentensor of A_R, for every $R>0$, is invariant under $\mathfrak{G}$. Moreover,*

$$\operatorname{ind}^- A_R = \#\left\{ t\in(0,R) \,:\, \frac{a'(t)}{a(t)} = \frac{b'(t)}{b(t)} \right\}. \tag{1.12}$$

At a degenerate endpoint the zero eigenvalue is not counted. On the complete manifold,

$$\operatorname{ran} \mathbf{1}_{(-\infty,0)}(L_X) \subset L^2(S^2T^*X)^{\mathfrak{G}}. \tag{1.13}$$

Every complete negative eigenvalue has a one-dimensional full eigenspace. In particular,

$$\operatorname{ind}^- A_R = \frac{\sqrt{(m-1)(9-m)}}{2\pi}\log R + O(1) \qquad (R\longrightarrow\infty).$$

1.3. **Filling multiplicity and nonlinear symmetry.**

Theorem E (Persistence of filling multiplicity). *Set $M = D^{p+1}\times S^q$ and $\gamma_* = (p-1)\gamma_p + (q-1)\gamma_q$ on ∂M. For every $N, k\geq 1$ there is a C^∞-open neighborhood $\mathcal{U}_{N,k}$ of γ_* such that every metric in $\mathcal{U}_{N,k}$ admits at least N pairwise nonisometric smooth Ricci-flat fillings on M, each with strictly convex boundary and reduced Morse index at least k. The neighborhood contains a nonempty open subset on which the boundary metrics and all these fillings have trivial isometry group.*

The initial-index calculation in Sections 18 to 20 uses the conformally reduced form, including its harmonic trace. For each fixed irreducible representation ϖ, there is $R_\varpi>0$ such that the corresponding cap form is positive for $0<R<R_\varpi$. The radius need not be uniform in ϖ. The full kernel classification then transports this positivity on every nontrivial representation to all radii; the trivial representation supplies the initial value in (1.12). Complete simplicity additionally uses the invariant-sector theorem of Angenent–Knopf [6, Main Theorem II(f)]. Theorem E concerns a finite collection on an open set depending on N and k, not infinitely many branches on one fixed open set.

Gibbons, Hartnoll and Pope [11, Section 6] proposed a relation between negative Lichnerowicz modes and nonuniqueness of Einstein fillings. Their discussion concerns generic creation of pairs of branches; it also includes infinite global filling multiplicity. Angenent and Knopf [6, Main Theorem II] proved infinite discrete negative spectrum for the complete metrics (1.1), in the opposite operator sign convention, as well as simplicity in the invariant sector. Neither assertion classifies the full kernel (1.3). For compact positive-Einstein Böhm metrics, Verger [15] proves divergence of the negative TT index along the degenerating sequences. The compact negative-mode results in [14] also concern closed positive-Einstein manifolds. Neither result is used here. The assertions proved below concern complete Ricci-flat metrics, their geometric boundary kernels, and the exclusion of non-invariant negative modes. The boundary problem in this memoir fixes the induced metric; it is not componentwise tensor Dirichlet data and is not the mixed conformal-class/mean-curvature problem.

Theorem D follows from Theorems 14.2, 20.1, 21.2 and 21.4; Theorem E is Theorem 22.1. All Ricci-flat variation, gauge, and spectral arguments used in these implications are given below.

Homogeneous energy and the nonlinear equation. For the complete metric, put $\rho=(1+t^2)^{1/2}$ and let $\mathscr{E}$ be the completion of compactly supported smooth symmetric tensors in

$$\|h\|_{\mathscr{E}}^2 = \int_X \left(|\nabla h|^2 + \rho^{-2}|h|^2\right) dv_g. \tag{1.14}$$

For smooth tensors also put

$$\|h\|_{\mathscr{X}} = \sup_X \left(|h| + \rho|\nabla h| + \rho^2|\nabla^2 h|\right). \tag{1.15}$$

All derivatives and norms use the fixed background g.

Theorem F (Transverse homogeneous coercivity). *There is $c_g > 0$ such that*

$$Q_g(h) \geq c_g \|h\|_{\mathscr{E}}^2 \qquad (h \in \mathscr{E},\ \mathcal{P}h = 0). \tag{1.16}$$

In particular, a distributional solution $L_g h = 0$ with $h \in \mathscr{E}$ and $\mathcal{P}h = 0$ is zero. The estimate controls homogeneous energy; it does not assert a positive L^2 spectral gap.

For $\widetilde{g} = g + h$, define the background DeTurck vector field by

$$\mathcal{W}_g(\widetilde{g})^k = \widetilde{g}^{ij}\big(\Gamma(\widetilde{g})_{ij}^k - \Gamma(g)_{ij}^k\big).$$

Its linear contribution to $-2\operatorname{Ric}(\widetilde{g}) + \mathcal{L}_{\mathcal{W}_g(\widetilde{g})}\widetilde{g}$ is $-L_g h$.

Theorem G (Ancient symmetry). *There is $\varepsilon_g > 0$ with the following property. Let $g + h(\tau)$, $-\infty < \tau \leq 0$, be a smooth solution of*

$$\partial_\tau(g+h) = -2\operatorname{Ric}(g+h) + \mathcal{L}_{\mathcal{W}_g(g+h)}(g+h). \tag{1.17}$$

Put $w = (I - \mathcal{P})h$ and assume

$$w \in C^1_{\mathrm{loc}}((-\infty,0];L^2) \cap C^0_{\mathrm{loc}}((-\infty,0];H^2), \qquad \sup_{\tau\leq 0}\|h(\tau)\|_{\mathscr{X}} \leq \varepsilon_g, \qquad \sup_{\tau \leq 0}\|w(\tau)\|_2 < \infty.$$

Then $h(\tau)$ is invariant for every $\tau \leq 0$. No backward convergence is assumed. The invariant component need not belong to L^2. The DeTurck diffeomorphisms normalized by the identity at time zero commute with $\mathfrak{G}$, and the associated Ricci flow is invariant.

Theorem H (Stationary symmetry). *Let h be smooth with $\|h\|_{\mathscr{X}} \leq \varepsilon_g$ and $(I - \mathcal{P})h \in \mathscr{E}$, after decreasing the constant in Theorem G if necessary. If*

$$\operatorname{Ric}(g+h) = 0, \qquad \mathcal{W}_g(g+h) = 0,$$

then $g + h$ is invariant under $\mathfrak{G}$.

Theorems F–H are proved in Sections 23 to 27. Theorem F follows from exterior coercivity and the vanishing of the finite-energy transverse kernel. The sector identities in Sections 5, 6 and 8 prove this vanishing after the infinitesimal diffeomorphism terms have been removed. Theorem G uses the abstract ancient-solution lemma 27.2, in addition to the nonlinear difference estimate. Theorem H uses the same difference estimate in the energy dual. The fixed background gauge and all global norms in these statements are hypotheses. No assertion is made for arbitrary pointed ancient limits.

Angenent–Knopf [6, Main Theorem I] construct finite-dimensional ancient families after a positive spectral threshold has been chosen. Theorem G concerns every existing solution in its stated global class and requires no temporal rate. These are different hypotheses: the little-Hölder spaces in [6] are not identified here with the weighted spaces used above. Homogeneous coercivity and nonlinear estimates for linearly stable ALE Ricci-flat metrics are developed in [10]. The present metrics are asymptotically conical and have infinitely many negative eigenvalues. The cone indicial theory of [12] gives the corresponding indicial roots.

1.4. **A consequence for the full-tensor boundary operator.**

For the next statement, the boundary datum is a section of $S^2T^*X|_{\Sigma_R}$, not merely a tensor on $T\Sigma_R$. Let $\mathcal{T}_R = \ker \mathcal{P} \cap H^{1/2}(S^2T^*X|_{\Sigma_R})$. For $\lambda \geq 0$, let $M_R(\lambda)$ be the sum of the interior and exterior Dirichlet-to-Neumann maps for $L + \lambda$. The interior solution is smooth at the singular orbit; the exterior solution has finite homogeneous energy and also finite L^2 norm when $\lambda > 0$. The variational construction is given in Section 28.

Theorem I (A consequence for boundary traces). *The operator $M_R(\lambda)$ is a bounded symmetric isomorphism $\mathcal{T}_R \to \mathcal{T}_R^*$ for every $R > 0$ and $\lambda \geq 0$. There are $c, R_0 > 0$ such that*

$$\langle M_R(\lambda)f, f\rangle \geq c(R^{-1} + \sqrt{\lambda})\|f\|^2_{L^2(\Sigma_R)} \qquad (R \geq R_0,\ \lambda \geq 0).$$

For each $f \in \mathcal{T}_R$, a positive measure ν_f on $(0,\infty)$ satisfies

$$\langle M_R(\lambda)f, f\rangle = \langle M_R(0)f, f\rangle + \int_{(0,\infty)} \frac{\lambda}{\lambda + \xi}\, d\nu_f(\xi), \qquad \int \frac{d\nu_f(\xi)}{1+\xi} < \infty.$$

Theorem I is a consequence of transverse coercivity and the spectral theorem. It concerns the finite-energy realization on the fixed exterior. General Dirichlet-to-Neumann coupling formulas are studied in [2]. An ancient solution on an unspecified exterior is not identified here with that realization.

1.5. **Organization and dependence of the results.** The boundary conditions in this memoir define four distinct objects:

Object	Space	Boundary condition or realization
K_R	Infinitesimal Ricci-flat metrics	$h^\top = 0$, modulo Lie derivatives of vector fields vanishing on Σ_R.
A_R	Closure of $\mathcal{W}(M_R, g)$ in L^2	The conformally reduced form q_{M_R}, with its harmonic trace correction.
L_X	$L^2(S^2T^*X)$	The Friedrichs realization on the complete manifold.
$M_R(\lambda)$	$\mathcal{T}_R \to \mathcal{T}_R^*$	All tensor components prescribed on Σ_R; finite homogeneous energy on the exterior.

In particular, the form defining A_R is not the componentwise Dirichlet form of L. Its kernel is identified with K_R only after the conformal and infinitesimal-diffeomorphism terms have been eliminated.

1.6. **Structure of the proofs.** Fix a nontrivial irreducible representation ϖ of $\mathfrak{G}$. The harmonic decomposition reduces the Dirichlet kernel to tensor, vector, and scalar systems. Their boundary conditions are derived from $h^\top = 0$ before an auxiliary gauge is chosen. In the two-factor scalar system, the elimination of the constraints gives a positive leading matrix and a completed quadratic form. Its derivative term has zero limits at the singular orbit and at the Dirichlet boundary. The remaining coefficients are positive, so the two gauge-invariant unknowns vanish. The low scalar degrees and the vector and tensor systems are treated separately. This proves $K_{R,\varpi} = 0$ for every $R > 0$; the invariant calculation gives the one-dimensional kernel precisely at the umbilic radii.

The kernel calculation alone does not determine a negative index. The reduced form must first be realized on its own constrained domain. In the compatible-pair coordinates of Sections 12 and 13, its mass is the norm of the TT representative. The right inverse of the divergence constraint gives exact projections and local identifications of these domains. The flat normal model is strictly positive. After ϖ has been fixed, its angular derivatives are bounded operators on the finite angular spaces, and the rescaled constrained form converges in bilinear-form norm to that model. Thus

$$\forall\varpi \neq \mathbf{1} \quad \exists R_\varpi > 0 \quad q_R|_\varpi > 0 \qquad (0 < R < R_\varpi).$$

The vanishing of $K_{R,\varpi}$ makes the negative index locally constant on $(0, \infty)$. It is therefore zero at every radius. A negative cap eigenspace is finite-dimensional; any non-invariant vector in it would generate a nontrivial irreducible summand with negative form, a contradiction. This argument does not require a radius uniform in ϖ. The trivial representation gives the initial index zero, and each umbilic crossing increases that index by one. Equivariant spectral exhaustion then gives the complete negative spectral classification.

For homogeneous coercivity, three separate assertions are needed: nonnegativity on the transverse space, a coercive estimate outside a compact set, and vanishing of its finite-energy distributional kernel. The cone angular calculation proves the exterior estimate, including the critical indicial value. The sector identities, evaluated in the original smooth gauge, exclude finite-energy zero modes. A compactness argument and the localization identity then give $Q \geq c_g\|\cdot\|_{\mathscr{E}}^2$ on $\ker\mathcal{P}$. The nonlinear difference estimate is bounded by $C\varepsilon\|w\|_{\mathscr{E}}\|v\|_{\mathscr{E}}$. Choosing $C\varepsilon < c_g$ gives transverse dissipation. The ancient-solution lemma uses the spectral square-function identity, not an L^2 spectral gap, to conclude that a bounded ancient transverse component vanishes.

Finally, the curvature boundary coordinates identify the full induced-metric degeneracies with folds of an elliptic map. The regular normalized truncations give local inverses on unrestricted smooth boundary data. A finite intersection of their neighborhoods preserves strict convexity, the chosen finite-dimensional negative subspaces, and distinct values of $\int_\Sigma H\,dA$. This proves pairwise nonisometry and the precise quantifiers in Theorem E.

Conclusion	Inputs used in its proof
Geometric kernel and folds	Sections 4 to 6, 8 and 9.
Reduced form and index	Sections 10, 12 to 14 and 17 to 20.
Complete spectral symmetry	Sections 16 and 21.
Finite filling multiplicity	Sections 9 and 22.
Homogeneous coercivity	Sections 23 to 25.
Nonlinear and boundary consequences	Sections 26 to 28.

1.7. **The case $p = q = 2$.** Here $X = \mathbb{R}^3 \times S^2$, $M_R \simeq D^3 \times S^2$, and the principal orbit is $S^2 \times S^2$. Write γ_P, γ_Q for its two curvature-one factor metrics. The limiting cone and the normalized boundary metric are, respectively,

$$g_C = dt^2 + \frac{t^2}{3}(\gamma_P + \gamma_Q), \qquad \gamma_* = \gamma_P + \gamma_Q.$$

They are different normalizations. In particular, $\lambda_1(\gamma_*) = 2 > 4/3$, whereas the first positive eigenvalue of the cone link is 6.

The scale and shape coordinates for a scaled truncation $c^2 g|_{M_R}$ are

$$s = \log c + \frac{\log a(R) + \log b(R)}{2}, \qquad z = \log a(R) - \log b(R),$$

and its boundary metric is

$$e^{2s}\big(e^z \gamma_P + e^{-z}\gamma_Q\big).$$

The umbilic condition is $z'(R) = u(R) - v(R) = 0$. At such a radius,

$$z''(R) = a(R)^{-2} - b(R)^{-2} \neq 0.$$

Indeed, equality would make the Ricci-flat Cauchy data those of a shifted cone, contrary to $a(0) = 0$ and $b(0) > 0$; this is the uniqueness argument in Proposition 9.1. Theorem A therefore gives the fold and the one-dimensional geometric kernel there. A representative is $\mathcal{L}_Y g - 2g$, where Y is a smooth radial vector field with $Y|_{\Sigma_R} = \kappa^{-1}\nu$ and $u(R) = v(R) = \kappa$.

By contrast, the normalized fillings of γ_* occur when $z = 0$, not when $z' = 0$. At a zero of z, one has $z' \neq 0$ by the same uniqueness argument. Taking $c = (a(R)b(R))^{-1/2}$ gives $s = 0$ and boundary metric γ_*. These regular normalized truncations are the starting points for the filling branches in Theorem E.

Here $\alpha = 3/2$ and $\beta = \sqrt{15}/2$ in Lemma 3.3. Substitution in Theorem D gives

$$\operatorname{ind}^- A_R = \#\{t \in (0, R) : u(t) = v(t)\} = \frac{\sqrt{15}}{2\pi}\log R + O(1).$$

The group in the spectral and nonlinear statements is $O(3) \times O(3)$. Theorem E gives, for every N, k, an open neighborhood of $\gamma_P + \gamma_Q$ with at least N pairwise nonisometric fillings of reduced index at least k. No numerical location of an umbilic radius is needed for any of these conclusions.

1.8. **Principal notation.** The following symbols retain these meanings throughout the memoir. A scalar letter explicitly redefined inside a matrix calculation is local to that calculation. The notation A_R refers to the reduced operator; II_R denotes the second fundamental form in the boundary displacement calculation. Irreducible representations are denoted by ϖ, and the global energy weight by ρ.

Symbol	Meaning	Definition
X, g	Complete Ricci-flat Böhm manifold and fixed metric	(1.1)
M_R, Σ_R, γ_R	Truncation, its boundary, and induced metric	Section 1

Symbol	Meaning	Definition
p, q, m, d	Factor dimensions, $m = p + q$, $d = m + 1$	Section 1
u, v, H, W	a'/a, b'/b, $pu + qv$, $a^p b^q$	(3.1)
r_P, r_Q, D_0	$(p-1)/a^2$, $(q-1)/b^2$, $pr_P + qr_Q$	(3.4)
$A, A_0; \mathrm{II}_R$	Shape operator, its trace-free part, and the second fundamental form on Σ_R	(3.2)
$\mathfrak{G}, \mathcal{P}$	Compact isometry group and normalized group average	Section 1
K_R	Geometric Dirichlet kernel modulo boundary-fixing Lie derivatives	(1.3)
L, L_X	Lichnerowicz expression and complete Friedrichs realization	(1.8)
δ, δ^*, β	Divergence, symmetrized derivative, Bianchi operator	Section 2
$\mathcal{K}$	Trace-free symmetrized covariant derivative	(10.9)
$\mathcal{W}, f_T$	TT representatives and harmonic trace correction	(1.6)–(1.7)
q_R, A_R	Reduced quadratic form and its self-adjoint operator	Section 13
B	Geometric Hessian, with $q_R = 2B$ on representatives	(10.2)
$\mathscr{P}_R$	Compatible-pair Hilbert form domain	Section 13
$\mathcal{N}_R$	Scalar Dirichlet-to-Neumann operator on M_R	Section 11
$\Pi_D, \Theta, \mathcal{B}_\partial$	Induced-metric map, curvature boundary coordinates, elliptic boundary map	Section 9
$\mathscr{F}$	Full Bianchi-gauged interior and boundary map	(9.8)
$\rho, \mathscr{E}, \mathscr{X}$	Weight $(1 + t^2)^{1/2}$, homogeneous energy, weighted C^2 norm	(1.14)–(1.15)
$\mathcal{W}_g, \mathcal{F}_g^{\mathrm{RD}}, \mathcal{N}$	DeTurck vector, Ricci–DeTurck expression, nonlinear remainder	Section 26
$\mathcal{T}_R, M_R(\lambda)$	Full transverse trace space and sum of the two conormal maps	Section 28
ζ, c, χ	Scalar gauge-invariant coefficient functions	Lemma 4.4
D, G, N, V	Constraint determinant and scalar quadratic-form matrices	Sections 7 and 8
Π_i, F_i	Dimension polynomials with exact integer coefficients	Sections G and I

The scalar weight $\varrho_0 = W/J_0$ in Section 6 and the auxiliary derivative $v = s'$ in Section 7 are distinct from the global weight ρ.

I. Geometric Dirichlet deformations

2. Variation formulas and boundary gauge

Our curvature convention is

$$R(U,V)W=\nabla_U\nabla_V W-\nabla_V\nabla_U W-\nabla_{[U,V]}W,\qquad \sec(U\wedge V)=\langle R(U,V)V,U\rangle$$

for an orthonormal pair. The curvature action is

$$(\mathring{R}h)(U,V)=\sum_i h(R(e_i,U)V,e_i).$$

Thus $\mathring{R}g=\mathrm{Ric}$. The nonnegative scalar Laplacian, divergence, and symmetrized covariant derivative are

$$\Delta=-\operatorname{div}\nabla,\qquad (\delta h)_j=-\nabla^i h_{ij},\qquad (\delta^*\omega)_{ij}=\tfrac{1}{2}(\nabla_i\omega_j+\nabla_j\omega_i).$$

The operators δ and δ^* are formal adjoints in the absence of a boundary term. All integrals use the Riemannian density. The second fundamental form is $A(U,V)=g(\nabla_U\nu,V)$.

Lemma 2.1. *On a Ricci-flat manifold,*

$$2\,\mathrm{Ric}'(h)=Lh-2\delta^*\delta h-\nabla^2\operatorname{tr}h,\tag{2.1}$$

$$r(h):=DR(h)=\delta\delta h+\Delta\operatorname{tr}h,\tag{2.2}$$

$$\operatorname{tr}Lh=\Delta\operatorname{tr}h,\qquad \delta Lh=\nabla^*\nabla\delta h,\tag{2.3}$$

$$L(\delta^*\omega)=\delta^*(\nabla^*\nabla\omega),\qquad L(fg)=(\Delta f)g.\tag{2.4}$$

Furthermore,

$$2\delta\delta^*\omega=\nabla^*\nabla\omega+d\delta\omega.\tag{2.5}$$

Proof. Differentiation of the Levi–Civita connection gives

$$2\dot\Gamma^k_{ij}=\nabla_i h_j{}^k+\nabla_j h_i{}^k-\nabla^k h_{ij},\qquad \mathrm{Ric}'(h)_{ij}=\nabla_k\dot\Gamma^k_{ij}-\nabla_j\dot\Gamma^k_{ik}.$$

Commuting the derivatives yields

$$2\,\mathrm{Ric}'(h)_{ij}=-\nabla^k\nabla_k h_{ij}+\nabla_i\nabla^k h_{kj}+\nabla_j\nabla^k h_{ki}-\nabla_i\nabla_j\operatorname{tr}h-2(\mathring{R}h)_{ij},$$

which proves (2.1). Its trace is (2.2). Direct contraction of $\delta\delta^*\omega$ gives (2.5); the Ricci commutator term is zero. Naturality gives $\mathrm{Ric}'(\delta^*\omega)=0$. Insert this identity and (2.5) in (2.1), and use $\operatorname{tr}\delta^*\omega=-\delta\omega$, to obtain the first equation in (2.4). The second follows from $\nabla g=0$ and $\mathring{R}g=0$. Formal adjoints of these local identities give (2.3); equivalently they hold as identities of differential expressions on every interior coordinate chart. □

Let $\beta h=\delta h+\frac{1}{2}d\operatorname{tr}h$. Then

$$2\,\mathrm{Ric}'(h)=Lh-2\delta^*\beta h,\qquad \beta(\mathcal{L}_Y g)=\nabla^*\nabla Y^\flat.\tag{2.6}$$

Lemma 2.2. *Every smooth tensor h on a compact connected Ricci-flat manifold with nonempty boundary has a unique boundary-fixing correction $\mathcal{L}_Y g$ such that $\beta(h+\mathcal{L}_Y g)=0$. If a compact group preserves the metric, the correction commutes with its action.*

Proof. Solve

$$\nabla^*\nabla Y^\flat=-\beta h,\qquad Y|_{\partial M}=0.$$

For $Y\in H^1_0(TM)$, the scalar Poincaré inequality and $|d|Y||\le|\nabla Y|$ give

$$\|Y\|^2_{L^2}\le C\|\nabla Y\|^2_{L^2},\qquad \|Y\|^2_{H^1}\le(1+C)\|\nabla Y\|^2_{L^2}.$$

The functional $Z\mapsto-\langle\beta h,Z^\flat\rangle$ is bounded on H^1_0. The coercive variational theorem therefore gives a unique weak solution. Dirichlet regularity gives a smooth solution for smooth h. If two corrections satisfied the equation, their difference Z would obey $\int_M|\nabla Z|^2=0$ and $Z|_{\partial M}=0$, hence $Z=0$. Equation (2.6) proves the assertion. Pullback by an isometry preserves the equation and its boundary data; uniqueness proves equivariance. □

We shall use the following precise external uniqueness statement [5, Theorem 2.2 and Corollary 2.4]:

$$\operatorname{Ric}'(h) = 0, \quad h^\top = 0, \quad A_g'(h) = 0, \quad \pi_1(M, \partial M) = 0 \quad \Longrightarrow \quad h = \mathcal{L}_Y g, \quad Y|_{\partial M} = 0. \tag{2.7}$$

For the present truncations, $M_R \simeq D^{p+1} \times S^q$ and $\Sigma_R \simeq S^p \times S^q$ are simply connected, since $p, q \geq 2$, and Σ_R is connected. The relative homotopy sequence therefore gives $\pi_1(M_R, \Sigma_R) = 0$. The metrics and variations are smooth, so they also satisfy the $C^{5,\alpha}$ hypothesis in the quoted uniqueness result. In the Bianchi representative of Lemma 2.2, the conclusion of (2.7) is $h = 0$. For the finite angular sums used in the kernel calculation, Lemma 4.6 gives an independent proof of this uniqueness conclusion.

3. The Böhm equations

Write

$$u = \frac{a'}{a}, \quad v = \frac{b'}{b}, \quad H = pu + qv, \quad W = a^p b^q, \quad r_P = \frac{p-1}{a^2}, \quad r_Q = \frac{q-1}{b^2}. \tag{3.1}$$

Here primes denote t derivatives. In a radially parallel orthonormal frame, the orbit shape operator is $A = u \operatorname{Id}_P + v \operatorname{Id}_Q$. Its tangential trace-free part is

$$A_0 = \frac{q}{m}(u-v)\operatorname{Id}_P - \frac{p}{m}(u-v)\operatorname{Id}_Q, \qquad |A_0|^2 = \frac{pq}{m}(u-v)^2. \tag{3.2}$$

We identify endomorphisms and covariant tensors using the current orbit metric when taking contractions.

Lemma 3.1. *The equations* (1.1) *imply*

$$u' = r_P - Hu, \qquad v' = r_Q - Hv, \qquad W' = HW, \tag{3.3}$$

$$r_P' = -2ur_P, \qquad r_Q' = -2vr_Q, \qquad D_0 := pr_P + qr_Q = H^2 - pu^2 - qv^2. \tag{3.4}$$

On $t > 0$,

$$0 < a'(t) < 1, \qquad 0 < b'(t) < 1. \tag{3.5}$$

Proof. In orbit coordinates the nonzero Christoffel symbols, in addition to those of the two unit spheres, are

$$\Gamma^0_{ij} = -aa'\gamma_{p,ij}, \quad \Gamma^i_{0j} = u\delta^i_j, \quad \Gamma^0_{\alpha\beta} = -bb'\gamma_{q,\alpha\beta}, \quad \Gamma^\alpha_{0\beta} = v\delta^\alpha_\beta.$$

Contraction gives

$$\operatorname{Ric}_{00} = -p(u' + u^2) - q(v' + v^2), \quad \operatorname{Ric}^\sharp_{PP} = (r_P - u' - Hu)\operatorname{Id}_P,$$

and

$$\operatorname{Ric}^\sharp_{QQ} = (r_Q - v' - Hv)\operatorname{Id}_Q.$$

These yield (3.3)–(3.4). Put $b_0 = b(0)$ and $\nu_0 = (q-1)/((p+1)b_0^2) > 0$. The initial equations give

$$a = t - \frac{q\nu_0}{6p}t^3 + O(t^5), \qquad b = b_0\bigl(1 + \tfrac{1}{2}\nu_0 t^2 + O(t^4)\bigr).$$

In particular both slopes lie in $(0,1)$ for small positive t. At a first zero of a',

$$a'' = \frac{p-1}{a} > 0,$$

contradicting first contact from above. At a first contact of a' with 1 from below,

$$a'' = -q\frac{b'}{b} < 0.$$

For b' the corresponding second derivatives are $b'' = (q-1)/b > 0$ at $b' = 0$ and $b'' = -pa'/a < 0$ at $b' = 1$. Consider the first contact of the pair (a', b') with the boundary of $(0,1)^2$. A zero coordinate is excluded by the first inequalities. If neither coordinate is zero, a coordinate equal to one is excluded by the second inequalities. Thus no first contact occurs on a compact regular interval, proving (3.5) for all $t > 0$. ☐

Lemma 3.2. *Define*

$$X_0 = p(p-1)(a^{-2} - u^2), \quad Y_0 = q(q-1)(b^{-2} - v^2), \quad \theta = \frac{X_0}{X_0 + Y_0}.$$

Then

$$\frac{p-1}{2p} < \theta < \frac{q+1}{2q} \qquad (t > 0). \tag{3.6}$$

Proof. The two quantities are positive by (3.5); the constraint gives $X_0 + Y_0 = 2pq\,uv$. Differentiation by (3.3) gives

$$\theta' = -u(2p\theta - p + 1)(1-\theta) + v(q + 1 - 2q\theta)\theta. \tag{3.7}$$

Put $\theta_- = (p-1)/(2p)$ and $\theta_+ = (q+1)/(2q)$. At the two endpoints it has the values

$$\theta'|_{\theta_-} = v(q+1-2q\theta_-)\theta_- > 0, \qquad \theta'|_{\theta_+} = -u(2p\theta_+ - p + 1)(1-\theta_+) < 0.$$

The initial expansions give

$$\theta = \theta_- + ct^2 + O(t^4), \qquad c = \frac{v_0\theta_-(q+1-2q\theta_-)}{p+3} > 0.$$

The coefficient is obtained by substituting the expansion in (3.7): its terms of order t give $2c = -2p(1-\theta_-)c + v_0\theta_-(q+1-2q\theta_-)$, and $2 + 2p(1-\theta_-) = p+3$.

At a first contact with either endpoint at positive time, the derivative has the sign opposite to that required for exit. Thus there is no exit from (θ_-, θ_+). ☐

3.1. **The conical end.** Böhm's existence theorem gives the conical asymptotics

$$a(t) = \sqrt{\frac{p-1}{m-1}}\, t + o(t), \qquad b(t) = \sqrt{\frac{q-1}{m-1}}\, t + o(t). \tag{3.8}$$

The first-derivative convergence is given in [6, Section 6.4]. All higher derivatives on rescaled compact annuli follow from the Ricci-flat equations. Indeed, on a fixed interval $[a_0, a_1] \subset (0, \infty)$, put $a_R(x) = R^{-1}a(Rx)$ and $b_R(x) = R^{-1}b(Rx)$. Their C^1 limits are positive multiples of x, and

$$a_R'' = \frac{(p-1)(1-(a_R')^2)}{a_R} - q\frac{a_R' b_R'}{b_R}, \qquad b_R'' = \frac{(q-1)(1-(b_R')^2)}{b_R} - p\frac{a_R' b_R'}{a_R}.$$

The denominators are uniformly bounded away from zero on this interval. The right sides are smooth functions of (a_R, b_R, a_R', b_R'). Convergence in C^j, for $j \geq 1$, therefore gives convergence of the right sides in C^{j-1} and hence convergence of (a_R, b_R) in C^{j+1}. Induction proves convergence through every finite order. The decay estimate below is derived from the equations. Define

$$s = \frac{1}{m}\left(p \log\frac{a}{\sqrt{p-1}} + q\log\frac{b}{\sqrt{q-1}}\right), \quad z = \log\frac{a}{\sqrt{p-1}} - \log\frac{b}{\sqrt{q-1}}, \quad w = \frac{dz}{ds}.$$

Then $ds/dt = H/m > 0$. Set

$$c_0 = \frac{pq}{m}, \quad K(w) = m(m-1) - c_0 w^2, \quad V(z) = pe^{-2qz/m} + qe^{2pz/m}.$$

The constraint and the difference of the tangential equations give

$$\left(\frac{dt}{ds}\right)^2 = e^{2s}\frac{K(w)}{V(z)}, \tag{3.9}$$

$$z_s = w, \qquad w_s = -K(w)\left(\frac{w}{m} + \frac{V'(z)}{2c_0 V(z)}\right), \tag{3.10}$$

$$\frac{d}{ds}\log\frac{V(z)}{K(w)} = -\frac{2pq}{m^2}w^2. \tag{3.11}$$

Lemma 3.3. *Put $\alpha = (m-1)/2$ and $\beta = \sqrt{(m-1)(9-m)}/2 > 0$. In radial parallel identification, there are a nonzero constant multiple D of $(q/m)\,\mathrm{Id}_P - (p/m)\,\mathrm{Id}_Q$ and a real number ϑ such that*

$$A_0(t) = t^{-1-\alpha}\big(D\cos(\beta\log t + \vartheta) + o(1)\big), \qquad |\nabla_\nu A_0| \le C t^{-2-\alpha}. \tag{3.12}$$

In particular, for all sufficiently large R,

$$\int_{R/2}^{R} t^{m+2}|A_0(t)|^2\,dt \ge cR^2. \tag{3.13}$$

There are infinitely many umbilic radii, with no finite accumulation point.

Proof. Equation (3.8) gives $(z,w)\to(0,0)$. The linearization of (3.10) is

$$\frac{d}{ds}\begin{pmatrix} z \\ w\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2(m-1) & -(m-1)\end{pmatrix}\begin{pmatrix} z \\ w\end{pmatrix} + O(z^2+w^2).$$

After a real linear change of variables the polar equations are

$$\varrho_s = -\alpha\varrho + O(\varrho^2), \qquad \vartheta_s = \beta + O(\varrho).$$

Choose s_0 so large that the error in ϱ_s/ϱ has absolute value at most $\alpha/2$ for $s \ge s_0$. The orbit is nonstationary, so $\varrho(s_0) > 0$, and uniqueness prevents ϱ from vanishing at a finite parameter. Integration gives

$$0 < \varrho(s) \le \varrho(s_0)e^{-\alpha(s-s_0)/2}, \qquad \int_{s_0}^{\infty}\varrho(s)\,ds < \infty.$$

There are functions e_1, e_2, with $|e_i(s)| \le C\varrho(s)$, such that

$$e^{\alpha s}\varrho(s) = e^{\alpha s_0}\varrho(s_0)\exp\left(\int_{s_0}^{s} e_1(v)\,dv\right), \qquad \vartheta(s) - \beta s = \vartheta(s_0) - \beta s_0 + \int_{s_0}^{s} e_2(v)\,dv.$$

Both integrals converge. The first formula gives a finite strictly positive limiting amplitude; the second gives a limiting phase. The two formulas first give $\varrho(s) = O(e^{-\alpha s})$. Hence $\int_s^\infty |e_i(v)|\,dv = O(e^{-\alpha s})$. Replacing the amplitude and phase by their limits therefore changes (z,w) by $O(e^{-2\alpha s})$. Differentiating its autonomous equation proves the same remainder bound for each fixed number of s derivatives. Equation (3.9) gives $t = \sqrt{m-1}\,e^s + t_0 + O(e^{-(2\alpha-1)s})$. Since $u - v = wH/m$, (3.12) follows. The continuous function

$$\varphi \longmapsto \int_{1/2}^{1} x\cos^2(\beta\log x + \varphi)\,dx$$

is strictly positive for every φ, and has a positive minimum on a period. Rescaling the integral proves (3.13). For large s, $\vartheta_s \ge \beta/2 > 0$ and $\vartheta(s) \to +\infty$. The invertible linear change to polar coordinates maps the line $w = 0$ to a line through the origin. Its two directions have arguments ϑ_0 and $\vartheta_0 + \pi$. Thus $w = 0$ exactly when $\vartheta(s) \in \vartheta_0 + \pi\mathbb{Z}$, and there are infinitely many such parameters. At a zero of w, (3.10) makes $w_s \ne 0$, since $V'(z) = 0$ only at $z = 0$ and the orbit is not stationary. Thus all zeros are isolated. □

4. Harmonic decomposition and the singular orbit

For a unit sphere S^n, let $\mathcal{H}_\ell(S^n)$ be the real eigenspace

$$\Delta Y = \ell(\ell+n-1)Y.$$

A one-form is a sum of an exact form and a coclosed form. A symmetric tensor is a sum of a trace, a trace-free Hessian, a symmetrized derivative of a coclosed one-form, and a TT tensor. The trace-free Hessian of $Y \in \mathcal{H}_\ell$ is

$$\nabla^2 Y + \frac{\ell(\ell+n-1)}{n}Y\gamma_n.$$

It vanishes when $\ell = 0$ or 1.

Lemma 4.1. *These decompositions give a complete orthogonal decomposition of the tensor components on every principal orbit into scalar-derived, coclosed-vector-derived, and factor-TT sectors. The linearized Einstein equations preserve the resulting finite systems of radial coefficient functions. The equations for the scalar degrees* 0 *and* 1 *are obtained after removing their zero or repeated tensor components.*

Proof. On S^n put $\mathcal{K}\omega = \delta^*\omega + (\delta\omega/n)\gamma_n$. Its adjoint on trace-free tensors is δ. The symbol of $\delta\mathcal{K}$ is

$$\frac{1}{2}|\xi|^2\,\mathrm{Id} + \left(\frac{1}{2} - \frac{1}{n}\right)\xi \otimes \xi,$$

which is positive for $\xi \neq 0$. If k_0 is trace-free, then δk_0 is orthogonal to $\ker\mathcal{K}$, by integration by parts. Solving $\delta\mathcal{K}\omega = \delta k_0$ on the orthogonal complement of this finite-dimensional kernel gives $k_0 = k_{\mathrm{TT}} + \mathcal{K}\omega$. The Hodge decomposition of ω has no harmonic summand, since $H^1(S^n;\mathbb{R}) = 0$. The exact and coclosed summands give the asserted scalar and vector terms. The TT summand is orthogonal to $\operatorname{im}\mathcal{K}$ because $\langle k_{\mathrm{TT}}, \mathcal{K}\omega\rangle = \langle \delta k_{\mathrm{TT}}, \omega\rangle = 0$. The trace is orthogonal to every trace-free summand. If ω is coclosed and $\nabla^*\nabla\omega = \mu\omega$, the curvature-one commutator gives $\delta\mathcal{K}\omega = (\mu - n + 1)\omega/2$. Hence

$$\langle \mathcal{K}(dY), \mathcal{K}\omega\rangle = \frac{\mu - n + 1}{2}\langle dY, \omega\rangle = 0.$$

The last equality follows from $\delta\omega = 0$. In particular, for a scalar eigenfunction of eigenvalue λ,

$$\nabla^*\nabla dY = (\lambda - n + 1)dY, \qquad \delta\nabla^2 Y = (\lambda - n + 1)dY.$$

The corresponding contractions of symmetrized coclosed derivatives are multiples of the original one-form. Round curvature acts algebraically on the trace-free tensor summands. Expansion in the scalar, coclosed-one-form, and TT eigenspaces is complete by the compact elliptic spectral theorem. Apply these decompositions to each factor; use the one-form decomposition for mixed components. Covariant differentiation and contraction preserve their listed generating collections. The Christoffel symbols in the proof of Lemma 3.1 have no angular dependence except that of the two sphere connections. Hence the Ricci variation preserves each radial system. The first-harmonic identity $\nabla^2 Y = -Y\gamma_n$ explains the repeated components which must be removed. □

Lemma 4.2 (Normal gauge). *A smooth tensor on M_R can be changed by a smooth Lie derivative, not necessarily boundary-fixing, so that*

$$h_{00} = h_{0A} = 0 \tag{4.1}$$

on the regular part. The construction is equivariant under $\mathfrak{G}$. The resulting tensor remains smooth at the singular orbit.

Proof. Extend the tensor to $M_{R+\eta}$ for some $\eta > 0$, and consider $g_s = g + sh$. Keep the embedded singular orbit $B = \{0\} \times S^q$ fixed. Identify its g_s-normal bundle isometrically with its g-normal bundle by the graph projection followed by the positive square root of the induced positive bundle endomorphism. This identification depends smoothly on s and commutes with isometries of the initial data. Pull back g_s by its normal exponential map, using this fixed normal bundle.

For a variation $c(r, z)$ of unit-speed normal geodesics, let $J = \partial_z c$ and $T = \partial_r c$. Torsion-freeness and the geodesic equation give

$$\partial_r\langle T, J\rangle = \langle T, \nabla_r J\rangle = \langle T, \nabla_z T\rangle = \tfrac{1}{2}\partial_z|T|^2 = 0.$$

At $r = 0$, this inner product is zero both for variations of the basepoint along the singular orbit and for variations of its unit normal. Hence it vanishes for every r. The pullback metric has the form $dr^2 + \gamma_s(r)$, with no mixed radial terms. At $s = 0$ the normal exponential map is the original polar parametrization. On the compact normal disc of radius $R + \eta/2$ its differential is nonsingular and it is injective. Nonsingularity persists by a finite coordinate cover. Pairs of points outside these coordinate neighborhoods have images a positive distance apart; sufficiently small perturbations preserve injectivity for these pairs as well. The maps therefore give diffeomorphisms onto their images. Their derivative in s is a smooth vector field in Cartesian normal coordinates, including at the zero section. Differentiating the pullback gives (4.1). The construction uses only equivariant bundle operations and the geodesic equation, proving the last assertions. □

Let $Y \in \mathcal{H}_\ell(S^p)$ and $Z \in \mathcal{H}_j(S^q)$. In a two-factor scalar sector, a tensor in (4.1) is written

$$\begin{aligned} h_{PP} &= 2a^2(AYZ\gamma_p + CZ\nabla^2 Y), \\ h_{QQ} &= 2b^2(BYZ\gamma_q + DY\nabla^2 Z), \\ h_{i\alpha} &= 2e\nabla_i Y\nabla_\alpha Z. \end{aligned} \tag{4.2}$$

The functions A, B, C, D, e depend only on t.

Lemma 4.3. *For $\ell, j \geq 2$, the coefficients of the tangential part of a smooth tensor, written as in* (4.2)*, satisfy, near the collapsing S^p,*

$$A, C = O(t^{\ell-2}), \qquad B, D, e = O(t^\ell). \tag{4.3}$$

Each differentiation lowers the displayed power by at most one. The same bounds, interpreted for the nonredundant combinations, hold in a degree-one sector.

Proof. Use coordinates $x \in \mathbb{R}^{p+1}$ normal to the singular orbit and coordinates on the singular orbit. Taylor expansion in x has smooth coefficients on the singular orbit. Restriction of a polynomial of degree k to $|x| = t$ has only scalar harmonics of degree at most k, with the same parity as k. Contracting one normal covariant index with a vector tangent to the sphere $|x| = t$ contributes one factor t and one angular vector. With two normal covariant indices the angular degree can therefore exceed the Taylor degree by at most two. Consequently a normal–normal tensor of scalar degree ℓ, after division by $a^2 \asymp t^2$, has coefficients $O(t^{\ell-2})$. Components with both covariant indices tangent to the singular orbit have scalar degree ℓ first at order t^ℓ. For a normal–singular orbit component, contraction with a sphere tangent vector contributes the additional factor t; a scalar gradient of degree ℓ can first occur at polynomial degree $\ell - 1$. Its coordinate coefficient is therefore $O(t^\ell)$. Taylor remainders satisfy the corresponding estimates after any fixed number of radial derivatives. Projection to a fixed finite harmonic space is a bounded angular integral and preserves the estimates. For degree one, only the difference of the trace and Hessian coefficients is intrinsic, since $\nabla^2 Y = -Y\gamma_p$; the argument applies to that difference and to the remaining components. ☐

Lemma 4.4 (Scalar gauge invariants). *Suppose $\ell, j \geq 2$. The functions*

$$\zeta = uB - vA, \qquad Z_0 = \zeta/u, \qquad c = 2e - a^2C - b^2D \tag{4.4}$$

are unchanged by Lie derivatives in the same scalar representation. Consequently their singular-orbit estimates may be obtained from the original smooth tensor. If that tensor satisfies $h^\top = 0$ on Σ_R, then every gauge representative satisfies

$$\zeta(R) = c(R) = 0. \tag{4.5}$$

If $\ell \geq 2$ and $j = 1$, use $D = 0$; the corresponding invariant is $\chi = uZ_0 + (M_0u/q)c$, with $M_0 = q/b^2$. If $\ell = 1$ and $j \geq 2$, use $C = 0$ and $\chi = uZ_0 - (v/a^2)c$.

Proof. For a vector field $X = \xi YZ\partial_t + \eta Z\nabla Y + \omega Y\nabla Z$, the tangential Lie derivative gives

$$(A, B, C, D, 2e) \longmapsto (A + u\xi, B + v\xi, C + \eta, D + \omega, 2e + a^2\eta + b^2\omega). \tag{4.6}$$

Thus $\delta\zeta = \delta c = 0$. These equalities require smoothness of ξ, η, ω only on the regular part. Since the tangential generators are independent for $\ell, j \geq 2$, the zero boundary tensor has zero coefficients; this proves (4.5). The estimates of Lemma 4.3 can therefore be applied before changing gauge.

For $j = 1$, the identity $\nabla^2 Z = -Z\gamma_q$ gives, in the convention $D = 0$, $\delta B = v\xi - \omega$ and $\delta c = b^2\omega$. Hence $\delta\chi = -u\omega + (M_0u/q)b^2\omega = 0$. For $\ell = 1$, one has $\delta A = u\xi - \eta$ and $\delta c = a^2\eta$; therefore $\delta\chi = v\eta - (v/a^2)a^2\eta = 0$. ☐

The two invariants have the following quotient interpretation. At a fixed regular radius use the coefficient vector $\mathbf{a} = (A, B, C, D, 2e)^T$ and the maps

$$\mathcal{G}_t = \begin{pmatrix} u & 0 & 0 \\ v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & a^2 & b^2 \end{pmatrix}, \qquad \mathcal{I}_t = \begin{pmatrix} -v & u & 0 & 0 & 0 \\ 0 & 0 & -a^2 & -b^2 & 1 \end{pmatrix}. \tag{4.7}$$

One has $\mathcal{I}_t\mathcal{G}_t = 0$, $\operatorname{rank}\mathcal{G}_t = 3$, and $\operatorname{rank}\mathcal{I}_t = 2$, since $u > 0$. Thus

$$0 \longrightarrow \mathbb{R}^3 \xrightarrow{\mathcal{G}_t} \mathbb{R}^5 \xrightarrow{\mathcal{I}_t} \mathbb{R}^2 \longrightarrow 0$$

is exact. In particular, $\zeta = c = 0$ is equivalent to

$$(A, B, C, D, 2e) = \mathcal{G}_t(A/u, C, D)^T.$$

This assertion concerns the tangential tensor at a fixed radius. The radial equations determine whether these boundary displacements extend to a Lie derivative of the full tensor.

When the first degree is one, the coefficient relation is $\mathbf{r}_P = (1, 0, 1, 0, 0)^T$ and $\mathcal{I}_t\mathbf{r}_P = (-v, -a^2)^T$. The quotient by this line is represented by $\zeta - vc/a^2$. When the second degree is one, $\mathbf{r}_Q = (0, 1, 0, 1, 0)^T$ and $\mathcal{I}_t\mathbf{r}_Q = (u, -b^2)^T$; the quotient is represented by $\zeta + uc/b^2$. If both degrees are one, these two image vectors are independent, since $a^2u + b^2v > 0$. This accounts for the separate low-degree calculation in Lemma 5.3.

Lemma 4.5. *Let S be a closed embedded submanifold of codimension at least two in a smooth Riemannian manifold (M, g). Suppose h is a smooth symmetric tensor on M and X is a smooth vector field on $M \setminus S$ such that $\mathcal{L}_X g = h$. Then X extends uniquely to a smooth vector field on M.*

Proof. Put $s = h/2$, let X_j denote the metric dual of X, and set $A_{ij} = \nabla_i X_j - s_{ij}$. Then $A_{ij} = -A_{ji}$. Define

$$C_{ijk}(X) = (\nabla_i\nabla_j - \nabla_j\nabla_i)X_k.$$

The commutator is a smooth bundle map, linear in X. Differentiating $\nabla_i X_j + \nabla_j X_i = 2s_{ij}$ and permuting the three indices gives

$$\begin{aligned} \nabla_i X_j &= s_{ij} + A_{ij}, \\ \nabla_i A_{jk} &= \nabla_j s_{ik} - \nabla_k s_{ij} + \tfrac{1}{2}\big(C_{ijk}(X) - C_{ikj}(X) - C_{jki}(X)\big). \end{aligned}$$

In fixed coordinates this is a system $\partial_i U = B_i U + F_i$ for $U = (X, A)$, with smooth coefficients B_i, F_i across S.

Choose a coordinate product neighborhood in which $S = \{0\} \times \mathbb{R}^{\dim S}$. On a smaller neighborhood, any two points outside S can be joined outside S by a path of length at most C times their coordinate distance, contained in the larger neighborhood. Indeed, a line segment meeting S is replaced, in a two-dimensional normal plane, by a semicircle of arbitrarily small radius. The codimension assumption supplies this normal plane. Let K bound the coefficient matrices and the inhomogeneous terms in these coordinates. Along a path of length ℓ beginning at x_0, Grönwall's inequality gives

$$|U(x)| \le (|U(x_0)| + K\ell)e^{K\ell}.$$

The path lengths from one fixed x_0 are uniformly bounded in the smaller neighborhood. Thus U is bounded there. The equations now give $|\partial_i U| \le C$, and the paths of length at most $C_0|x - y|$ give $|U(x) - U(y)| \le CC_0|x - y|$. Consequently U has a unique continuous extension across S.

Integrate $\partial_i U = B_i U + F_i$ along coordinate segments avoiding S, and pass to limits of translated segments. Almost every translate avoids S, since its codimension is at least two. Continuity gives the integral identity on every segment in the smaller neighborhood. Thus the extension is C^1 and satisfies the same system. Induction using the smoothness of B_i, F_i gives $U \in C^\infty$. The extensions agree on overlapping neighborhoods by continuity. □

Lemma 4.6 (Cauchy uniqueness in a finite angular sum). *Let h be smooth on M_R, contained in a finite sum of angular types. If $\mathrm{Ric}'_g(h) = 0$, $h^\top|_{\Sigma_R} = 0$, and $A'_g(h)|_{\Sigma_R} = 0$, then $h = \mathcal{L}_Y g$ for a smooth vector field Y with $Y|_{\Sigma_R} = 0$.*

Proof. On $(0, R] \times S^p \times S^q$, write $g = dt^2 + \gamma(t)$. Solve, with zero terminal values at R,

$$\partial_t Z^0 = -\tfrac{1}{2} h_{00}, \qquad \partial_t Z^a = -\gamma^{ab}(h_{0b} + \partial_b Z^0). \tag{4.8}$$

The coefficients are smooth on every $[\varepsilon, R]$. Integration gives a smooth field Z on the regular part and commutes with the angular projections. The formulas

$$(\mathcal{L}_Z g)_{00} = 2\partial_t Z^0, \qquad (\mathcal{L}_Z g)_{0a} = \gamma_{ab}\partial_t Z^b + \partial_a Z^0$$

show that $k = h + \mathcal{L}_Z g$ has $k_{00} = k_{0a} = 0$. Its induced variation and its second fundamental form variation at R are unchanged, since Z vanishes there. Hence $k^\top(R) = \partial_t k^\top(R) = 0$. In the finite angular sum the tangential Ricci equation is a system

$$-\tfrac{1}{2}\mathbf{k}'' + P_1(t)\mathbf{k}' + P_0(t)\mathbf{k} = 0$$

with smooth matrix coefficients on $[\varepsilon, R]$. The leading coefficient follows directly by differentiating the normal-coordinate Ricci formula. Uniqueness for this linear system gives $\mathbf{k} = 0$ on $[\varepsilon, R]$. Since $\varepsilon > 0$ is arbitrary, $h = -\mathcal{L}_Z g$ on the regular part. The preceding removability lemma extends $-Z$ across the singular orbit, of codimension $p + 1 \geq 3$. The extension retains its zero value on Σ_R. □

5. Tensor, vector, and low scalar degrees

Proposition 5.1. *A factor-TT sector contains no nonzero class in K_R.*

Proof. Let k be TT on S^p, with $\nabla^*\nabla k = \sigma k$, $\sigma \geq 0$, and let Z be a scalar harmonic of eigenvalue μ on S^q. Normalize their squared angular norms to one. The tensor $T = a^2 F(t) k Z$ is TT on the full space. Its energy is

$$Q(T) = \int_0^R W\left[(F')^2 + \left(\frac{\sigma + 2}{a^2} + \frac{\mu}{b^2}\right) F^2\right] dt. \tag{5.1}$$

To check the constant $2/a^2$, the radial tensor slots in the covariant derivative contribute $2u^2|T|^2$. The factor curvature contributes $2(1 - a'^2)a^{-2}|T|^2$ to $-2\langle \mathring{R}T, T\rangle$. Their sum is $2a^{-2}|T|^2$. The other derivative terms give exactly the displayed scalar and tensor eigenvalues. A geometric kernel in this sector satisfies $LT = 0$ by (2.1); its induced boundary condition gives $F(R) = 0$. To justify the inner endpoint, integrate first on $\{\varepsilon < t < R\}$. Smoothness gives bounded $|T| + |\nabla T|$ near the singular orbit, while $\mathrm{Area}(\Sigma_\varepsilon) = O(\varepsilon^p)$. Therefore

$$\left|\int_{\Sigma_\varepsilon} \langle \nabla_\nu T, T\rangle \, dA\right| \leq C\varepsilon^p \longrightarrow 0.$$

Hence (5.1) is zero and $F = 0$. Interchanging the factors proves the other case. □

Proposition 5.2. *Every sector containing a coclosed-vector type on either sphere has zero geometric Dirichlet kernel.*

Proof. Regard the metric as $g_B + r^2\gamma_n$, with the selected sphere as fibre. Let ω be a coclosed one-form on this unit sphere, with $\nabla^*\nabla\omega = k_V^2\omega$. Write the vector-type coefficients as

$$h_{ai} = r^2 B_a \omega_i, \qquad h_{ij} = 2r^2 C \nabla_{(i}\omega_{j)}.$$

A fibre vector field $\phi\omega^\sharp$ changes B to $B + d\phi$ and C to $C + \phi$. Thus $\mathcal{A} = B - dC$ is invariant. In the Killing case, the second tensor is zero; $\mathcal{A} = B$ is then defined modulo exact forms.

The Christoffel symbols involving the fibre are

$$\Gamma^a_{ij} = -rD^a r\,\gamma_{n,ij}, \qquad \Gamma^i_{aj} = r^{-1} D_a r\,\delta^i_j.$$

The complete covariant calculation in Lemma E.1 gives the necessary mixed-component equation

$$\delta_B(r^{n+2} d\mathcal{A}) + (k_V^2 - n + 1) r^n \mathcal{A} = 0. \tag{5.2}$$

The angular contraction is $\nabla^i(\nabla_i\omega_j+\nabla_j\omega_i)=-(k_V^2-n+1)\omega_j$; the base and fibre divergences are displayed separately in Section E.1. Their sum, after use of the vertical Ricci-flat equation, gives (5.2) with the indicated sign and weight. The other Ricci components are included in (E.1). The same equation is the zero-source specialization of [8, Section IV.B] (equation (4.9) in arXiv:1306.4927v1), with positive-definite base metric and $F_a=r\mathcal{A}_a$.

Integration on the unit sphere gives

$$(k_V^2-n+1)\|\omega\|_{L^2}^2=2\|\delta^*\omega\|_{L^2}^2\geq 0.$$

For a non-Killing harmonic, $h^\top=0$ gives $C=0$ and $B^\top=0$ at the outer boundary, hence $\mathcal{A}^\top=0$. In the Killing case this condition follows directly from the mixed components. Pairing (5.2) with $\mathcal{A}$ gives

$$\int_B r^{n+2}|d\mathcal{A}|^2\,dv_B+(k_V^2-n+1)\int_B r^n|\mathcal{A}|^2\,dv_B=0. \tag{5.3}$$

The boundary integrand, up to its orientation sign, is $r^{n+2}\langle \iota_\nu d\mathcal{A},\mathcal{A}\rangle$. The one-form $\iota_\nu d\mathcal{A}$ annihilates ν because $d\mathcal{A}$ is alternating. Thus it pairs only the tangential component of $\mathcal{A}$, which is zero at the outer boundary. If the fibre collapses, smooth tensor coefficients give the sufficient estimates $\mathcal{A}=O(t^{-1})$ and $d\mathcal{A}=O(t^{-2})$; the inner flux is $O(t^{n-1})$ and tends to zero for $n\geq 2$. These estimates follow directly from $h_{ai}=r^2B_a\omega_i$, $r\asymp t$, and the normal Taylor expansion in Lemma 4.3. If the base itself closes at the singular orbit, no inner boundary is present.

If $k_V^2>n-1$, then $\mathcal{A}=0$. Otherwise $d\mathcal{A}=0$. The base is simply connected, so $\mathcal{A}=d\phi$; the outer condition makes ϕ constant on its connected boundary, and that constant can be subtracted. In both cases the deformation is boundary-fixing pure gauge near the outer boundary. Apply Lemma 4.6. □

Lemma 5.3. *The scalar sectors of degrees* $(1,0)$, $(0,1)$, *and* $(1,1)$ *have zero geometric kernel.*

Proof. Put $a_h=A_g'(h)$. For a geometric kernel, the linearized Gauss and Codazzi equations are

$$\operatorname{div}_\Sigma(a_h-(\operatorname{tr}a_h)\gamma)=0,\qquad \langle H\gamma-A,a_h\rangle=0. \tag{5.4}$$

In degree $(1,0)$, the first-harmonic Hessian is $-Y\gamma_p$, so $a_h=Y(\alpha\gamma_P+\beta\gamma_Q)$. The two equations in (5.4) become

$$(p-1)\alpha+q\beta=0,\qquad p(H-u)\alpha+q(H-v)\beta=0.$$

They imply $(m-1)v\alpha=0$. Since $v>0$, $a_h=0$. In degree $(0,1)$ the equations are

$$p\alpha+(q-1)\beta=0,\qquad p(H-u)\alpha+q(H-v)\beta=0.$$

Eliminating α gives $(m-1)u\beta=0$; hence $\alpha=\beta=0$. In degree $(1,1)$, the additional component is $(a_h)_{i\alpha}=c\nabla_iY\nabla_\alpha Z$. Let $\lambda_P=p/a^2$ and $\lambda_Q=q/b^2$. The momentum equations are

$$(p-1)\alpha+q\beta+c\lambda_Q=0,\qquad p\alpha+(q-1)\beta+c\lambda_P=0.$$

Multiply the first by pu and the second by qv, add, and use the second equation of (5.4). This gives $c(pu\lambda_Q+qv\lambda_P)=0$. Thus $c=0$, and the remaining matrix has determinant $1-m\neq 0$, so $a_h=0$. In all three cases Lemma 4.6 gives the assertion. □

6. Scalar harmonics on one factor

Let $Z\in\mathcal{H}_j(S^q)$, $j\geq 2$, with eigenvalue $\mu=j(j+q-1)$. A normally gauged tensor in this sector is

$$h=2a^2AZ\gamma_p+2b^2(BZ\gamma_q+C\nabla^2Z). \tag{6.1}$$

Write

$$Z_0=B-\frac{v}{u}A,\qquad \vartheta=\mu-q>0,\quad s_0=pu+(q-1)v,$$

$$F=p(m-1)u^2+\vartheta s_0^2,\quad J_0=F/u^2,\quad \varrho_0=W/J_0,\quad \sigma=r_Q/s_0. \tag{6.2}$$

All these denominators are positive.

Lemma 6.1. *Every geometric deformation in* (6.1) *satisfies*

$$(\varrho_0 Z_0')' - \varrho_0 V_\mu Z_0 = 0, \qquad V_\mu = r_Q \left(\frac{\mu - 2(q-1)}{q-1} + \frac{J_0'}{s_0 J_0} \right). \tag{6.3}$$

Its induced-metric boundary condition is $Z_0(R) = 0$.

Proof. Put $\Xi = pA' + qB' - \mu C'$. The PP equation, the radial–Q momentum equation, and the Hamiltonian equation are, respectively,

$$A'' + HA' + u\Xi = (\mu/b^2 - 2r_P)A, \tag{6.4}$$

$$pA' + (q-1)B' - (q-1)C' + p(u-v)A = 0, \tag{6.5}$$

$$p(\mu/b^2 - r_P)A + (\mu - q)r_Q B = p(H-u)A' + (H-v)(qB' - \mu C'). \tag{6.6}$$

These are the constant-first-factor specialization of the intrinsic coefficient table and the Codazzi and Gauss contractions proved in Lemma E.2. In particular the pure Hessian is a tangential Lie derivative and has zero intrinsic scalar-curvature variation. Set $y = A/u$. Eliminate C' from (6.5)–(6.6), using (3.4). The result is

$$y' = -\frac{(q-1)\vartheta}{F}(s_0 Z_0' + r_Q Z_0). \tag{6.7}$$

Substitution in (6.4) gives

$$y'' + \left(\frac{2r_P}{u} - \frac{\mu s_0}{q-1} \right) y' - \vartheta Z_0' = 0.$$

The two coefficient identities in (E.11), obtained by differentiating (6.7), give (6.3). The leading coefficient used in that elimination is strictly negative and hence nonzero. At the boundary a radial displacement changes (A, B) by $(u\xi, v\xi)$, while an angular gradient changes only C. Thus $Z_0(R) = 0$ follows from the original zero induced variation. □

Proposition 6.2. *Every scalar sector constant on one factor and nonconstant on the other has zero geometric kernel.*

Proof. Degree one was treated in Lemma 5.3. For degree at least two, the exact identity is

$$V_\mu + \sigma' + \frac{\varrho_0'}{\varrho_0}\sigma - \sigma^2 = \frac{r_Q F}{(q-1)s_0^2} > 0. \tag{6.8}$$

Indeed, $s_0' = pr_P + (q-1)r_Q - Hs_0$, $r_Q' = -2vr_Q$, and $F = \mu s_0^2 - (q-1)D_0$. Also $\varrho_0'/\varrho_0 = H - J_0'/J_0$. The J_0' terms cancel in the left side of (6.8); the remaining expression is $r_Q[\mu/(q-1) - D_0/s_0^2]$. Equation (6.3) therefore gives

$$\frac{d}{dt}\left[\varrho_0 Z_0 (Z_0' + \sigma Z_0)\right] = \varrho_0 \left[(Z_0' + \sigma Z_0)^2 + \frac{r_Q F}{(q-1)s_0^2} Z_0^2 \right]. \tag{6.9}$$

When the a factor collapses, $\varrho_0 \asymp t^p$, $\sigma = O(t)$, and smoothness gives $Z_0 = O(1)$, $Z_0' = O(t)$. The inner flux is zero. In the exchanged case the sufficient estimates are $\varrho_0 \asymp t^{q+4}$, $\sigma = O(t^{-1})$, $Z_0 = O(t^{-2})$, and $Z_0' = O(t^{-3})$; the flux is $O(t^{q-1})$ and again tends to zero. These estimates follow from Lemma 4.3 and the factors v/u at the singular orbit. The quantity Z_0 is unchanged by scalar Lie derivatives, so the estimates may be computed from the original smooth tensor; see Lemma 4.4. The outer flux is zero by $Z_0(R) = 0$. Integrating (6.9) gives $Z_0 = 0$. Equation (6.7) implies $y' = 0$, hence $A = cu$, $B = cv$. Equation (6.5) then gives $C' = -c/b^2$. Thus the deformation is the Lie derivative of $cZ\partial_t + C\nabla Z$ on the regular part. After undoing the auxiliary normal gauge, the zero induced boundary variation forces its normal coefficient, and then its gradient coefficient, to vanish at R. It is boundary-fixing pure gauge in an outer collar. Apply Lemma 4.6. □

7. The two-factor scalar system

Let $\ell, j \geq 2$, $\lambda = \ell(\ell + p - 1)$, $\mu = j(j + q - 1)$, and use (4.2). Set

$$L_0 = \lambda/a^2, \quad M_0 = \mu/b^2, \quad s_P = H - u, \quad s_Q = H - v,$$

$$Z_0 = B - (v/u)A, \quad c = 2e - a^2C - b^2D, \quad \mathbf{z} = \begin{pmatrix} Z_0 \\ c \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \zeta \\ c \end{pmatrix} = \begin{pmatrix} uZ_0 \\ c \end{pmatrix}.$$

The letters L_0, M_0 in this section are scalar coefficients, not differential operators. Put

$$s = A/u, \quad v = s', \quad U = s + a^2C', \quad V = s + b^2D', \qquad \mathbf{w} = (v, U, V)^T.$$

Lemma 7.1. *The momentum and Hamiltonian constraints are*

$$\mathbb{A}\mathbf{w} = T_1\mathbf{z}' + T_0\mathbf{z}, \tag{7.1}$$

where

$$\mathbb{A} = \begin{pmatrix} s_P & M_0/2 - r_P & -M_0/2 \\ s_Q & -L_0/2 & L_0/2 - r_Q \\ D_0 & -L_0 s_P & -M_0 s_Q \end{pmatrix}, \tag{7.2}$$

$$T_1 = \begin{pmatrix} -q & -M_0/2 \\ -(q-1) & -L_0/2 \\ -qs_Q & 0 \end{pmatrix}, \qquad T_0 = \begin{pmatrix} q(u-v) & M_0 u \\ 0 & L_0 v \\ qL_0 + (q-1)M_0 - qr_Q & L_0M_0 \end{pmatrix}. \tag{7.3}$$

The determinant $\mathsf{D} = \det \mathbb{A}$ *is positive. With* $x = (\lambda - p)r_P = (p-1)L_0 - pr_P$ *and* $y = (\mu - q)r_Q = (q-1)M_0 - qr_Q$, *its factorization is*

$$\begin{aligned} 2pq\mathsf{D} = {} & (m-1)(qL_0v + pM_0u)^2 \\ & + (qD_0L_0 + 2p(m-1)M_0u^2)x \\ & + (2q(m-1)L_0v^2 + pD_0M_0)y + 2D_0xy. \end{aligned} \tag{7.4}$$

Proof. For $g = dt^2 + \gamma(t)$ the coordinate expressions are

$$\mathrm{Ric}_{00} = -\tfrac{1}{2}\operatorname{tr}_\gamma \gamma'' + \tfrac{1}{4}\operatorname{tr}(\gamma^{-1}\gamma'\gamma^{-1}\gamma'),$$

$$\mathrm{Ric}_{0a} = \tfrac{1}{2}(\nabla^b\gamma'_{ab} - \nabla_a \operatorname{tr}_\gamma \gamma'),$$

$$\mathrm{Ric}_{ab} = (\mathrm{Ric}_\gamma)_{ab} - \tfrac{1}{2}\gamma''_{ab} - \tfrac{1}{4}(\operatorname{tr}_\gamma \gamma')\gamma'_{ab} + \tfrac{1}{2}(\gamma'\gamma^{-1}\gamma')_{ab}.$$

The coefficients of the independent angular one-forms $Z\,dY$ and $Y\,dZ$ in the two mixed components of the linearized Ricci tensor are

$$\begin{aligned} 0 &= -(p-1)A' - qB' + (p-1)C' + \mu D' - M_0e' + 2M_0ue + (u-v)(qB - \mu D), \\ 0 &= -(q-1)B' - pA' + (q-1)D' + \lambda C' - L_0e' + 2L_0ve + (v-u)(pA - \lambda C). \end{aligned} \tag{7.5}$$

The coefficient of $2YZ$ in the scalar constraint is

$$\begin{aligned} & [(p-1)L_0 + pM_0 - pr_P]A + [qL_0 + (q-1)M_0 - qr_Q]B + L_0M_0c \\ & = s_P(pA' - \lambda C') + s_Q(qB' - \mu D'). \end{aligned} \tag{7.6}$$

To verify its left side, use $DR_\gamma(h) = \delta\delta h + \Delta \operatorname{tr} h - \langle \mathrm{Ric}_\gamma, h\rangle$ on the product orbit. The two pure Hessian terms cancel their factor-curvature terms, leaving the combination $2e - a^2C - b^2D = c$. The right side is $H\dot{H} - \langle A, \dot{A}\rangle$, where $\dot{A}$ is the variation of the shape endomorphism and $\dot{H} = \operatorname{tr}\dot{A}$. The normal Ricci component itself is

$$\mathrm{Ric}'(h)_{00} = \left[-\Xi'' - 2u(pA' - \lambda C') - 2v(qB' - \mu D')\right]YZ, \qquad \Xi = pA + qB - \lambda C - \mu D.$$

Now replace $B = Z_0 + vs$, $A = us$, $a^2C' = U - s$, $b^2D' = V - s$, and $2e = c + a^2C + b^2D$. The coefficients of (v, U, V) in these three equations are the rows of (7.2); the terms containing Z_0', c', Z_0, c are the rows of (7.3). The contractions used here are $\Delta Y = \lambda Y$, $\operatorname{div}\nabla^2 Y = -(\lambda - p + 1)dY$, and the corresponding identities on the other sphere. All terms containing the undifferentiated s cancel by (3.3)–(3.4). Expansion of the 3×3 determinant, replacement of $(p-1)L_0$ by $x + pr_P$ and $(q-1)M_0$ by

$y + qr_Q$, and use of (3.4) give (7.4). Each term is nonnegative; the first is positive. This also proves invertibility when exactly one of x, y is zero. □

Lemma 7.2. *Let $\Xi = pA + qB - \lambda C - \mu D$. The tangential equations are*

$$\begin{aligned} A'' + HA' + u\Xi' &= (L_0 + M_0 - 2r_P)A, \\ B'' + HB' + v\Xi' &= (L_0 + M_0 - 2r_Q)B, \\ C'' + HC' &= (M_0 - L_0)C + (2A - \Xi - 2M_0 e)/a^2, \\ D'' + HD' &= (L_0 - M_0)D + (2B - \Xi - 2L_0 e)/b^2, \\ e'' + (H - 2u - 2v)e' + 4uve &= -(p-1)(A - C) - (q-1)(B - D). \end{aligned} \tag{7.7}$$

Together with (7.1) *they imply*

$$Z_0'' + HZ_0' = (L_0 + M_0 - 2r_Q)Z_0 + 2(vr_P/u - r_Q)v, \tag{7.8}$$

$$c'' + (H - 2u - 2v)c' = (L_0 + M_0 - 4uv)c - 2(u - v)(U - V). \tag{7.9}$$

Proof. Write $\gamma = \gamma(t)$ for the orbit metric and k for its variation in normal gauge. Differentiating the tangential Ricci formula in the preceding proof gives

$$\begin{aligned} (\mathrm{Ric}_g' \, k)_{ab} &= (\mathrm{Ric}_\gamma' \, k)_{ab} - \tfrac12 k_{ab}'' - \tfrac14\big(\mathrm{tr}_\gamma k' - \langle k, \gamma'\rangle_\gamma\big)\gamma_{ab}' - \tfrac14(\mathrm{tr}_\gamma \gamma')k_{ab}' \\ &\quad + \tfrac12\big(k'\gamma^{-1}\gamma' + \gamma'\gamma^{-1}k' - \gamma'\gamma^{-1}k\gamma^{-1}\gamma'\big)_{ab}. \end{aligned}$$

The intrinsic term is determined by

$$2\,\mathrm{Ric}_\gamma' \, k = \nabla_\gamma^*\nabla_\gamma k + \mathrm{Ric}_\gamma \circ k + k \circ \mathrm{Ric}_\gamma - 2\mathring{R}_\gamma k - 2\delta_\gamma^*\delta_\gamma k - \nabla_\gamma^2 \, \mathrm{tr}_\gamma k.$$

The trace and divergence coefficients are (E.4); substituting them in the displayed intrinsic variation gives the five rows of (E.5), as proved in Lemma E.2. The two Ricci endomorphism eigenvalues of the orbit are r_P and r_Q. The coefficients of the radial second derivatives are $-a^2$, $-b^2$, $-a^2$, $-b^2$, and -1, respectively; dividing by these nonzero factors gives (7.7). Its terms proportional to Ξ' arise from the differentiated orbit mean curvature. The first two equations and $A = us$ give (7.8). Differentiate $c = 2e - a^2C - b^2D$ twice, substitute the last three equations of (7.7), and use $(a^2)' = 2ua^2$, $(b^2)' = 2vb^2$. The terms containing C, D, e combine to $(L_0 + M_0 - 4uv)c$; the remaining terms are $-2(u - v)(U - V)$. This proves (7.9). □

7.1. **The constraint adjugate.** In the following identity only, write $s_0 = s_P$, $t_0 = s_Q$, $P_0 = r_P$, $Q_0 = r_Q$, and $F_0 = L_0 s_0 + M_0 t_0$. Then

$$\operatorname{adj} \mathbb{A} = \begin{pmatrix} \frac{L_0}{2}(F_0 - 2Q_0 s_0) & \frac{M_0}{2}(F_0 - 2P_0 t_0) & P_0Q_0 - \frac12(P_0L_0 + Q_0M_0) \\ (L_0/2 - Q_0)D_0 + M_0 t_0^2 & M_0(D_0/2 - s_0 t_0) & Q_0 s_0 - F_0/2 \\ L_0(D_0/2 - s_0 t_0) & (M_0/2 - P_0)D_0 + L_0 s_0^2 & P_0 t_0 - F_0/2 \end{pmatrix}. \tag{7.10}$$

Each entry is its transposed signed 2×2 minor. Expansion along a row gives $\mathbb{A} \operatorname{adj} \mathbb{A} = \mathsf{D} I_3$. The relation $s_P s_Q = D_0 - (m-1)uv$ follows from (3.4); it will be used in the matrix products below.

7.2. **A symmetric form for the system.** Set

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 1 \\ -L_0 & 0 & 0 \\ 0 & -M_0 & 0 \end{pmatrix}, \quad \mathbb{C} = \mathbb{P}\mathbb{A}, \quad B_i = \mathbb{P}T_i \quad (i = 0, 1).$$

The matrix $\mathbb{C}$ is symmetric and invertible. Define

$$G_0 = \begin{pmatrix} -2q(q-1) & 0 \\ 0 & L_0 M_0 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} 2[-q(q-1)L_0 - (q-1)(q-2)M_0 + q(q-2)r_Q] & -2(q-1)L_0M_0 \\ -2(q-1)L_0M_0 & -4uvL_0M_0 \end{pmatrix}. \tag{7.11}$$

Put

$$\bar G = G_0 + 2B_1^T \mathbb{C}^{-1} B_1, \quad \bar N = 2B_1^T \mathbb{C}^{-1} B_0, \quad \bar V = V_0 + 2B_0^T \mathbb{C}^{-1} B_0. \tag{7.12}$$

Since $\mathbf{z} = \mathbb{T}\mathbf{y}$, with $\mathbb{T} = \operatorname{diag}(u^{-1}, 1)$, define

$$\begin{aligned} \mathsf{G} &= \mathbb{T}^T \bar{G} \mathbb{T}, \\ \mathsf{N} &= \mathbb{T}^T \bar{G} \mathbb{T}' + \mathbb{T}^T \bar{N} \mathbb{T}, \\ \mathsf{V} &= \mathbb{T}'^T \bar{G} \mathbb{T}' + \mathbb{T}'^T \bar{N} \mathbb{T} + \mathbb{T}^T \bar{N}^T \mathbb{T}' + \mathbb{T}^T \bar{V} \mathbb{T}. \end{aligned} \tag{7.13}$$

The matrices above also follow from an algebraic elimination. Set $\mathbf{b} = B_1\mathbf{z}' + B_0\mathbf{z}$ and define

$$\mathscr{L}(\mathbf{z}, \mathbf{z}', \mathbf{w}) = \mathbf{z}'^T G_0 \mathbf{z}' + \mathbf{z}^T V_0 \mathbf{z} - 2\mathbf{w}^T \mathbb{C}\mathbf{w} + 4\mathbf{w}^T \mathbf{b}.$$

Its derivative in $\mathbf{w}$ is $4(\mathbf{b} - \mathbb{C}\mathbf{w})$. Thus its stationary equation in $\mathbf{w}$ is precisely (7.1), and

$$\begin{aligned} \mathscr{L} &= \mathbf{z}'^T \bar{G} \mathbf{z}' + 2\mathbf{z}'^T \bar{N} \mathbf{z} + \mathbf{z}^T \bar{V} \mathbf{z} \\ &\quad - 2(\mathbf{w} - \mathbb{C}^{-1}\mathbf{b})^T \mathbb{C}(\mathbf{w} - \mathbb{C}^{-1}\mathbf{b}). \end{aligned} \tag{7.14}$$

Indeed, expansion of the last term leaves $2\mathbf{b}^T\mathbb{C}^{-1}\mathbf{b}$; its three coefficients are those defining $\bar{G}, \bar{N}, \bar{V}$. No sign of $\mathbb{C}$ is assumed in this elimination. Its relation to the reduced field equations is the pair of coefficient identities (7.19)–(7.20) below.

Lemma 7.3. *Every solution of* (7.1), (7.8), *and* (7.9) *satisfies*

$$\mathsf{G}\mathbf{y}'' + (\mathsf{G}' + H\mathsf{G} + \mathsf{N} - \mathsf{N}^T)\mathbf{y}' + (\mathsf{N}' + H\mathsf{N} - \mathsf{V})\mathbf{y} = 0. \tag{7.15}$$

Furthermore,

$$\mathsf{G}_{11} = \frac{(m-1)(qL_0x + pM_0y + 2xy)}{\mathsf{D}} > 0, \tag{7.16}$$

$$\det \mathsf{G} = \frac{2L_0M_0(m-1)xy}{\mathsf{D}} > 0. \tag{7.17}$$

Proof. Put $U_i = \mathbb{A}^{-1}T_i$ for $i = 0, 1$, and set

$$D_1 = \begin{pmatrix} H & 0 \\ 0 & H - 2u - 2v \end{pmatrix}, \qquad D_2 = \begin{pmatrix} L_0 + M_0 - 2r_Q & 0 \\ 0 & L_0 + M_0 - 4uv \end{pmatrix},$$

$$E_0 = \begin{pmatrix} 2(vr_P/u - r_Q) & 0 & 0 \\ 0 & -2(u-v) & 2(u-v) \end{pmatrix}.$$

Elimination of $\mathbf{w}$ gives

$$\mathbf{z}'' + F_1\mathbf{z}' + F_0\mathbf{z} = 0, \qquad F_1 = D_1 - E_0U_1, \qquad F_0 = -D_2 - E_0U_0. \tag{7.18}$$

The two identities needed for its symmetric form are

$$\bar{G}' + H\bar{G} + \bar{N} - \bar{N}^T = \bar{G}F_1, \tag{7.19}$$

$$\bar{N}' + H\bar{N} - \bar{V} = \bar{G}F_0. \tag{7.20}$$

In every occurrence of $\mathbb{A}^{-1}$ substitute (7.10) divided by D, and use $\mathbb{C}^{-1} = \mathbb{A}^{-1}\mathbb{P}^{-1}$. Differentiate products entry by entry, with

$$(A^{-1})' = -A^{-1}A'A^{-1}, \qquad L_0' = -2uL_0, \quad M_0' = -2vM_0, \quad s_P' = H' - u', \quad s_Q' = H' - v'.$$

Replace u', v', r_P', r_Q', H' using (3.3) and $H' = pu' + qv'$. The first-derivative terms in the Euler expression then have coefficient $\bar{G}(D_1 - E_0\mathbb{A}^{-1}T_1)$, and the zeroth-order terms have coefficient $-\bar{G}(D_2 + E_0\mathbb{A}^{-1}T_0)$. These are precisely the right sides of (7.19)–(7.20); the scalar reduction in the products is $s_Ps_Q = D_0 - (m-1)uv$. Multiplying (7.18) by $\bar{G}$ now gives the Euler equation of $\int W(\mathbf{z}'^T\bar{G}\mathbf{z}' + 2\mathbf{z}'^T\bar{N}\mathbf{z} + \mathbf{z}^T\bar{V}\mathbf{z})\,dt$. Substitute $\mathbf{z} = \mathbb{T}\mathbf{y}$ and expand its two derivatives. The three resulting coefficients are (7.13), which proves (7.15).

The cofactor products also give

$$
\begin{aligned}
\mathsf{D}\mathsf{G}_{11} &= (m-1)(qL_0x + pM_0y + 2xy), \\
\mathsf{D}\mathsf{G}_{12} &= (m-1)L_0M_0(ux - vy), &(7.21)\\
\mathsf{D}\mathsf{G}_{22} &= \frac{L_0M_0}{pq}\big((m-1)(pM_0u^2x + qL_0v^2y) + D_0xy\big). &(7.22)
\end{aligned}
$$

Expansion of $\mathsf{G}_{11}\mathsf{G}_{22} - \mathsf{G}_{12}^2$ and (7.4) give (7.17). The quantities $x, y, L_0, M_0, \mathsf{D}$ are positive, so (7.16) and (7.17) imply $\mathsf{G} > 0$. ☐

8. Positive decompositions of the scalar form

In this section write $L = L_0$, $M = M_0$, $G = \mathsf{G}$, $N = \mathsf{N}$, and $V = \mathsf{V}$. The determinant is still D. These letters denote the matrices and coefficients in (7.2)–(7.13), not geometric differential operators. The background identities used below are (3.3)–(3.4) and the interval in Lemma 3.2. The scalar quadratic form is

$$\mathcal{Q}_{\mathrm{sc}}[\mathbf{y}] = \int W\big(\mathbf{y}'^TG\mathbf{y}' + 2\mathbf{y}'^TN\mathbf{y} + \mathbf{y}^TV\mathbf{y}\big)\,dt. \tag{8.1}$$

For the algebraic assertions, $p, q \geq 2$ may be real. Besides (3.3)–(3.4), their hypotheses are

$$
\begin{aligned}
&r_P = (p-1)u^2 + 2quv\theta, \qquad r_Q = (q-1)v^2 + 2puv(1-\theta),\\
&u, v, r_P, r_Q > 0, \qquad \theta_- := \frac{p-1}{2p} < \theta < \frac{q+1}{2q} =: \theta_+.
\end{aligned}
\tag{8.2}
$$

For the Böhm background these relations follow from the definition of θ and Lemma 3.2. The geometric applications retain the integral dimension hypotheses in Section 1.

For a symmetric matrix $S(t)$,

$$(W\mathbf{y}^TS\mathbf{y})' = W\{2\mathbf{y}'^TS\mathbf{y} + \mathbf{y}^T(S' + HS)\mathbf{y}\}. \tag{8.3}$$

Consequently subtraction of such a derivative changes only the symmetric part of the mixed coefficient. Positivity of G alone does not control either $N - N^T$ or the zeroth-order term.

Suppose $G = \sum_i \omega_i b_i^T b_i$, where $\omega_i > 0$ and b_i are rows. Given real functions σ_i, put $d_i = b_i' + \sigma_i b_i$ and $N_* = \sum_i \omega_i b_i^T d_i$. Then

$$N_* - N_*^T = \sum_i \omega_i(b_i^Tb_i' - b_i'^Tb_i).$$

Thus the skew identity in (8.7) is exactly the condition that $S_* = N - N_*$ be symmetric. Formula (8.3) then reduces the form to four derivative squares and the matrix $\mathcal{R} = V - \sum_i \omega_i d_i^T d_i - S_*' - HS_*$. The rows and weights below are obtained by separating the three terms in the numerator of G_{11} and the remaining c'^2 term. Their moving coefficients Mu/q and $-Lv/p$ also give the required skew part. The choices $\sigma_1 = v$, $\sigma_2 = u$, $\sigma_3 = \sigma_4 = 0$ are used throughout; positivity of the resulting residual is proved rather than assumed.

In the variables $F_1 = b_1\mathbf{y}$, $F_2 = b_2\mathbf{y}$, the sufficient sign conditions are

$$(R_*)_{12} < 0, \qquad (R_*)_{11} + (R_*)_{12} > 0, \qquad (R_*)_{22} + (R_*)_{12} > 0.$$

They give precisely the three remaining squares in (8.10). The higher-degree proof establishes these inequalities on the whole parameter region. If a degree is one, the exact quotient in (4.7) has dimension one; the independent scalar equation is therefore treated separately in Lemma 8.8.

8.1. Rank-one decompositions and skew parts.

Put $A_+ = v(H + 2u) - r_Q$ and $B_+ = u(H + 2v) - r_P$.

Lemma 8.1. *The skew matrix $J = (N - N^T)/2$ satisfies*

$$J_{12} = \frac{(m-1)LM}{2\mathsf{D}}(A_+y - B_+x). \tag{8.4}$$

Proof. The homogeneity of both sides reduces the identity to $u = 1$. In the remaining algebra put $r = v$, $P = r_P$, $Q = r_Q$ and $H = p + qr$. Substitution of (7.10) gives the following identity before imposing the scalar constraint:

$$\begin{aligned} &\mathsf{D}(N_{12} - N_{21}) - (m-1)LM(A_+y - B_+x) \\ &\qquad = -LM\big(pP + qQ - H^2 + p + qr^2\big)\,\mathcal{T}, \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} \mathcal{T} &= (H-P)\big((q-1)P + qQ - qL - (q-1)M\big) \\ &\quad + 2r\big((q-1)P - qL\big) + 2\big(qQ - (q-1)M\big). \end{aligned} \tag{8.6}$$

The first factor on the right is zero. Division by $2\mathsf{D}$ proves (8.4). □

Proposition 8.2. *If $x, y > 0$, define positive weights*

$$\omega_1 = \frac{(m-1)qLx}{\mathsf{D}}, \quad \omega_2 = \frac{(m-1)pMy}{\mathsf{D}}, \quad \omega_3 = \frac{2(m-1)xy}{\mathsf{D}}, \quad \omega_4 = \frac{LMD_0xy}{pq\mathsf{D}},$$

and row vectors

$$b_1 = (1, Mu/q), \quad b_2 = (1, -Lv/p), \quad b_3 = (1,0), \quad b_4 = (0,1).$$

Then

$$G = \sum_{i=1}^{4} \omega_i b_i^T b_i, \qquad \frac{1}{2}(N - N^T) = \frac{1}{2}\sum_{i=1}^{4} \omega_i (b_i^T b_i' - b_i'^T b_i). \tag{8.7}$$

Proof. The coefficients of the first identity are exactly (7.16)–(7.22). The background equations imply

$$(Mu/q)' = -MB_+/q, \qquad (-Lv/p)' = LA_+/p.$$

Since $b_3', b_4' = 0$, the upper-right entry in the second identity is

$$\frac{1}{2}\left(-\omega_1 MB_+/q + \omega_2 LA_+/p\right) = \frac{(m-1)LM}{2\mathsf{D}}(A_+y - B_+x).$$

Apply Lemma 8.1. The diagonal entries of both skew matrices vanish. □

The four weights are positive. Thus (8.7) proves positivity of the leading coefficient for real $p, q \geq 2$.

8.2. **The residual form.** Choose

$$\sigma_1 = v, \quad \sigma_2 = u, \quad \sigma_3 = \sigma_4 = 0, \qquad d_i = b_i' + \sigma_i b_i.$$

Put

$$\begin{aligned} N_* &= \sum_i \omega_i b_i^T d_i, \qquad S_* = N - N_*, \\ \mathcal{R} &= V - \sum_i \omega_i d_i^T d_i - S_*' - HS_*. \end{aligned} \tag{8.8}$$

Proposition 8.2 makes S_* symmetric. Thus $\mathcal{R}$ is symmetric.

Proposition 8.3. *For every smooth* $\mathbf{y}$ *one has the pointwise identity*

$$\begin{aligned} &W\big(\mathbf{y}'^T G\mathbf{y}' + 2\mathbf{y}'^T N\mathbf{y} + \mathbf{y}^T V\mathbf{y}\big) \\ &\quad = \frac{d}{dt}(W\mathbf{y}^T S_*\mathbf{y}) + W\left\{\sum_{i=1}^{4}\omega_i\big((b_i\mathbf{y})' + \sigma_i b_i\mathbf{y}\big)^2 + \mathbf{y}^T\mathcal{R}\mathbf{y}\right\}. \end{aligned} \tag{8.9}$$

Proof. Expansion of the four squares gives the leading coefficient G, the mixed coefficient N_*, and the zeroth-order coefficient $\sum \omega_i d_i^T d_i$. The remaining mixed coefficient is symmetric and satisfies

$$2W\mathbf{y}'^T S_*\mathbf{y} = (W\mathbf{y}^T S_*\mathbf{y})' - W\mathbf{y}^T(S_*' + HS_*)\mathbf{y}.$$

Substitution proves (8.9). □

Set $k_1 = Mu/q$, $k_2 = Lv/p$ and

$$C = \frac{1}{k_1 + k_2}\begin{pmatrix} k_2 & k_1 \\ 1 & -1 \end{pmatrix}, \qquad R_* = C^T \mathcal{R} C.$$

Then $\mathbf{y} = C(F_1, F_2)^T$, where $F_i = b_i\mathbf{y}$ for $i = 1, 2$. Define

$$r_1 = (R_*)_{11} + (R_*)_{12}, \qquad r_2 = (R_*)_{22} + (R_*)_{12}, \qquad r_{12} = -(R_*)_{12}.$$

The exact identity

$$\mathbf{y}^T \mathcal{R}\mathbf{y} = r_1F_1^2 + r_2F_2^2 + r_{12}(F_1 - F_2)^2 \tag{8.10}$$

reduces residual positivity to the next theorem.

Theorem 8.4 (Positivity for higher scalar harmonics). *Let $p, q \geq 2$ be real and let the positive background quantities satisfy* (3.3), (3.4), *and* (8.2). *If $\lambda \geq 2p$ and $\mu \geq 2q$, then*

$$r_1 > 0, \qquad r_2 > 0, \qquad r_{12} > 0.$$

In particular $\mathcal{R}$ is positive definite. The inequalities include continuous spectral parameters in these half-lines.

An exact specialization. Consider the Ricci-flat product cone $dt^2 + (t^2/3)(\gamma_2 + \gamma_2)$, and degree two on both factors. At $t = 1$ the coefficient data are

$$p = q = 2, \quad u = v = 1, \quad r_P = r_Q = 3, \quad \lambda = \mu = 6, \quad L = M = 18, \quad \theta = \tfrac{1}{2}.$$

The background derivatives, taken before evaluation, are $u' = v' = -1$, $r_P' = r_Q' = -6$, and $L' = M' = -36$. The formulas in Section 7 give

$$\mathsf{D} = 4320, \qquad G = \begin{pmatrix} 4/5 & 0 \\ 0 & 81 \end{pmatrix}, \quad N = \begin{pmatrix} -236/5 & -324 \\ -324 & -1701 \end{pmatrix}, \quad V = \begin{pmatrix} -1076/5 & -648 \\ -648 & 4293 \end{pmatrix}. \tag{8.11}$$

In particular, the sign of V is not the required positivity assertion. The four weights are $(\omega_1, \omega_2, \omega_3, \omega_4) = (3/10, 3/10, 1/5, 162/5)$, and $b_1 = (1, 9)$, $b_2 = (1, -9)$. The prescribed completion gives

$$N_* = \begin{pmatrix} 3/5 & 0 \\ 0 & -486/5 \end{pmatrix}, \qquad \mathcal{R} = \begin{pmatrix} 116/5 & 0 \\ 0 & 12474/5 \end{pmatrix}.$$

Since $\zeta = (F_1 + F_2)/2$ and $c = (F_1 - F_2)/18$, one obtains

$$\mathbf{y}^T\mathcal{R}\mathbf{y} = \frac{58}{5}F_1^2 + \frac{58}{5}F_2^2 + \frac{19}{10}(F_1 - F_2)^2. \tag{8.12}$$

This specialization is an identity on the cone, not an assertion that a smooth Böhm filling passes through these cone Cauchy data. The argument for arbitrary p, q and all regular radii uses the coefficient identities below, not this single specialization.

8.3. **The residual numerators.** Under

$$(u, v, r_P, r_Q, L, M) \longmapsto (cu, cv, c^2r_P, c^2r_Q, c^2L, c^2M), \quad c > 0,$$

the degrees of the entries of $G, N, V, \mathcal{R}$ are, respectively,

$$\begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}.$$

Since k_1, k_2 have degree three, the functions r_1, r_2, r_{12} have degree zero. We may normalize $u = 1$ at the point in question. Write $r = v/u$ and set

$$P = p - 1 + 2qr\theta, \qquad Q = (q-1)r^2 + 2pr(1-\theta), \tag{8.13}$$

$$H = p + qr, \qquad D = pP + qQ, \qquad a_0 = P - H, \tag{8.14}$$

$$x = (p-1)L - pP, \qquad y = (q-1)M - qQ, \tag{8.15}$$

$$\mathcal{T} = pM + qLr. \tag{8.16}$$

The normalized background derivative on functions of (r,θ,L,M) is

$$\begin{aligned}\mathscr{D} &= (Q-rP)\partial_r + \left[-(2p\theta-p+1)(1-\theta)+r(q+1-2q\theta)\theta\right]\partial_\theta \\ &\quad -2(a_0+1)L\partial_L - 2(a_0+r)M\partial_M. \end{aligned}\tag{8.17}$$

For an expression of degree e, the normalized radial derivative is $ea_0+\mathscr{D}$. This follows from $u' = u^2 a_0$, $L' = -2uL$, and $M' = -2vM$.

Lemma 8.5. *There are polynomials* $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ *in* $\mathbb{Q}[p,q,r,\theta,L,M]$ *such that*

$$r_1 = \frac{(m-1)qLx}{2\mathsf{D}^2\mathcal{T}}\,\mathfrak{a}, \qquad r_2 = \frac{(m-1)pMy}{2\mathsf{D}^2\mathcal{T}}\,\mathfrak{b}, \tag{8.18}$$

$$r_{12} = \frac{pqLMxy}{2\mathsf{D}^2\mathcal{T}^2}\,\mathfrak{c}. \tag{8.19}$$

They satisfy

$$\begin{aligned}\mathfrak{b}_{p,q}(r,\theta,L,M) &= r^{11}\mathfrak{a}_{q,p}(r^{-1},1-\theta,M/r^2,L/r^2), \\ \mathfrak{c}_{p,q}(r,\theta,L,M) &= r^{10}\mathfrak{c}_{q,p}(r^{-1},1-\theta,M/r^2,L/r^2).\end{aligned}\tag{8.20}$$

The total degrees in (L,M) *of* $\mathfrak{a}$ *and* $\mathfrak{c}$ *are four and three.*

Proof. The following formulas specify the polynomials without a choice of normalization of factors. In the normalized variables, form $\mathcal{R}$ by (8.8), using (7.10) and (8.17). Then set

$$\begin{aligned}\mathfrak{a} &= \frac{2\mathsf{D}^2}{(m-1)qLx}(qLr\mathcal{R}_{11}+pq\mathcal{R}_{12}), \\ \mathfrak{b} &= \frac{2\mathsf{D}^2}{(m-1)pMy}(pM\mathcal{R}_{11}-pq\mathcal{R}_{12}), \\ \mathfrak{c} &= \frac{2pq\mathsf{D}^2}{LMxy}\left(\mathcal{R}_{22}-\frac{pM-qLr}{pq}\mathcal{R}_{12}-\frac{LMr}{pq}\mathcal{R}_{11}\right).\end{aligned}\tag{8.21}$$

Substitution of (7.10) clears the denominators in these expressions. A coefficient form of the first and third expressions is given in (8.22) and Appendices F–G. It verifies their polynomial character as well as their degrees. The formula for $\mathfrak{b}$ is obtained by the indicated exchange. To check the exchange directly, replace each monomial $p^a q^b r^c \theta^d L^e M^f$ by

$$q^a p^b r^{11-c-2e-2f}(1-\theta)^d M^e L^f$$

in $\mathfrak{a}$, and use degree ten for $\mathfrak{c}$. Collecting gives precisely the second and third expressions of (8.21). The factor formulas follow by multiplication with the displayed C in (8.10). ☐

Put

$$\xi = L - \frac{2pP}{p-1} \geq 0, \qquad \eta = M - \frac{2qQ}{q-1} \geq 0, \qquad s = \frac{\theta-\theta_-}{\theta_+-\theta} > 0.$$

Thus

$$\theta = \Theta(s) := \frac{\theta_- + \theta_+ s}{1+s}.$$

Let $\mathfrak{n}_A = \mathfrak{a}$, $\mathfrak{n}_C = \mathfrak{c}$, and $d_A = 4$, $d_C = 3$. The two rational identities used in the proof are

$$\begin{aligned}&\mathfrak{n}_H\left(r,\Theta(s),\frac{2pP}{p-1}+\xi,\frac{2qQ}{q-1}+\eta\right) \\ &= \sum_{k+\ell\leq d_H}\frac{\xi^k\eta^\ell \sum_{(i,j)\in I^H_{k\ell}} c^H_{k\ell ij}(p-2,q-2)r^i s^j}{(p-1)^{d_H-k}(q-1)^{d_H-\ell}(2pq)^{D^H_{k\ell}}(1+s)^{D^H_{k\ell}}}, \qquad H\in\{A,C\}.\end{aligned}\tag{8.22}$$

Here P,Q on the left are evaluated at $\theta=\Theta(s)$. The indices $D^H_{k\ell}$, sets $I^H_{k\ell}$, and all coefficients are printed in Appendix F. There are 341 nonzero coefficient functions for $H=A$ and 210 for $H=C$.

8.4. **Finite coefficient comparison.** Write

$$\mathfrak{n}_H = \sum_{a+b\le d_H} n^H_{ab}(p,q,r,\theta)L^aM^b,$$

where the coefficients n^H_{ab} are obtained from (7.10), (8.17), and (8.21). Matrix products use

$$(AB)_{ij} = \sum_k A_{ik}B_{kj}, \qquad (F/G)' = F'/G - FG'/G^2.$$

For fixed k,ℓ, the numerator of the $(\xi^k\eta^\ell)$ coefficient after clearing $(p-1)^{d_H-k}(q-1)^{d_H-\ell}$ is exactly

$$F^H_{k\ell} = \sum_{\substack{a\ge k,\ b\ge \ell\\ a+b\le d_H}} \binom{a}{k}\binom{b}{\ell} n^H_{ab}(2pP)^{a-k}(2qQ)^{b-\ell}(p-1)^{d_H-a}(q-1)^{d_H-b}. \tag{8.23}$$

Its degree in θ is the tabulated integer $D^H_{k\ell}$. If $F^H_{k\ell} = \sum_z f_z(p,q,r)\theta^z$, then its cleared interval substitution is

$$\sum_{z=0}^{D^H_{k\ell}} f_z(p,q,r)\big(q(p-1)+p(q+1)s\big)^z\big(2pq(1+s)\big)^{D^H_{k\ell}-z}. \tag{8.24}$$

For each z, the coefficient of s^j in the two powers in (8.24) is

$$\sum_{e=\max(0,j-D+z)}^{\min(z,j)} \binom{z}{e}\binom{D-z}{j-e}[q(p-1)]^{z-e}[p(q+1)]^e(2pq)^{D-z}, \qquad D = D^H_{k\ell}. \tag{8.25}$$

Collecting powers of r gives the coefficients in Appendix F. Each is factored into polynomials printed in Appendix G; the convention for every row is stated there. Equations (8.23)–(8.25), together with the full coefficient tables, are finite rational coefficient-comparison identities. All indices and coefficient polynomials in these two identities are specified in Sections F and G.

Lemma 8.6. *For every real $p,q\ge 2$, $r,s>0$, and $\xi,\eta\ge 0$, both expressions in* (8.22) *are strictly positive. More precisely,*

$$\mathfrak{a} \ge \frac{4mp^4q(p-1)^2}{q-1}\frac{r}{(1+s)^5}, \tag{8.26}$$

$$\mathfrak{c} \ge \frac{4p^4(p-1)^3}{(1+s)^4}. \tag{8.27}$$

Proof. Every Π_i in Appendix G is a polynomial in $a=p-2$, $b=q-2$ with nonnegative coefficients. Hence both $\Pi_i(a,b)$ and $\Pi_i(b,a)$ are nonnegative. Every rational multiplier in Appendix F is positive, and all denominators in (8.22) are positive. Thus every summand in that identity is nonnegative.

For the A identity, the coefficient at $(k,\ell,i,j)=(0,0,1,0)$ before division by the denominator is

$$128(q-1)^3q^6m(p-1)^6p^9.$$

Its denominator is $(p-1)^4(q-1)^4(2pq)^5(1+s)^5$. Their quotient proves (8.26). For the C identity, the coefficient at $(0,0,0,0)$ is

$$64(q-1)^3q^4(p-1)^6p^8.$$

Division by $(p-1)^3(q-1)^3(2pq)^4(1+s)^4$ proves (8.27). Both lower bounds are strictly positive. ☐

Proof of Theorem 8.4. The spectral bounds give $x\ge pP>0$ and $y\ge qQ>0$. Therefore every prefactor in (8.18)–(8.19) is positive. Lemma 8.6 proves $r_1>0$ and $r_{12}>0$.

The transformation $(p,q,r,\theta,L,M)\mapsto(q,p,r^{-1},1-\theta,M/r^2,L/r^2)$ preserves the hypotheses. The interval coordinate changes from s to s^{-1}, and the excess parameters change from (ξ,η) to $(\eta/r^2,\xi/r^2)$. The first identity in (8.20) therefore makes $\mathfrak{b}$ positive. This proves $r_2>0$.

Since $k_1+k_2>0$, the matrix C is invertible. Formula (8.10) now shows that $\mathbf{y}^T\mathcal{R}\mathbf{y}>0$ for every nonzero $\mathbf{y}$. ☐

Reading the coefficient data. In the dimension-polynomial tables write $a = p - 2$, $b = q - 2$. The notation $a^i b^j [c_0, \dots, c_s]_b$ means

$$a^i b^j \sum_{\ell=0}^{s} c_\ell b^\ell.$$

These two letters are local dimension shifts, not the warping functions. A table factor $\Pi_i(b, a)$ is obtained by interchanging the two shifts in the entire polynomial. All factors and their multiplicities in a coefficient row are multiplied before the spectral and interval monomials are attached.

For example, the higher-mode row $(k, \ell, i, j) = (0, 0, 1, 0)$ in the $H = A$ table contributes exactly

$$\begin{aligned} &\frac{128(q-1)^3 q^6 (p+q)(p-1)^6 p^9}{(p-1)^4 (q-1)^4 (2pq)^5 (1+s)^5}\, r \\ &\qquad = \frac{4(p+q)p^4 q(p-1)^2}{q-1} \frac{r}{(1+s)^5}. \end{aligned} \tag{8.28}$$

All remaining summands are nonnegative, which gives the first explicit lower bound. The second lower bound uses the $H = C$ row $(0, 0, 0, 0)$ in the same manner. In the exceptional table the row $(i, j, k) = (8, 0, 0)$ contributes $(q-1)^2(q+1)(q+2)^2 r^8$ to $(1+s)^4 \mathcal{Z}$.

Each displayed coefficient identity is an equality in a rational polynomial ring. After multiplying by its stated denominator, one compares the coefficient of every monomial in the union of the two supports. A monomial absent from either side has coefficient zero on that side. This rule verifies both the printed coefficients and the absence of any additional terms.

8.5. **Reconstruction of the coefficient identities.** The tables encode exact rational functions, not values sampled from the parameter region. For each identity, the following algebraic operations specify an independent reconstruction. Form the constraint matrix and its adjugate from (7.2) and (7.10); form G, N, V from (7.12)–(7.13); apply the homogeneous derivative (8.17); and form the three numerators in (8.21). The divisions are identities in a rational polynomial ring. The resulting numerators are then subjected to the spectral shift and interval substitution in (8.23)–(8.25). Their coefficients must equal the expanded table factors, with zero coefficients at every omitted monomial. Comparing only the listed nonzero coefficients would not suffice.

The first-harmonic calculation is separate. It starts from the independent second-factor equation in Lemma 8.8, verifies its three multiplier identities, and forms (8.39). The interval substitution is (8.41). The exceptional table is compared after multiplication by its positive dimension denominator. In both calculations the variables p, q remain symbolic. Positivity on the closed dimension faces $p = 2$ or $q = 2$ follows from the shifted integer coefficients and the explicit strictly positive terms, not from a continuity argument that could lose strictness.

The accompanying verification directory contains a generator that reads no coefficient table, a separate checker of the generated polynomials against the typeset data, and checks of the omitted monomials, denominator signs, and strictly positive terms. The command `python verification/run_all.py` regenerates the expressions from their matrix definitions before comparing them. The differential geometric reductions, the boundary conditions, and the functional analytic arguments remain the mathematical proofs of the corresponding sections; they are not assertions proved by a coefficient comparison.

8.6. **A monotone boundary expression.** For every smooth $\mathbf{y}$, (8.9) and Theorem 8.4 give the exact identity

$$\begin{aligned} &W\big(\mathbf{y}'^T G \mathbf{y}' + 2\mathbf{y}'^T N \mathbf{y} + \mathbf{y}^T V \mathbf{y}\big) \\ &= \big(W \mathbf{y}^T S_* \mathbf{y}\big)' + W \left\{ \sum_{i=1}^{4} \omega_i \big((b_i \mathbf{y})' + \sigma_i b_i \mathbf{y}\big)^2 + r_1 F_1^2 + r_2 F_2^2 + r_{12}(F_1 - F_2)^2 \right\}. \end{aligned} \tag{8.29}$$

All seven weights on the right are strictly positive. In particular the form is positive on nonzero compactly supported smooth $\mathbf{y}$.

If $\mathbf{y}$ solves the Euler equation of (8.1), then

$$\big(W(G\mathbf{y}' + N\mathbf{y})\big)' = W(N^T\mathbf{y}' + V\mathbf{y}).$$

Multiply by $\mathbf{y}^T$, use the product rule, and subtract the derivative term in (8.29). The resulting flux is

$$\mathcal{F}_* = W\mathbf{y}^T(G\mathbf{y}' + N_*\mathbf{y}), \tag{8.30}$$

not merely $W\mathbf{y}^TG\mathbf{y}'$. It satisfies

$$\mathcal{F}_*' = W\left\{\sum_{i=1}^{4}\omega_i\big((b_i\mathbf{y})' + \sigma_ib_i\mathbf{y}\big)^2 + r_1F_1^2 + r_2F_2^2 + r_{12}(F_1 - F_2)^2\right\}. \tag{8.31}$$

Proposition 8.7. *Every two-factor scalar sector with $\ell, j \geq 2$ has zero geometric Dirichlet kernel.*

Proof. The spherical eigenvalues satisfy $\lambda \geq 2(p+1) > 2p$ and $\mu \geq 2(q+1) > 2q$. Thus every coefficient on the right of (8.31) is positive. At the singular orbit, (1.2) and Lemma 4.3 give

$$u = t^{-1} + O(t), \quad v = O(t), \quad r_P, L, x \asymp t^{-2}, \quad r_Q, M, y \asymp 1, \quad \mathsf{D} \asymp t^{-6}, \quad W \asymp t^p,$$

$$\zeta = O(t^{\ell-1}), \quad c = O(t^{\ell}), \quad \zeta' = O(t^{\ell-2}), \quad c' = O(t^{\ell-1}).$$

Equations (7.16), (7.21), and (7.22) imply

$$G = \begin{pmatrix} O(t^2) & O(t) \\ O(t) & O(1) \end{pmatrix}.$$

The rank-one weights and their differentiated rows satisfy

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (O(t^2), O(t^6), O(t^4), O(1)),$$

$$b_1, b_2 = (1, O(t^{-1})), \quad d_1 = (O(t), O(t^{-2})), \quad d_2 = (O(t^{-1}), O(t^{-2})).$$

Since $d_3 = d_4 = 0$, these estimates give

$$N_* = \begin{pmatrix} O(t^3) & O(1) \\ O(t^2) & O(t^{-1}) \end{pmatrix}, \qquad \mathcal{F}_* = O(t^{p+2\ell-1}) \longrightarrow 0. \tag{8.32}$$

At the outer endpoint $\mathbf{y}(R) = 0$ by (4.5), so $\mathcal{F}_*(R) = 0$. Integrating (8.31) over $[\varepsilon, R]$ and letting $\varepsilon \downarrow 0$ makes the integral of its nonnegative right side zero. The strict inequalities $r_1, r_2 > 0$ imply $F_1 = F_2 = 0$. The change of variables in (8.10) is invertible, so $\mathbf{y} = 0$.

The constraints now give $v = U = V = 0$. Thus s is constant, $C' = -s/a^2$, and $D' = -s/b^2$. The tensor is the Lie derivative of

$$sYZ\partial_t + CZ\nabla Y + DY\nabla Z$$

on the regular part. Undo the normal gauge. At R, the independent trace-free Hessians force the two angular coefficients of the total vector field to vanish, and the traces force its normal coefficient to vanish. Thus the original tensor is boundary-fixing pure gauge in a collar of Σ_R. Lemma 4.6 gives the global assertion. □

8.7. **A degree-one factor.** If $\lambda = p$ and $\mu \geq 2(q+1)$, adding the same function k to A and C does not change (4.2). It changes (ζ, c) by $(-vk, -a^2k)$. The nonredundant variable is

$$\chi = \zeta - \frac{v}{a^2}c. \tag{8.33}$$

Choose the representative $c = 0$. If $\mu = q$ and $\lambda \geq 2(p+1)$, the corresponding variable is $\chi = \zeta + (u/b^2)c$, and again one may take $c = 0$. In either case $\chi(R) = 0$ follows from the geometric boundary condition.

Lemma 8.8. *In either exceptional family the geometric equation implies*

$$-(Wg_1\chi')' + WP\chi = 0, \qquad g_1 = \mathsf{G}_{11} > 0, \qquad P = \mathsf{V}_{11} - \mathsf{N}_{11}' - H\mathsf{N}_{11}. \tag{8.34}$$

The matrices are evaluated at the exceptional spectral value.

Proof. It suffices to consider $\lambda = p$; the other case follows by interchanging the two factors. The variable c is set to zero by the trace–Hessian redundancy. Put

$$(\alpha_0, \alpha_1, \alpha_2)^T = \mathbb{A}^{-1} T_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (\beta_0, \beta_1, \beta_2)^T = \mathbb{A}^{-1} T_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then $(v, U, V)^T = (\alpha_i Z_0' + \beta_i Z_0)_{i=0}^2$. The $YZ\gamma_q$ equation is independent of the first-factor trace–Hessian relation, because $j \geq 2$. Substituting $A = us$, $B = vs + Z_0$ in that equation, and using $s' = v$, gives

$$Z_0'' + (H + qv)Z_0' - (L + M - 2r_Q)Z_0 + vv' + 2r_Q v - vLU - vMV = 0.$$

Thus its three coefficients in (Z_0'', Z_0', Z_0) are

$$\begin{aligned} a_2 &= 1 + v\alpha_0, \\ a_1 &= H + qv + v(\alpha_0' + \beta_0) + 2r_Q\alpha_0 - vL\alpha_1 - vM\alpha_2, \\ a_0 &= -(L + M - 2r_Q) + v\beta_0' + 2r_Q\beta_0 - vL\beta_1 - vM\beta_2. \end{aligned}$$

Writing $Z_0 = u^{-1}\chi$ gives coefficients

$$e_2 = a_2/u, \quad e_1 = 2a_2(u^{-1})' + a_1/u, \quad e_0 = a_2(u^{-1})'' + a_1(u^{-1})' + a_0/u.$$

The adjugate (7.10), $pr_P = (p-1)L$, and (3.4) give the following identities:

$$e_2 = \frac{M}{2(q-1)\mathsf{D}}\big((q-1)(m-1)Lv + pys_Q + p(m-1)r_Q u\big) > 0, \tag{8.35}$$

$$g_1 e_1 = e_2(g_1' + Hg_1),$$

$$g_1 e_0 = e_2(\mathsf{N}_{11}' + H\mathsf{N}_{11} - \mathsf{V}_{11}). \tag{8.36}$$

For the first equality, expansion of the first row of $\operatorname{adj}\mathbb{A}\, T_1$ yields

$$2\mathsf{D}(1 + v\alpha_0) = uM\big((m-1)Lv + p(Ms_Q - r_Q s_P)\big).$$

Use $(q-1)(Ms_Q - r_Q s_P) = ys_Q + (m-1)r_Q u$. For the other two equalities, differentiate the displayed expressions for α_i, β_i by the quotient rule and substitute the background identities. The first coefficient reduces to $e_2(g_1' + Hg_1)$ and the second to $e_2(\mathsf{N}_{11}' + H\mathsf{N}_{11} - \mathsf{V}_{11})$. Division by $e_2/g_1 > 0$ proves (8.34). No dependent first-harmonic Hessian equation was imposed. ☐

Suppose first that

$$\lambda = p, \qquad \mu = 2(q+1) + h, \qquad h \geq 0.$$

Let $g_1 = G_{11}$, $n_1 = N_{11}$ and $v_1 = V_{11}$, specialized at these spectral values. Define

$$P_u = v_1 - n_1' - Hn_1 + (g_1 u)' + Hg_1 u - g_1 u^2. \tag{8.37}$$

Since $x = 0$ and $y > 0$, (7.4) and (7.16) give $\mathsf{D} > 0$ and $g_1 > 0$.

Theorem 8.9. *Under* (3.3)–(3.4) *and* (8.2), $P_u > 0$. *This holds for all real* $p, q \geq 2$, *without a restriction on* $p + q$. *The exchanged assertion, with* $\mu = q$, $\lambda \geq 2(p+1)$ *and completion by* v, *also holds.*

Proof. Normalize $u = 1$ and write $r = v/u$. Then

$$R_P = p - 1 + 2qr\theta, \quad R_Q = (q-1)r^2 + 2pr(1-\theta), \quad H = p + qr, \quad a_0 = R_P - H.$$

For a normalized function define

$$\begin{aligned} \mathcal{D} = {} & (R_Q - rR_P)\partial_r \\ & + \big[-(2p\theta - p + 1)(1-\theta) + r(q + 1 - 2q\theta)\theta\big]\partial_\theta. \end{aligned} \tag{8.38}$$

An expression homogeneous of degree e has radial derivative $ea_0 + \mathcal{D}$ after this normalization. In particular the degrees of $g_1, n_1, v_1, \mathsf{D}$ are respectively $-2, -1, 0, 6$.

Define the rational expression

$$\mathcal{Z} = \frac{\mathsf{D}}{g_1}\left[v_1 - (H - a_0)n_1 - \mathcal{D}n_1 + \mathcal{D}g_1 + (H - a_0 - 1)g_1\right]. \tag{8.39}$$

The derivative rule and (8.37) give

$$\mathcal{Z} = \left.\frac{\mathsf{D}P_u}{g_1}\right|_{u=1}. \tag{8.40}$$

All entries in this expression are defined by (7.12)–(7.13).

Put

$$\Theta(s) = \frac{\theta_- + \theta_+ s}{1+s}, \qquad a = p-2, \quad b = q-2.$$

The complete polynomial identity used in the proof is

$$(1+s)^4 \mathcal{Z}(r, \Theta(s), h) = \sum_{(i,j,k)\in\mathcal{I}} c_{ijk}(a,b)\, r^i s^j h^k. \tag{8.41}$$

The set $\mathcal{I}$ and every coefficient are specified in Sections H and I. There are 135 entries. The coefficient convention is ascending powers of r, s, h.

Identity (8.41) is checked as follows. Substitute (7.10) into the entries defining g_1, n_1, v_1. Differentiate each rational entry by (8.39). Replace R_P, R_Q, H by their displayed polynomials, and use $\lambda = p$, $\mu = 2(q+1)+h$. After cancellation of all factors depending on r, θ, h, the expression belongs to $\mathbb{Q}(p,q)[r,\theta,h]$. Its remaining denominator depends only on p, q and is nonzero for $p, q \geq 2$. The degrees in θ, r, h are at most four, eight, and three, respectively. For a monomial $r^i \theta^j h^k$ of that quotient, the substitution in (8.41) replaces its θ part by

$$(\theta_- + \theta_+ s)^j (1+s)^{4-j}.$$

The binomial formula gives the coefficient multiplying s^t as

$$\sum_{e=\max(0,t-4+j)}^{\min(j,t)} \binom{j}{e}\binom{4-j}{t-e} \theta_-^{j-e}\theta_+^e. \tag{8.42}$$

Expansion and collection by (8.42) give exactly the printed coefficient table. The coefficient table is the finite expansion determined by (7.10), (8.39), and (8.42).

All factors in the table are polynomials in a, b with nonnegative coefficients and strictly positive constant term. All denominators have the same property. Thus every c_{ijk} is positive for $a, b \geq 0$. In particular,

$$c_{8,0,0} = (q-1)^2(q+1)(q+2)^2 > 0.$$

Since $r, s > 0$ and $h \geq 0$, the right side of (8.41) is strictly positive even for $h = 0$. Equations (8.40) and $g_1, \mathsf{D} > 0$ prove $P_u > 0$.

For the other exceptional family exchange (p, u, r_P, λ) with (q, v, r_Q, μ). The scalar variable changes sign, which leaves its form unchanged, and the completion changes from u to v. In normalized variables the corresponding numerator is

$$r^8 \mathcal{Z}_{q,p}(r^{-1}, 1-\theta, h).$$

The power eight is the degree of $\mathsf{D}P_u/g_1$. The interval for $1-\theta$ is precisely the exchanged interval, so the preceding positivity applies. □

For later use, write both conclusions as

$$P_\sigma = P + (g_1\sigma)' + Hg_1\sigma - g_1\sigma^2 > 0, \qquad \sigma = \begin{cases} u, & \lambda = p, \\ v, & \mu = q. \end{cases} \tag{8.43}$$

Proposition 8.10. *The scalar sectors $(1, j)$ and $(\ell, 1)$, with the other degree at least two, have zero geometric kernel.*

Proof. Equation (8.34) gives

$$\frac{d}{dt}[Wg_1\chi(\chi' + \sigma\chi)] = W[g_1(\chi'+\sigma\chi)^2 + P_\sigma\chi^2]. \tag{8.44}$$

When the collapsing factor has degree one, the intrinsic combinations in Lemma 4.3 give $\chi = O(1)$, $\chi' = O(t^{-1})$, $g_1 = O(t^2)$, and $\sigma = O(t^{-1})$. The inner flux is $O(t^{p+1})$ and tends to zero. If the collapsing factor has degree $\ell \geq 2$ and the other factor has degree one, the intrinsic coefficients give

$$\chi = O(t^{\ell-1}), \quad \chi' = O(t^{\ell-2}), \quad g_1 = O(t^2), \quad \sigma = v = O(t).$$

Thus $W g_1 \chi(\chi' + \sigma\chi) = O(t^{p+2\ell-1}) \to 0$. The outer flux is zero. Thus $\chi = 0$. Choose the representative $c = 0$. The constraints give $v = U = V = 0$, so s is constant, $C' = -s/a^2$, and $D' = -s/b^2$. The tensor on the regular part is the Lie derivative of $sYZ\partial_t + CZ\nabla Y + DY\nabla Z$. When $\lambda = p$, independence of the trace and Hessian on the q factor forces the normal and q-gradient boundary coefficients to vanish; the remaining p-gradient coefficient then vanishes. The argument is interchanged when $\mu = q$. Apply Lemma 4.6. □

9. The full kernel and the induced-metric map

Proposition 9.1. *For every $R > 0$, the kernel in* (1.3) *is invariant and satisfies* (1.4).

Proof. The harmonic decomposition in Lemma 4.1 is exhausted by Propositions 5.1, 5.2, 6.2, 8.7 and 8.10 and Lemma 5.3 and the invariant sector. Whenever a cited sector calculation gives $h = \mathcal{L}_X g$ on the regular part, Lemma 4.5 extends X across the singular orbit, whose codimension is $p + 1 \geq 3$. This does not change its prescribed outer boundary value. To pass to an arbitrary smooth deformation, use its unique Bianchi representative from Lemma 2.2. The gauge projection commutes with $\mathfrak{G}$. Each non-invariant harmonic component is boundary-fixing pure gauge by the cited propositions. Its Bianchi representative is zero, since its gauge field solves a zero-boundary connection-Laplacian equation. Completeness of the angular expansion therefore leaves an invariant tensor. No summation of separately chosen gauge fields is required.

An invariant infinitesimal deformation is

$$h = 2c\, g + \mathcal{L}_{f(t)\partial_t} g. \tag{9.1}$$

To prove this, put it in radial Gaussian gauge and write $h = 2\alpha a^2 \gamma_p + 2\beta b^2 \gamma_q$. The two tangential equations are

$$\alpha'' + (2pu + qv)\alpha' + qu\beta' + 2r_P\alpha = 0,$$
$$\beta'' + (pu + 2qv)\beta' + pv\alpha' + 2r_Q\beta = 0.$$

Scaling $a_s(t) = sa(t/s)$, $b_s(t) = sb(t/s)$ gives the solution $(\alpha, \beta) = (1 - tu, 1 - tv)$. Subtract $\beta(0)$ times this solution. Smoothness gives a residual $\alpha, \beta = O(t^2)$. For $Y = (\alpha, t\alpha', \beta, t\beta')^T$, its equation is

$$tY' = MY + O(t^2)Y, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2(p-1) & 1-2p & 0 & -q \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-p \end{pmatrix}.$$

The characteristic polynomial is $z(z+1)(z+p-1)(z+2p-2)$. Its zero root is simple and all other roots are negative. Each negative Jordan block contributes a polynomial times a decaying exponential. Thus $\|e^{Mv}\| \leq C$ for $v \geq 0$. Write the differential equation as $tY' = MY + E(t)Y$, where $\|E(t)\| \leq Ct^2$. Variation of constants gives

$$Y(t) = (t/\varepsilon)^M Y(\varepsilon) + \int_\varepsilon^t (t/s)^M E(s)Y(s)\,\frac{ds}{s}.$$

The first term tends to zero because $Y(\varepsilon) = O(\varepsilon^2)$. The integral is absolutely convergent at zero. Hence $|Y(t)| \leq C\int_0^t s|Y(s)|\,ds$. If $F(t) = \int_0^t s|Y(s)|\,ds$, then $F(0) = 0$ and $F' \leq CtF$, so $F = 0$. Uniqueness for the ordinary differential equation extends this vanishing along the regular interval. Undoing the radial gauge proves (9.1).

Modulo boundary-fixing radial fields, only c and $F = f(R)$ remain. The boundary equations are

$$c + u(R)F = 0, \qquad c + v(R)F = 0. \tag{9.2}$$

They give dimensions zero and one as stated, except that the latter class must be shown nonzero. Suppose $u = v = \kappa$ at R. Then $r_P \neq r_Q$. If they were equal, (3.4) would give $r_P = r_Q = (m-1)\kappa^2$, and the Cauchy data (a, b, a', b') would be those of the shifted product cone

$$a_C(t) = \sqrt{\frac{p-1}{m-1}}(t-t_0), \quad b_C(t) = \sqrt{\frac{q-1}{m-1}}(t-t_0), \quad t_0 = R - \kappa^{-1}.$$

Uniqueness for the regular ordinary differential system identifies the solutions while both warping functions are positive. If $t_0 > 0$, this contradicts $b(t_0) > 0$; if $t_0 = 0$, it contradicts $b(0) > 0$; if $t_0 < 0$, it contradicts $a(0) = 0$.

Choose a smooth radial field X with $X|_{\Sigma_R} = \kappa^{-1}\nu$ and put $h_0 = \mathcal{L}_X g - 2g$. Then $h_0^\top = 0$, and the normal variation of the second fundamental form gives

$$A'_g(h_0) = \kappa^{-1}(\mathrm{Ric}_\gamma - (m-1)\kappa^2\gamma) \neq 0. \tag{9.3}$$

Indeed, for constant normal speed f the variation is $f(A^2 - \mathcal{R}_\nu)$, and $\mathcal{R}_\nu = HA - A^2 - \mathrm{Ric}_\gamma$. The constant rescaling $-2g$ contributes $-A$. This proves (9.3). A boundary-fixing Lie derivative has zero variation of the induced second fundamental form, so $[h_0] \neq 0$. □

9.1. **Boundary coordinates near an umbilic metric**. The nonlinear conclusion uses two external boundary-moduli results. Under $\pi_1(M, \partial M) = 0$, the space of Einstein metrics with fixed Einstein constant, modulo boundary-fixing diffeomorphisms, is a smooth Banach manifold in $C^{k,\alpha}$, $k \geq 3$, $0 < \alpha < 1$, with tangent space the infinitesimal Einstein deformations. The mixed map

$$\Psi : [g] \longmapsto ([g^\top], H_g) \tag{9.4}$$

is Fredholm of index zero; its gauge-fixed linearized boundary problem is elliptic. These are [4, Theorems 1.1–1.2] and [5, Section 3]. The relative fundamental-group hypothesis holds for the present caps. We give the boundary change of coordinates needed below; in particular, we do not treat the induced-metric map as Fredholm in unchanged finite-regularity Hölder spaces.

Lemma 9.2. *Let γ be a smooth metric on a closed m-manifold, $m \geq 3$, with scalar curvature $m(m-1)\kappa^2$, $\kappa > 0$. Assume*

$$m\kappa^2 \notin \sigma(\Delta_\gamma). \tag{9.5}$$

On smooth metrics near γ, the map

$$\Theta : \widehat{\gamma} \longmapsto ([\widehat{\gamma}], R_{\widehat{\gamma}})$$

is a local smooth change of coordinates. It and its inverse satisfy the elliptic estimates with the two-derivative shift in the scalar-curvature coordinate.

Proof. Represent a conformal class by the unique metric $\bar{\gamma}$ with volume density equal to that of γ. Write $\widehat{\gamma} = e^{2f}\bar{\gamma}$. The scalar-curvature equation is

$$R_{e^{2f}\bar{\gamma}} = e^{-2f}\left(R_{\bar{\gamma}} + 2(m-1)\Delta_{\bar{\gamma}} f - (m-1)(m-2)|df|^2_{\bar{\gamma}}\right). \tag{9.6}$$

Its derivative in f at $(\bar{\gamma}, f) = (\gamma, 0)$ is $2(m-1)(\Delta_\gamma - m\kappa^2)$. By (9.5), this is an isomorphism from $C^{k,\alpha}$ to $C^{k-2,\alpha}$, with elliptic inverse estimates. The Banach implicit function theorem therefore solves (9.6) uniquely for f, depending smoothly on $(\bar{\gamma}, R_{\widehat{\gamma}})$. Differentiating that equation and applying its elliptic estimates gives the same construction at each higher regularity. Smooth data give smooth f. The inverses agree wherever both are defined, by uniqueness in the lowest fixed regularity. The scalar instance of the product and inverse-derivative formulas in Section E.5 gives the estimates at each order, with conformal data in $C^{k,\alpha}$ and scalar data in $C^{k-2,\alpha}$. Their estimates give the stated smooth coordinate change on the intersection of these Hölder scales. □

Lemma 9.3. *At a Ricci-flat metric with $A = \kappa\gamma$, $\kappa > 0$, the map*

$$\mathcal{B}_\partial : [g] \longmapsto ([g^\top], H_g^2 - |A_g|^2_{g^\top}) \tag{9.7}$$

from the $C^{k,\alpha}$ Einstein moduli space to $\mathrm{Conf}^{k,\alpha}(\Sigma) \times C^{k-1,\alpha}(\Sigma)$ is a smooth Fredholm map of index zero. Its gauge-fixed boundary problem is elliptic near this metric. On smooth Einstein metrics, $\mathcal{B}_\partial = \Theta \circ \Pi_D$.

Proof. The second component of (9.7) is a first-order expression in the interior metric, so the indicated map is smooth. At $A = \kappa\gamma$, its variation is

$$(H^2 - |A|^2)' = 2(m-1)\kappa H'.$$

Indeed $H' = \operatorname{tr} A' - \kappa \operatorname{tr} h^\top$ and $(|A|^2)' = 2\kappa \operatorname{tr} A' - 2\kappa^2 \operatorname{tr} h^\top = 2\kappa H'$. Thus $D\mathcal{B}_\partial$ is $D\Psi$ followed by multiplication of the scalar target by the nonzero constant $2(m-1)\kappa$. It is Fredholm of index zero. Its gauge-fixed boundary symbol is the mixed symbol of (9.4) with this same invertible row operation. Here is its complementing-condition calculation. Use an inward normal coordinate x_0 and an orthonormal tangential coordinate system at the boundary point. Put $\tau = |\xi| > 0$. A decaying solution of the interior principal equation has the form $e^{i\langle \xi, x\rangle - \tau x_0} c_{ab}$. The zero trace-free boundary data give $c_{ij} = \phi\delta_{ij}$ for $1 \le i, j \le m$. Write $C = c_{00}$. The tangential Bianchi equations and the normal Bianchi equation are

$$\tau c_{0j} + \frac{i\xi_j}{2}\big(C + (m-2)\phi\big) = 0, \qquad \frac{\tau}{2}(C - m\phi) - i\xi^j c_{0j} = 0.$$

The first equations imply $i\xi^j c_{0j} = \frac{\tau}{2}(C + (m-2)\phi)$; the last then gives $(m-1)\tau\phi = 0$. Thus $\phi = 0$. Since the outward normal is $-\partial_{x_0}$, the principal mean-curvature variation is

$$H'_{\mathrm{pr}} = \frac{\tau m}{2}\phi + i\xi^j c_{0j}.$$

Its zero boundary value gives $C = 0$, and the tangential Bianchi equations give $c_{0j} = 0$. All components vanish. The number of scalar boundary conditions is $(m+1) + (m(m+1)/2 - 1) + 1 = (m+1)(m+2)/2$, the rank of the interior symmetric-tensor bundle. This verifies the complementing condition. The complementing condition and the associated boundary estimates [1] persist under sufficiently small perturbations of the coefficients. This proves ellipticity in a neighborhood. Finally the Ricci-flat Gauss constraint is $R_{g^\top} = H^2 - |A|^2$, giving the last identity. ☐

9.2. **The full elliptic map and its inverse.** The following construction is used for Proposition 9.5 and Lemma 22.3. Constants in this subsection depend on the fixed compact cap and its fixed smooth background metric g_0. No estimate uniform in a sequence of normalized caps is asserted. Fix $0 < \alpha < 1$, and write $\|\cdot\|_j$ for $C^{j,\alpha}$ norms in fixed finite interior and boundary atlases. Fix a low order $r_0 \ge 4$.

Let $B_0(q)$ denote the conformal-class boundary coordinate, using the representative with the volume density of $g_0^\top$, and let $B_1(q) = H_q^2 - |A_q|^2$. Define the full gauged map

$$\mathscr{F}(q) = \big(\Phi(q),\ \beta_{g_0} q|_\Sigma,\ B_0(q),\ B_1(q)\big), \qquad \Phi(q) = \operatorname{Ric}_q + \delta_q^* \beta_{g_0} q. \tag{9.8}$$

Its natural target norm at order $r \ge r_0$ is

$$|(f, b, c, h)|_r = \|f\|_{r-2} + \|b\|_{r-1} + \|c\|_r + \|h\|_{r-1}. \tag{9.9}$$

Conformal-class coordinates are local coordinates in the bundle of positive definite forms modulo positive scalar multiplication. Thus B_0 is a pointwise smooth algebraic map.

At an umbilic background the last boundary derivative is $2(m-1)\kappa H'$. The full linearized system in (9.8) is therefore the Bianchi, conformal-class and mean-curvature system, with one boundary row multiplied by a nonzero constant. It is elliptic of index zero by [4, Proposition 3.1]. On a sufficiently late fixed normalized cap, the scalar principal row is a sufficiently small perturbation of this row and is joined to it through elliptic rows. Thus the same assertion holds there. This concerns the full interior and boundary operator, not an unchanged-regularity Dirichlet map.

Proposition 9.4 (Elliptic coordinates on smooth scales). *Suppose $D\mathscr{F}_{g_0}$ is completed to an isomorphism by a finite number of smooth target vectors and continuous finite-rank source coordinates. The resulting* *nonlinear bordered map has a local inverse $\mathscr{S}$ on a fixed $C^{r_0,\alpha}$ neighborhood. It maps smooth data to smooth metrics. For every $r \geq r_0$ and every $n \geq 1$,*

$$\|D^n\mathscr{S}(y)[y_1,\dots,y_n]\|_r \leq C_{r,n}\Bigg[\sum_{i=1}^{n}|y_i|_r\prod_{j\neq i}|y_j|_{r_0} + (1+|y-y_0|_r)\prod_{i=1}^{n}|y_i|_{r_0}\Bigg]. \tag{9.10}$$

Finite-dimensional parameters are included in both norms. Moreover

$$\|\mathscr{S}(y)-\mathscr{S}(y_0)\|_r \leq C_r|y-y_0|_r. \tag{9.11}$$

The neighborhood is restricted only in the fixed low norm; its smooth elements need not be small in all higher norms.

The proof, with estimates for each derivative of the inverse, is in Section E.5. In the fold application the source has one kernel coordinate and the target one cokernel coordinate; in the regular filling application neither is needed. The scalar curvature change of boundary coordinates loses two derivatives, and the resulting filling map retains the one-derivative shift in (E.32).

9.3. **The fold of the induced-metric map.**

Proposition 9.5. *At every umbilic radius the full smooth induced-metric map is a fold.*

Proof. Let $u(R) = v(R) = \kappa > 0$. The curvature-ratio bounds in Lemma 3.2 give

$$\frac{p}{a^2} = p\kappa^2 + \frac{2pq}{p-1}\kappa^2\theta > m\kappa^2, \qquad \frac{q}{b^2} > m\kappa^2.$$

These are the lowest positive scalar eigenvalues on the two factors. The product spectrum consists of their sums, including zero on either factor. Hence (9.5) holds; the constant eigenfunction has eigenvalue zero, not $m\kappa^2$. By Lemmas 9.2 and 9.3, the full induced-metric map in smooth boundary coordinates has the same local singularity as $\mathcal{B}_\partial$.

The kernel of $D\mathcal{B}_\partial$ is K_R, since $D\Theta$ is injective on smooth data. Weak kernel vectors are smooth by the elliptic boundary estimates for $\mathcal{B}_\partial$. By Proposition 9.1, the kernel is one-dimensional. Index zero makes the cokernel one-dimensional as well.

Consider the invariant family of scaled truncations $c^2 g|_{M_R}$. On the boundary use scale and shape coordinates

$$s_0(R) = \frac{1}{m}\left(p\log\frac{a}{\sqrt{p-1}} + q\log\frac{b}{\sqrt{q-1}}\right), \qquad z(R) = \log\frac{a}{\sqrt{p-1}} - \log\frac{b}{\sqrt{q-1}}.$$

Replacing $\log c$ by $s = \log c + s_0(R)$ makes the invariant induced-metric map $(s,R) \mapsto (s, z(R))$. At the umbilic radius,

$$z' = u - v = 0, \qquad z'' = r_P - r_Q \neq 0 \tag{9.12}$$

by the cone-uniqueness argument in Proposition 9.1. The resulting second derivative is nonzero in the full cokernel of $D\Pi_D$: if an invariant boundary tensor were in its image, averaging a preimage would give an invariant preimage. The invariant image at the umbilic point has only the scale direction, whereas (9.12) has a nonzero shape component. The same statement holds for $\mathcal{B}_\partial$, because $D\Theta$ is invertible and the kernel's first boundary derivative is zero.

Use the full gauge-fixed operator (9.8) and its bordered map (E.31). The source projection is pairing with the smooth kernel vector, and the target border is the smooth nonzero cokernel vector supplied by (9.12). The derivative of the bordered map is an isomorphism. By Proposition 9.4, its inverse satisfies (9.11) and (9.10) on a fixed low-order neighborhood, with every finite-dimensional projection bounded in the indicated scales. Setting the interior and Bianchi data to zero gives Ricci-flat metrics, as verified in Section E.5. The resulting smooth tame coordinates write $\mathcal{B}_\partial$ as

$$(y,s) \longmapsto (y, \varphi(y,s)), \qquad \varphi_s(0,0) = 0, \quad \varphi_{ss}(0,0) \neq 0.$$

The regularities of the conformal and scalar components are those in (9.9); no unchanged-Hölder Dirichlet inverse is used.

Solve $\varphi_s(y, s_*(y)) = 0$. Taylor's integral formula gives

$$\varphi(y,s) - \varphi(y, s_*(y)) = (s - s_*(y))^2 c(y,s), \quad c(y,s) = \int_0^1 (1-t)\varphi_{ss}(y, s_*(y) + t(s - s_*(y)))\, dt.$$

Put $a_* = |\varphi_{ss}(0,0)| > 0$. By continuity, choose the neighborhood so that $|\varphi_{ss}(y,s) - \varphi_{ss}(0,0)| < a_*/2$. Taylor's integral then gives $|c(y,s)| \geq a_*/4$, with constant sign. At $s = s_*(y)$,

$$\partial_s\big((s - s_*(y))\sqrt{|c(y,s)|}\big) = \sqrt{|c(y, s_*(y))|} \geq \sqrt{a_*}/2.$$

The scalar inverse function theorem therefore applies. Replacing $s - s_*(y)$ by $(s - s_*(y))\sqrt{|c(y,s)|}$ gives (1.5), after a sign change in the last target coordinate. Differentiation of the scalar critical-point equation and of this square-root change gives the tame estimates stated at the end of Section E.5, since their scalar denominators stay nonzero. The coordinate change Θ^{-1}, with its retained two-derivative scalar shift, transfers the normal form to the full induced-metric map. □

Remark 9.6. The boundary coordinates used in Lemma 9.2 are compatible with smooth boundary metrics but distinguish the regularities of their conformal and scalar-curvature components. This is necessary: the Einstein Dirichlet problem is not Fredholm between unchanged finite-regularity Hölder spaces. A general smooth tame formulation under definiteness of $H\gamma - A$ is discussed in [3, Section 1]. The fold proof above uses the explicit elliptic map (9.7) near its umbilic point.

Remark 9.7. The normal form counts fillings in one neighborhood of one moduli point, with fixed boundary markings. It gives two, one, or zero nearby fillings according to the sign of the last target coordinate. It is not a global bound. In particular, the oscillations in Lemma 3.3 give infinitely many scaled truncations with the fixed boundary metric $(p-1)\gamma_p + (q-1)\gamma_q$; this phenomenon is already discussed in [11, Section 6].

II. The reduced Hessian and its index

10. Conformal reduction of the Dirichlet Hessian

In this section (M^d, g) is compact, Ricci-flat, connected, and has nonempty smooth boundary; $d \geq 3$. Let

$$\mathcal{S}(g) = -\int_M R_g \, dv_g - 2\int_{\partial M} H_g \, dA_g, \qquad E(g) = \mathrm{Ric}_g - \tfrac{1}{2} R_g g.$$

For variations fixing the induced metric, integration of the scalar-curvature variation gives

$$D\mathcal{S}_g(h) = \int_M \langle E(g), h\rangle \, dv_g. \tag{10.1}$$

Here is the boundary calculation, with all normal derivatives taken in radial parallel identification. Put $k = h^\top$, $s = h(\nu, \nu)$, and $\eta = h(\nu, \cdot)^\top$. The boundary term obtained by integrating $DR_g(h)$ is

$$(\nabla^j h_{ij} - \nabla_i \operatorname{tr} h)\nu^i = \mathrm{div}_\Sigma \, \eta + Hs - \langle A, k\rangle - \nabla_\nu \operatorname{tr}_\Sigma k.$$

Differentiating the unit normal and the second fundamental form gives

$$\dot{\nu} = -\eta^\sharp - \tfrac{1}{2} s\nu, \qquad 2\dot{H} = \nabla_\nu \operatorname{tr}_\Sigma k - 2\,\mathrm{div}_\Sigma \, \eta - Hs.$$

Also $D(dA_g)[h] = \frac{1}{2} \operatorname{tr}_\Sigma k \, dA_g$. Thus the boundary contribution to $D\mathcal{S}_g(h)$ is

$$\int_{\partial M} \big(\mathrm{div}_\Sigma \, \eta + \langle A - H\gamma, k\rangle\big) \, dA_g = \int_{\partial M} \langle A - H\gamma, h^\top\rangle \, dA_g.$$

The divergence integrates to zero on the closed boundary. This proves (10.1) when $h^\top = 0$. At the Ricci-flat metric, the Hessian is

$$B(h, k) = \int_M \langle E'(h), k\rangle \, dv_g, \qquad h^\top = k^\top = 0. \tag{10.2}$$

It is symmetric, since it is the second derivative of (10.1) on the affine space of fixed tangential boundary values.

Put

$$\mathcal{V} = \{h \in C^\infty(S^2 T^* M) \,:\, h^\top = 0\}, \quad \mathcal{Z} = \{h \in \mathcal{V} \,:\, r(h) = 0\}, \quad \mathcal{G}_0 = \{\mathcal{L}_Y g \,:\, Y|_{\partial M} = 0\}.$$

Let $D = \Delta_D^{-1}$ be the scalar Dirichlet inverse.

Proposition 10.1. *The map*

$$Ph = h - \frac{1}{d-1} Dr(h)\, g \tag{10.3}$$

is a projection from $\mathcal{V}$ onto $\mathcal{Z}$. Moreover,

$$B(Ph, Pk) = B(h, k) + \frac{d-2}{2(d-1)} \langle r(h), Dr(k)\rangle_{L^2}. \tag{10.4}$$

The radical of $B|_{\mathcal{Z}}$, modulo $\mathcal{G}_0$, is precisely the geometric Einstein Dirichlet kernel.

Proof. Equation (2.2) gives $r(fg) = (d-1)\Delta f$. Thus $r(Ph) = 0$, and Ph has the same induced boundary variation as h. The trace of $E'(h)$ is $-(d-2)r(h)/2$, whence

$$B(h, fg) = -\frac{d-2}{2} \int_M f r(h) \, dv_g, \quad B(fg, \psi g) = -\frac{(d-1)(d-2)}{2} \int_M \langle df, d\psi\rangle \, dv_g$$

for zero-boundary f, ψ. Expansion of $B(Ph, Pk)$ proves (10.4). In particular,

$$B(Ph, Ph) = \sup_{f|_{\partial M} = 0} B(h + fg, h + fg). \tag{10.5}$$

More precisely, for every f with zero boundary value,

$$B(h + fg, h + fg) = B(Ph, Ph) - \frac{(d-1)(d-2)}{2} \left\| d\left(f + \frac{1}{d-1} Dr(h)\right)\right\|_2^2.$$

The last term is nonpositive and vanishes for $f = -(d-1)^{-1}Dr(h)$, proving (10.5).

If $h \in \mathcal{Z}$ annihilates B on $\mathcal{Z}$, it also annihilates every zero-boundary conformal variation by the first displayed formula. Every $k \in \mathcal{V}$ is a sum of Pk and such a conformal variation. Hence $B(h,k) = 0$ for all compactly supported k, so $E'(h) = 0$. Since $r(h) = 0$, this is $\mathrm{Ric}'(h) = 0$. Conversely such an h is in the radical. Naturality gives $E'(\mathcal{L}_Y g) = 0$ and $r(\mathcal{L}_Y g) = 0$, proving the quotient statement. □

10.1. **The nonlinear reduction.** For a metric $\widehat{g}$ close to g, with the same induced boundary metric, solve

$$\frac{4(d-1)}{d-2}\Delta_{\widehat{g}} u + R_{\widehat{g}} u = 0, \qquad u|_{\partial M} = 1. \tag{10.6}$$

At $(\widehat{g}, u) = (g, 1)$ the derivative in the zero-boundary unknown is an invertible positive multiple of Δ_D. The implicit function theorem and elliptic regularity give a unique nearby positive smooth solution. The metric $u^{4/(d-2)}\widehat{g}$ is scalar-flat and induces the same boundary metric. The differential of this operation is (10.3). Let q be any nearby scalar-flat metric, not assumed Ricci-flat. For a fixed-boundary variation v put

$$P_q v = v - \frac{1}{d-1}\Delta_{q,D}^{-1}(DR_q(v))q.$$

Since $R_q = 0$, the conformal variation formula gives $DR_q(fq) = (d-1)\Delta_q f$. Consequently $DR_q(P_q v) = 0$, $(P_q v)^\top = 0$, and $v - P_q v = fq$ with $f|_{\partial M} = 0$. Moreover $DS_q(fq) = -(d-2)\int_M f R_q \, dv_q/2 = 0$. If q is a critical point on the scalar-flat fixed-boundary metrics, then $DS_q(P_q v) = 0$, so $DS_q(v) = 0$ for every fixed-boundary v. Testing compactly supported v gives $E(q) = 0$. Its trace, $-(d-2)R_q/2$, and $d \geq 3$ imply $\mathrm{Ric}_q = 0$. Conversely a Ricci-flat metric annihilates the first variation. This proves the critical-point assertion without applying a Ricci-flat Hessian formula at a non-Ricci-flat point.

Proposition 10.2. *Each class in $\mathcal{Z}/\mathcal{G}_0$ has a unique representative of the form*

$$h = T + fg, \qquad T \in \mathcal{W}(M,g), \qquad \Delta f = 0, \qquad T^\top = -f\gamma. \tag{10.7}$$

The representative is linear in the class, and

$$2B(h,h) = \int_M \langle LT - (d-2)\nabla^2 f, T\rangle \, dv_g = q_M(T). \tag{10.8}$$

Proof. Define the conformal-Killing operator

$$\mathcal{K}\omega = \delta^*\omega + \frac{\delta\omega}{d} g.$$

On Ricci-flat M,

$$2\delta\mathcal{K}\omega = \nabla^*\nabla\omega + (1 - 2/d)d\delta\omega. \tag{10.9}$$

For $\omega|_{\partial M} = 0$, its energy is

$$2\|\mathcal{K}\omega\|_{L^2}^2 = \|\nabla\omega\|_{L^2}^2 + (1-2/d)\|\delta\omega\|_{L^2}^2.$$

The Dirichlet Poincaré inequality gives coercivity. If $h_0 = h - (\mathrm{tr}\, h/d)g$, solve $\delta\mathcal{K}\omega = \delta h_0$ with zero boundary values. Then

$$T = h_0 - \mathcal{K}\omega, \qquad f = \frac{\mathrm{tr}\, h + \delta\omega}{d}, \qquad h = T + \delta^*\omega + fg.$$

The tensor T is TT. Naturality and $r(h) = 0$ give $\Delta f = 0$. The tangential part of $\delta^*\omega$ is zero at the boundary, so $T^\top = -f\gamma$. Subtracting $\delta^*\omega$ is a boundary-fixing gauge change. For two representatives, write $T_1 - T_2 + (f_1 - f_2)g = \delta^*\eta$, with $\eta|_{\partial M} = 0$. Pair this identity with $T_1 - T_2$ and integrate. Its trace vanishes, and integration by parts gives

$$\|T_1 - T_2\|_2^2 = \langle \delta(T_1 - T_2), \eta\rangle_{L^2} = 0.$$

Thus $T_1 = T_2$. Their boundary conditions imply $f_1 - f_2 = 0$ on ∂M; uniqueness for the scalar Dirichlet problem gives $f_1 = f_2$ on M. The tensor $2E'(T + fg) = LT - (d-2)\nabla^2 f$ is trace-free and divergence-free by (2.3); pairing it with fg gives zero. This proves (10.8). □

Lemma 10.3. *For a smooth trace-free tensor U compactly supported in the interior of M, let $T(U)$ be the TT representative of PU. Then*

$$q_M(T(U)) \le Q_M(U) - \frac{d}{d-1}\|\delta U\|_{L^2}^2 \le Q_M(U), \qquad \|T(U)\|_{L^2} \le \|U\|_{L^2}. \tag{10.10}$$

Proof. Since U is compactly supported and trace-free, integration of (2.1) gives $2B(U,U) = Q_M(U) - 2\|\delta U\|_{L^2}^2$ and $r(U) = \delta\delta U$. If $z = Dr(U)$, then

$$\|dz\|_{L^2}^2 = \langle r(U), z\rangle = \langle \delta U, dz\rangle \le \|\delta U\|_{L^2}\|dz\|_{L^2}.$$

Thus $\langle r(U), Dr(U)\rangle \le \|\delta U\|_{L^2}^2$. Apply (10.4). The TT representative is $U - \mathcal{K}\omega$, with ω solving the vector Dirichlet equation in Proposition 10.2. Integration by parts makes $U - \mathcal{K}\omega$ orthogonal to $\mathcal{K}\omega$; hence its norm is at most $\|U\|_{L^2}$. □

11. Conformal-Killing estimates on expanding caps

The estimates in this section concern arbitrary smooth vector fields, not Einstein deformations. The operator on vector fields is

$$\mathcal{K}X = \frac{1}{2}\mathcal{L}_X g - \frac{\operatorname{div} X}{d} g.$$

We use the same symbol for the operator on one-forms under metric duality. Set $A_0 = A - (H/m)\gamma$ on an orbit. The tensors A, A_0 and the function H are parallel along that orbit.

11.1. Coercivity on every cap.

Lemma 11.1. *For every $R > 0$, a conformal Killing field on M_R whose tangential boundary value is zero vanishes. Consequently,*

$$\|X\|_{H^1(M_R)} \le C_R\|\mathcal{K}X\|_{L^2(M_R)}, \qquad X^\top|_{\Sigma_R} = 0. \tag{11.1}$$

After smooth radial identification of the caps, C_R is bounded on each compact subinterval of $(0,\infty)$.

Proof. Suppose $\mathcal{L}_X g = 2\sigma g$. By (2.1), or by its pure-trace specialization, Ricci-flatness gives

$$0 = -(d-2)\nabla^2\sigma + (\Delta\sigma)g.$$

Taking the trace gives $2(d-1)\Delta\sigma = 0$, and hence $\nabla^2\sigma = 0$. Thus $\nabla\sigma$ is parallel.

Let $B = \{t = 0\}$ be the singular orbit. Its tangent and normal spaces have dimensions q and $p+1$. Write

$$\nu_0 = \frac{q-1}{(p+1)b(0)^2} > 0.$$

The singular orbit expansions give sectional curvature $-\nu_0$ on every mixed plane spanned by a normal vector N and a tangent vector U. In the orthogonal splitting $T_xX = N_xB \oplus T_xB$, the full mixed curvature operator is the corresponding multiple of $N \wedge U$. This follows either from the warped connection or by taking the limit of the radial and mixed factor curvature formulas. There are no additional mixed components: the isotropy group $O(p+1) \times O(q)$ fixes the two summands and acts independently on them. Choose orthonormal bases $(N_a)_{a=1}^{p+1}$ of N_xB and $(U_i)_{i=1}^q$ of T_xB. For $V = V_N + V_B$, the mixed curvature formula gives

$$\sum_{a=1}^{p+1}\sum_{i=1}^{q} |R(N_a, U_i)V|^2 = \nu_0^2\big(q|V_N|^2 + (p+1)|V_B|^2\big).$$

A vector annihilated by every $R(N_a, U_i)$ is therefore zero. The identity $R(N,U)\nabla\sigma = 0$ therefore gives $\nabla\sigma = 0$ at x, and parallelness gives $\nabla\sigma = 0$ everywhere.

The singular orbit is totally geodesic. Restriction of the homothety equation to TB gives

$$\mathcal{L}_{X^\top|_B} g_B = 2\sigma g_B.$$

Its trace integrates over the closed singular orbit to $0 = 2q\sigma \operatorname{Vol}(B)$, so $\sigma = 0$. Therefore X is Killing. On Σ_R, the tangential Killing equation and $X^\top = 0$ give $2X_\nu A = 0$. Both principal curvatures u, v are positive; hence $X_\nu = 0$. Tangential derivatives of X vanish at the boundary, and the remaining Killing equations give $\nabla_\nu X = 0$. Along a normal geodesic, a Killing field is a Jacobi field. Uniqueness gives its vanishing on a collar. The same Jacobi identity along successive geodesic segments in the connected interior gives $X = 0$ on M_R.

Dain's inequality [9], applied in finitely many interior and boundary charts and summed with a partition of unity, gives

$$\|X\|_{H^1} \le C_R\big(\|\mathcal{K}X\|_{L^2} + \|X\|_{L^2}\big).$$

If (11.1) failed, there would be fields of H^1 norm one, with zero tangential boundary value, whose conformal strains tend to zero. A weakly convergent subsequence is strongly convergent in L^2. Its limit is a weak conformal Killing field with the same boundary condition. Interior elliptic regularity gives smoothness away from the boundary. Its divergence has zero Hessian, so the preceding singular-orbit argument makes it zero. The resulting Killing field satisfies the first-order system for (X, A) in Lemma 4.5, with $s = 0$. Its coefficients are smooth up to the boundary. Solving this linear system along the inward normal geodesics extends (X, A) smoothly to the boundary. Its trace has the prescribed zero tangential value, and the first part of the proof applies. The preceding argument gives zero limit. The displayed inequality then contradicts the normalization.

For local uniformity, pull M_R to M_1 by $(r, \omega, y) \mapsto (Rr, \omega, y)$. In Cartesian normal coordinates this map is $(z, y) \mapsto (Rz, y)$, so the pulled-back metrics are smooth up to the singular orbit and smooth in $R > 0$. On a compact interval of radii the chart inequalities are uniform. A contradicting sequence of radii has a positive limiting radius; the preceding compactness argument at that radius gives the same contradiction. □

Lemma 11.2. *The problem*

$$\delta\mathcal{K}\omega = f, \qquad \omega^\top = 0, \qquad (\mathcal{K}\omega)_{\nu\nu} = 0 \tag{11.2}$$

has a unique solution $\omega \in H^2$ for each $f \in L^2$. It satisfies

$$\|\omega\|_{H^2(M_R)} \le C_R\|f\|_{L^2(M_R)}.$$

The solution operators, after radial pullback, depend smoothly on $R > 0$ between these Sobolev spaces. Smooth data give smooth solutions.

Proof. The weak problem is

$$\int_{M_R} \langle \mathcal{K}\omega, \mathcal{K}\eta\rangle \, dv_g = \int_{M_R} \langle f, \eta\rangle \, dv_g, \qquad \eta^\top|_{\Sigma_R} = 0.$$

The coercivity in Lemma 11.1 and the Hilbert-space variational theorem give existence and uniqueness in H^1. Compactly supported tests give the interior equation. Allowing arbitrary normal boundary values gives the natural condition $(\mathcal{K}\omega)_{\nu\nu} = 0$.

The complementing condition is checked as follows. Fix a boundary point, replace the principal coefficients by their values there, and take the tangential Fourier transform with nonzero frequency ξ. A decaying solution of the homogeneous principal equations in the half-space, with zero tangential Dirichlet data and zero normal traction, has zero energy by integration by parts. Its principal conformal strain therefore vanishes. Its divergence is an affine function by the flat version of the pure-trace Ricci identity. Decay forces this divergence to vanish, and the remaining Killing equations force the solution to vanish. The system has d scalar boundary conditions, one for each component of the second-order strongly elliptic system

$$2\delta\mathcal{K} = \nabla^*\nabla + (1 - 2/d)d\delta.$$

The boundary estimates for elliptic systems [1] consequently give $\|\omega\|_{H^2} \le C_R(\|f\|_{L^2} + \|\omega\|_{H^1})$. Coercivity removes the last term. The same estimates at every order give smoothness.

The pullback operators and boundary data are smooth in R. Trivialize the tangential Dirichlet subspace by the smooth positive square root of the boundary metric. In the weak problem the remaining boundary condition is natural and the domain is fixed. The inverse identity

$$A_R^{-1} - A_a^{-1} = A_R^{-1}(A_a - A_R)A_a^{-1}$$

is understood on the variational space

$$\mathscr{V}_\partial = \{\omega \in H^1(T^*M) \,:\, \omega^\top|_{\partial M} = 0\}.$$

After pullback, $A_R : \mathscr{V}_\partial \to \mathscr{V}_\partial^*$ is the operator of the displayed coercive bilinear form. The inverse identity gives continuity from $\mathscr{V}_\partial^*$ to $\mathscr{V}_\partial$; differentiation gives $(A_R^{-1})' = -A_R^{-1}A_R'A_R^{-1}$ and smooth dependence. Here $\mathscr{V}_\partial^*$ is the dual of the mixed test space, not a notation for the dual of H_0^1. The H^2 estimates applied to the differentiated equations give smoothness in $L^2 \to H^2$. All bounds are local in $R > 0$. □

Lemma 11.3. *On a collar of the cone over*

$$h = \frac{p-1}{m-1}\gamma_p + \frac{q-1}{m-1}\gamma_q,$$

every conformal Killing field whose tangential part is zero on the outer boundary is a multiple of $r\partial_r$.

Proof. Suppose $\mathcal{L}_Y g_C = 2\sigma g_C$. Differentiating the Ricci tensor and using $\mathrm{Ric}_{g_C} = 0$ gives

$$0 = -(d-2)\nabla^2\sigma + (\Delta\sigma)g_C.$$

Its trace first gives $\Delta\sigma = 0$, and then $\nabla^2\sigma = 0$. The radial and mixed components imply $\sigma = r\psi(z) + b$, with b constant. The tangential component is $\nabla_h^2\psi + \psi h = 0$. Thus $\Delta_h\psi = m\psi$. The first positive scalar eigenvalue of the product link is

$$\min\left\{\frac{(m-1)p}{p-1}, \frac{(m-1)q}{q-1}\right\} > m.$$

Therefore $\psi = 0$, and σ is constant. The field $Z = Y - \sigma r\partial_r$ is Killing and has zero tangential boundary value. The tangential Killing equation on the outer boundary is $2Z_\nu A = 0$. Since $A = r^{-1}\gamma$, its normal value is also zero. Tangential derivatives of Z vanish there. The remaining Killing equations give $\nabla_\nu Z = 0$ there. Along each radial geodesic a Killing field satisfies the Jacobi equation; uniqueness with zero Cauchy data gives $Z = 0$ on the collar. □

Let $\overline{X}$ be the average of X under $\mathfrak{G}$. It is radial, and $X^\perp = X - \overline{X}$ has zero group average. If $X^\top = 0$ at Σ_R, put $\phi = X_\nu|_{\Sigma_R}$ and $\phi^\perp = \phi - \overline{\phi}$. The average $\overline{\phi}$ is constant on Σ_R.

Proposition 11.4. *There are $R_0, C, c > 0$ such that, for $R \geq R_0$ and every smooth X with $X^\top = 0$ on Σ_R,*

$$\int_{R/2\leq t\leq R} \big(|\nabla X^\perp|^2 + R^{-2}|X^\perp|^2\big)\, dv_g \leq C\int_{R/2\leq t\leq R} |\mathcal{K}X^\perp|^2\, dv_g, \tag{11.3}$$

$$\|\phi^\perp\|_{L^2(\Sigma_R)}^2 \leq CR\|\mathcal{K}X^\perp\|_{L^2}^2, \tag{11.4}$$

$$\langle\phi^\perp, \mathcal{N}_R\phi^\perp\rangle_{L^2(\Sigma_R)} \leq C\|\mathcal{K}X^\perp\|_{L^2}^2, \tag{11.5}$$

$$\|\mathcal{K}\overline{X}\|_{L^2}^2 \geq c\overline{\phi}^2. \tag{11.6}$$

Here $\mathcal{N}_R$ is the scalar Dirichlet-to-Neumann operator of M_R; the norms without a specified domain are on M_R.

Proof. The trace-free Korn inequality [9, Theorem 1.1 and Corollary 1.2] on a fixed smooth collar is

$$\|Y\|_{H^1} \leq C\big(\|\mathcal{K}Y\|_{L^2} + \|Y\|_{L^2}\big).$$

On the rescaled collars $R^{-2}g$, these constants can be chosen uniformly. To see this, use one finite chart cover of the limiting cone collar and a partition of unity. The principal coefficients converge in C^1 and the zeroth-order coefficients remain bounded. Differences of the principal parts are absorbed in the left side of the inequality; derivatives of the fixed partition are bounded zeroth-order terms.

If (11.3) failed, there would be fields of zero group average on the rescaled collars, with tangential boundary value zero, unit H^1 norm, and conformal-Killing norm tending to zero. Rellich compactness gives a subsequence converging strongly in L^2 and weakly in H^1 to a conformal Killing field on the cone collar. The uniform inequality makes the limit nonzero. Its boundary trace is tangentially zero and its group average is zero. This contradicts Lemma 11.3. Rescaling gives (11.3). The trace inequality on the same collar gives

$$\|X_\nu^\perp\|^2_{L^2(\Sigma_R)} \le C \int_{R/2\le t\le R} \left(R|\nabla X^\perp|^2 + R^{-1}|X^\perp|^2\right) dv_g,$$

which proves (11.4).

Choose a smooth radial cutoff which is zero near $R/2$, one near R, and has derivative bounded by C/R. For $e = \chi\langle X^\perp, \nu\rangle$ on the collar, extended by zero, $e|_{\Sigma_R} = \phi^\perp$. Since $|d\chi| \le C/R$ and $|\nabla\nu| = |A| \le C/R$ there,

$$|de| \le C\left(|\nabla X^\perp| + R^{-1}|X^\perp|\right).$$

Integration and (11.3) bound the Dirichlet energy of this extension by $C\|\mathcal{K}X^\perp\|_2^2$. The harmonic extension minimizes the scalar Dirichlet energy. This proves (11.5).

For the radial estimate write

$$\overline{X} = \psi(t)\partial_t, \qquad \varrho = W^{1/m}, \qquad \psi = \varrho v_0.$$

The radial eigenvalue of $\mathcal{K}\overline{X}$ is $\frac{m}{m+1}(\psi' - (H/m)\psi)$. Its tangential trace part is the negative of this eigenvalue divided by m, and its tangential trace-free part is ψA_0. Since $\varrho'/\varrho = H/m$,

$$\|\mathcal{K}\overline{X}\|^2_{L^2} = c_{p,q}\int_0^R \varrho^{m+2}\left(\frac{m}{m+1}(v_0')^2 + |A_0|^2 v_0^2\right) dt, \tag{11.7}$$

where $c_{p,q} = \mathrm{Vol}(S^p, \gamma_p)\,\mathrm{Vol}(S^q, \gamma_q) > 0$. On $[R/2, R]$, ϱ is comparable to R, and $v_0(R) = \overline{\phi}/\varrho(R)$. If $\overline{\phi} = 0$, the right side of (11.6) is zero and the left side is nonnegative. Otherwise, if $|v_0(t) - v_0(R)| \ge |v_0(R)|/2$ somewhere in this interval, weighted Cauchy–Schwarz gives

$$\int_{R/2}^R \varrho^{m+2}(v_0')^2 dt \ge cR^{m+1}|v_0(R)|^2 \ge cR^{m-1}\overline{\phi}^2.$$

If there is no such point, then $|v_0| \ge |v_0(R)|/2$ throughout the collar. The oscillatory lower bound in Lemma 3.3 gives

$$\int_{R/2}^R \varrho^{m+2}|A_0|^2 v_0^2 dt \ge cR^2|v_0(R)|^2 \ge c\overline{\phi}^2.$$

These alternatives prove (11.6). □

Corollary 11.5. *For $R \ge R_0$, a conformal Killing field on M_R with zero tangential boundary value is zero. For each such fixed cap,*

$$\|X\|_{H^1(M_R)} \le C_R\|\mathcal{K}X\|_{L^2}, \qquad X^\top|_{\Sigma_R} = 0. \tag{11.8}$$

Proof. If $\mathcal{K}X = 0$, averaging commutes with $\mathcal{K}$, and the two averaged summands are orthogonal. Equations (11.4) and (11.6) give $X_\nu = 0$ on the boundary. Thus X has full zero boundary value. The energy identity (10.9) gives $\nabla X = 0$, hence $X = 0$. The fixed-cap Korn inequality with an L^2 remainder, compactness, and this vanishing result give (11.8) by the same normalized contradiction argument used above. □

12. A mixed decomposition of the boundary Hessian

Fix $R > 0$. All traces and normal components below are at Σ_R unless specified otherwise.

Proposition 12.1. *For every $T \in \mathcal{W}(M_R, g)$ there is a unique smooth vector field X satisfying*

$$\delta\mathcal{K}X = 0, \qquad X^\top = 0, \qquad (\mathcal{K}X)(\nu,\nu) = T(\nu,\nu). \tag{12.1}$$

If

$$S = T - \mathcal{K}X, \qquad \phi = X_\nu|_{\Sigma_R}, \qquad \eta = S(\nu,\cdot)^\top, \tag{12.2}$$

then

$$\delta S = 0, \quad \operatorname{tr} S = 0, \quad S_{\nu\nu} = 0, \quad S^\top = -\phi A_0, \tag{12.3}$$

and

$$\|T\|_{L^2}^2 = \|S\|_{L^2}^2 + \|\mathcal{K}X\|_{L^2}^2. \tag{12.4}$$

Moreover, with $h = T + f_T g$, there is a harmonic function F such that

$$h = S + \delta^* X^\flat + Fg, \qquad F|_{\Sigma_R} = -\frac{H}{m}\phi. \tag{12.5}$$

Proof. On the space of H^1 vector fields with tangential boundary value zero, solve

$$\int_{M_R} \langle \mathcal{K}X, \mathcal{K}Y\rangle \, dv_g = \int_{M_R} \langle T, \mathcal{K}Y\rangle \, dv_g \qquad (Y^\top|_{\Sigma_R} = 0). \tag{12.6}$$

Coercivity follows from (11.1); the right side is continuous. The Hilbert-space variational theorem gives a unique weak solution. Integration by parts first with compactly supported Y, and then with arbitrary normal boundary value, gives (12.1).

The principal symbol of $2\delta\mathcal{K}$ is $|\xi|^2 \operatorname{Id} + (1 - 2/d)\xi \otimes \xi$, which is positive definite. The mixed boundary conditions satisfy the complementing condition. Indeed, in the constant-coefficient half-space problem, tangential Dirichlet data and zero normal traction make the boundary term in the energy identity vanish. A decaying Fourier solution consequently has $\mathcal{K}X = 0$. Take its nonzero tangential frequency along the first coordinate. A second tangential coordinate exists since $d \geq 3$. The $(2,2)$ equation gives $\operatorname{div} X = 0$, the $(1,1)$ equation gives $X_1 = 0$, the $(1,j)$ equations give the other tangential components zero, and the $(1,\nu)$ equation gives $X_\nu = 0$. Thus there is no nonzero decaying solution. The elliptic boundary estimates for this verified symbol [1], followed by differentiation and interior regularity, make the weak solution smooth.

The tensor S is TT, and its normal component is zero by (12.1). Put $\sigma = \operatorname{div} X/d$. The equation $\delta\mathcal{K}X = 0$ and (10.9), followed by divergence, give $\Delta\sigma = 0$. Since $X^\top = 0$,

$$\operatorname{div} X = \nabla_\nu X_\nu + H\phi, \qquad (\mathcal{K}X)_{\nu\nu} = \nabla_\nu X_\nu - \sigma.$$

Using $T_{\nu\nu} = m f_T$ gives

$$\sigma|_{\Sigma_R} = f_T + \frac{H}{m}\phi, \qquad (\mathcal{K}X)^\top = -f_T\gamma + \phi A_0.$$

This proves (12.3) and (12.5) with $F = f_T - \sigma$. Finally,

$$\int_{M_R} \langle S, \mathcal{K}X\rangle \, dv_g = \int_{M_R} \langle \delta S, X^\flat\rangle \, dv_g + \int_{\Sigma_R} S(\nu, X) \, dA_\gamma = 0,$$

since $X^\top = 0$ and $S_{\nu\nu} = 0$. This proves (12.4). □

Set

$$\mathcal{V}_R = \langle \nabla_\nu A, A_0\rangle + \frac{H}{m}|A_0|^2, \qquad \Pi = H\gamma - A. \tag{12.7}$$

The normal derivative of a tangential tensor is taken in radial parallel identification.

Proposition 12.2. *In the notation of Proposition 12.1,*

$$\begin{aligned} q_R(T) = Q_R(S) + \int_{\Sigma_R} \Big[& 2(H\gamma + A)(\eta, \eta) \\ & + 2\left(\frac{(m-1)H}{m}\gamma - 2A_0\right)(\eta, d_\Sigma\phi) \\ & + \Pi(d_\Sigma\phi, d_\Sigma\phi) - \mathcal{V}_R\phi^2\Big] \, dA_\gamma - \frac{(m-1)H^2}{m}\langle \phi, \mathcal{N}_R\phi\rangle_{L^2(\Sigma_R)}. \end{aligned} \tag{12.8}$$

Here $q_R = q_{M_R}$.

Proof. The tensor $2E'(h) = LS - (m-1)\nabla^2 F$ is TT. Pair it with (12.5), integrate by parts, and use $X^\top = 0$ and $S_{\nu\nu} = 0$. This gives

$$q_R(T) = Q_R(S) - \int_{\Sigma_R} \langle \nabla_\nu S, S\rangle \, dA_\gamma - (m-1)\int_{\Sigma_R} \langle \eta, d_\Sigma F\rangle \, dA_\gamma + \int_{\Sigma_R} \phi (LS)_{\nu\nu} \, dA_\gamma - (m-1)\int_{\Sigma_R} \phi F_{\nu\nu} \, dA_\gamma. \tag{12.9}$$

For the component calculation write $s = S_{\nu\nu}$, $K = S^\top$, and use a prime for ∇_ν. In a radial parallel frame the two components of $\delta S = 0$ are

$$s' + Hs + \operatorname{div}_\Sigma \eta - A : K = 0, \qquad \eta' + (H\gamma + A)\eta + \operatorname{div}_\Sigma K = 0.$$

At the boundary, $s = 0$ and $K = -\phi A_0$. Since A_0 is parallel along the orbit,

$$s' = -\operatorname{div}_\Sigma \eta - \phi |A_0|^2, \qquad \eta' = A_0 d_\Sigma \phi - (H\gamma + A)\eta. \tag{12.10}$$

The normal component of LS is

$$(LS)_{\nu\nu} = -s'' - Hs' + \Delta_\Sigma s + 4A : \nabla_\Sigma \eta + 2|A|^2 s + 2A' : K.$$

In obtaining this identity, the two tangential slots produced by differentiating the normal vector contribute $4A : \nabla_\Sigma \eta$; the Ricci-flat radial equation replaces the remaining curvature contraction by $2A' : K$. The three first-derivative component formulas in (E.12) give the full rough normal Laplacian (E.13); its $A^2 : K$ term cancels the radial curvature term. This proves the displayed identity independently of the subsequent boundary substitutions. The term $2(\operatorname{div}_\Sigma A)\cdot \eta$ vanishes on a product orbit. Differentiate the first divergence equation and use

$$(\operatorname{div}_\Sigma \eta)' = \operatorname{div}_\Sigma \eta' - A : \nabla_\Sigma \eta, \qquad \operatorname{tr} K' = -s'.$$

Substitution of (12.10) gives

$$(LS)_{\nu\nu} = A_0 : \nabla^2_\Sigma \phi - \frac{(m-1)H}{m} \operatorname{div}_\Sigma \eta + 2A_0 : \nabla_\Sigma \eta - \mathcal{V}_R \phi - A_0 : K'. \tag{12.11}$$

The term $-\phi A_0 : K'$ in (12.11) cancels the tangential part of $-\langle S', S\rangle$ in (12.9). The remaining mixed part is $-2\langle \eta', \eta\rangle$. Since F is harmonic,

$$F_{\nu\nu} = \Delta_\Sigma F - H\mathcal{N}_R F, \qquad F|_{\Sigma_R} = -\frac{H}{m}\phi.$$

Substitute these identities in (12.9). Integrating the $\operatorname{div}_\Sigma \eta$ and $\nabla^2_\Sigma \phi$ terms by parts on the closed orbit gives respectively the mixed coefficient in (12.8) and the coefficient $((m-1)H/m)\gamma - A_0 = H\gamma - A$ of $|d_\Sigma \phi|^2$. The scalar normal derivative gives the final Dirichlet-to-Neumann term. The four remaining groups and their coefficients are written separately in Section E.3; their sum is (12.8). ☐

Proposition 12.3. *Let $R > 0$. Denote by $\mathcal{B}_R(\eta, \zeta)$ the quadratic part of (12.8) in η and $\zeta = d_\Sigma \phi$. Then*

$$\mathcal{B}_R(\eta, \zeta) \geq c_m H\big(|\eta|^2 + |\zeta|^2\big), \qquad c_m = \frac{1}{4}\left(1 - \frac{2}{m} - \frac{1}{m^2}\right) > 0. \tag{12.12}$$

Consequently, for every fixed $R > 0$,

$$q_R(T) \geq Q_R(S) + c_m H\left(\|\eta\|^2_{L^2(\Sigma_R)} + \|d_\Sigma \phi\|^2_{L^2(\Sigma_R)}\right) - C_R \|\mathcal{K}X\|^2_{L^2(M_R)}. \tag{12.13}$$

The constants C_R may be chosen locally bounded on $(0, \infty)$.

Proof. Diagonalize A in the orthogonal product splitting. If $k \in \{u, v\}$ is its eigenvalue in one direction, put $z = k/H$. Since $p, q \geq 2$ and $u, v > 0$, one has $0 < z < 1/2$. In this direction the coefficient matrix of $H^{-1}\mathcal{B}_R$ is

$$M_m(z) = \begin{pmatrix} 2(1+z) & (m+1)/m - 2z \\ (m+1)/m - 2z & 1 - z \end{pmatrix}.$$

Its determinant has the positive decomposition

$$\det M_m(z) = (1-2z)\left(1 - \frac{2}{m} - \frac{1}{m^2}\right) + 2z\left(\frac{3}{2} - \frac{1}{m^2}\right) + 6z\left(\frac{1}{2} - z\right). \tag{12.14}$$

Expansion proves the identity. The first two coefficients form a convex combination of two positive numbers, whose smaller value is $1 - 2/m - 1/m^2$. Also $\operatorname{tr} M_m(z) = 3 + z < 4$. The larger eigenvalue is at most the trace, so the smaller eigenvalue is at least $\det M_m(z)/\operatorname{tr} M_m(z) \geq c_m$. Summing over the orthogonal directions proves (12.12).

For the vector Dirichlet solution with value $\phi\nu$, Lemma 11.1 and the trace estimate give

$$\|\phi\|_{H^{1/2}(\Sigma_R)} \leq C_R \|\mathcal{K}X\|_{L^2(M_R)}.$$

The scalar Dirichlet-to-Neumann map is bounded from $H^{1/2}$ to $H^{-1/2}$: extend its boundary value in H^1, use the minimizing property of its harmonic extension, and use the weak Green identity to bound the conormal derivative. Hence the absolute value of the final term of (12.8) is at most $C_R\|\mathcal{K}X\|^2$. The bounded function $\mathcal{V}_R$ gives the same estimate for its term. This proves (12.13). Smooth radial pullback and compactness of a radius interval make all coefficients and elliptic constants locally uniform. □

Theorem 12.4. *There are constants $R_1, C, c > 0$ such that, for $R \geq R_1$ and every $T \in \mathcal{W}(M_R, g)$,*

$$q_R(T) \geq Q_R(S) + \frac{c}{R}\left(\|\eta\|^2_{L^2(\Sigma_R)} + \|d_\Sigma\phi\|^2_{L^2(\Sigma_R)}\right) - \frac{C}{R^2}\|\mathcal{K}X\|^2_{L^2}. \tag{12.15}$$

Proof. In radial orthonormal identification, $RA \to \gamma$, $RH \to m$, and $RA_0 \to 0$. Multiplying the quadratic part of (12.8) in $(\eta, d_\Sigma\phi)$ by R gives in the limit

$$2(m+1)|\eta|^2 + 2(m-1)\langle \eta, d_\Sigma\phi\rangle + (m-1)|d_\Sigma\phi|^2.$$

Its coefficient matrix has positive diagonal entries and determinant $(m-1)(m+3) > 0$. Thus it is bounded below, for all sufficiently large R, by $c(|\eta|^2 + |d_\Sigma\phi|^2)/R$ before multiplication by R.

The estimates for A_0 in Lemma 3.3 give

$$|\mathcal{V}_R| \leq CR^{-m-2}, \qquad |\mathcal{V}_R|\operatorname{Vol}(\Sigma_R) \leq CR^{-2}. \tag{12.16}$$

Indeed $\langle A', A_0\rangle = \langle A_0', A_0\rangle$, and both summands in (12.7) have order $R^{-3-2\alpha} = R^{-m-2}$. Constants are annihilated by $\mathcal{N}_R$, and $H = O(R^{-1})$. Equation (11.5) therefore bounds the absolute value of the last term in (12.8) by $CR^{-2}\|\mathcal{K}X^\perp\|^2_{L^2}$. The function $\mathcal{V}_R$ is constant along Σ_R. Its mean-zero contribution is bounded using (11.4); its constant contribution is bounded using (11.6) and (12.16). This gives

$$\left|\int_{\Sigma_R} \mathcal{V}_R\phi^2\, dA_\gamma\right| \leq CR^{-2}\left(\|\mathcal{K}X^\perp\|^2_{L^2} + \|\mathcal{K}\overline{X}\|^2_{L^2}\right).$$

Averaging is an orthogonal projection and commutes with $\mathcal{K}$, so the parenthesis equals $\|\mathcal{K}X\|^2_{L^2}$. These estimates prove (12.15). □

13. The closed reduced form

The quadratic form will be closed in the norm of its TT representative. No norm on the unreduced trace component is added.

Theorem 13.1. *For each fixed $R > 0$, q_R is closable and bounded below on $\mathscr{H}_R$. Its closure has compact embedding of its form domain into $\mathscr{H}_R$ and therefore defines a self-adjoint operator A_R with compact resolvent.*

Proof. Consider smooth pairs (S, ϕ) satisfying

$$\delta S = 0, \quad \operatorname{tr} S = 0, \quad S_{\nu\nu} = 0, \quad S^\top = -\phi A_0. \tag{13.1}$$

For each such pair solve the vector Dirichlet problem

$$\delta\mathcal{K}X = 0, \qquad X|_{\Sigma_R} = \phi\nu, \tag{13.2}$$

and put $T = S + \mathcal{K}X$. The full Dirichlet energy identity gives uniqueness and existence by subtracting an H^1 extension of the boundary data and solving in H^1_0. Elliptic regularity makes X smooth. The tangential trace-free part of $\mathcal{K}X$ is ϕA_0, so $T \in \mathcal{W}$. Conversely, Proposition 12.1 gives this pair for every $T \in \mathcal{W}$, and uniqueness in (13.2) recovers the same X.

For this fixed cap, (11.1) and the trace theorem give

$$\|\phi\|_{H^{1/2}(\Sigma_R)} \le C_R \|\mathcal{K}X\|_{L^2}. \tag{13.3}$$

A bounded H^1 extension of $\phi\nu$, followed by energy minimization for (13.2), gives the reverse bound

$$\|\mathcal{K}X\|_{L^2} \le C_R \|\phi\|_{H^{1/2}(\Sigma_R)}. \tag{13.4}$$

The potential $-2\mathring{R}$ is bounded on the fixed cap. The lower estimate (12.13), the mass identity (12.4), and (13.3) imply, for a sufficiently large fixed c_R,

$$q_R(T) + c_R\|T\|^2_{L^2} \ge c'_R \left(\|S\|^2_{H^1(M_R)} + \|\phi\|^2_{H^1(\Sigma_R)} \right).$$

For the reverse inequality use (12.8), the trace bound $\|S|_{\Sigma_R}\|_{L^2} \le C_R\|S\|_{H^1}$, the bounded map $\mathcal{N}_R : H^{1/2} \to H^{-1/2}$, and (13.4). Enlarging the constants gives

$$q_R(T) + c_R\|T\|^2_{L^2} \asymp_R \|S\|^2_{H^1(M_R)} + \|\phi\|^2_{H^1(\Sigma_R)}. \tag{13.5}$$

Define $\mathscr{P}_R$ to be the closure of the *smooth* compatible pairs (13.1) in the product norm on the right. The identification with all weak compatible pairs and the local dependence on R are proved in Section 14. For $(S, \phi) \in \mathscr{P}_R$, solve (13.2) weakly and define $T = S + \mathcal{K}X$. The map into L^2 is continuous. The identity (12.4) passes to the closure. If its image is zero, then $S = 0$ and $\mathcal{K}X = 0$. Equation (13.3) gives $\phi = 0$. Thus the map is injective.

The form-norm completion is consequently a subspace of $\mathscr{H}_R$, not a completion with an additional null component. For closability, let $T_n \to 0$ in L^2 and suppose that (T_n) is Cauchy in the shifted form norm. Equation (13.5) makes the associated pairs converge in $\mathscr{P}_R$ to a pair (S, ϕ). Continuity of the map $(S, \phi) \mapsto S + \mathcal{K}X$ gives image zero. Injectivity, proved above, implies $(S, \phi) = 0$. The same norm equivalence therefore gives convergence to zero in the shifted form norm. This is closability. The extension on the complete space $\mathscr{P}_R$ is closed. It also proves a finite lower bound in the L^2 norm.

Finally, a sequence bounded in this form norm has S precompact in $L^2(M_R)$ and ϕ precompact in $H^{1/2}(\Sigma_R)$. The latter follows from compactness of $H^1(\Sigma_R) \hookrightarrow H^{1/2}(\Sigma_R)$. Equation (13.4) then makes $\mathcal{K}X$ precompact in $L^2(M_R)$. Thus $T = S + \mathcal{K}X$ is precompact in $\mathscr{H}_R$. The representation theorem for closed lower-bounded symmetric forms gives A_R; compactness of the shifted resolvent follows from this compact embedding. □

Proposition 13.2. *Every eigentensor of A_R is smooth up to Σ_R. Its zero eigenspace is canonically isomorphic to K_R.*

Proof. Let T be an eigentensor with eigenvalue λ. Its eigenspace is finite-dimensional and invariant under $\mathfrak{G}$. The decomposition $T \mapsto (S, \phi)$ is linear, continuous in the form norm, and equivariant. The boundary function ϕ therefore belongs to a finite-dimensional $\mathfrak{G}$-invariant subspace of $H^1(\Sigma_R)$. Such functions are smooth: the orbit is homogeneous, and the matrix coefficients of a finite-dimensional continuous representation of the compact Lie group are smooth. Equivalently, convolution with the smooth character projections identifies this subspace with finitely many spherical representation types.

The vector field X in (13.2) and the harmonic function F with boundary value $-H\phi/m$ are smooth. In the interior, the eigenvalue equation implies

$$LS - (d-2)\nabla^2 F = \lambda(S + \mathcal{K}X). \tag{13.6}$$

For the distributional identity, let k be a compactly supported smooth symmetric tensor. Project it to the scalar-flat fixed-boundary space by (10.3), and then take its TT representative by Proposition 10.2.

Call that representative V_k. It is an admissible smooth form test. For smooth compatible pairs, the variation formula and integration by parts give

$$q_R(T, V_k) = \langle LS - (d-2)\nabla^2 F, k\rangle.$$

The trace and the boundary-fixing gauge terms introduced by the two projections pair to zero. The same equality passes to the form closure, since $S \in H^1$ and k is compactly supported. Also $\langle T, V_k\rangle = \langle T, k\rangle$: T is distributionally TT, and the discarded vector field has zero boundary trace. Approximation of that field by H^1_0 fields justifies the pairing. The form eigenvalue identity proves (13.6).

Interior elliptic estimates for L make S smooth away from the boundary, including at the singular orbit, which is an interior submanifold. In a collar of Σ_R, S belongs to finitely many angular representation types by equivariance of the decomposition. In radial parallel identification, (13.6) is then a finite system of ordinary differential equations with leading term $-\partial_t^2$ times the identity and with smooth coefficients up to $t = R$. Starting with H^1 regularity, the equation gives second derivatives in L^2, then all higher radial derivatives successively. Thus S, and hence T, is smooth up to the boundary.

At $\lambda = 0$, (13.6) is precisely $\mathrm{Ric}'(T + f_T g) = 0$. Conversely a smooth geometric kernel class lies in the radical of the Hessian by Proposition 10.1; its TT representative lies in the operator kernel by the definition of the associated operator. Uniqueness of the representative proves the asserted isomorphism. □

14. Local form domains and the relative Morse index

All local trivializations in this section commute with $\mathfrak{G}$. Sobolev orders are taken on a fixed pulled-back cap; no assertion is made that their constants remain bounded as the radius tends to zero.

Lemma 14.1. *Let $\mathscr{V}_R$ be the space of pairs $(S, \phi) \in H^1(S^2T^*M_R) \times H^1(\Sigma_R)$ satisfying*

$$\operatorname{tr} S = 0, \qquad S_{\nu\nu} = 0, \qquad S^\top = -\phi A_0.$$

The map

$$C_R : \mathscr{V}_R \longrightarrow L^2(T^*M_R), \qquad C_R(S,\phi) = \delta S,$$

has a bounded right inverse which is smooth in R after pullback. Its kernel is exactly the closure of the smooth compatible pairs in (13.1).

Proof. For $f \in L^2(T^*M_R)$, let ω solve (11.2) and set

$$B_R f = (\mathcal{K}\omega, -\omega_\nu|_{\Sigma_R}). \tag{14.1}$$

The trace and normal-normal conditions follow from the definition of $\mathcal{K}$ and the natural boundary condition. Since $\omega^\top = 0$,

$$\operatorname{div}\omega = \nabla_\nu \omega_\nu + H\omega_\nu.$$

The equation $(\mathcal{K}\omega)_{\nu\nu} = 0$ gives $\nabla_\nu\omega_\nu = H\omega_\nu/m$, and therefore

$$(\mathcal{K}\omega)^\top = \omega_\nu\left(A - \frac{H}{m}\gamma\right) = \omega_\nu A_0.$$

Thus $B_R f \in \mathscr{V}_R$ and $C_R B_R f = f$. The estimates in Lemma 11.2 give $\mathcal{K}\omega \in H^1$ and $\omega_\nu|_\Sigma \in H^{3/2} \subset H^1$, with the required norm bound and parameter dependence.

Here is a precise density construction before imposing divergence. If $\eta = S(\nu,\cdot)^\top$, the full boundary trace of S is

$$b_R(\eta,\phi) = \nu^\flat \otimes \eta + \eta \otimes \nu^\flat - \phi A_0.$$

It is trace-free. Choose a bounded right inverse $\mathscr{E}_R$ of the trace map from trace-free H^1 tensors to trace-free $H^{1/2}$ boundary tensors, taking smooth data to smooth tensors. Such an extension is obtained in a collar by the Fourier multiplier $e^{-t\sqrt{1+|\xi|^2}}$, followed by a collar cutoff, a partition of unity, and the pointwise trace-free projection. For $c(\xi) = (1+|\xi|^2)^{1/2}$,

$$\int_0^\infty \left(|\partial_t e^{-tc(\xi)}|^2 + c(\xi)^2|e^{-tc(\xi)}|^2\right) dt = c(\xi).$$

Plancherel's identity gives the $H^{1/2} \to H^1$ bound; the collar cutoff and the finite partition of unity preserve it. The boundary trace is the identity. To ensure equivariance, replace the extension by

$$\mathscr{E}_R^{\mathfrak{G}} = \int_{\mathfrak{G}} a^* \mathscr{E}_R (a^{-1})^* \, da,$$

where da is normalized Haar measure. The trace map commutes with pullback, so $\operatorname{Tr} \mathscr{E}_R^{\mathfrak{G}} = I$. Compactness of $\mathfrak{G}$ gives the same boundedness, and differentiation under the integral preserves smoothness. Haar invariance proves that $\mathscr{E}_R^{\mathfrak{G}}$ commutes with the action. We use this averaged extension below. Consequently

$$S = S_0 + \mathscr{E}_R b_R(\eta, \phi), \qquad S_0 \in H_0^1(S_0^2 T^* M_R).$$

Approximate S_0 by compactly supported smooth trace-free tensors, η by smooth one-forms in $H^{1/2}$, and ϕ by smooth functions in H^1. Their displayed reconstructions converge in the product norm and satisfy all boundary constraints exactly.

The bounded projection

$$\Pi_R = I - B_R C_R \tag{14.2}$$

onto $\ker C_R$ takes smooth pairs to smooth pairs by elliptic regularity. Applying it to the preceding approximants proves density of the smooth compatible pairs in $\ker C_R$. The reverse inclusion follows by continuity of divergence and the trace maps. □

Theorem 14.2. *For every $a > 0$ there is an interval I about a and a fixed Hilbert space $\mathscr{V}$ on which the closed forms q_R, $R \in I$, are represented by a smooth family of bounded symmetric forms b_R. Their mass forms $\mathfrak{m}_R$ are smooth, positive definite, and compact on $\mathscr{V}$. For constants $c, C > 0$ independent of $R \in I$,*

$$b_R(v, v) + C\mathfrak{m}_R(v, v) \geq c\|v\|_{\mathscr{V}}^2. \tag{14.3}$$

The corresponding bounded self-adjoint operators representing b_R on $\mathscr{V}$ are Fredholm. Their kernels and negative indices agree with those of A_R. The same assertions hold after restriction to the kernel of group averaging.

Proof. Pull every cap to the fixed manifold M_a by radial dilation. The positive square root of the metric change identifies the trace-free bundles and their normal and tangential slots smoothly. The reconstruction in the preceding proof identifies the spaces before imposing the divergence constraint with the fixed Hilbert space

$$H_0^1(S_0^2 T^* M_a) \oplus H^{1/2}(T^* \Sigma_a) \oplus H^1(\Sigma_a).$$

After the bundle identification, choose the averaged fixed collar extension from Lemma 14.1 and conjugate it back to define $\mathscr{E}_R$. Radial dilation and the positive bundle identifications commute with $\mathfrak{G}$, so the resulting family is equivariant and smooth in R. The coefficients of b_R depend smoothly on R. The reconstruction and its inverse therefore depend smoothly on R in the indicated operator norms. The projection (14.2) becomes a smooth family of bounded projections on this space.

After shrinking I, the operator

$$U_R = \Pi_R \Pi_a + (I - \Pi_R)(I - \Pi_a) = I + (\Pi_R - \Pi_a)(2\Pi_a - I) \tag{14.4}$$

is invertible by its norm-convergent Neumann series. The identity $\Pi_R U_R = U_R \Pi_a$ shows that it maps $\ker C_a$ onto $\ker C_R$. Take $\mathscr{V} = \ker C_a$. The density statement in Lemma 14.1 identifies these kernels with the closed form domains constructed in Theorem 13.1.

Pull the form and its mass back by U_R. In compatible-pair coordinates the mass is

$$\mathfrak{m}_R(S, \phi) = \|S\|_{L^2}^2 + \|\mathcal{K}_R X_R^D(\phi)\|_{L^2}^2, \tag{14.5}$$

where $X_R^D(\phi)$ solves the vector Dirichlet equation with boundary value $\phi\nu$. The Dirichlet solution operators and the scalar harmonic extensions depend smoothly on R, by the inverse identity and the boundary estimates. Formula (12.8) consequently gives bounded smooth bilinear forms on the common Hilbert space. The mass is positive definite by (12.4) and (13.3). Its compactness follows from

$$H^1(M_a) \Subset L^2(M_a), \qquad H^1(\Sigma_a) \Subset H^{1/2}(\Sigma_a),$$

and the bounded Dirichlet extension $H^{1/2}(\Sigma_a) \to H^1(M_a)$.

The locally uniform form-norm estimates in Theorem 13.1 give (14.3). Choose the fixed Hilbert scalar product to be $\mathfrak{G}$-invariant by Haar averaging. Relative to it, let Q_R and M_R represent b_R and $\mathfrak{m}_R$. Then $Q_R + CM_R$ is boundedly invertible and M_R is compact. Thus Q_R is Fredholm with finite negative index. Its radical is the radical of the closed form, and hence equals $\ker A_R$. The negative index is the same variational index in either realization, since the transformation of form domains is bijective. Every construction commutes with the group action, so the invariant subspace and its averaging-zero complement are preserved. □

Theorem 14.3. *Let $0 < a < b$ be nonumbilic radii. Then*

$$\operatorname{ind}^- A_b - \operatorname{ind}^- A_a = \#\{R \in (a, b) \,:\, u(R) = v(R)\}. \tag{14.6}$$

There is an integer $\kappa_\perp \geq 0$, independent of $R > 0$, such that

$$\operatorname{ind}^-(A_R|_{\ker \mathcal{P}}) = \kappa_\perp, \qquad \mathcal{P}T = \int_{\mathfrak{G}} U^* T \, dU. \tag{14.7}$$

Here dU is normalized Haar measure. Moreover,

$$\operatorname{ind}^- A_R = \frac{\sqrt{(m-1)(9-m)}}{2\pi} \log R + O(1) \qquad (R \longrightarrow \infty). \tag{14.8}$$

Proof. In the local form chart of Theorem 14.2, a radius with zero kernel is represented by an invertible bounded self-adjoint Fredholm operator. Its spectrum has a gap about zero, and norm-small perturbations preserve its negative index. At a radius R_* with one-dimensional kernel, choose a unit kernel vector e and write the bounded form operator in the splitting $\mathbb{R}e \oplus e^\perp$ as

$$Q_R = \begin{pmatrix} a_R & b_R^* \\ b_R & C_R \end{pmatrix}.$$

The operator C_{R_*} is invertible. Continuity and the Neumann series make C_R invertible nearby. Completing the square gives

$$\langle Q_R(se+v), se+v\rangle = \langle C_R(v + sC_R^{-1}b_R), v + sC_R^{-1}b_R\rangle + (a_R - \langle C_R^{-1}b_R, b_R\rangle)s^2.$$

At R_*, $a_{R_*} = 0$ and $b_{R_*} = 0$. The derivative of the last scalar coefficient is consequently $\langle \dot{Q}_{R_*} e, e\rangle$, the crossing form on the kernel. Theorem C makes that derivative negative. Thus one positive direction becomes negative as R increases through R_*, and the total negative index increases by one.

The umbilic radii are simple zeros of $u - v$, since at such a zero $(u - v)' = r_P - r_Q \neq 0$ by Proposition 9.1. They cannot accumulate in a compact subinterval: an accumulation would give both $u - v = 0$ and $(u - v)' = 0$ at its limiting point. Cover $[a, b]$ by finitely many of the local form charts. Summing their index changes proves (14.6).

The averaging-zero restriction has zero kernel at every radius by Theorem A. Its negative index is locally constant by the first paragraph and is finite by Theorem 14.2. The connectedness of $(0, \infty)$ proves (14.7). The value $\kappa_\perp = 0$ is proved in Theorem 21.2 by an initial-index calculation in the same form domain.

For (14.8), the polar angle of the autonomous system in Lemma 3.3 has derivative $\beta + O(e^{-\alpha s})$, where

$$\alpha = (m-1)/2, \qquad \beta = \sqrt{(m-1)(9-m)}/2.$$

The integral of the angular error converges. For some fixed ϑ_0, the condition $w = 0$ is equivalent to $\vartheta(s) \in \vartheta_0 + \pi\mathbb{Z}$. On a sufficiently late interval, ϑ is strictly increasing and $\vartheta(s) = \beta s + \vartheta_\infty + O(e^{-\alpha s})$. Counting integers between its endpoint values gives $\beta s/\pi + O(1)$ zeros up to s. Also $s = \log R + O(1)$ on the asymptotically conical end. Apply (14.6) from one fixed nonumbilic radius. At an endpoint crossing the strict negative index differs from either neighboring index by at most one, which does not change the error term. □

15. Negative spectrum on the complete metric

This section records the analytic facts needed for comparison, independently of the compact negative-mode manuscript. The infinite negative spectrum for the particular Böhm metrics is already established in [6]; the proof of its existence given below is included to identify the operator and threshold.

Proposition 15.1. *Let (X, g) be complete, Ricci-flat, and without boundary. Assume that curvature is bounded and tends to zero at infinity. The Friedrichs realization of $L = \nabla^*\nabla - 2\mathring{R}$ is bounded below and has no negative essential spectrum. There are finitely many eigenvalues below each fixed negative threshold. Every negative eigentensor is smooth and TT.*

Proof. Let $A = \nabla^*\nabla$ be the nonnegative Friedrichs realization and let $V = -2\mathring{R}$. The bundle endomorphism V is bounded and tends to zero. Multiplication by a compactly supported smooth cutoff, followed by $(A + 1)^{-1}$, is compact as an operator into L^2: interior elliptic estimates give local H^2 bounds, and the compact Sobolev embedding on a fixed compact set gives compactness. The omitted tail in $V(A + 1)^{-1}$ has norm bounded by the supremum of $|V|$ there. Hence $V(A + 1)^{-1}$ is compact. The resolvent identity, or the compact-perturbation theorem for a self-adjoint operator, gives equality of the essential spectra of $A + V$ and A. Since $A \geq 0$, negative spectrum is discrete. The lower bound $L \geq -\|V\|_\infty$ then makes the count below each negative threshold finite.

Let $LH = \mu H$, $\mu < 0$, $H \in L^2$. The form domain gives $\nabla H \in L^2$, and elliptic regularity gives smoothness. The identities (2.3) imply

$$\Delta \operatorname{tr} H = \mu \operatorname{tr} H, \qquad \nabla^*\nabla\delta H = \mu\delta H.$$

Both unknowns are in L^2. If u denotes either one, test its equation with $\chi_s^2 u$, where $\chi_s(x) = \chi(d(x, x_0)/s)$, $\chi = 1$ on $[0, 1]$ and $\chi = 0$ on $[2, \infty)$. Completeness makes its support compact. This cutoff is Lipschitz and satisfies $|d\chi_s| \leq C/s$ almost everywhere; it is an admissible compactly supported H^1 test multiplier. Integration gives

$$\|\nabla(\chi_s u)\|_2^2 - \mu\|\chi_s u\|_2^2 = \int_X |d\chi_s|^2|u|^2\, dv_g.$$

The right side tends to zero. Since $-\mu > 0$, this forces $u = 0$. Thus H is TT. This test uses local smoothness and does not presuppose a global H^1 bound on δH. □

Lemma 15.2. *On the cone $g_C = dr^2 + r^2h$ over the product link in Lemma 11.3, let*

$$k = \frac{1}{p}h_P - \frac{1}{q}h_Q,$$

where h_P, h_Q are its two factor metrics. For compactly supported smooth f,

$$Q_C(r^2 fk) = \|k\|_{L^2(h)}^2 \int_0^\infty \left(r^m(f')^2 - 2(m-1)r^{m-2}f^2\right) dr. \tag{15.1}$$

Proof. The tensor k is parallel, trace-free, and divergence-free on the link. Its curvature contraction is $\mathring{R}_h k = (m - 1)k$, separately on each factor. In a radially parallel orthonormal cone frame, the tangential components of $r^2 fk$ are fk. Radial differentiation contributes $(f')^2|k|^2$. Tangential covariant differentiation into a radial slot contributes $2r^{-2}f^2|k|^2$. The cone curvature contraction on a tangential trace-free tensor is $r^{-2}(\mathring{R}_h k + k)$. Its k contribution cancels the preceding $2r^{-2}$ term after multiplication by -2. The remaining potential is $-2(m - 1)r^{-2}f^2|k|^2$. Multiplication by the cone volume density $r^m dr\, dv_h$ gives (15.1). □

Corollary 15.3. *For the metrics* (1.1), *L_X has infinitely many negative eigenvalues, counted with multiplicities. They can accumulate only at zero.*

Proof. Set $\alpha = (m - 1)/2$ and $\beta^2 = (m - 1)(9 - m)/4 > 0$. If $s = \log r$ and $f = r^{-\alpha}\chi(s)$, integration of the cross term gives

$$Q_C(r^2 fk) = \|k\|_2^2 \int_{\mathbb{R}} \left((\chi')^2 - \beta^2\chi^2\right) ds.$$

Choose an interval of length greater than π/β. The first Dirichlet sine has negative value of the displayed form; approximation in H_0^1 gives a compactly supported smooth function with the same strict sign. It defines a negative tensor supported in a fixed compact cone annulus. Smooth conical convergence transfers this strict inequality to every sufficiently distant dilate of the annulus in X. Choose disjoint such dilates. Their tensors have zero mutual form and L^2 pairings, so every finite span is negative definite. The variational principle and Proposition 15.1 give infinitely many discrete negative eigenvalues. The same proposition excludes every other negative accumulation point. □

16. Spectral exhaustion of the reduced form

For $\varepsilon > 0$, put

$$N_X(\varepsilon) = \dim \mathbf{1}_{(-\infty,-\varepsilon)}(L_X)L^2.$$

As an auxiliary comparison, let L_R^N be the operator of the symmetric-tensor form Q_R on the whole space $H^1(S^2T^*M_R)$, with no trace or divergence constraint. Its natural boundary condition is the connection-Neumann condition. Denote its count below $-\varepsilon$ by $N_R^N(\varepsilon)$.

Proposition 16.1. *For $\varepsilon > 0$ with $-\varepsilon \notin \sigma(L_X)$,*

$$N_R^N(\varepsilon) = N_X(\varepsilon) \tag{16.1}$$

for all sufficiently large R.

Proof. For the lower bound, choose a basis of the finite-dimensional global spectral subspace below $-\varepsilon$. Smooth compactly supported approximation in the form norm makes the matrix of $Q + \varepsilon\|\cdot\|_2^2$ converge to a negative-definite matrix. The approximants therefore span a negative subspace of dimension $N_X(\varepsilon)$, supported in a fixed cap. They are admissible Neumann form tests on every larger cap.

For the upper bound we prove tightness at a fixed negative threshold. Let H_R be a normalized Neumann eigentensor with eigenvalue $\lambda_R \le -\varepsilon$. Since $V = -2\mathring{R}$ is bounded,

$$\|\nabla H_R\|_2^2 = \lambda_R - \langle VH_R, H_R\rangle \le \|V\|_\infty.$$

Take s large and a radial cutoff χ_s which is zero for $t \le s$, one for $t \ge 2s$, and satisfies $|d\chi_s| \le C/s$. If $R > 2s$, testing the Neumann form equation by $\chi_s^2 H_R$ gives

$$Q_R(\chi_s H_R) = \lambda_R\|\chi_s H_R\|_2^2 + \int_{M_R} |d\chi_s|^2|H_R|^2\, dv_g.$$

For sufficiently large s, $V \ge -\varepsilon/2$ on the support of χ_s. Consequently

$$\frac{\varepsilon}{2}\|\chi_s H_R\|_2^2 \le \frac{C}{s^2}, \tag{16.2}$$

uniformly in R. This is a bound on the entire exterior $t \ge 2s$, not only on the cutoff annulus.

If (16.1) failed in the upper direction along $R_i \to \infty$, choose $N_X(\varepsilon) + 1$ orthonormal eigentensors with eigenvalues below $-\varepsilon$ on each cap. Their eigenvalues lie in the fixed interval $[-\|V\|_\infty, -\varepsilon]$. Pass to a subsequence for which every eigenvalue converges. On each compact subset, the gradient bound and Rellich compactness give strong L^2 convergence after diagonal extraction. Let $\widehat{H}_i$ denote extension by zero as an L^2 section, and let H be one of the local limits. Fatou's lemma and (16.2) give the same tail bound for H. Splitting the difference at $t = 2s$ yields

$$\limsup_{i\to\infty} \|\widehat{H}_i - H\|_2^2 \le \frac{C}{\varepsilon s^2}.$$

Letting $s \to \infty$ proves global L^2 convergence. It preserves norms and pairwise inner products. On every fixed compact subset the gradient bounds pass to the limit weakly; exhaustion gives $H \in H^1(X)$. For $\varphi \in C_c^\infty$, passage in the weak equation gives $Q(H,\varphi) = \lambda\langle H, \varphi\rangle$ at the limiting eigenvalue. Both sides are continuous in the H^1 form norm, so density extends this identity to the form domain. By the definition of the associated operator, $H \in D(L_X)$ and $L_X H = \lambda H$. Each limiting eigenvalue is at most

$-\varepsilon$ and belongs to the global spectrum. By the hypothesis on the threshold it is strictly less than $-\varepsilon$. The resulting $N_X(\varepsilon)+1$ orthonormal negative eigentensors contradict the definition of $N_X(\varepsilon)$. ☐

Proof of Theorem B. The realization, compact resolvent, regularity, and kernel statement are Theorem 13.1 and Proposition 13.2. For the count, write $N_R(\varepsilon) = \#\{\lambda(A_R) < -\varepsilon\}$. Equations (12.15) and (12.4) give

$$q_R(T) + \varepsilon\|T\|_{L^2}^2 \geq Q_R(S) + \varepsilon\|S\|_{L^2}^2 + \left(\varepsilon - \frac{C}{R^2}\right)\|\mathcal{K}X\|_{L^2}^2. \tag{16.3}$$

The inequality extends to the closed form domain. If $R^2 > C/\varepsilon$, projection $T \mapsto S$ is injective on every negative subspace of the form on the left, and its image is negative for $Q_R + \varepsilon\|\cdot\|_2^2$. Hence

$$N_R(\varepsilon) \leq N_R^N(\varepsilon). \tag{16.4}$$

For the reverse inequality choose smooth compactly supported trace-free tensors spanning a negative subspace of $Q_X + \varepsilon\|\cdot\|_2^2$ of dimension $N_X(\varepsilon)$. Such tensors are obtained by cutting off a basis of the global negative eigenspace, whose members are TT by Proposition 15.1, and approximating the cutoffs if necessary. Fix these tensors before increasing R. For all sufficiently large caps, Lemma 10.3 gives

$$q_R(T_R(U)) + \varepsilon\|T_R(U)\|_{L^2}^2 \leq Q_X(U) + \varepsilon\|U\|_{L^2}^2. \tag{16.5}$$

The map to the reduced TT representative is injective on this negative subspace: a zero image would contradict strict negativity in (16.5). Therefore $N_R(\varepsilon) \geq N_X(\varepsilon)$. Combine this inequality with (16.4) and Proposition 16.1 to obtain (1.10). ☐

Corollary 16.2. *Let $\mu_1 \leq \mu_2 \leq \cdots < 0$ be the complete negative eigenvalues with multiplicities. Let $\lambda_j(R)$ be the eigenvalues of A_R, arranged increasingly. For every fixed j,*

$$\lim_{R\to\infty} \lambda_j(R) = \mu_j. \tag{16.6}$$

Moreover $\operatorname{ind}^- A_R \to +\infty$.

Proof. Choose negative numbers $a < \mu_j < b < 0$ outside the global spectrum, as close to μ_j as desired. The count below a is less than j, and the count below b is at least j. Eventual equality of these two counts on the caps gives $a \leq \lambda_j(R) < b$. This proves (16.6). For every positive integer k, Corollary 15.3 supplies a negative threshold with at least k eigenvalues below it. Theorem B gives $\operatorname{ind}^- A_R \geq k$ on every sufficiently large cap. ☐

17. The geometric crossing form

We compute the variation on the kernel of the Hessian. This calculation does not replace the separate operator construction of Section 13.

Proposition 17.1. *Let a smooth boundary in a fixed Ricci-flat manifold move with outward normal speed $v_* > 0$. At a parameter where h is an infinitesimal Ricci-flat deformation with $h^\top = 0$, set $a_h = A_g'(h)$. The crossing form of B on the geometric kernel is*

$$\Gamma_B(h,k) = -2\int_\Sigma v_*\big(\langle a_h, a_k\rangle - \operatorname{tr} a_h \ \operatorname{tr} a_k\big)\, dA_\gamma. \tag{17.1}$$

It descends to the boundary-fixing gauge quotient.

Proof. A boundary-fixing infinitesimal diffeomorphism puts h in Gaussian gauge in a collar: $h(\nu,\cdot) = 0$. Since $h^\top = 0$ at the boundary, all components vanish there, and $a_h = \frac{1}{2}\nabla_\nu h^\top$. Choose a family of fixed-induced-metric variations on the moving domains. Differentiating its boundary condition gives

$$\dot{h}^\top = -v_*\nabla_\nu h^\top = -2v_* a_h. \tag{17.2}$$

At the crossing $E'(h) = 0$. Differentiation of $B(h,h) = \int \langle E'(h), h\rangle$ therefore gives $\Gamma_B(h,h) = \int \langle E'(\dot h), h\rangle$; the integrand in the domain-variation term is zero. Green's identity for the linearized Einstein tensor, with h in the preceding boundary gauge, is

$$\int_M \langle E'(k), h\rangle\, dv_g - \int_M \langle E'(h), k\rangle\, dv_g = \int_\Sigma \langle k^\top, a_h - (\operatorname{tr} a_h)\gamma\rangle\, dA_\gamma.$$

Its normal second-derivative part on tangential components is $-\frac{1}{2}\partial_\nu^2 h^\top + \frac{1}{2}\gamma\partial_\nu^2 \operatorname{tr}_\gamma h^\top$; integration of these two terms gives the displayed boundary expression. All other boundary terms contain a boundary value of h and vanish. Substitute (17.2) to obtain the quadratic case of (17.1); polarization gives the bilinear case.

The conformal correction in (10.4) has zero first derivative at the crossing, since $r(h) = 0$. Thus the same form is obtained from the conformally reduced Hessian. Boundary-fixing diffeomorphisms do not change either linearized Cauchy datum, so the form descends to the quotient. □

Lemma 17.2. *Suppose $R_\gamma > 0$ and the infinitesimal Cauchy uniqueness conclusion* (2.7) *holds. Then* (17.1) *is negative definite on every nonzero geometric kernel.*

Proof. The Gauss constraint is $R_\gamma = H^2 - |A|^2$. Its derivative in a geometric kernel direction, whose induced metric variation is zero, is $H \operatorname{tr} a_h = \langle A, a_h\rangle$. Decompose

$$A = \frac{H}{m}\gamma + A_0, \qquad a_h = \frac{\tau}{m}\gamma + a_0.$$

Since $R_\gamma > 0$, $H \neq 0$. The constraint gives $\frac{m-1}{m} H\tau = \langle A_0, a_0\rangle$. Hence

$$\begin{aligned} |a_h|^2 - (\operatorname{tr} a_h)^2 &= |a_0|^2 - \frac{m-1}{m}\tau^2 \\ &\geq \frac{mR_\gamma}{(m-1)H^2}|a_0|^2. \end{aligned}$$

Equality at zero forces $a_0 = 0$ and then $\tau = 0$. If the integral in (17.1) were zero, a_h would vanish everywhere. Cauchy uniqueness would make h boundary-fixing pure gauge. Thus every nonzero quotient class has a strictly negative crossing value. □

Proof of Theorem C. For each Böhm orbit,

$$R_{\gamma_R} = \frac{p(p-1)}{a(R)^2} + \frac{q(q-1)}{b(R)^2} > 0.$$

By Theorem A, each geometric kernel class has an invariant Bianchi representative. Lemma 4.6 therefore gives the required Cauchy uniqueness on this kernel. Apply Proposition 17.1 and Lemma 17.2. At an umbilic orbit the linearized constraint also gives $\operatorname{tr} a_h = 0$, so the quadratic value is $-2\int v_*|a_h|^2$. Theorem A gives the one-dimensional kernel and the full fold statement. □

Remark 17.3. The geometric kernel is the kernel of the induced-metric differential. The reduced self-adjoint form has this kernel at every radius and recovers the complete negative-threshold counts at infinity. Its crossing sign, together with the local form domains of Section 14, gives the relative index formula (14.6). The initial-index calculation in Sections 18 to 20 gives the absolute count in Theorem 21.4. The filling count in (1.5) remains local in the boundary-marked moduli space.

18. The reduced form on the flat normal model

In this section and the next two sections, let $p, q \geq 2$ be arbitrary, put $N = p + 1$, $m = p + q$, and $d = N + q$. No assumption $m \leq 8$ is needed for the model or its fixed-representation perturbation. Set

$$B = B_1^N \subset \mathbb{R}^N, \qquad M_0 = B \times T^q,$$

where T^q is a fixed flat torus and M_0 carries the product metric g_0.

Theorem 18.1 (Flat normal-model positivity). *The form* (1.8) *on* M_0*, restricted to tensors invariant under translations of* T^q*, has a closed lower-bounded realization with compact resolvent. This realization is strictly positive: for some* $c_{p,q} > 0$*,*

$$q_0(T) \geq c_{p,q}\|T\|_{L^2}^2 \tag{18.1}$$

on its closed form domain.

The form realization is proved below. Its nonpositive spectrum is excluded in Section 19. All tensors in this section are invariant under translations of T^q. Dividing every integral by its volume, one may regard them as tensors on B with values in the constant bundle $\mathbb{R}^N \oplus \mathbb{R}^q$. Derivatives occur only in the $\mathbb{R}^N$ directions. Greek capital indices I, J refer to $\mathbb{R}^q$; indices A, B refer to $\mathbb{R}^N$.

Set

$$\mathcal{W}_0 = \{T \in C^\infty(S^2T^*(B \times T^q)) \,:\, \delta T = 0,\ \operatorname{tr} T = 0,\ (T^\top)_0 = 0\}.$$

For $T \in \mathcal{W}_0$, put $f_\partial = T_{\nu\nu}/m$. Then

$$T^\top = -f_\partial \gamma.$$

Let f be the harmonic extension of f_∂. Define

$$q_0(T) = \int_B \langle LT - (m-1)\nabla^2 f, T\rangle. \tag{18.2}$$

By Section 10, this is twice the Hessian of $-\int R\,dv - 2\int H\,dA$ on scalar-flat fixed-boundary metrics, modulo boundary-fixing diffeomorphisms. In the present notation, if $r(h) = DR(h)$ and $D = \Delta_D^{-1}$, then

$$Ph = h - \frac{1}{m}Dr(h)g, \qquad r(Ph) = 0,$$

and the scalar Schur complement is

$$B(Ph, Pk) = B(h,k) + \frac{m-1}{2m}\langle r(h), Dr(k)\rangle.$$

The vector Dirichlet York decomposition gives a unique representative $h = T + fg$, with T TT and f harmonic. Since

$$2E'(T + fg) = LT - (m-1)\nabla^2 f, \qquad E = \operatorname{Ric} - \tfrac{1}{2}Rg,$$

formula (18.2) follows. All operations commute with torus translations.

18.1. **The form realization.** For a vector field Y define

$$\mathcal{K}_0 Y = \tfrac{1}{2}\mathcal{L}_Y g_0 - \frac{\operatorname{div} Y}{d} g_0.$$

On fields with $Y^\top = 0$ at $\partial B \times T^q$, there is a coercive estimate

$$\|Y\|_{H^1(B)} \leq C\|\mathcal{K}_0 Y\|_{L^2(B)}. \tag{18.3}$$

To verify the absence of a kernel, suppose $\mathcal{K}_0 Y = 0$. The IJ components give $\operatorname{div}_{\mathbb{R}^N} Y_{\mathbb{R}^N} = 0$, since $q > 0$. The AI components give $\partial_A Y_I = 0$. The AB components give the Euclidean Killing equation. Differentiating it and permuting indices shows that all second derivatives of $Y_{\mathbb{R}^N}$ vanish, so

$$Y_{\mathbb{R}^N} = a + Bx, \qquad B^T = -B.$$

For each unit vector x, tangential vanishing gives $(I - x \otimes x)a + Bx = 0$. The equation at $-x$ gives $(I - x\otimes x)a - Bx = 0$. Subtraction yields $Bx = 0$ for every unit x, so $B = 0$. The remaining equations force a to be parallel to every unit vector, hence $a = 0$ because $N \geq 3$. The vertical constants vanish by the tangential boundary condition. The trace-free Korn estimate with an added L^2 term, applied on this fixed flat product, and Rellich compactness now imply (18.3): a normalized contradicting sequence would converge to a nonzero kernel field. The underlying Korn estimate is the one in [9].

The mixed vector problem

$$\delta\mathcal{K}_0 Y = 0, \quad Y^\top = 0, \quad (\mathcal{K}_0 Y)_{\nu\nu} = T_{\nu\nu}$$

has a unique solution. Its variational form is

$$\int_B \langle \mathcal{K}_0 Y, \mathcal{K}_0 Z\rangle = \int_B \langle T, \mathcal{K}_0 Z\rangle, \qquad Z^\top = 0.$$

The energy estimate (18.3) gives existence. Its boundary symbol has no nonzero decaying Fourier solution: the zero tangential data and zero normal traction give zero $\mathcal{K}_0$ energy, and the preceding component equations at a nonzero tangential frequency force the solution to vanish. Elliptic boundary estimates give smoothness for smooth data.

Put $S = T - \mathcal{K}_0 Y$ and $\phi = Y_\nu|_\partial$. Then

$$\delta S = 0, \quad \operatorname{tr} S = 0, \quad S_{\nu\nu} = 0, \quad S^\top = -\phi A_0, \qquad \|T\|^2 = \|S\|^2 + \|\mathcal{K}_0 Y\|^2. \tag{18.4}$$

Here A has eigenvalues 1 in the p sphere directions and 0 in the q torus directions, $H = p$, and $A_0 = A - (p/m)\gamma$.

For $\eta = S(\nu,\cdot)^\top$, integration of the Ricci variation gives

$$\begin{aligned} q_0(T) = \int_B |\nabla S|^2 + \int_{\partial B} \Big[& 2(H\gamma + A)(\eta,\eta) + 2\big(\tfrac{(m-1)H}{m}\gamma - 2A_0\big)(\eta, d\phi) \\ & + (H\gamma - A)(d\phi, d\phi) - V_0\phi^2\Big] - \frac{(m-1)H^2}{m}\langle \phi, \mathcal{N}_0\phi\rangle, \end{aligned} \tag{18.5}$$

where $\mathcal{N}_0$ is the ball Dirichlet-to-Neumann map and $V_0 = -pq^2/m^2$. This follows from (12.8): here $\langle \nabla_\nu A, A_0\rangle = -pq/m$ and $H|A_0|^2/m = p^2q/m^2$, whose sum is $V_0 = -pq^2/m^2$. The boundary derivatives have no vertical component.

The matrix of the quadratic part in one direction, divided by H, is

$$M_m(z) = \begin{pmatrix} 2(1+z) & (m+1)/m - 2z \\ (m+1)/m - 2z & 1 - z \end{pmatrix}, \qquad z \in \{0, 1/p\}.$$

For $0 \le z \le 1/2$,

$$\det M_m(z) = (1-2z)(1 - 2/m - 1/m^2) + 2z(3/2 - 1/m^2) + 6z(1/2 - z) > 0.$$

The trace is bounded by 4. Thus this quadratic part is positive definite. The trace theorem, (18.3), and the minimizing property of harmonic extension give

$$\|\phi\|_{H^{1/2}} \asymp \|\mathcal{K}_0 Y\|_{L^2}, \qquad |\langle \phi, \mathcal{N}_0\phi\rangle| \le C\|\phi\|^2_{H^{1/2}}.$$

Consequently, for a sufficiently large C,

$$q_0(T) + C\|T\|^2 \asymp \|S\|^2_{H^1(B)} + \|\phi\|^2_{H^1(\partial B)}. \tag{18.6}$$

Completing smooth compatible pairs gives a closed form. The mass embedding is compact by $H^1(B) \Subset L^2(B)$ and $H^1(\partial B) \Subset H^{1/2}(\partial B)$. The mixed inverse also gives density in the weak constraint space: solve $\delta\mathcal{K}_0\omega = \delta S$ with $\omega^\top = 0$ and $(\mathcal{K}_0\omega)_{\nu\nu} = 0$, and subtract $(\mathcal{K}_0\omega, -\omega_\nu)$ from smooth approximants. This proves closability, compact resolvent, and the asserted form domain.

18.2. **A necessary eigenvalue equation.** If T is an eigenvector with eigenvalue $-x$, $x \ge 0$, its interior equation is

$$(-\Delta_{\mathrm{e}} + x)T = (m-1)\nabla^2 f, \quad \Delta_{\mathrm{e}} f = 0, \quad \delta T = 0, \quad \operatorname{tr} T = 0, \quad T^\top = -f\gamma. \tag{18.7}$$

Here $\Delta_{\mathrm{e}} = \sum_A \partial_A^2$ is the Euclidean, nonpositive Laplacian. To obtain the interior equation, test the eigenvalue equation against the TT representative of Pk for an arbitrary compactly supported symmetric tensor k. The conformal and boundary-fixing gauge corrections pair to zero against both T and $2E'(T + fg)$. This yields (18.7) distributionally. Interior regularity and the finite angular decomposition of an eigenspace give smoothness up to the boundary. Alternatively, in each spherical type the equation is a radial elliptic system with smooth coefficients near $r = 1$.

It suffices below to show that (18.7) has no nonzero smooth solution. No converse sufficiency of these boundary conditions is used.

19. Exclusion of nonpositive model eigenvalues

Suppose (18.7) holds. The vertical components satisfy

$$(-\Delta_e + x)T_{IJ} = 0, \qquad T_{IJ}|_{r=1} = -f\delta_{IJ}.$$

Hence $T_{IJ} = -v\delta_{IJ}$, where $(-\Delta_e + x)v = 0$ and $v|_{r=1} = f$. For each I, the vector $U_A = T_{AI}$ is divergence-free, solves $(-\Delta_e + x)U = 0$, and has zero tangential boundary value. Its divergence equation gives $\partial_\nu U_\nu = -pU_\nu$ at the boundary. Therefore

$$0 = \int_B (|\nabla U|^2 + x|U|^2) + p\int_{\partial B} |U_\nu|^2,$$

so $U = 0$.

If $f = 0$, the remaining normal block S is Euclidean TT and satisfies $(-\Delta_e + x)S = 0$. At the boundary $S^\top = S_{\nu\nu} = 0$. With $\eta = S(\nu, \cdot)^\top$, its tangential divergence equation gives $\nabla_\nu \eta = -(p+1)\eta$. Integration by parts gives

$$0 = \int_B (|\nabla S|^2 + x|S|^2) + 2(p+1)\int_{\partial B} |\eta|^2.$$

It follows that $S = 0$. Thus any nonzero solution must have $f \neq 0$.

Decompose under $O(N)$ and select a nonzero boundary scalar harmonic Y of degree $\ell \geq 0$. Write

$$\Lambda = \ell(\ell + p - 1), \qquad f(r, \omega) = r^\ell Y(\omega).$$

All equations preserve its scalar-derived tensor type. Other angular types have zero scalar boundary value and vanish by the preceding energy argument.

19.1. The regular scalar solution.

Lemma 19.1. *Let $\ell \geq 0$ and $x \geq 0$. The regular solution of* (19.1)*, normalized by $v_x(1) = 1$, is analytic in x. Its boundary logarithmic derivative satisfies* (19.3).

Proof. Let v_x be the radial solution normalized by $v_x(1) = 1$:

$$v_x'' + \frac{p}{r}v_x' - \frac{\Lambda}{r^2}v_x = xv_x, \qquad v_x(r) = r^\ell w_x(r), \quad w_x(0) > 0. \tag{19.1}$$

Its power series, up to normalization, is

$$r^\ell \sum_{k=0}^{\infty} \frac{x^k r^{2k}}{4^k k!(\ell + (p+1)/2)_k}.$$

It is analytic in $x \geq 0$. Put

$$z = z_\ell(x) = v_x'(1) - \ell. \tag{19.2}$$

Since

$$(r^{2\ell+p}w_x')' = xr^{2\ell+p}w_x,$$

positivity and monotonicity of w_x give

$$0 < z_\ell(x) < \frac{x}{2\ell + p + 1} \quad (x > 0), \qquad z_\ell(0) = 0, \quad z_\ell'(0) = \frac{1}{2\ell + p + 1}. \tag{19.3}$$

The strict upper bound follows by replacing $w_x(r)$ by $w_x(1)$ in the integral from zero to one; the series gives the derivative at zero. ☐

19.2. **The normal TT homogeneous solution.**

Lemma 19.2. *For each fixed scalar harmonic Y of degree $\ell \geq 2$, the regular solutions of $(-\Delta_{\mathrm{e}} + x)Z = 0$ in the Euclidean TT scalar type generated by Y form a one-dimensional space. A generator has the coefficients* (19.4)*, with $A = r^{-2}v_x$, and trace-free tangential coefficient* (19.5)*. For $\ell = 0, 1$ the regular scalar-derived TT solution space is zero.*

Proof. For $\ell \geq 2$, set $\mathcal{H}_Y = \nabla^2_{S^p} Y + (\Lambda/p)Y\gamma_p$. A scalar-type Euclidean TT tensor has the form

$$Z = A(r)Y\,dr^2 + 2rB(r)\,dr\,dY + r^2\big(C(r)Y\gamma_p + D(r)\mathcal{H}_Y\big).$$

Here $2dr\,dY = dr \otimes dY + dY \otimes dr$. Trace and divergence give

$$C = -A/p, \quad B = \frac{rA' + (p+1)A}{\Lambda}, \quad D = \frac{p}{(p-1)(\Lambda - p)}\big(rB' + (p+1)B - A/p\big). \tag{19.4}$$

The normal-normal component of $(-\Delta_{\mathrm{e}} + x)Z = 0$ is

$$A'' + \frac{p+4}{r}A' - \frac{\Lambda - 2(p+1)}{r^2}A = xA.$$

Its regular solution is $A = r^{-2}v_x$. Formulas (19.4) determine a tensor Z_x. The remaining equation components vanish: their residual is TT, is scalar-type, and has zero normal-normal component; (19.4) then forces every component to vanish.

The resulting tensor is smooth at zero. Here is an explicit check. For the term $r^{\ell+2k-2}$ of A, set $h = \ell + 2k$, $b = (h+p-1)/\Lambda$, and

$$d_k = \frac{p(h+p-1)^2 - \Lambda}{(p-1)\Lambda(\Lambda - p)}.$$

Let $P_\ell = r^\ell Y$ be the harmonic homogeneous polynomial. The tensor term is a combination of

$$r^{2k}\nabla^2 P_\ell, \quad r^{2k-2}(x^\flat \otimes dP_\ell + dP_\ell \otimes x^\flat), \quad r^{2k-2}P_\ell g_{\mathrm{e}}, \quad r^{2k-4}P_\ell x^\flat \otimes x^\flat.$$

The coefficients in this order are

$$d_k, \qquad b - (\ell-1)d_k, \qquad -\frac{1}{p} + \frac{\ell(\ell-1)}{p}d_k, \qquad \frac{p+1}{p} - 2\ell b + (\ell^2 - \Lambda/p)d_k.$$

At $k = 0$, $b = 1/\ell$ and $d_0 = 1/(\ell(\ell-1))$; hence the second and third coefficients vanish. The last coefficient is

$$\frac{p+1}{p} - 2\ell b + (\ell^2 - \Lambda/p)d_k = \frac{4k(k-1)}{(\ell+p)(\ell+p-1)},$$

so it vanishes at $k = 0, 1$. All remaining powers are nonnegative even integers. The convergent series therefore gives a smooth tensor.

At $r = 1$, $A = 1$, $C = -1/p$, and

$$D = d_\ell(x) = \frac{\Lambda + p(\ell + z + p - 1) + px/(p-1)}{\Lambda(\Lambda - p)} = \frac{K + pz + px/(p-1)}{\ell(\ell-1)K}, \quad K = (\ell+p)(\ell+p-1). \tag{19.5}$$

In particular $d_\ell(x) > 0$. For $\ell = 0$, the divergence equation is $rA' + (p+1)A = 0$, so $A = cr^{-p-1}$ and regularity forces $c = 0$. For $\ell = 1$, the trace-free Hessian vanishes and $\Lambda = p$. The two divergence equations become

$$pB = rA' + (p+1)A, \qquad rB' + (p+1)B = A/p.$$

Eliminating B gives

$$r^2A'' + (2p+3)rA' + p(p+2)A = 0.$$

Its indicial polynomial is $(z+p)(z+p+2)$. Both powers are negative, so regularity gives $A = B = C = 0$. ☐

19.3. **The boundary residual for positive x.**

Proposition 19.3. *The boundary system* (18.7) *has no nonzero smooth solution when $x > 0$.*

Proof. The normal block of (18.7) is divergence-free and has trace $qv_x Y$. A particular solution is

$$T^{\mathrm{part}}_{AB} = -\frac{q}{px}\nabla_A\nabla_B(v_x Y) + \frac{q}{p}v_x Y\,\delta_{AB} + \frac{m-1}{x}\nabla_A\nabla_B(r^\ell Y). \tag{19.6}$$

The trace and divergence identities follow from $\Delta_{\mathrm{e}}(v_x Y) = xv_x Y$ and $\Delta_{\mathrm{e}}(r^\ell Y) = 0$. Every other scalar-type solution differs by a multiple of Z_x when $\ell \geq 2$, and agrees with (19.6) when $\ell = 0, 1$.

For $\ell = 0, 1$, the boundary coefficient of $Y\gamma_p$ is

$$C_\ell(x) = \frac{q}{p} - \frac{qz_\ell(x)}{px}.$$

The required value is -1. But (19.3) gives

$$C_\ell(x) + 1 \geq \frac{m}{p} - \frac{q}{p(2\ell + p + 1)} > 0. \tag{19.7}$$

Let $\ell \geq 2$. The trace-free tangential coefficient of (19.6) is c/x, where $c = (p-1)m/p$. The zero trace-free boundary condition therefore forces addition of $-cZ_x/(xd_\ell(x))$. The residual in the scalar tangential boundary equation is

$$C_\ell(x) + 1 = \frac{m}{px}\left[x - \frac{q}{m}z - \frac{\ell(\ell-1)(x + (p-1)z)}{K + pz + px/(p-1)}\right]. \tag{19.8}$$

Put $a_\ell = 2\ell + p + 1$ and $D_\ell = K + pz + px/(p-1)$. Using $z \leq x/a_\ell$ gives

$$\begin{aligned} C_\ell(x) + 1 &\geq \frac{m}{pa_\ell D_\ell}\left[(a_\ell - q/m)D_\ell - 2(\ell+p)\ell(\ell-1)\right] \\ &\geq \frac{m(\ell+p)(3\ell+p-1)}{a_\ell D_\ell} > 0. \end{aligned} \tag{19.9}$$

The second inequality uses $q/m < 1$, $D_\ell \geq K$, and the identity

$$(2\ell + p)K - 2(\ell+p)\ell(\ell-1) = p(\ell+p)(3\ell+p-1).$$

Thus the boundary equation cannot hold for $x > 0$. ☐

19.4. **The value $x = 0$.**

Proposition 19.4. *The boundary system* (18.7) *has no nonzero smooth solution when $x = 0$.*

Proof. Put $f = r^\ell Y$ and $c = (m-1) - q/p = (p-1)m/p$. The normalized radial solution v_x is analytic at $x = 0$ by Lemma 19.1. The tensor Z_x and its boundary coefficient $d_\ell(x)$ are consequently analytic there. For $\ell \geq 2$, the formulas at zero give

$$Z_0 = \frac{\nabla^2 f}{\ell(\ell-1)}, \qquad d_\ell(0) = \frac{1}{\ell(\ell-1)}, \qquad xT^{\mathrm{part}}\big|_{x=0} = c\nabla^2 f.$$

Thus the analytic tensor $xT^{\mathrm{part}} - cZ_x/d_\ell(x)$ vanishes at $x = 0$. It equals xU_x for an analytic tensor family U_x, for example by integrating its x derivative from 0 to x. For $x > 0$,

$$U_x = T^{\mathrm{part}} - \frac{c}{xd_\ell(x)}Z_x.$$

Passage to $x = 0$ in the component equations shows that U_0 solves the zero-parameter interior system, has the prescribed trace, and has zero trace-free tangential boundary value. It is the unique regular scalar-type solution with these properties: the difference of two such solutions is a multiple of Z_0, and its trace-free boundary coefficient is that multiple times $d_\ell(0) \neq 0$.

For $\ell = 0, 1$, $\nabla^2 f = 0$ and $\nabla^2(v_x Y)|_{x=0} = 0$. The numerator of the apparent pole in (19.6) therefore vanishes at zero. The same integral argument removes it. Uniqueness follows from the absence of a regular TT homogeneous solution in these degrees, proved in Lemma 19.2.

For $\ell = 0, 1$, (19.7) has a strictly positive limit. For $\ell \geq 2$, use $z_\ell(0) = 0$ and $z'_\ell(0) = (2\ell + p + 1)^{-1}$ in (19.8). The remaining scalar boundary residual at zero is

$$\frac{3\ell p + 3\ell q + \ell + p^2 + pq - q - 1}{(\ell + p - 1)(2\ell + p + 1)} > 0.$$

Its numerator equals

$$(3\ell + p - 1)(p + q) + (\ell + p - 1) > 0 \qquad (p, q \geq 2,\ \ell \geq 2).$$

Thus the final boundary equation cannot hold at $x = 0$. □

Proof of Theorem 18.1. The realization in Section 18 is lower bounded and has compact resolvent. Every nonpositive eigenvector would be a smooth solution of (18.7). Propositions 19.3 and 19.4 exclude all such solutions. The lowest eigenvalue is therefore positive, which proves (18.1). If $q_0 + C\mathfrak{m}_0 \geq c\|\cdot\|^2_{\mathscr{P}_0}$ is (18.6) and $q_0 \geq \lambda_0 \mathfrak{m}_0$ with $\lambda_0 > 0$, then

$$q_0 \geq \frac{c}{1 + C/\lambda_0}\|\cdot\|^2_{\mathscr{P}_0}. \tag{19.10}$$

Thus the gap is also coercive in the compatible-pair form norm. □

20. The fixed-representation limit at the singular orbit

Let g satisfy (1.1) and (1.2) near its smooth singular orbit. In this section $p, q \geq 2$ are unrestricted. For an irreducible real representation ϖ of $\mathfrak{G}$, the ϖ-isotypic subspace is the closed sum of all subrepresentations isomorphic to ϖ.

Theorem 20.1 (Initial positivity in a fixed representation). *For each ϖ there is $R_\varpi > 0$ such that*

$$q_R(T) > 0 \qquad (0 < R < R_\varpi, \quad 0 \neq T \in \mathcal{D}(q_R)_\varpi). \tag{20.1}$$

No uniformity in ϖ is asserted.

Proof. Pull M_R to $B_1^N \times S^q$ by $t = Rr$, and put

$$\widehat{g}_R = R^{-2}g = dr^2 + \left(\frac{a(Rr)}{R}\right)^2 \gamma_p + \left(\frac{b(Rr)}{R}\right)^2 \gamma_q.$$

Multiplication of the volume density by $(R/b_0)^q$ does not change the form index or generalized eigenvalues. In Cartesian normal variables its resulting density is

$$\left(\frac{a(Rr)}{Rr}\right)^p \left(\frac{b(Rr)}{b_0}\right)^q dx\, dv_{\gamma_q} = (1 + O(R^2))dx\, dv_{\gamma_q}. \tag{20.2}$$

Smoothness at the singular orbit makes this an estimate with any fixed finite number of derivatives.

Identify the tangent bundle isometrically with $E = TB^N \oplus TS^q$ equipped with its fixed product fibre metric, using the positive square root in the normal block and multiplication by $R/b(Rr)$ in the vertical block. The normal metric coefficients differ from the Euclidean coefficients by $O(R^2)$ in Cartesian coordinates. In this identification,

$$\nabla^{(R)}_A = \partial_A + O(R^2)\partial_x + O(R^2), \qquad \nabla^{(R)}_I = \frac{R}{b_0}\nabla^{S^q}_I + O(R^2), \qquad |\mathrm{Rm}_{\widehat{g}_R}| = O(R^2). \tag{20.3}$$

For example, the mixed connection coefficient is $\partial_r \log b(Rr) = Rb'(Rr)/b(Rr) = O(R^2)$; the other statements follow by differentiating the displayed metric. Write $\widehat{A}_R$ and $\widehat{H}_R$ for the second fundamental form and mean curvature of $\widehat{g}_R$ at $r = 1$. Then

$$\widehat{A}_R = \mathrm{diag}(1 \cdot \gamma_p, 0 \cdot \gamma_q) + O(R^2), \quad \widehat{H}_R = p + O(R^2), \quad \nabla_\nu \widehat{A}_R = \mathrm{diag}(-\gamma_p, 0) + O(R^2). \tag{20.4}$$

There is no assertion that $\widehat{g}_R$ converges as a nondegenerate metric on the whole product. Under a constant metric scaling $\widetilde{g} = c^2 g$, the tangent scaling $T \mapsto c^2 T$ satisfies

$$q_{\widetilde{g}}(c^2 T) = c^{d-2} q_g(T), \qquad \|c^2 T\|^2_{L^2(\widetilde{g})} = c^d \|T\|^2_{L^2(g)}. \tag{20.5}$$

Indeed $f_{c^2T,\widetilde{g}} = f_{T,g}$, the Levi–Civita connection is unchanged, and each contraction in the operator contains one inverse metric. Multiplying both integrals by the positive normalization factor in (20.2) leaves their signs and generalized Rayleigh quotients unchanged.

20.1. **Bounded angular derivatives.** Fix an irreducible $\mathfrak{G}$-type ϖ. For each of the finitely many tensor bundles used below, its angular isotypic space on $S^p \times S^q$ is finite-dimensional. For the multiplicity bound, evaluate an equivariant map from the finite-dimensional representation at one point of the homogeneous space; its values are constrained by the isotropy representation in the finite-rank fibre. Differentiation in S^q therefore satisfies

$$\|\nabla^j_{S^q} U\|_{L^2} \le C_{\varpi,j}\|U\|_{L^2} \tag{20.6}$$

for every fixed j. Normal derivatives retain the prescribed $O(q+1)$ type. For each normal derivative order, apply the same finite-dimensional estimate to the resulting tensor bundle. Hence, for integer $s \ge 0$, the anisotropic norm

$$\|U\|^2_{s,B} = \sum_{j=0}^{s} \int_{B\times S^q} |\nabla^j_B U|^2$$

is equivalent on this fixed type to the full reference-product H^s norm. The boundary norms are obtained in the same manner. Constants may depend on ϖ and s, but not on R.

By (20.3), the first-order operators δ_R and $\mathcal{K}_R$ converge, in $H^1_\varpi \to L^2_\varpi$ norm, to the operators δ_0 and $\mathcal{K}_0$ which differentiate only in B^N. Their second-order mixed and Dirichlet operators converge in $H^2_\varpi \to L^2_\varpi$ norm. Their boundary operators converge in the corresponding trace norms. Every difference is $O_\varpi(R)$. The limiting operator acts fibrewise over S^q as the product-model operator in the preceding sections, with the vertical space T_yS^q in place of $\mathbb{R}^q$.

20.2. **Uniform inverses and exact constraints.** For the limiting mixed vector problem, (18.3) holds pointwise over S^q. Integrating it and using (20.6) gives coercivity in H^1_ϖ. Equation (20.3) permits absorption of its $O_\varpi(R)$ perturbation, proving the same estimate uniformly for small R. The full Dirichlet and tangential-Dirichlet/normal-traction inverses consequently converge in norm. More explicitly, regard a second-order operator together with its boundary operator as a map

$$\mathfrak{L}_R : H^2_\varpi \longrightarrow L^2_\varpi \oplus H^{3/2}_\varpi(\text{Dirichlet data}) \oplus H^{1/2}_\varpi(\text{traction data}).$$

The limiting constant-coefficient system on the normal ball is elliptic and invertible, by the energy and boundary-symbol verification in Section 18. It has a bounded inverse. Since $\|\mathfrak{L}_R - \mathfrak{L}_0\| = O_\varpi(R)$,

$$\mathfrak{L}_R^{-1} = \mathfrak{L}_0^{-1}\left[I + (\mathfrak{L}_R - \mathfrak{L}_0)\mathfrak{L}_0^{-1}\right]^{-1}$$

converges by its Neumann series. For $H^{1/2}$ Dirichlet data, subtract the solutions with the same boundary trace. Their difference has zero trace and satisfies the variational difference equation in Section E.4. Coercivity gives (E.22), for both the vector Dirichlet solution and the scalar harmonic extension.

The divergence constraint can be trivialized without an approximate projection. Let $\mathscr{V}_R$ consist of pairs $(S,\phi) \in H^1_\varpi \times H^1_\varpi(\partial)$ satisfying

$$\operatorname{tr} S = 0, \qquad S_{\nu\nu} = 0, \qquad S^\top = -\phi(\widehat{A}_R)_0.$$

A fixed normal-collar extension of the boundary tensor

$$\nu^\flat \otimes \eta + \eta \otimes \nu^\flat - \phi(\widehat{A}_R)_0$$

identifies these spaces by norm-continuous bounded maps. Average the extension over $\mathfrak{G}$; its boundary trace remains the identity and its operator bounds remain finite. All resulting maps preserve the specified isotypic subspace. For $h \in L^2_\varpi(E)$ solve

$$\delta_R\mathcal{K}_R\omega = h, \qquad \omega^\top = 0, \qquad (\mathcal{K}_R\omega)_{\nu\nu} = 0,$$

and put

$$B_R h = (\mathcal{K}_R\omega, -\omega_\nu|_\partial), \qquad C_R(S,\phi) = \delta_R S.$$

The boundary divergence formula gives $(\mathcal{K}_R\omega)^\top = \omega_\nu(\widehat{A}_R)_0$, hence $C_RB_R = I$. The inverse estimates just proved show norm convergence of B_R. Thus

$$P_R = I - B_RC_R$$

is a norm-continuous family of projections, including at $R = 0$. The operator

$$U_R = P_RP_0 + (I - P_R)(I - P_0) = I + (P_R - P_0)(2P_0 - I)$$

is invertible for small R and maps $\ker C_0$ onto $\ker C_R$. Smoothing the free interior and boundary data, followed by P_R, proves that these kernels are exactly the completed smooth compatible-pair domains.

20.3. **Convergence of the forms.** Pull the rescaled form and mass to the fixed space $\mathscr{P}_{0,\varpi} = \ker C_0$ using U_R. The mixed boundary identity (18.5), with its R-dependent coefficients, gives

$$\|b_R - b_0\|_{\mathcal{L}_{\mathrm{sym}}(\mathscr{P}_{0,\varpi},\mathscr{P}^*_{0,\varpi})} \longrightarrow 0. \tag{20.7}$$

Indeed the interior derivative coefficients and densities converge by (20.2)–(20.3); the boundary shape coefficients by (20.4); the boundary derivatives by (20.6); and both the scalar Dirichlet-to-Neumann and vector Dirichlet solution operators by the preceding inverse identities. For $v, w \in \mathscr{P}_{0,\varpi}$ the contributions satisfy

$$\begin{aligned}
|(b_R - b_0)_{\mathrm{int}}(v, w)| &\le C_\varpi R\|v\|_{\mathscr{P}_{0,\varpi}}\|w\|_{\mathscr{P}_{0,\varpi}},\\
|(b_R - b_0)_{\mathrm{boundary}}(v, w)| &\le C_\varpi R\|v\|_{\mathscr{P}_{0,\varpi}}\|w\|_{\mathscr{P}_{0,\varpi}},\\
|\langle\phi, (\mathcal{N}_R - \mathcal{N}_0)\psi\rangle| &\le C_\varpi R\|\phi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}.
\end{aligned}$$

The first two bounds use the $H^1 \to H^{1/2}$ trace map, the coefficient estimates, and $U_R - I = O_\varpi(R)$. For the third bound, the weak scalar Dirichlet inverse gives $\|\mathcal{H}_R - \mathcal{H}_0\|_{H^{1/2}\to H^1} = O_\varpi(R)$ for harmonic extension. Insert these extensions in the polarized Dirichlet-energy identity defining $\mathcal{N}_R$. The triangle inequality and the uniform energy bound give the displayed estimate. For each of these contributions, and for the mass form, Lemma E.3 proves the full estimate (E.23) on a specified common compatible-pair space. The inverse estimates, the exact projections and every boundary term are given in Section E.4. This proves convergence in bilinear-form norm, not only coefficient convergence on individual smooth tests.

The limiting form is the integral over S^q of the product-model form. Its energy is unchanged under orthogonal changes of frame in T_yS^q, so this description is globally well-defined. By Theorem 18.1, $b_0 \ge c_{p,q}\mathfrak{m}_0$. Combining this with (18.6) and (20.6) gives

$$b_0(v, v) \ge c_\varpi\|v\|^2_{\mathscr{P}_{0,\varpi}}$$

for some $c_\varpi > 0$. Equation (20.7) gives $b_R(v, v) \ge (c_\varpi/2)\|v\|^2$ for all sufficiently small R. Normal rescaling and the density normalization multiply forms and masses by positive factors; they do not change the sign index. This proves Theorem 20.1. ☐

21. The initial index and the full negative spectrum

We return to $p, q \ge 2$ and $p + q \le 8$. All operators and group actions in this section are those defined in Section 1. Let $\mathcal{P}$ be normalized averaging under $\mathfrak{G}$. Isometric pullback preserves the closed cap forms and the complete form and commutes with their resolvents. It therefore commutes with every spectral projection. The invariant subspace and its orthogonal complement are reducing subspaces.

Proposition 21.1. *The negative-threshold equality in Theorem B holds separately on* $\operatorname{ran}\mathcal{P}$ *and on* $\ker\mathcal{P}$.

Proof. The mixed vector problem, the scalar Dirichlet projection, the boundary harmonic extension, and the radial cutoff functions commute with $\mathcal{P}$. The comparison $T \mapsto S$ in (12.15) therefore takes averaging-zero tensors to averaging-zero tensors, and invariant tensors to invariant tensors. The same upper-bound proof compares with the corresponding restriction of the tensor Neumann form. In the Neumann compactness argument, a strongly L^2 limit retains either averaging condition, since

$\mathcal{P}$ is a bounded orthogonal projection. The lower bound uses cutoff global eigentensors in the chosen subspace, followed by the same equivariant conformal and York projections. Every dimension inequality in the proof of Theorem B consequently holds in each reducing subspace. The resulting equalities are the asserted ones. □

Theorem 21.2 (Vanishing of the non-invariant index). *For every $R > 0$,*

$$\kappa_\perp = 0, \qquad q_R(T) \ge 0 \quad (T \in \mathcal{D}(q_R),\ \mathcal{P}T = 0). \tag{21.1}$$

Every negative cap eigentensor is invariant. Moreover,

$$\operatorname{ran} \mathbf{1}_{(-\infty,0)}(L_X) \subset L^2(S^2T^*X)^{\mathfrak{G}}. \tag{21.2}$$

Proof. Fix a nontrivial irreducible real representation ϖ of $\mathfrak{G}$. The constructions of Theorem 14.2 commute with $\mathfrak{G}$. They restrict to local Hilbert form charts on the ϖ-isotypic subspaces. In such a chart the form is represented by a bounded self-adjoint Fredholm operator: its shift by a sufficiently large multiple of the compact mass form is coercive.

Its radical is zero at every radius. Indeed Proposition 13.2 identifies it with the ϖ-part of K_R, and Theorem A makes that part zero. A self-adjoint Fredholm operator with zero kernel is invertible. Its negative index is locally constant: if its inverse norm is C, a perturbation of operator norm smaller than $1/C$ remains invertible; a homotopy in this ball has no zero eigenvalue, and its finite-dimensional negative spectral projection has constant rank. Connectedness of $(0, \infty)$ makes this index independent of R.

Theorem 20.1 gives zero index for this fixed ϖ at small positive radii. Therefore the ϖ-index is zero at every radius. This argument fixes ϖ before sending R to zero.

If a cap eigenspace at a negative eigenvalue had a non-invariant vector, its finite dimension and compact-group invariance would give a nontrivial irreducible summand. The form is negative definite on that summand, contradicting the preceding conclusion. Thus every negative cap eigentensor is invariant. The spectral theorem for the compact-resolvent operator now gives (21.1) on its entire closed form domain.

For $\varepsilon > 0$ with $-\varepsilon \notin \sigma(L_X)$, Proposition 21.1 on $\ker \mathcal{P}$ gives

$$\dim \operatorname{ran} \mathbf{1}_{(-\infty,-\varepsilon)}(L_X|_{\ker \mathcal{P}}) = \#\{\lambda(A_R|_{\ker \mathcal{P}}) < -\varepsilon\} = 0$$

for all sufficiently large R. Choose a sequence of such ε decreasing to zero. The associated spectral projections increase strongly to $\mathbf{1}_{(-\infty,0)}(L_X|_{\ker \mathcal{P}})$. Each is zero; hence their limit is zero. This proves (21.2). □

Corollary 21.3 (Complete simplicity). *Every negative L^2 eigenvalue of L_X has a one-dimensional full eigenspace. The spectrum in $(-\infty, 0)$ is an infinite sequence of simple eigenvalues accumulating only at zero.*

Proof. Every negative full eigenspace is invariant by Theorem 21.2. Angenent–Knopf's invariant-sector simplicity theorem [6, Main Theorem II(f)], with the opposite operator sign, gives its dimension one. Discreteness and the location of accumulation follow from Proposition 15.1; infinitude is Corollary 15.3. □

Theorem 21.4 (Absolute fold count). *For every $R > 0$,*

$$\operatorname{ind}^- A_R = \#\{t \in (0, R) \,:\, u(t) = v(t)\}. \tag{21.3}$$

In particular $\operatorname{ind}^- A_R = 0$ *for all sufficiently small positive R. If R is umbilic, its one-dimensional zero eigenspace is not included in* (21.3).

Proof. Apply Theorem 20.1 to the trivial representation. Its restricted form has zero negative index on some interval $(0, R_0)$. Every other representation has zero negative index at every radius by Theorem 21.2. Thus the full initial index is zero on $(0, R_0)$; a radius uniform over the nontrivial representations is not needed.

The singular orbit expansion gives $u - v = t^{-1} + O(t) > 0$ near zero. Each later zero is simple by Proposition 9.1, and the zeros cannot accumulate in any compact subinterval of $(0, \infty)$. For a nonumbilic endpoint R, choose $a \in (0, \min\{R_0, R\})$ before the first zero and apply (14.6) on $[a, R]$. This gives (21.3).

At an umbilic endpoint, the complementary form to the one-dimensional kernel is invertible. Its negative index is constant nearby. The scalar Schur complement vanishes at the crossing and has negative derivative by theorem C. It is positive immediately before the crossing, zero at the crossing, and negative immediately after it. The negative index at the crossing therefore equals its left limit, proving the strict endpoint convention. □

Proof of Theorem D. The local form statement is Theorem 14.2. Theorems 21.2 and 21.4 give invariance and the absolute count; Corollary 21.3 gives complete simplicity. The logarithmic asymptotic follows from (14.8). □

Remark 21.5. The index in (21.3) is the index of the same closed form used in the spectral exhaustion theorem. No estimate for componentwise tensor-Dirichlet data is substituted. The geometric folds are local in the boundary-marked moduli space. Neither a bound of two on the global number of fillings nor a fold theorem for arbitrary Einstein manifolds follows from these statements.

22. Open sets of boundary data with many fillings

For a compact Ricci-flat metric h, let $m_{\mathrm{red}}(h)$ be the variational negative index of its scalar-flat, fixed-induced-metric Einstein–Hilbert Hessian, modulo boundary-fixing diffeomorphisms. This definition is the supremum of the dimensions of negative-definite subspaces. It does not require a spectral realization for a nearby metric with no symmetry.

Theorem 22.1. *Set*

$$M = D^{p+1} \times S^q, \qquad \Sigma = S^p \times S^q, \qquad \gamma_* = (p-1)\gamma_p + (q-1)\gamma_q.$$

For every pair of positive integers N, k, there is a C^∞-open neighborhood $\mathcal{U}_{N,k}$ of γ_* such that every $\gamma \in \mathcal{U}_{N,k}$ has at least N pairwise nonisometric smooth Ricci-flat fillings $h_1(\gamma), \dots, h_N(\gamma)$ on M, satisfying

$$h_i(\gamma)^\top = \gamma, \qquad A_{h_i(\gamma)} > 0, \qquad m_{\mathrm{red}}(h_i(\gamma)) \geq k. \tag{22.1}$$

These finite collections of branches depend smoothly on the boundary data in local smooth boundary-moduli charts. The neighborhood contains a nonempty open subset of metrics with trivial isometry group. For every boundary metric in that subset, all the displayed fillings also have trivial isometry group.

Lemma 22.2. *There are parameters $s_j \to \infty$ and radii $R_j \to \infty$ for which the normalized caps*

$$\widehat{g}_j = e^{-2s_j} g|_{M_{R_j}} \tag{22.2}$$

satisfy

$$\widehat{g}_j^\top = \gamma_*, \qquad K(\widehat{g}_j) = 0, \qquad A_{\widehat{g}_j} \longrightarrow \kappa_* \gamma_*, \quad \kappa_* = (m-1)^{-1/2}.$$

Their boundary mean curvatures are pairwise distinct. Their reduced negative indices tend to infinity.

Proof. Use s, z, w from the autonomous system of Lemma 3.3. The polar angle in Lemma 3.3 is eventually strictly increasing and unbounded. The line $z = 0$ corresponds under the fixed invertible linear change of variables to arguments $\vartheta_1 + \pi\mathbb{Z}$. Choose the successive parameters $s_j \to \infty$ attaining those arguments and put $R_j = t(s_j)$. Then $z(s_j) = 0$. The normalized boundary metric is then γ_*. If $w(s_j) = 0$, the trajectory reaches the equilibrium $(z, w) = (0, 0)$ at finite time; uniqueness would make it stationary, contrary to the singular orbit behavior. Thus $u(R_j) \neq v(R_j)$ and Theorem A gives $K(\widehat{g}_j) = 0$.

Write $N_s = dt/ds$. The Ricci-flat constraint gives $N_s^2 = e^{2s} K(w)/V(z)$. The two normalized principal curvatures at $z = 0$ are

$$\sqrt{\frac{V(0)}{K(w)}} \left(1 + \frac{q}{m} w\right), \qquad \sqrt{\frac{V(0)}{K(w)}} \left(1 - \frac{p}{m} w\right).$$

Since $w(s_j) \to 0$, both tend to κ_*. The normalized boundary mean curvature is

$$\widehat{H}(s) = m\sqrt{\frac{V(z(s))}{K(w(s))}}.$$

The autonomous equations give

$$\frac{d}{ds}\log \widehat{H}(s) = -\frac{pq}{m^2}w(s)^2. \tag{22.3}$$

The integral of w^2 on any nontrivial interval is positive; otherwise the trajectory is stationary on that interval and hence everywhere. Therefore $\widehat{H}$ is strictly decreasing, proving distinctness.

The index divergence follows from (14.8). It is unchanged by constant scaling: the negative Einstein–Hilbert action with its boundary term scales by $c^{d-2} > 0$ under $g \mapsto c^2 g$. This multiplication preserves the signature of its reduced Hessian and its boundary-fixing gauge quotient. □

Lemma 22.3. *For every sufficiently large j, the induced-metric map at $[\widehat{g}_j]$ has a local smooth inverse on all smooth boundary metrics near γ_*. The target is not restricted to invariant metrics.*

Proof. Consider the elliptic boundary coordinates

$$\mathcal{B}_\partial(h) = ([h^\top], H_h^2 - |A_h|^2), \qquad \Theta(\gamma) = ([\gamma], R_\gamma)$$

from Section 9. Fix a Hölder exponent $0 < \alpha < 1$ and an integer $r \geq 3$. The map $\mathcal{B}_\partial$ is smooth from the $C^{r,\alpha}$ Ricci-flat moduli space to $\mathrm{Conf}^{r,\alpha}(\Sigma) \times C^{r-1,\alpha}(\Sigma)$.

The highest-order part of its scalar boundary derivative is

$$2\langle H\gamma - A, A'(h)\rangle.$$

At the normalized caps, its coefficient tends to $2(m-1)\kappa_*\gamma_*$. The boundary metric is exactly γ_*, and both normalized principal curvatures tend to κ_*. The principal boundary problem is therefore a small perturbation of the conformal-class/mean-curvature problem, with its scalar row multiplied by $2(m-1)\kappa_*$. The complementing condition is open in its coefficients. A straight homotopy of these scalar principal rows stays in that open set for sufficiently large j. The interior principal operator is unchanged in this homotopy. The boundary estimates [1] give a continuous path of Fredholm operators, so its index agrees with the index zero of the mixed problem [4]. These are boundary-symbol assertions on each fixed cap; they require no uniform curvature bound at the normalized singular orbit.

One has $R_{\gamma_*} = m$. On pure conformal boundary variations, the scalar derivative of Θ is an invertible multiple of

$$\Delta_{\gamma_*} - \frac{m}{m-1}.$$

Indeed

$$\lambda_1(\gamma_*) = \min\left\{\frac{p}{p-1}, \frac{q}{q-1}\right\} > \frac{m}{m-1}, \tag{22.4}$$

and the constant eigenvalue is zero. The boundary coordinate lemma consequently applies. The identity $\mathcal{B}_\partial = \Theta \circ \Pi_D$ shows that the smooth kernel of $D\mathcal{B}_\partial$ is $K(\widehat{g}_j) = 0$. Its weak kernel is smooth by elliptic boundary regularity. Index zero therefore makes its differential an isomorphism.

Apply Proposition 9.4 to the full unbordered operator (9.8). Its derivative is invertible by the preceding kernel and index calculation. Set the interior and Bianchi data to zero and substitute the boundary data $\Theta(\gamma)$. The inverse gives a smooth Ricci-flat metric; the scalar coordinate inverse identifies its induced metric with γ. For a fixed low order $r_0 \geq 4$, smallness in $C^{r_0+1,\alpha}$ of $\gamma - \gamma_*$ defines the required C^∞-open neighborhood. At every higher order the derivatives of the filling map satisfy (E.32); the map has the displayed one-derivative shift from boundary metric to interior metric. All higher-regularity inverses agree by uniqueness in the first Banach chart. Constants may depend on this fixed normalized cap, and no uniformity in j is asserted. □

Lemma 22.4. *If $m_{\mathrm{red}}(h_0) \geq k$ at a compact Ricci-flat metric h_0, then $m_{\mathrm{red}}(h) \geq k$ for every sufficiently nearby smooth Ricci-flat metric on the same manifold.*

Proof. By the definition of the variational index, choose smooth representatives $v_1, \dots, v_k$ of a reduced negative subspace at h_0, with $v_i^\top = 0$. For a nearby Ricci-flat metric h, define

$$P_h v = v - \frac{1}{d-1}\Delta_{h,D}^{-1}(DR_h(v))h.$$

The Dirichlet solution vanishes on the boundary, so $(P_h v)^\top = 0$. The identity $DR_h(fh) = (d-1)\Delta_h f$ gives $DR_h(P_h v) = 0$. In fixed charts the scalar Dirichlet inverse depends continuously on the coefficients, by its inverse identity and elliptic estimates. Hence the matrix

$$\left(B_h(P_h v_i, P_h v_j)\right)_{1\le i,j\le k}$$

is continuous in h. Denote this matrix by $C(h)$. Choose $\varepsilon > 0$ such that

$$c^T C(h_0) c \le -2\varepsilon |c|^2 \qquad (c \in \mathbb{R}^k).$$

Restrict the neighborhood until $\|C(h) - C(h_0)\|_{\mathrm{op}} < \varepsilon$. Then $c^T C(h) c \le -\varepsilon |c|^2$ for every c. In particular, the vectors $P_h v_i$ are independent. Their span maps injectively to the quotient by boundary-fixing gauge, since a pure gauge vector belongs to the radical. Thus the variational index is at least k; no spectral realization for the nonsymmetric metric is required. □

Proof of Theorem 22.1. By Lemma 22.2, choose N sufficiently late caps with index at least k. For each cap, Lemma 22.3 gives a local filling branch over a neighborhood of γ_*. Shrink these finitely many neighborhoods so that all second fundamental forms stay positive and the index bound persists by Lemma 22.4.

At the chosen caps the real numbers

$$\mathcal{J}(h) = \int_\Sigma H_h \, dA_{h^\top}$$

are distinct, because their common boundary metric is γ_* and their mean curvatures are distinct. Choose pairwise disjoint intervals around these N numbers. The functional $\mathcal{J}$ is continuous on smooth metrics, so each branch can be restricted further until its values remain in the designated interval. Isometries of compact manifolds with boundary preserve $\mathcal{J}$. Hence no two of these fillings are isometric, even after their boundary markings are forgotten. The finite intersection of the resulting neighborhoods is $\mathcal{U}_{N,k}$.

Metrics with trivial isometry group form an open dense set in the smooth metrics on the closed manifold Σ, by the Riemannian genericity theorem recalled in [13]. Since $\dim \Sigma \ge 4$, the dimension hypothesis is satisfied. Its intersection with $\mathcal{U}_{N,k}$ contains a nonempty open set. Every isometry of a filling restricts to a boundary isometry. If this restriction is the identity, its derivative fixes the tangent space and also the inward unit normal, so its derivative is the identity at every boundary point. The isometry then fixes all sufficiently short inward normal geodesics, and hence a collar. At an interior point of that collar its value and derivative determine it along geodesics; connectedness gives the identity everywhere. Thus restriction to the boundary is injective, and trivial boundary isometry group implies trivial filling isometry group. □

Remark 22.5. The quantifiers in Theorem 22.1 are $\forall N, k\ \exists \mathcal{U}_{N,k}\ \forall \gamma \in \mathcal{U}_{N,k}$. No single open neighborhood carrying infinitely many simultaneous branches is asserted. The symmetric infinite fibre at γ_* is already part of the phenomenon discussed in [11, Section 6]; the conclusion here is persistence of each arbitrarily large finite collection, together with arbitrarily large reduced index, under arbitrary sufficiently small boundary perturbations.

III. Symmetry on the complete metric

23. Homogeneous energy and a coercivity criterion

Fix the scale of the complete metric (1.1). In the remainder of the memoir, t is the distance to the singular orbit and τ is a time variable. Put

$$\rho = (1+t^2)^{1/2}, \qquad \|h\|_{\mathscr{E}}^2 = \int_X \left(|\nabla h|^2 + \rho^{-2}|h|^2\right) dv_g. \tag{23.1}$$

The function t^2 is smooth at the singular orbit, so ρ is a smooth proper invariant function with $|d\rho| \le 1$. Let $\mathscr{E}$ be the completion of $C_c^\infty(S^2T^*X)$ in (23.1), and let $\mathscr{E}_\perp = \ker \mathcal{P} \cap \mathscr{E}$. Conical convergence and compactness of the interior give

$$|\operatorname{Rm}_g| \le C_g \rho^{-2}. \tag{23.2}$$

The form

$$Q(h,k) = \int_X \left(\langle \nabla h, \nabla k\rangle - 2\langle \mathring{R}h, k\rangle\right) dv_g$$

is consequently bounded on $\mathscr{E}\times\mathscr{E}$. On each compact set the energy norm controls the H^1 norm. Group averaging is an orthogonal projection for the energy inner product as well as for the L^2 inner product. On the energy dual, averaging is defined by $\langle \mathcal{P}\ell, v\rangle = \langle \ell, \mathcal{P}v\rangle$. A functional on $\mathscr{E}_\perp$ is extended to $\mathscr{E}$ by composition with $I-\mathcal{P}$; this identifies $\mathscr{E}_\perp^*$ with the averaging-zero functionals.

Lemma 23.1. *Every smooth tensor with finite norm* (23.1) *belongs to* $\mathscr{E}$. *Multiplication by a smooth bounded function with* $|d\chi| \le C\rho^{-1}$ *is bounded on* $\mathscr{E}$. *If* $\chi^2+\eta^2 = 1$, *then*

$$Q(h) = Q(\chi h) + Q(\eta h) - \int_X (|d\chi|^2 + |d\eta|^2)|h|^2\, dv_g. \tag{23.3}$$

The identity holds whenever the multiplications are bounded on $\mathscr{E}$.

Proof. Choose $\chi_R = \chi(\rho/R)$, where $\chi = 1$ on $[0,1]$, $\chi = 0$ on $[2,\infty)$, and $|\chi'| \le C$. Then

$$\|(1-\chi_R)h\|_{\mathscr{E}}^2 \le C\int_{\{\rho>R\}} (|\nabla h|^2 + \rho^{-2}|h|^2)\, dv_g \longrightarrow 0.$$

A compactly supported H^1 tensor is approximated by smooth tensors using convolution in a finite coordinate cover and a partition of unity. This proves the first assertion, also for locally H^1 tensors with finite energy. The product rule proves the multiplier assertion. Expanding the two gradient squares in the right side of (23.3), the mixed terms cancel because $\chi\, d\chi + \eta\, d\eta = 0$. The curvature terms use $\chi^2+\eta^2 = 1$. Approximation extends the identity from smooth compactly supported tensors. ☐

By Theorem 21.2, the spectral measure of L_X on $\ker \mathcal{P} \subset L^2$ is supported in $[0,\infty)$. Hence

$$Q(h) \ge 0 \qquad (h \in \mathscr{E}_\perp). \tag{23.4}$$

Indeed it holds first on compactly supported averaging-zero tensors, and then by energy completion. It implies

$$|Q(h,k)|^2 \le Q(h)Q(k) \qquad (h,k\in\mathscr{E}_\perp), \tag{23.5}$$

by considering $Q(h+sk)\ge 0$ for real s.

Proposition 23.2 (Coercivity criterion). *Suppose* (23.2) *and* (23.4) *hold. Suppose that for some invariant compact set* K *and* $c_\infty > 0$,

$$Q(h) \ge c_\infty \|h\|_{\mathscr{E}}^2 \quad (h \in C_c^\infty(X\setminus K; S^2T^*X),\ \mathcal{P}h = 0). \tag{23.6}$$

If

$$h\in\mathscr{E}_\perp, \quad Lh = 0 \text{ in distributions} \quad \Longrightarrow \quad h = 0, \tag{23.7}$$

then there is $c_g > 0$ *such that*

$$Q(h) \ge c_g\|h\|_{\mathscr{E}}^2 \qquad (h\in\mathscr{E}_\perp). \tag{23.8}$$

Conversely, (23.8) *implies* (23.7).

Proof. If (23.8) fails, choose $h_i \in C_c^\infty \cap \mathscr{E}_\perp$ with $\|h_i\|_{\mathscr{E}} = 1$ and $Q(h_i) \to 0$. A subsequence converges weakly in $\mathscr{E}$ to h and strongly in L^2 on each compact set. The latter follows from the compact embedding of H^1 into L^2 in finitely many relatively compact coordinate charts and a diagonal subsequence. Equation (23.5) implies $Q(h_i, \varphi) \to 0$ for every $\varphi \in C_c^\infty \cap \ker \mathcal{P}$. Boundedness of the form gives $Q(h, \varphi) = 0$. For an invariant test tensor the same pairing is zero by averaging. Thus $Lh = 0$ distributionally and $h = 0$ by (23.7).

Choose a smooth invariant function ϑ equal to zero on a neighborhood of K and equal to $\pi/2$ outside a larger compact set. Set $\chi = \cos\vartheta$ and $\eta = \sin\vartheta$, taking the transition in the exterior region of (23.6). Both multipliers preserve averaging-zero tensors. Equations (23.3), (23.4), and (23.6) give

$$c_\infty \|\eta h_i\|_{\mathscr{E}}^2 \le Q(h_i) + \int_X |d\vartheta|^2 |h_i|^2 \, dv_g \longrightarrow 0.$$

Local L^2 convergence then gives $\|\rho^{-1} h_i\|_2 \to 0$. By (23.2),

$$\|\nabla h_i\|_2^2 = Q(h_i) + 2\int_X \langle \mathring{R} h_i, h_i\rangle \, dv_g \longrightarrow 0,$$

contradicting $\|h_i\|_{\mathscr{E}} = 1$.

Conversely, let h satisfy the hypotheses in (23.7). Pair $Lh = 0$ first with compactly supported tests and then with energy approximations to h. This gives $Q(h) = 0$, and (23.8) gives $h = 0$. ☐

The proposition applies without a cohomogeneity assumption. Its proof requires a complete manifold, a compact isometric group action, a proper invariant weight with bounded gradient, the curvature bound, exterior coercivity, nonnegativity, and the stated distributional kernel condition. The next two sections verify its last two geometric hypotheses for (1.1).

24. A strict Hardy gap on the non-invariant cone tensors

Let

$$h_Z = \frac{p-1}{m-1}\gamma_p + \frac{q-1}{m-1}\gamma_q, \qquad g_C = dr^2 + r^2 h_Z, \qquad \alpha = \frac{m-1}{2}. \tag{24.1}$$

All angular bundles in this section are identified by radial parallel transport. An angular symmetric tensor is a triple in

$$\mathbb{R} \oplus T^*Z \oplus S^2T^*Z; \qquad (a, \omega, k) \longleftrightarrow a\, dr^2 + r(dr \otimes \omega + \omega \otimes dr) + r^2 k.$$

The notation $r^s\tau$ uses this identification, so its pointwise norm is $r^{\operatorname{Re} s}|\tau|$.

Lemma 24.1. *There are self-adjoint elliptic angular operators B_2 on the preceding bundle and B_1 on $\mathbb{R} \oplus T^*Z$ such that*

$$L_C = -\partial_r^2 - \frac{m}{r}\partial_r + r^{-2}B_2, \qquad \nabla_C^*\nabla_C|_{T^*C} = -\partial_r^2 - \frac{m}{r}\partial_r + r^{-2}B_1. \tag{24.2}$$

They have compact resolvent, commute with $\mathfrak{G}$, and $B_1 \ge 0$. There are angular maps P_0, Q_0, independent of s, for which

$$\delta_C(r^s\tau) = r^{s-1}(sP_0\tau + Q_0\tau). \tag{24.3}$$

Here $P_0\tau$ is minus contraction with the radial vector. If $P_0\tau = 0$ and $\tau = k$ is tangential, then $Q_0 k = (\operatorname{tr}_{h_Z} k, \delta_{h_Z} k)$.

Proof. Choose an h_Z-orthonormal frame $\bar{e}_A$ and put $e_0 = \partial_r$, $e_A = r^{-1}\bar{e}_A$. The cone connection satisfies

$$\nabla_{e_0} e_A = 0, \qquad \nabla_{e_A} e_0 = r^{-1} e_A, \qquad \nabla_{e_A} e_B = r^{-1}\sum_C \bar{\Gamma}_{AB}^C e_C - r^{-1}\delta_{AB} e_0.$$

Consequently radial differentiation of tensor components is ordinary differentiation, and angular differentiation is $r^{-1}\mathfrak{D}_A$ for a first-order operator $\mathfrak{D}_A$ on the angular bundle. The connection energy is the sum of the radial derivative square and $r^{-2}\sum_A |\mathfrak{D}_A\tau|^2$. Cone curvature is r^{-2} times an angular endomorphism. Polarization and integration with density $r^m\, dr\, dv_{h_Z}$ give (24.2). The one-form angular form is $\sum_A \|\mathfrak{D}_A\tau\|_2^2$, proving its nonnegativity. The angular principal symbols are $|\xi|^2\,\mathrm{Id}$. Compactness of Z gives self-adjoint elliptic realizations with compact resolvent. The displayed connection formulas also give (24.3): only the radial derivative differentiates r^s, with coefficient minus radial contraction. For a tangential tensor, the radial divergence component is $\operatorname{tr} k$, and the tangential component is δk. Every operation commutes with isometries of the link. $\square$

The cone indicial decomposition for general Einstein links is treated in [12]. The threshold exclusion needed here follows from the preceding formulas.

Lemma 24.2. *Every eigentensor $B_2\tau = \nu\tau$ with $\nu \le -\alpha^2$ is tangential TT. On tangential TT tensors,*

$$B_2 = L_{h_Z} - 2(m-1), \qquad L_{h_Z} = \nabla_{h_Z}^*\nabla_{h_Z} + 2(m-1) - 2\mathring{R}_{h_Z}. \tag{24.4}$$

Proof. Complexify the bundles. If $\nu < -\alpha^2$, put $b = \sqrt{-\alpha^2 - \nu}$ and $s_\pm = -\alpha \pm ib$. Equation (24.2) gives $L_C(r^{s_\pm}\tau) = 0$. By (2.3) and (24.3),

$$B_1(s_\pm P_0\tau + Q_0\tau) = (s_\pm - 1)(s_\pm + m - 2)(s_\pm P_0\tau + Q_0\tau).$$

The scalar coefficient equals $\nu + 1 \mp 2ib$, which is nonreal. Self-adjointness gives $s_\pm P_0\tau + Q_0\tau = 0$. Subtraction gives $P_0\tau = 0$, and then $Q_0\tau = 0$.

If $\nu = -\alpha^2$, both $r^{-\alpha}\tau$ and $r^{-\alpha}\log r\,\tau$ solve the cone equation. The divergence of the first would be a one-form eigenmode with angular eigenvalue $1 - \alpha^2 < 0$, unless $-\alpha P_0\tau + Q_0\tau = 0$. Nonnegativity of B_1 gives this equality. The divergence of the logarithmic solution is now $r^{-\alpha-1}P_0\tau$, with the same negative angular eigenvalue. Thus $P_0\tau = Q_0\tau = 0$. The last assertion of Lemma 24.1 proves the TT condition.

For a tangential TT tensor k, $\delta_C(r^s k) = 0$ for every s. Applying the divergence commutation to $L_C(r^s k)$ shows

$$(s-2)P_0(B_2k) + Q_0(B_2k) = 0 \qquad \text{for every } s.$$

Hence the tangential TT subspace is invariant. The angular derivative into the two radial tensor slots contributes $2|k|^2$ to the connection energy. On a tangential trace-free tensor the cone curvature action is $\mathring{R}_{h_Z}k + k$ at radius one. The two identity terms cancel in the Lichnerowicz form. The remaining angular form is $\langle(\nabla_{h_Z}^*\nabla_{h_Z} - 2\mathring{R}_{h_Z})k, k\rangle$. Polarization on the invariant subspace gives (24.4). $\square$

Lemma 24.3. *The kernel of L_{h_Z} on all symmetric tensors is the span of its two parallel factor metrics. Every nonzero eigenvalue is at least*

$$c_Z = \min\left\{\frac{(m-1)p}{p-1}, \frac{(m-1)q}{q-1}\right\} > m. \tag{24.5}$$

Moreover,

$$c_Z > \frac{(m-1)(9-m)}{4}. \tag{24.6}$$

Proof. The product connection preserves the two factor-trace subbundles, their trace-free complements, and the mixed subbundle. On the trace subbundles L_{h_Z} is the scalar product Laplacian on the coefficients. Its first positive eigenvalue is (24.5), from the spherical eigenvalues $\ell(\ell + p - 1)$ and $j(j + q - 1)$ and the two metric scales. Since $p, q < m$, each number in the minimum is greater than m.

Set $K_P = (m-1)/(p-1)$ and $K_Q = (m-1)/(q-1)$. For a trace-free tensor supported in the first factor the operator is $\nabla^*\nabla + 2pK_P$; for the second it is $\nabla^*\nabla + 2qK_Q$. On a mixed tensor it is $\nabla^*\nabla + 2(m-1)$. These terms are positive and at least c_Z, because $p/(p-1), q/(q-1) \le 2$. This proves the kernel and spectral assertions. For $5 \le m \le 8$ the right side of (24.6) is at most $m - 1$. For $m = 4$ one has $p = q = 2$, $c_Z = 6$, and the right side is $15/4$. $\square$

Proposition 24.4 (Exterior transverse coercivity). *There are $R_0, c_\infty > 0$ such that*

$$Q(h) \ge c_\infty\|h\|_{\mathscr{E}}^2 \quad (h \in C_c^\infty(\{t > R_0\}; S^2T^*X),\ \mathcal{P}h = 0). \tag{24.7}$$

Proof. If a non-invariant eigenvector of B_2 had eigenvalue at most $-\alpha^2$, Lemma 24.2 would give a non-invariant tangential TT eigentensor of L_{h_Z} with eigenvalue at most

$$2(m-1) - \alpha^2 = \frac{(m-1)(9-m)}{4}.$$

This contradicts Lemma 24.3; the TT part of its zero eigenspace is the invariant relative-factor line. Compact resolvent of B_2 therefore gives some $\eta > 0$ with

$$B_2 + \alpha^2 \geq \eta\,\mathrm{Id} \quad \text{on } \ker \mathcal{P}. \tag{24.8}$$

For a compactly supported radial coefficient f,

$$\int_0^\infty r^m |f'|^2\, dr - \alpha^2 \int_0^\infty r^{m-2}|f|^2\, dr = \int_{\mathbb{R}} \left| \frac{d}{ds}(e^{\alpha s} f(e^s)) \right|^2 ds \geq 0.$$

Applying this identity to angular coefficients and then using (24.8) gives $Q_C(h) \geq \eta \int r^{-2}|h|^2$. Since $|\mathrm{Rm}_C| \leq Cr^{-2}$, the identity defining Q_C also bounds $\int |\nabla^C h|^2$ by a constant times $Q_C(h)$.

Identify the bundles of g and g_C on the end by the positive square root of $g_C^{-1}g$. This is an equivariant fibre isometry. Differentiated conical convergence gives, beyond a sufficiently large radius, a bound ε for the relative metric error, εr^{-1} for the connection error, and εr^{-2} for the curvature error. For the induced tensor identification I,

$$|Q_g(h) - Q_C(Ih)| \leq C\varepsilon \int (|\nabla^C Ih|^2 + r^{-2}|Ih|^2)\, dv_C.$$

The same estimates compare the two homogeneous energy norms. If $Q_C(v) \geq c_C\|v\|^2_{\mathscr{E}_C}$ and the displayed error bound has constant C_0, choose $C_0\varepsilon \leq c_C/2$. Then $Q_g(h) \geq (c_C/2)\|Ih\|^2_{\mathscr{E}_C}$. Equivariance preserves $\mathcal{P}h = 0$, and $\rho \asymp r$ on the end. This proves (24.7). ☐

25. Finite-energy solutions at zero

Lemma 25.1. *Suppose $h \in \mathscr{E}$ and $Lh = 0$ distributionally. Then h is smooth and TT. For each integer $j \geq 0$,*

$$\sup_{\{R<t<2R\}} R^{\alpha+j}|\nabla^j h| \leq C_j \left(\int_{\{R/2<t<4R\}} t^{-2}|h|^2\, dv_g \right)^{1/2} \longrightarrow 0. \tag{25.1}$$

Proof. Local elliptic regularity for L applies because $h \in H^1_{\mathrm{loc}}$; it gives smoothness. Set $u = \operatorname{tr} h$. The commutation (2.3) gives $\Delta u = 0$. Testing against $\chi_R^2 u$, with the cutoffs in Lemma 23.1, yields

$$\int \chi_R^2 |du|^2 \leq 4 \int |d\chi_R|^2 u^2 \leq C \int_{\{R<\rho<2R\}} \rho^{-2}|h|^2 \longrightarrow 0.$$

Thus u is constant. The conical volume growth gives $\int_X \rho^{-2}\, dv_g = \infty$, so $u = 0$. Next $\omega = \delta h$ belongs to L^2, and $\nabla^*\nabla\omega = 0$. Testing with $\chi_R^2\omega$ gives

$$\int \chi_R^2 |\nabla\omega|^2 \leq 4 \int |d\chi_R|^2|\omega|^2 \leq \frac{C}{R^2}\|\omega\|_2^2 \longrightarrow 0.$$

Thus ω is parallel. Its pointwise norm is constant; infinite volume and $\omega \in L^2$ force that constant to be zero.

Pull $\{R/2 < t < 4R\}$ to a fixed cone annulus and multiply the metric and covariant two-tensor by R^{-2}. The tensor's pointwise norm is unchanged. The operator is uniformly elliptic in a finite collection of charts, with all fixed-order coefficients uniformly bounded by differentiated conical convergence. The interior H^{j+s} estimate, followed by Sobolev embedding for $s > d/2$, gives

$$\sup_{\{R<t<2R\}} R^j|\nabla^j h| \leq C_j \left(R^{-d} \int_{\{R/2<t<4R\}} |h|^2\, dv_g \right)^{1/2}.$$

Since $d - 2 = 2\alpha$ and $t \asymp R$ on this annulus, this is (25.1). The tail integral tends to zero by finite energy. ☐

The next estimate concerns only boundary tensors that are in the image of a boundary displacement. It is not a surjectivity assertion for the induced-metric map.

Lemma 25.2 (Boundary displacements at large radius). *For sufficiently large R, let*

$$\mathfrak{D}_R(\xi, Y) = 2\xi \mathrm{II}_R + \mathcal{L}_Y \gamma_R, \qquad \gamma_R = g^\top|_{\Sigma_R},$$

where Y is tangential. If $b \in \operatorname{im} \mathfrak{D}_R$, a preimage can be chosen so that, for every fixed integer $s > m/2$,

$$\sup_{\Sigma_R} |\xi \nu + Y|_g \le C_s R \sum_{j=0}^{s} R^j \sup_{\Sigma_R} |\nabla_\Sigma^j b|_{\gamma_R}. \tag{25.2}$$

A preimage of data in a fixed nontrivial angular representation can be chosen in that representation, with a constant allowed to depend on it.

Proof. On the fixed product Z, set $\bar\gamma_R = R^{-2}\gamma_R$, $\overline{\mathrm{II}}_R = R^{-1}\mathrm{II}_R$, $\bar H_R = \operatorname{tr}_{\bar\gamma_R} \overline{\mathrm{II}}_R$, $\bar b = R^{-2} b$, and $f = \xi/R$. Then

$$\bar b = 2\delta^*_{\bar\gamma_R} Y^\flat + 2f \overline{\mathrm{II}}_R, \qquad (\bar\gamma_R, \overline{\mathrm{II}}_R, \bar H_R) \longrightarrow (h_Z, h_Z, m)$$

in every C^s norm. Write $\mathcal{K}_h Y = \frac{1}{2}\mathcal{L}_Y h - (\operatorname{div}_h Y/m)h$.

Every conformal Killing field of h_Z is Killing. Indeed, if $\mathcal{L}_Y h_Z = 2\sigma h_Z$, differentiation of $\mathrm{Ric}_{h_Z} = (m-1)h_Z$ and the conformal variation formula give $\Delta_{h_Z}\sigma = m\sigma$. Equation (24.5) implies $\sigma = 0$. Denote the space of Killing fields by $\mathfrak{k}$. For $Y \perp \mathfrak{k}$ in $L^2(h_Z)$, ellipticity of $\delta\mathcal{K}$ and its kernel $\mathfrak{k}$ give

$$\|Y\|_{H^{s+1}} \le C_s \|\mathcal{K}_{h_Z} Y\|_{H^s}. \tag{25.3}$$

The elliptic estimate with an additional $\|Y\|_2$ term follows by applying the second-order elliptic estimate to $\delta\mathcal{K}$. Suppose the L^2 term could not be removed on $\mathfrak{k}^\perp$. There would be $Y_i \perp \mathfrak{k}$ with $\|Y_i\|_{H^1} = 1$ and $\|\mathcal{K}Y_i\|_2 \to 0$. A subsequence converges weakly in H^1 and strongly in L^2 to Y. Then $\mathcal{K}Y = 0$ and $Y \perp \mathfrak{k}$, so $Y = 0$. The elliptic estimate with the L^2 term now implies $\|Y_i\|_{H^1} \to 0$, a contradiction. For $s \ge 1$, the second-order elliptic estimate gives

$$\|Y\|_{H^{s+1}} \le C_s\big(\|\delta\mathcal{K}Y\|_{H^{s-1}} + \|Y\|_2\big) \le C_s'\big(\|\mathcal{K}Y\|_{H^s} + \|Y\|_2\big).$$

The estimate just proved absorbs the last term.

All fields in $\mathfrak{k}$ are sums of Killing fields of the two factors. To see this, decompose $Y = Y_P + Y_Q$. The pure-factor Killing equations say that Y_P is Killing on the first sphere for each point of the second, and conversely for Y_Q. Take the first-factor divergence of the mixed Killing equation. Since $\operatorname{div}_P Y_P = 0$, this gives the scalar equation $\Delta_P (Y_Q)_\beta = 0$ for each second-factor covariant component. Compactness of the first sphere makes Y_Q independent of that factor. The mixed equation then makes Y_P independent of the second. Such fields preserve every product metric $\bar\gamma_R$. Subtracting one of these fields from a chosen preimage changes neither b nor ξ. Arrange $Y \perp \mathfrak{k}$. Taking trace and trace-free parts gives

$$f = \frac{\operatorname{tr}_{\bar\gamma_R} \bar b - 2\operatorname{div}_{\bar\gamma_R} Y}{2\bar H_R},$$

$$\mathcal{K}_{\bar\gamma_R} Y = \frac{1}{2}\left(\bar b_0 - \frac{\operatorname{tr}\bar b}{\bar H_R}\overline{\mathrm{II}}_{R,0}\right) + \frac{\operatorname{div}_{\bar\gamma_R} Y}{\bar H_R}\overline{\mathrm{II}}_{R,0}.$$

Equation (25.3), convergence of $\bar\gamma_R$, and $\overline{\mathrm{II}}_{R,0} \to 0$ imply

$$\|Y\|_{H^{s+1}} \le C_s \|\bar b\|_{H^s} + o(1)\|Y\|_{H^{s+1}}.$$

Write the coefficient of the last term as ε_R, with $\varepsilon_R \to 0$. For R large enough that $\varepsilon_R \le 1/2$,

$$\|Y\|_{H^{s+1}} \le 2C_s \|\bar b\|_{H^s}.$$

The trace equation, $\bar H_R \to m > 0$, and boundedness of divergence from H^{s+1} to H^s give $\|f\|_{H^s} \le C_s'\|\bar b\|_{H^s}$. Sobolev embedding on the fixed compact link proves (25.2); a tangential vector has g-norm R times its $\bar\gamma_R$-norm. Isotypic projection commutes with $\mathfrak{D}_R$ and preserves the estimate up to a fixed representation-dependent constant. □

Proposition 25.3 (Vanishing of the finite-energy transverse kernel). *If $h \in \mathscr{E}_\perp$ and $Lh = 0$ in distributions, then $h = 0$.*

Proof. By Lemma 25.1, h is smooth and TT and therefore $\mathrm{Ric}'(h) = 0$. Project onto a fixed nontrivial angular representation. The projection is bounded on $\mathscr{E}$, preserves the equations, and has the decay (25.1). We retain the notation h for this projection. The harmonic decomposition of Lemma 4.1 has only finitely many tensor generators in the chosen representation. We first prove that

$$h^{\mathsf{T}}|_{\Sigma_R} \in \operatorname{im} \mathfrak{D}_R \qquad \text{for every sufficiently large } R. \tag{25.4}$$

Two scalar degrees at least two. The quantities $\zeta = uB - vA$ and $c = 2e - a^2C - b^2D$ depend only on tangential coefficients and are invariant under every infinitesimal coordinate change (4.6). They can therefore be estimated in the original TT gauge and can be substituted in the equations obtained in normal gauge. Equation (25.1), applied to tangential components and their derivatives, gives

$$\zeta = o(t^{-\alpha-1}), \quad \zeta' = o(t^{-\alpha-2}), \qquad c = o(t^{2-\alpha}), \quad c' = o(t^{1-\alpha}). \tag{25.5}$$

In fact $A, B, C, D = o(t^{-\alpha})$ in each fixed angular basis and $e = o(t^{2-\alpha})$. The connection coefficients are $O(t^{-1})$, which gives the differentiated bounds.

The entries of the leading coefficient matrix in Section 7 satisfy

$$\mathsf{G}_{11} = O(t^2), \qquad \mathsf{G}_{12} = O(t^{-1}), \qquad \mathsf{G}_{22} = O(t^{-4}). \tag{25.6}$$

To verify the orders, use $L_0, M_0, r_P, r_Q = O(t^{-2})$, $u, v = O(t^{-1})$, and $\mathsf{D} \asymp t^{-6}$ in the explicit matrix formulas. The coefficient of t^{-6} in D is positive by (7.4). Equations (7.16) and (7.21) give $\mathsf{G}_{11} = O(t^2)$ and $\mathsf{G}_{12} = O(t^{-1})$. For G_{22}, use its explicit numerator rather than dividing an upper bound by G_{11}. In (7.22), $L_0M_0 = O(t^{-4})$, and each term inside the parentheses is $O(t^{-6})$. Since $\mathsf{D} \asymp t^{-6}$, $\mathsf{G}_{22} = O(t^{-4})$.

The same rank-one formulas give

$$N_* = \begin{pmatrix} O(t) & O(t^{-2}) \\ O(t^{-2}) & O(t^{-5}) \end{pmatrix}. \tag{25.7}$$

Indeed $\omega_1, \omega_2, \omega_3 = O(t^2)$, while $b_1, b_2 = (1, O(t^{-3}))$ and $d_1, d_2 = (O(t^{-1}), O(t^{-4}))$; the other two differentiated rows vanish. Hence

$$\mathcal{F}_* = W\mathbf{y}^T(\mathsf{G}\mathbf{y}' + N_*\mathbf{y}) = o(1), \qquad \mathbf{y} = (\zeta, c)^T. \tag{25.8}$$

For example $W\zeta^2(N_*)_{11} = o(t^{m-2\alpha-1})$ and $Wc^2(N_*)_{22} = o(t^{m-2\alpha-1})$; the mixed and derivative terms have the same exponent, which is zero. The identity (8.31) is valid on every finite interval. Its inner flux tends to zero by (8.32), using a smooth normal gauge and the intrinsic character of ζ, c. Its outer flux tends to zero by (25.8). Integration makes its nonnegative right side zero. Since $r_1, r_2 > 0$, one has $F_1 = F_2 = 0$, and hence $\zeta = c = 0$. At a fixed outer radius, the equations

$$\xi = A/u, \qquad \eta = C, \qquad \omega = D$$

then solve (4.6) for the tangential tensor. This proves (25.4) in this case.

One scalar degree equal to one. Use the intrinsic scalar χ of Section 8.7. Its definition and (25.5) give $\chi = o(t^{-\alpha-1})$ and $\chi' = o(t^{-\alpha-2})$. Its leading coefficient is $g_1 = O(t^2)$, and its completion coefficient is $\sigma = O(t^{-1})$. Hence the outer flux in (8.44) is $o(1)$. Its inner flux is zero by Proposition 8.10. Integration gives $\chi = 0$. Choose the redundant trace–Hessian representative so that $c = 0$; then $\zeta = 0$ as well, and (4.6) again generates the tangential tensor. Equivalently, when the first degree is one, use the intrinsic first-factor coefficient $A - C$, take $\xi = B/v$, $\omega = D$, and $\eta = u\xi - (A - C)$. The equation $\chi = 0$ is exactly the remaining mixed-component equation. Interchange the factors for the other exceptional case.

One nonconstant scalar factor. The scalar invariant of Section 6 satisfies $Z_0 = o(t^{-\alpha})$, $Z_0' = o(t^{-\alpha-1})$. Its weight ϱ_0 is comparable to t^m, and its completion coefficient is $O(t^{-1})$. Therefore $\varrho_0 Z_0(Z_0' + \sigma Z_0) = o(1)$. The singular orbit calculation and (6.9) imply $Z_0 = 0$. The boundary traces and Hessian are then generated by a normal displacement and a tangential scalar gradient.

Factor-TT tensors. For the radial coefficient in (5.1), $F = o(t^{-\alpha})$ and $F' = o(t^{-\alpha-1})$. Thus $WFF' = o(1)$. Integration of its radial equation from the singular orbit to infinity makes the positive energy (5.1) zero. Hence $F = 0$.

Vector tensors. Use the fibre notation $g_B + r_f^2 \gamma_n$ and the coefficients of Proposition 5.2; $r_f \asymp t$ at infinity. For a non-Killing harmonic, $\mathcal{A} = B - dC$ satisfies

$$|\mathcal{A}| = o(t^{-\alpha-1}), \qquad |d\mathcal{A}| = |dB| = o(t^{-\alpha-2}).$$

These estimates follow from $|B| \le Ct^{-1}|h|$, $|C| \le C|h|$, and the first two derivative bounds in (25.1). In the Killing case take $\mathcal{A} = B$; the same bounds hold. The area of the base boundary is $O(t^{m-n})$. The flux from (5.2) is bounded by

$$Ct^{m-n}t^{n+2}|\mathcal{A}|\,|d\mathcal{A}| = o(1).$$

Its inner flux is zero by the smoothness estimates in Proposition 5.2. The weighted energy is nonnegative. It follows that $\mathcal{A} = 0$ for a non-Killing harmonic and $d\mathcal{A} = 0$ in the Killing case. In the first case $B = dC$, so the tangential tensor is generated by the fibre field $C\omega^\sharp$. In the second case the restriction of $\mathcal{A}$ to the remaining sphere is closed and therefore exact, since that sphere has dimension at least two. Its scalar potential generates the tangential mixed tensor. This proves (25.4).

The degrees $(1,0), (0,1), (1,1)$. In degree $(1,0)$, write the tangential tensor as $2AY\gamma_P + 2BY\gamma_Q$. It is generated by $\xi Y v + \eta \nabla Y$ with $\xi = B/v$, $\eta = u\xi - A$. The other single degree is interchanged. In degree $(1,1)$, with mixed coefficient $2e$, solve

$$\xi = \frac{2e + a^2A + b^2B}{a^2u + b^2v}, \qquad \eta = u\xi - A, \qquad \omega = v\xi - B.$$

The denominator is positive. These formulas follow from $\nabla^2 Y = -Y\gamma_p$ and $\nabla^2 Z = -Z\gamma_q$. Thus every tangential tensor in these three sectors belongs to the image in (25.4); no field equation or decay assertion for a redundant coefficient is used.

The preceding cases exhaust every nontrivial angular representation. By Lemma 25.2, (25.1), and the identities converting ambient to tangential derivatives, choose a boundary vector V_R^∂ in this representation with

$$\mathfrak{D}_R(V_R^\partial) = h^\top, \qquad \sup_{\Sigma_R} |V_R^\partial| = o(R^{1-\alpha}) \longrightarrow 0. \tag{25.9}$$

Extend it smoothly into M_R by a collar cutoff, and project the extension onto the same angular representation. The tensor $h - \mathcal{L}_{\widetilde{V}_R} g$ fixes the induced boundary metric. Its class is non-invariant, so Theorem A makes it boundary-fixing pure gauge. There is therefore a smooth vector field V_R on M_R such that

$$h = \mathcal{L}_{V_R} g, \qquad V_R|_{\Sigma_R} = V_R^\partial.$$

The TT condition and (2.6) give $\nabla^*\nabla V_R^\flat = 0$. The function $|V_R|^2$ is subharmonic for $\operatorname{div}\nabla$; hence

$$\sup_{M_R} |V_R| \le \sup_{\Sigma_R} |V_R^\partial| \longrightarrow 0.$$

On a fixed relatively compact set, interior estimates for this vector equation give $\nabla V_R \to 0$. Thus $h = 0$ there. The set was arbitrary. Every nontrivial angular projection of the original tensor is zero; completeness of the harmonic decomposition and $\mathcal{P}h = 0$ give $h = 0$. ☐

Theorem 25.4 (Transverse homogeneous coercivity). *There is $c_g > 0$ such that*

$$\int_X \big(|\nabla h|^2 - 2\langle \mathring{R}h, h\rangle\big)\, dv_g \ge c_g \int_X \big(|\nabla h|^2 + \rho^{-2}|h|^2\big)\, dv_g \qquad (h \in \mathscr{E}_\perp). \tag{25.10}$$

Consequently $L : \mathscr{E}_\perp \to \mathscr{E}_\perp^*$ is a bounded isomorphism, in the weak energy sense.

Proof. Equations (23.2) and (23.4), Proposition 24.4, and Proposition 25.3 verify every hypothesis of Proposition 23.2. This proves (25.10). The form is bounded and coercive on the real Hilbert space $\mathscr{E}_\perp$. For $\ell \in \mathscr{E}_\perp^*$, minimization of $\frac{1}{2}Q(h) - \ell(h)$, or the Riesz representation theorem applied to the equivalent inner product Q, gives a unique h with $Q(h,k) = \ell(k)$ for every k. Its norm is at most $c_g^{-1}\|\ell\|_{\mathscr{E}_\perp^*}$. ☐

Inequality (25.10) is not an L^2 spectral gap. Its lower-order term has the inverse-square weight ρ^{-2}. The finite-energy condition at zero is used in Proposition 25.3; absence of negative spectrum alone would not imply this inequality.

26. The Ricci–DeTurck nonlinear remainder

For a smooth tensor h, put

$$\|h\|_{\mathscr{X}} = \sup_X\bigl(|h| + \rho|\nabla h| + \rho^2|\nabla^2 h|\bigr). \tag{26.1}$$

Norms, covariant derivatives, and integrals in this section use the fixed background g. If $|h|_g \le 1/2$, then $\widetilde{g} = g + h$ is positive definite. Let G^{ij} denote its inverse, and define

$$C^k_{ij}(h) = \tfrac{1}{2}G^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}), \qquad \mathcal{W}(h)^k = G^{ij}C^k_{ij}(h). \tag{26.2}$$

The Ricci–DeTurck expression is

$$\mathcal{F}^{\mathrm{RD}}_g(h) = -2\operatorname{Ric}(g+h) + \mathcal{L}_{\mathcal{W}(h)}(g+h), \qquad \mathcal{N}(h) = \mathcal{F}^{\mathrm{RD}}_g(h) + Lh. \tag{26.3}$$

Thus Ricci–DeTurck flow is $\partial_\tau h = -Lh + \mathcal{N}(h)$. The sign of $\mathcal{W}$ in (26.2) is fixed by this identity.

Lemma 26.1. *The expression $\mathcal{F}^{\mathrm{RD}}_g$ is equivariant under isometries of g, $D\mathcal{F}^{\mathrm{RD}}_g(0) = -L$, and*

$$\mathcal{N}(h) = a(h)^{ab}\nabla_a\nabla_b h + \mathcal{B}(h)(\nabla h, \nabla h) + \mathcal{C}(h)(\mathrm{Rm}_g), \qquad a(h)^{ab} = G^{ab} - g^{ab}. \tag{26.4}$$

Here $\mathcal{B}(h)$ is a fibrewise bilinear map on the two first-derivative tensors and $\mathcal{C}(h)$ is a fibrewise linear map on curvature tensors. They are smooth algebraic functions of h on $|h| \le 1/2$, with bounds

$$\begin{aligned} &|a(h)| \le C|h|, && |Da(h)| + |D\mathcal{B}(h)| + |\mathcal{B}(h)| \le C,\\ &|\mathcal{C}(h)| \le C|h|^2, && |D\mathcal{C}(h)| \le C|h|. \end{aligned} \tag{26.5}$$

The constant depends only on the dimension and the fixed tensor norm convention.

Proof. All expressions in (26.2) are natural with respect to background isometries, proving equivariance. The exact identities from the difference of two connections and the Lie derivative are

$$\operatorname{Ric}(g+h)_{ij} = \operatorname{Ric}(g)_{ij} + \nabla_k C^k_{ij} - \nabla_j C^k_{ik} + C^k_{k\ell}C^\ell_{ij} - C^k_{j\ell}C^\ell_{ik}, \tag{26.6}$$

$$(\mathcal{L}_{\mathcal{W}}(g+h))_{ij} = \mathcal{W}^k\nabla_k h_{ij} + (g+h)_{kj}\nabla_i\mathcal{W}^k + (g+h)_{ik}\nabla_j\mathcal{W}^k, \tag{26.7}$$

$$\nabla_a G^{ij} = -G^{ik}(\nabla_a h_{k\ell})G^{\ell j}. \tag{26.8}$$

Inserting the first derivatives in (26.2), the second-derivative terms in the sum (26.3) reduce to $G^{ab}\nabla_a\nabla_b h_{ij}$, apart from commutators of two covariant derivatives. Each commutator is the curvature action on one of the two tensor slots of h. Every other term from (26.6)–(26.8) contains two first derivatives of h, with coefficients that are products of inverse metrics and $g + h$.

At $h = 0$, (2.1) and $D\mathcal{W}_0(h)^\flat = -\delta h - \frac{1}{2}d\operatorname{tr}h$ give $D\mathcal{F}^{\mathrm{RD}}_g(0)[h] = -Lh$. The linear curvature term after the preceding commutations is therefore $2\mathring{R}h$. Subtracting it leaves a curvature-linear coefficient whose value and first derivative at $h = 0$ are zero. Define $\mathcal{B}$ by collecting the two-first-derivative terms, and $\mathcal{C}$ by collecting these remaining commutator terms. The full cancellation and quadratic expression are displayed in Section D. This defines the maps in (26.4) without a choice of coordinates. The inverse-metric entries and their first two h-derivatives are uniformly bounded for $|h| \le 1/2$. The derivative formula for an inverse is $DG(h)[v] = -GvG$. Taylor's formula gives (26.5). $\square$

The connection formulas in Lemma 26.1 also fix the relation with the Ricci–DeTurck convention in [6, Section 2]. Homogeneous-energy estimates in the dynamically stable ALE case are developed in [10]. No nonlinear stability theorem from that setting is assumed here.

Lemma 26.2 (Difference estimate). *There are $C_g > 0$ and $\varepsilon_1 > 0$ such that, for smooth h, k with*

$$\|h\|_{\mathscr{X}} + \|k\|_{\mathscr{X}} \le \varepsilon \le \varepsilon_1, \qquad w = h - k \in \mathscr{E},$$

the difference $\mathcal{N}(h) - \mathcal{N}(k)$ extends to an element of $\mathscr{E}^*$ satisfying

$$|\langle \mathcal{N}(h) - \mathcal{N}(k), v\rangle| \le C_g\varepsilon\|w\|_{\mathscr{E}}\|v\|_{\mathscr{E}} \qquad (v \in \mathscr{E}). \tag{26.9}$$

Proof. For a compactly supported smooth test v, separate the difference in (26.4) into

$$
\begin{aligned}
&a(h)\nabla^2 w + (a(h) - a(k))\nabla^2 k,\\
&\mathcal{B}(h)(\nabla w, \nabla h) + \mathcal{B}(h)(\nabla k, \nabla w) + (\mathcal{B}(h) - \mathcal{B}(k))(\nabla k, \nabla k),\\
&(\mathcal{C}(h) - \mathcal{C}(k))(\mathrm{Rm}_g).
\end{aligned}
$$

The order of the two arguments of $\mathcal{B}$ is retained, so no symmetry of that bilinear map is required. For the principal term, tensor contraction and compact support give

$$\int_X \langle a(h)^{ab}\nabla_a\nabla_b w, v\rangle\, dv_g = -\int_X \langle a(h)^{ab}\nabla_b w, \nabla_a v\rangle\, dv_g - \int_X \langle (\nabla_a a(h)^{ab})\nabla_b w, v\rangle\, dv_g.$$

By (26.1), (23.2), and (26.5),

$$|a(h)| \le C\varepsilon, \qquad |\nabla a(h)| \le C\varepsilon\rho^{-1},$$

and all terms other than $a(h)\nabla^2 w$ are bounded by

$$C_g\varepsilon\big(\rho^{-1}|\nabla w| + \rho^{-2}|w|\big).$$

The two integrands in the preceding integration-by-parts identity are bounded by

$$C\varepsilon|\nabla w|\,|\nabla v| + C\varepsilon\rho^{-1}|\nabla w|\,|v|.$$

Cauchy–Schwarz and (23.1) give (26.9). Every term is now a bounded linear functional of v in the energy norm. Density proves the claimed extension. This also justifies the calculation when $w \notin L^2$. □

27. Ancient and stationary symmetry rigidity

The metric g and the DeTurck vector field are fixed as in (26.2). Put

$$\mathscr{H}_\perp = \ker \mathcal{P} \cap L^2(S^2T^*X).$$

In this section $L_\perp$ denotes the restriction of the Friedrichs realization of L to $\mathscr{H}_\perp$.

Theorem 27.1 (Transverse dissipation). *There are $\varepsilon_g, c'_g > 0$ with the following property. Let I be a time interval and let $g + h(\tau)$ be a smooth Ricci–DeTurck solution. Suppose that $w = (I - \mathcal{P})h$ satisfies*

$$w \in C^1_{\mathrm{loc}}(I; L^2) \cap C^0_{\mathrm{loc}}(I; H^2), \qquad \sup_{\tau\in I}\|h(\tau)\|_{\mathscr{X}} \le \varepsilon_g. \tag{27.1}$$

Then, for $\tau_1 < \tau_2$ in I,

$$\|w(\tau_2)\|_2^2 + c'_g\int_{\tau_1}^{\tau_2}\|w(\tau)\|_{\mathscr{E}}^2\, d\tau \le \|w(\tau_1)\|_2^2. \tag{27.2}$$

Only the component w is required to belong to the indicated Sobolev spaces.

Proof. Set $k = \mathcal{P}h$. Invariance of g and ρ gives $\|k\|_{\mathscr{X}} \le \|h\|_{\mathscr{X}}$. Equivariance of $\mathcal{N}$ and commutation of L with $\mathcal{P}$ give

$$\partial_\tau w = -L_\perp w + f, \qquad f = (I - \mathcal{P})\big(\mathcal{N}(h) - \mathcal{N}(k)\big). \tag{27.3}$$

For $w \in H^2$, the differential expression Lw belongs to L^2. The integration-by-parts identity against compactly supported tensors extends to H^1 tests by approximation. Thus w belongs to the operator domain of $L_\perp$, and its operator value is the displayed differential expression. The pointwise bounds in the proof of Lemma 26.2 also give $f \in L^2$. The same lemma gives

$$|\langle f, v\rangle| \le 2C_g\varepsilon_g\|w\|_{\mathscr{E}}\|v\|_{\mathscr{E}} \qquad (v \in \mathscr{E}_\perp).$$

Pair (27.3) with w and apply Theorem 25.4:

$$\frac{1}{2}\frac{d}{d\tau}\|w\|_2^2 \le -(c_g - 2C_g\varepsilon_g)\|w\|_{\mathscr{E}}^2.$$

Choose $2\varepsilon_g \le \varepsilon_1$ and $2C_g\varepsilon_g \le c_g/2$. Integration gives the assertion with $c'_g = c_g$. All pairings are between the energy space and its dual; no integrability of k is used. □

Lemma 27.2 (Bounded ancient solutions of a dissipative equation). *Let A be a nonnegative self-adjoint operator on a real Hilbert space $\mathcal{H}$ with* $\ker A = 0$. *Let $\mathcal{V}$ be the completion of $D(A^{1/2})$ in the norm $\|v\|_{\mathcal{V}} = \|A^{1/2}v\|_{\mathcal{H}}$. Suppose*

$$u \in C^1_{\mathrm{loc}}((-\infty,0];\mathcal{H}) \cap C^0_{\mathrm{loc}}((-\infty,0];D(A)), \qquad u' + Au = f,$$

where $D(A)$ has its graph norm. Assume that, for fixed constants $C, c > 0$,

$$|\langle f(t), v\rangle| \le C\|u(t)\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \qquad\qquad v \in D(A^{1/2}), \tag{27.4}$$

$$\frac{d}{dt}\|u(t)\|^2_{\mathcal{H}} + c\|u(t)\|^2_{\mathcal{V}} \le 0, \tag{27.5}$$

$$\sup_{t\le 0}\|u(t)\|_{\mathcal{H}} < \infty. \tag{27.6}$$

Then $u = 0$.

Proof. Put $M = \sup_{s\le 0}\|u(s)\|_{\mathcal{H}} < \infty$. For $a < t \le 0$, integration of (27.5) gives

$$c\int_a^t \|u(s)\|^2_{\mathcal{V}}\,ds \le \|u(a)\|^2_{\mathcal{H}} - \|u(t)\|^2_{\mathcal{H}} \le M^2.$$

Monotone convergence as $a \to -\infty$ gives

$$\int_{-\infty}^t \|u(s)\|^2_{\mathcal{V}}\,ds < \infty \qquad (t \le 0). \tag{27.7}$$

Let E_A be the spectral resolution of A. For $\varphi \in \mathcal{H}$,

$$\|e^{-TA}\varphi\|^2_{\mathcal{H}} = \int_{[0,\infty)} e^{-2T\lambda}\,d\|E_A(\lambda)\varphi\|^2 \longrightarrow 0, \tag{27.8}$$

$$\int_0^\infty \|A^{1/2}e^{-sA}\varphi\|^2_{\mathcal{H}}\,ds = \int_{(0,\infty)}\left(\int_0^\infty \lambda e^{-2s\lambda}\,ds\right)d\|E_A(\lambda)\varphi\|^2 = \frac{1}{2}\|\varphi\|^2_{\mathcal{H}}. \tag{27.9}$$

The first limit is dominated convergence; the second identity is Tonelli's theorem. Both use $E_A(\{0\}) = 0$.

For $a < t$, the variation-of-constants identity gives

$$\langle u(t), \varphi\rangle = \langle u(a), e^{-(t-a)A}\varphi\rangle + \int_a^t \langle f(s), e^{-(t-s)A}\varphi\rangle\,ds. \tag{27.10}$$

One may first take $\varphi \in E_A([0,n])\mathcal{H}$, differentiate $\langle u(s), e^{-(t-s)A}\varphi\rangle$, and integrate. The limit $n \to \infty$ follows from (27.4), Cauchy–Schwarz, and (27.9). These estimates also show absolute convergence of the integral in (27.10) as $a \to -\infty$. The first term tends to zero by (27.6) and (27.8). Therefore

$$|\langle u(t), \varphi\rangle| \le \frac{C}{\sqrt{2}}\left(\int_{-\infty}^t \|u(s)\|^2_{\mathcal{V}}\,ds\right)^{1/2}\|\varphi\|_{\mathcal{H}}.$$

Taking the supremum over unit vectors yields

$$\|u(t)\|^2_{\mathcal{H}} \le \frac{C^2}{2}\int_{-\infty}^t \|u(s)\|^2_{\mathcal{V}}\,ds. \tag{27.11}$$

Its right side tends to zero as $t \to -\infty$, by (27.7). Equation (27.5) makes $\|u(t)\|_{\mathcal{H}}$ nonincreasing. For $s < t$, $\|u(t)\|_{\mathcal{H}} \le \|u(s)\|_{\mathcal{H}}$; letting $s \to -\infty$ proves $u(t) = 0$. ☐

Theorem 27.3 (Ancient symmetry under a bounded transverse norm). *Let $g + h(\tau)$, $-\infty < \tau \le 0$, satisfy* (27.1) *with the constant in Theorem 27.1. If*

$$\sup_{\tau\le 0}\|(I - \mathcal{P})h(\tau)\|_2 < \infty, \tag{27.12}$$

then $h(\tau)$ is invariant for every $\tau \le 0$. No backward convergence or prescribed temporal rate is assumed.

Proof. Take $\mathcal{H} = \mathscr{H}_\perp$ and $A = L_\perp$ in Lemma 27.2. The form domain of L is H^1, because its potential is bounded. The completion of $D(L_\perp^{1/2})$ in the form norm $Q^{1/2}$ is $\mathscr{E}_\perp$: compactly supported tensors are dense in that space, and Theorem 25.4 makes $Q^{1/2}$ equivalent to its energy norm. The same estimate gives $\ker L_\perp = 0$. Equation (27.3) and Lemma 26.2 verify (27.4); Theorem 27.1 verifies (27.5) after changing its constant by the equivalence of the two form norms. The remaining hypothesis is (27.12). Thus $(I - \mathcal{P})h = 0$ in L^2. Smoothness gives pointwise equality. □

Remark 27.4. The condition $\liminf_{\tau\to-\infty} \|(I - \mathcal{P})h(\tau)\|_2 = 0$ also implies symmetry, directly from (27.2). Indeed, for fixed τ, choose $s_i < \tau$ with the norms at s_i tending to zero and apply that inequality. In particular, backward L^2 convergence of the whole perturbation implies symmetry. The preceding theorem does not require such convergence, or finite energy of the invariant component.

Corollary 27.5. *For an ancient solution in Theorem 27.3, the equations*

$$\frac{d}{d\tau}\Phi_\tau = -\mathcal{W}(h(\tau)) \circ \Phi_\tau, \qquad \Phi_0 = \mathrm{Id},$$

define diffeomorphisms for every finite $\tau \le 0$. They commute with $\mathfrak{G}$, and $\Phi_\tau^*(g + h(\tau))$ is a $\mathfrak{G}$-invariant Ricci flow.

Proof. By (26.2), $|\mathcal{W}| \le C\varepsilon_g\rho^{-1}$ and $|\nabla\mathcal{W}| \le C\varepsilon_g\rho^{-2}$. An integral curve on a finite time interval has bounded g-length. Completeness and local existence for the vector-field equation extend it to the whole interval. Solving the reversed equation gives the inverse map. For $a \in \mathfrak{G}$, naturality and invariance give $\mathcal{W}(ax) = da_x\mathcal{W}(x)$. Uniqueness of integral curves implies $\Phi_\tau \circ a = a \circ \Phi_\tau$. Differentiating the pullback gives

$$\partial_\tau\big(\Phi_\tau^*(g+h)\big) = \Phi_\tau^*\big(-2\operatorname{Ric}(g+h) + \mathcal{L}_{\mathcal{W}}(g+h) - \mathcal{L}_{\mathcal{W}}(g+h)\big) = -2\operatorname{Ric}\big(\Phi_\tau^*(g+h)\big).$$

No limit of Φ_τ as $\tau \to -\infty$ is needed. □

Theorem 27.6 (Stationary symmetry). *Let h be smooth, let $\|h\|_{\mathscr{X}} \le \varepsilon_g$, and suppose $w = (I - \mathcal{P})h \in \mathscr{E}$. After decreasing ε_g if necessary,*

$$\|w\|_{\mathscr{E}} \le \frac{2}{c_g}\|(I - \mathcal{P})\mathcal{F}_g^{\mathrm{RD}}(h)\|_{\mathscr{E}_\perp^*}. \tag{27.13}$$

The projected expression on the right is defined distributionally and belongs to $\mathscr{E}_\perp^*$. In particular, $\mathcal{F}_g^{\mathrm{RD}}(h) = 0$ implies $h = \mathcal{P}h$. This applies when $\operatorname{Ric}(g+h) = 0$ *and $\mathcal{W}(h) = 0$.*

Proof. Put $k = \mathcal{P}h$. Equivariance gives, in distributions,

$$(I - \mathcal{P})\mathcal{F}_g^{\mathrm{RD}}(h) = -Lw + (I - \mathcal{P})\big(\mathcal{N}(h) - \mathcal{N}(k)\big).$$

Both terms on the right belong to $\mathscr{E}_\perp^*$, by boundedness of Q and Lemma 26.2. Pairing with w gives

$$(c_g - 2C_g\varepsilon_g)\|w\|_{\mathscr{E}}^2 \le \big|\langle (I - \mathcal{P})\mathcal{F}_g^{\mathrm{RD}}(h), w\rangle\big| \le \|(I - \mathcal{P})\mathcal{F}_g^{\mathrm{RD}}(h)\|_{\mathscr{E}_\perp^*}\|w\|_{\mathscr{E}}.$$

Choose $2C_g\varepsilon_g \le c_g/2$ and divide when $w \ne 0$. When $w = 0$, its left side is zero and its right side is nonnegative. □

Corollary 27.7 (Other Ricci-flat backgrounds). *Let a compact group act isometrically on a complete Ricci-flat manifold. Let $\rho \ge 1$ be a smooth proper invariant function with bounded gradient. Assume $|\mathrm{Rm}| \le C\rho^{-2}$, nonnegativity of the Lichnerowicz form on the averaging-zero energy space, coercivity outside a compact set, and vanishing of the distributional kernel in that energy space. Then Theorems 27.1, 27.3 and 27.6 hold for this background, group, and weight.*

Proof. Proposition 23.2 gives global energy coercivity. The connection identities (26.2), (26.6)–(26.8) and the proof of Lemma 26.2 then give the same dual estimate. The form completion, kernel assertion, and three proofs above use exactly these hypotheses. □

Proofs of Theorems F–H. Theorem F is Theorem 25.4. Theorem G is Theorem 27.3, followed by Corollary 27.5. Theorem H follows from Theorem 27.6. □

28. Boundary traces of the transverse operator

Fix $R > 0$ and write

$$X_- = M_R, \qquad X_+ = X \setminus \operatorname{int} M_R, \qquad \Sigma = \Sigma_R.$$

The boundary bundle in this section is $\mathcal{B}_\Sigma = S^2T^*X|_\Sigma$. A boundary value specifies every component of a symmetric tensor, including its normal components. It is not the induced-metric boundary datum in Eq. (1.3). Put

$$\mathcal{T}_R = \ker \mathcal{P} \cap H^{1/2}(\mathcal{B}_\Sigma), \qquad \mathscr{V}_0 = \mathscr{E}_\perp, \qquad \mathscr{V}_\lambda = \mathscr{E}_\perp \cap L^2 \quad (\lambda > 0),$$

with squared norm $\|u\|_{\mathscr{E}}^2 + \lambda\|u\|_2^2$ in the last space. For $\lambda \geq 0$ set

$$Q_\lambda(u, v) = Q(u, v) + \lambda\langle u, v\rangle_{L^2};$$

the second term is absent when $\lambda = 0$.

Lemma 28.1 (Prescribed trace). *For $f \in \mathcal{T}_R$ and $\lambda \geq 0$, there is a unique minimizer u_f^λ of $Q_\lambda(u, u)$ in*

$$\{u \in \mathscr{V}_\lambda \,:\, u|_\Sigma = f\}.$$

It depends linearly and continuously on f. Its restrictions to X_- and X_+ satisfy $(L + \lambda)u = 0$ weakly. For smooth f, both restrictions are smooth up to Σ.

Proof. The trace theorem on a fixed collar of Σ gives

$$\|u|_\Sigma\|_{H^{1/2}} \leq C_R\|u\|_{\mathscr{E}}.$$

A bounded $H^{1/2}$ extension into that collar, followed by a cutoff and averaging, gives a compactly supported H^1 extension e_f with $\mathcal{P}e_f = 0$, trace f, and $\|e_f\|_{H^1} \leq C_R\|f\|_{H^{1/2}}$. Here averaging means application of $I - \mathcal{P}$; it commutes with trace. The affine space is therefore nonempty and closed.

By Theorem 25.4,

$$Q_\lambda(u, u) \geq c_g\|u\|_{\mathscr{E}}^2 + \lambda\|u\|_2^2.$$

On the zero-trace subspace this is a coercive bounded bilinear form. There is a unique v in that subspace such that $Q_\lambda(v, w) = -Q_\lambda(e_f, w)$ for every zero-trace w. Then $u_f^\lambda = e_f + v$ is the unique minimizer. This construction also gives linearity and boundedness. For a test tensor φ supported in either open side, $(I - \mathcal{P})\varphi$ is supported in that side and has zero trace. The minimizing equation vanishes on this test; its pairing with $\mathcal{P}\varphi$ is zero by invariance of the form and $\mathcal{P}u_f^\lambda = 0$. Thus the field equation holds for all compactly supported tests, not only the averaging-zero ones. Dirichlet elliptic regularity on a collar, and interior regularity elsewhere, prove the smoothness assertion. □

Define $M_R(\lambda) \,:\, \mathcal{T}_R \to \mathcal{T}_R^*$ by

$$\langle M_R(\lambda)f, k\rangle = Q_\lambda(u_f^\lambda, u_k^\lambda). \tag{28.1}$$

If f is smooth, weak Green identities identify it with

$$M_R(\lambda)f = \nabla_{\nu_-}u_f^\lambda|_{X_-} + \nabla_{\nu_+}u_f^\lambda|_{X_+}, \qquad \nu_- = \partial_t, \quad \nu_+ = -\partial_t. \tag{28.2}$$

For the exterior conormal derivative, the weak definition in (28.1) replaces a boundary integration at infinity.

Theorem 28.2 (Coercivity of the boundary form). *For every $R > 0$ and $\lambda \geq 0$, the operator $M_R(\lambda)$ is a bounded symmetric isomorphism from $\mathcal{T}_R$ to its dual, and*

$$\langle M_R(\lambda)f, f\rangle = Q(u_f^\lambda) + \lambda\|u_f^\lambda\|_2^2. \tag{28.3}$$

In (28.3) *the L^2 term is omitted when $\lambda = 0$; no finite L^2 norm is asserted at that parameter. There are $R_0, c > 0$, independent of R and λ, such that*

$$\langle M_R(\lambda)f, f\rangle \geq c\big(R^{-1} + \sqrt{\lambda}\big)\|f\|_{L^2(\Sigma_R)}^2 \qquad (R \geq R_0,\ \lambda \geq 0). \tag{28.4}$$

Proof. The trace inequality and Theorem 25.4 give, for each fixed R,

$$\langle M_R(\lambda)f, f\rangle \geq c_R\|f\|^2_{H^{1/2}}.$$

The collar extension in Lemma 28.1 is an admissible competitor, so the reverse upper bound holds with a constant depending on R and λ. Cauchy–Schwarz for Q_λ gives boundedness of the polarized form. The Hilbert-space variational theorem gives the asserted isomorphism. Equation (28.3) is the definition.

For the uniform estimate, set

$$\ell = \min\{R/2, \lambda^{-1/2}\}$$

when $\lambda > 0$, and $\ell = R/2$ when $\lambda = 0$. Identify tensor fibres along radial geodesics by parallel transport. The one-dimensional trace inequality, integrated on Σ_R, gives

$$\|f\|^2_{L^2(\Sigma_R)} \leq C\left(\ell^{-1}\|u^\lambda_f\|^2_{L^2(R<t<R+\ell)} + \ell\|\nabla u^\lambda_f\|^2_{L^2(R<t<R+\ell)}\right). \tag{28.5}$$

To see the scalar inequality, write $F(0) = F(x) - \int_0^x F'(y)\,dy$, square, and integrate in $x \in (0, \ell)$. The radial volume densities at R and $R + x$ are uniformly comparable because $x \leq R/2$ and $a(t), b(t) \asymp t$ on the end. Thus the constant in (28.5) is independent of R, ℓ. Since

$$\ell^{-2} \leq 4R^{-2} + \lambda, \qquad \rho^{-2} \asymp R^{-2} \quad (R < t < R + \ell),$$

multiplication by ℓ^{-1} bounds its right side by $C(\|u^\lambda_f\|^2_{\mathscr{E}} + \lambda\|u^\lambda_f\|^2_2)$. Finally, $R^{-1} + \sqrt{\lambda} \leq 2\ell^{-1}$, and Theorem 25.4 proves (28.4). ☐

Theorem 28.3 (Positive measure representation). *Fix $R > 0$ and $f \in \mathcal{T}_R$. Put $m_f(\lambda) = \langle M_R(\lambda)f, f\rangle$. There is a positive measure ν_f on $(0,\infty)$ such that*

$$m_f(\lambda) = m_f(0) + \int_{(0,\infty)} \frac{\lambda}{\lambda + \xi}\,d\nu_f(\xi), \qquad \int_{(0,\infty)} \frac{d\nu_f(\xi)}{1 + \xi} < \infty. \tag{28.6}$$

For $\lambda > 0$ and $j \geq 1$,

$$(-1)^{j-1}m^{(j)}_f(\lambda) \geq 0. \tag{28.7}$$

If $f \neq 0$, then $m_f(0) > 0$ and $m'_f(\lambda) > 0$.

Proof. Let A_D be the self-adjoint operator associated with Q on the averaging-zero H^1 tensors whose trace on Σ_R is zero. Equivalently, it is the direct sum of the two zero-Dirichlet realizations. The form is nonnegative by Theorem 25.4. If $A_D u = 0$, its form identity and the same estimate give $u = 0$; hence $\ker A_D = 0$.

Fix $a > 0$ and set $u_a = u^a_f$. In the following resolvent identities assume $\lambda > 0$. Since $A_D \geq 0$, the spectral theorem gives $\|(A_D + \lambda)^{-1}\|_{L^2\to L^2} \leq \lambda^{-1}$. The difference $u^\lambda_f - u_a$ has zero trace. Its weak equation gives

$$u^\lambda_f = u_a - (\lambda - a)(A_D + \lambda)^{-1}u_a. \tag{28.8}$$

Both sides belong to the zero-trace form domain after subtracting u_a, so the resolvent identity is valid in L^2. The minimizing equations imply

$$Q_\lambda(u^\lambda_f, u_a) = m_f(\lambda), \qquad Q_a(u_a, u^\lambda_f) = m_f(a).$$

Subtracting gives

$$m_f(\lambda) - m_f(a) = (\lambda - a)\langle u^\lambda_f, u_a\rangle. \tag{28.9}$$

Let μ_f be the spectral measure of A_D associated with u_a. It is finite and has no atom at zero. Substitution of (28.8) into (28.9) gives

$$m_f(\lambda) - m_f(a) = (\lambda - a)\int_{[0,\infty)} \frac{\xi + a}{\xi + \lambda}\,d\mu_f(\xi), \tag{28.10}$$

$$m'_f(\lambda) = \int_{[0,\infty)} \frac{(\xi + a)^2}{(\xi + \lambda)^2}\,d\mu_f(\xi). \tag{28.11}$$

For $0 < \lambda_0 \le \lambda \le \lambda_1$ and $j \ge 1$, the function $(\xi + a)^2(\xi + \lambda)^{-j-1}$ is bounded uniformly for $\xi \ge 0$. The measure μ_f is finite, so differentiation under the integral gives

$$m_f^{(j)}(\lambda) = (-1)^{j-1} j! \int_{[0,\infty)} \frac{(\xi + a)^2}{(\xi + \lambda)^{j+1}} \, d\mu_f(\xi).$$

This proves (28.7). The value $\lambda = 0$ is obtained by the variational limit below, not by an L^2 inverse of A_D.

The variational characterization gives $m_f(\lambda) \ge m_f(0)$. For the reverse limiting inequality, choose a cutoff χ_T equal to one near Σ_R and vanishing outside a large compact set. Then $\chi_T u_f^0 \to u_f^0$ in $\mathscr{E}$, by Lemma 23.1. It is an admissible finite-mass competitor, so

$$m_f(\lambda) \le Q(\chi_T u_f^0) + \lambda \|\chi_T u_f^0\|_2^2.$$

First let $\lambda \downarrow 0$, then $T \to \infty$. This proves $m_f(\lambda) \to m_f(0)$. Integrating (28.11) from 0 to λ and applying Tonelli's theorem yields

$$m_f(\lambda) - m_f(0) = \int_{(0,\infty)} \frac{\lambda(\xi + a)^2}{\xi(\xi + \lambda)} \, d\mu_f(\xi).$$

Thus (28.6) holds with $d\nu_f(\xi) = (\xi + a)^2 \xi^{-1} d\mu_f(\xi)$. Setting $\lambda = 1$ proves the required integrability. If $f \ne 0$, the trace of u_a is nonzero, so $\mu_f \ne 0$ and (28.11) is strictly positive. The inequality $m_f(0) > 0$ follows from Theorem 28.2. □

Corollary 28.4 (A temporal quadratic form). *For fixed R and f, there is a nonnegative function k_f on $(0, \infty)$ such that*

$$m_f(\lambda) = m_f(0) + \int_0^\infty (1 - e^{-\lambda s}) k_f(s) \, ds, \qquad \int_0^\infty \min\{1, s\} k_f(s) \, ds < \infty.$$

For $a \in C_c^\infty(\mathbb{R})$, put

$$(\mathfrak{M}_f a)(t) = m_f(0) a(t) + \int_0^\infty k_f(s)(a(t) - a(t - s)) \, ds.$$

Then

$$\begin{aligned} \int_{\mathbb{R}} a(t)(\mathfrak{M}_f a)(t) \, dt = m_f(0) &\int_{\mathbb{R}} a(t)^2 \, dt \\ &+ \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} k_f(s)(a(t) - a(t - s))^2 \, dt \, ds. \end{aligned} \tag{28.12}$$

Proof. Take $k_f(s) = \int_{(0,\infty)} \xi e^{-\xi s} \, d\nu_f(\xi)$. The identity $\lambda/(\lambda + \xi) = \int_0^\infty (1 - e^{-\lambda s}) \xi e^{-\xi s} \, ds$ and Tonelli's theorem give the representation. For $s > 0$, the defining integral is finite because $(1 + \xi)\xi e^{-\xi s}$ is bounded. At $\lambda = 1$, the representation and comparison of $1 - e^{-s}$ with $\min\{1, s\}$ give the stated integrability. For translation by $s > 0$, the fundamental theorem of calculus and Minkowski's inequality give

$$\|a - a(\cdot - s)\|_2 \le \min\{s\|a'\|_2, 2\|a\|_2\}.$$

It follows that

$$\int_{\mathbb{R}} |a(t)(a(t) - a(t - s))| \, dt \le \|a\|_2 \min\{s\|a'\|_2, 2\|a\|_2\}.$$

The right side is at most $C_a \min\{s, 1\}$. The integrability of $\min\{s, 1\} k_f(s)$ therefore justifies Fubini in (28.12). For fixed s, translation invariance gives

$$\int_{\mathbb{R}} a(t)(a(t) - a(t - s)) \, dt = \frac{1}{2} \int_{\mathbb{R}} (a(t) - a(t - s))^2 \, dt.$$

Integration in s proves the formula. □

Proof of Theorem I. Apply Theorems 28.2 and 28.3. □

29. Analytic families and finite-energy coefficients

Proposition 29.1. *Let $J \subset \mathbb{R}$ be a connected open interval containing zero. Let $z \mapsto h(z)$ be a real-analytic map from J to $C^\infty_{\mathrm{loc}}(S^2T^*X)$, with $h(0) = 0$ and $g + h(z) > 0$. Suppose that, for some $\lambda > 0$,*

$$\lambda z\, \partial_z h(z) = \mathcal{F}^{\mathrm{RD}}_g(h(z)). \tag{29.1}$$

Write $h(z) = \sum_{j\geq 1} z^j h_j$ near zero, and assume

$$(I - \mathcal{P})h_j \in \mathscr{E} \qquad (j \geq 1).$$

Then $h(z)$ is invariant for every $z \in J$.

Proof. Put $w_j = (I - \mathcal{P})h_j$. Since $\mathcal{N}(0) = D\mathcal{N}_0 = 0$, the coefficient of z in (29.1) gives $(L + \lambda)w_1 = 0$. Suppose $w_1 = \cdots = w_{j-1} = 0$. The coefficient of z^j in $\mathcal{N}(h(z))$ depends only on $h_1, \dots, h_{j-1}$, and is invariant by equivariance. Therefore

$$(L + j\lambda)w_j = 0.$$

Let χ_R be the invariant cutoffs from Lemma 23.1. Testing the equation against $\chi_R^2 w_j$ gives

$$Q(\chi_R w_j) + j\lambda\|\chi_R w_j\|_2^2 = \int_X |d\chi_R|^2|w_j|^2\, dv_g \leq C \int_{\{R<\rho<2R\}} \rho^{-2}|w_j|^2\, dv_g.$$

The right side tends to zero. The first term on the left is nonnegative. On any fixed compact set, $\chi_R = 1$ for large R, so $w_j = 0$ there. Thus $w_j = 0$ on X. Induction proves this for all j. Analyticity gives $(I - \mathcal{P})h = 0$ near zero on every compact set; the identity theorem extends that equality to the connected interval J. □

Remark 29.2. The results on ancient solutions concern an existing solution in a fixed background gauge, with the stated global norms or coefficient conditions. They do not classify all ancient Ricci flows having g as a pointed smooth backward limit. Pointed convergence does not specify global diffeomorphisms or the behavior of the metric at spatial infinity. Nor does Proposition 29.1 assert that every ancient trajectory has an analytic parameterization at its backward limit. The invariant sector still has the infinitely many negative eigenvalues of Corollary 21.3; the symmetry assertions are not stability assertions for the full metric.

IV. Differential identities and coefficient data

Appendix A. Positive identities and interval coordinates

The leading-coefficient identity and the interval basis used in the scalar estimates are recorded here.

Use $L = L_0$, $M = M_0$, and $\Delta = \mathsf{D}$ in the notation of Section 7, and put $\mathcal{A} = qLx + pMy + 2xy$. The leading coefficient matrix has the exact square decomposition

$$(\zeta, c)\mathsf{G}\begin{pmatrix}\zeta \\ c\end{pmatrix} = \frac{(m-1)\mathcal{A}}{\Delta}\left(\zeta + \frac{LM(ux - vy)}{\mathcal{A}}c\right)^2 + \frac{2LMxy}{\mathcal{A}}c^2. \tag{A.1}$$

To check it, the coefficients of ζ^2 and $2\zeta c$ are the displayed entries of G. The remaining coefficient is $\mathsf{G}_{12}^2/\mathsf{G}_{11} + \det \mathsf{G}/\mathsf{G}_{11} = \mathsf{G}_{22}$. Every denominator is positive. This proves positivity of the leading coefficient directly; the restriction $m \leq 8$ is not used in this identity. The all-radius boundary principal form has the independent positive determinant identity (12.14).

For the interval substitution, let P have degree at most D in θ, and write

$$(1+s)^D P\left(\frac{\theta_- + \theta_+ s}{1+s}\right) = \sum_{j=0}^{D} C_j s^j.$$

Then

$$P(\theta) = \frac{1}{(\theta_+ - \theta_-)^D}\sum_{j=0}^{D} C_j(\theta - \theta_-)^j(\theta_+ - \theta)^{D-j}. \tag{A.2}$$

Indeed $s = (\theta - \theta_-)/(\theta_+ - \theta)$ and $1 + s = (\theta_+ - \theta_-)/(\theta_+ - \theta)$. Substitution and cancellation give (A.2). The coefficients may themselves be polynomials in the radial ratio and the nonnegative harmonic shifts. Nonnegative coefficients C_j therefore give positivity on the required interval as soon as one surviving term is strictly positive.

For the higher scalar form, (8.9) and (8.10) yield the seven-square identity (8.29). Its residual signs follow from the coefficient identities in Sections F and G. The exceptional scalar signs follow from Sections H and I. These are finite polynomial comparisons in the positive interval basis (A.2).

Appendix B. Nonconstant fibre harmonics in relative-factor variations

The following identity holds in arbitrary dimensions. It concerns the indicated trace-free test tensors, not the whole scalar-derived tensor sector.

Proposition B.1. *Let (B^n, g_B) be a smooth Riemannian manifold, $n \geq 2$, and let (F^q, g_F) be closed and connected, with $q \geq 2$ and $\mathrm{Ric}_{g_F} = (q-1)g_F$. Suppose $b > 0$ and*

$$g = g_B + b^2 g_F, \qquad \mathrm{Ric}_g = 0.$$

Set $d = n + q$ and $v = \log b$. Let Y be a nonconstant scalar eigenfunction on F, normalized by $\int_F Y^2 = 1$, with $\Delta_F Y = \lambda Y$. For $\phi \in C_c^\infty(\mathrm{int}\, B)$, put

$$T = \phi Y\left(\frac{1}{n}g_B - \frac{1}{q}b^2 g_F\right), \qquad \mathcal{J}(T) = Q(T) - \frac{d}{d-1}\|\delta T\|^2.$$

Then

$$\begin{aligned}\mathcal{J}(T) = \frac{d^2}{n^2 q(d-1)}\int_B b^q\Bigg[&(n-1)\left|d\phi + \frac{d-2}{n-1}\phi\, dv\right|^2 \\ &+ \frac{(d-2)(n-2)(q-1)}{n-1}\phi^2|dv|^2 \\ &+ \frac{n(q-1)}{q}(\lambda - q)b^{-2}\phi^2\Bigg]dv_{g_B} \geq 0.\end{aligned} \tag{B.1}$$

Neither the base nor its test function is required to be rotationally invariant.

Proof. All norms on the right are taken with respect to g_B. Put $\mathcal{D}_B = \mathrm{div}_B \nabla_B$. The vertical Ricci equation gives

$$\mathcal{D}_B v + q|dv|^2 = (q-1)b^{-2}. \tag{B.2}$$

Indeed its unnormalized expression is $\mathrm{Ric}_F = (b\mathcal{D}_B b + (q-1)|db|^2)g_F$; divide by b^2.

In an orthonormal horizontal/vertical frame, the tensor eigenvalues are $\phi Y/n$ and $-\phi Y/q$. Thus it is trace-free. The warped connection gives the horizontal and vertical divergence components

$$(\delta T)_B = -\frac{Y}{n}\big(d\phi + (n+q)\phi\, dv\big), \qquad (\delta T)_F = \frac{\phi}{q} d_F Y.$$

The two components are orthogonal. Since vertical covectors have squared norm b^{-2} times their fibre norm,

$$\|\delta T\|^2 = \int_B b^q \left[\frac{1}{n^2}|d\phi + (n+q)\phi\, dv|^2 + \frac{\lambda}{q^2 b^2}\phi^2\right] dv_{g_B}. \tag{B.3}$$

For the connection energy, derivatives of ϕY multiply the diagonal relative-factor tensor. The derivatives of that tensor are in the two mixed slots and are therefore orthogonal to those diagonal terms. Their contribution is $2d^2\phi^2 Y^2|dv|^2/(n^2 q)$. For a diagonal symmetric tensor with eigenvalues s_i, Ricci-flatness gives

$$-2\langle \mathring{R}T, T\rangle = 2\sum_{i<j} \sec(e_i \wedge e_j)(s_i - s_j)^2.$$

Only mixed pairs contribute. Their curvature sum is $-qb^{-1}\mathcal{D}_B b$. Adding the connection and curvature terms and integrating on F gives

$$Q(T) = \int_B b^q \left[\frac{d}{nq}|d\phi|^2 + \frac{d\lambda}{nqb^2}\phi^2 - \frac{2d^2}{n^2 q}\phi^2 \mathcal{D}_B v\right] dv_{g_B}. \tag{B.4}$$

Subtract $d/(d-1)$ times (B.3). First integrate the cross term in $\phi\, d\phi \cdot dv$. The identity

$$\mathrm{div}_B(b^q \nabla_B v) = (q-1)b^{q-2}$$

follows from (B.2); the compact support removes the boundary term. Collecting coefficients then gives

$$\begin{aligned}\frac{n^2 q(d-1)}{d^2}\mathcal{J}(T) = \int_B b^q \Big[&(n-1)|d\phi|^2 - (d-2)\phi^2 \mathcal{D}_B v \\ &+ \frac{n(q-1)}{q}(\lambda - q)b^{-2}\phi^2\Big] dv_{g_B}.\end{aligned} \tag{B.5}$$

A second integration by parts gives

$$-\int_B b^q \phi^2 \mathcal{D}_B v = \int_B b^q\big(2\phi\langle d\phi, dv\rangle + q\phi^2|dv|^2\big).$$

Complete the square in (B.5). The remaining coefficient is

$$q(d-2) - \frac{(d-2)^2}{n-1} = \frac{(d-2)(n-2)(q-1)}{n-1},$$

which proves the identity in (B.1).

Finally, the integrated Bochner formula on F gives

$$\lambda^2 \int_F Y^2 = \int_F |\nabla_F^2 Y|^2 + (q-1)\lambda \int_F Y^2 \geq \frac{\lambda^2}{q}\int_F Y^2 + (q-1)\lambda\int_F Y^2.$$

Since $\lambda > 0$, this implies $\lambda \geq q$. Every term on the right of (B.1) is consequently nonnegative. ☐

Remark B.2. For a Böhm metric, take $n = p + 1$, $B = (\mathbb{R}^{p+1}, dt^2 + a^2\gamma_p)$ and $F = S^q$. The proposition applies across the smooth singular orbit because $b > 0$ there. It rules out a negative value of the critical divergence expression in these nonconstant-fibre relative-factor test tensors. It does not identify every scalar-derived tensor with such a trial, and it does not bound the finite-boundary form q_R on its entire averaging-zero domain. The full initial-index assertion is instead proved in Theorems 20.1 and 21.2; it does not follow from this restricted family of test tensors.

Appendix C. The flat-model identities

The proof in Sections 18 to 20 does not use coefficient lists. Its sign assertions are (19.3), (19.9), and (19.10).

The estimate for the radial logarithmic derivative also follows by integrating its differential equation. Normalize the positive regular solution w_x by $w_x(1) = 1$. Integration of its equation gives

$$z_\ell(x) = x\int_0^1 r^{2\ell+p} w_x(r)\,dr.$$

For $x > 0$, the equation implies $0 < w_x(r) < 1$ on $[0, 1)$, and hence

$$0 < z_\ell(x) < \frac{x}{2\ell + p + 1}.$$

At $x = 0$, $w_0 = 1$. Differentiating the integral at zero gives $z_\ell'(0) = (2\ell + p + 1)^{-1}$. These are exactly the bounds used in the boundary residual; no numerical root exclusion is involved.

Appendix D. The covariant second-derivative cancellation

This appendix records the exact jet calculation underlying Lemma 26.1. Put

$$\Theta_{ij\ell} = \nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}, \qquad T_{ab\ell} = \nabla_a h_{b\ell} - \tfrac{1}{2}\nabla_\ell h_{ab}.$$

Thus $C^k_{ij} = \frac{1}{2}G^{k\ell}\Theta_{ij\ell}$ and $\mathcal{W}^k = G^{k\ell}G^{ab}T_{ab\ell}$. For a symmetric contravariant tensor A, define

$$\mathcal{U}_A(h)_{ij} = A^{ab}\big([\nabla_i, \nabla_a]h_{bj} + [\nabla_j, \nabla_a]h_{bi} + \tfrac{1}{2}[\nabla_j, \nabla_i]h_{ab}\big). \tag{D.1}$$

The commutator acts on the two covariant slots of h; thus it is linear in Rm_g and in h. Contraction with g^{ab} gives

$$\mathcal{U}_{g^{-1}}(h)_{ij} = 2(\mathring{R}h)_{ij} - \mathrm{Ric}_i{}^k h_{kj} - \mathrm{Ric}_j{}^k h_{ik} = 2(\mathring{R}h)_{ij}.$$

The terms containing two covariant derivatives of h in the Ricci–DeTurck expression are exactly

$$\begin{aligned}&G^{ab}\big(-\nabla_a\nabla_i h_{jb} - \nabla_a\nabla_j h_{ib} + \nabla_a\nabla_b h_{ij} + \nabla_j\nabla_i h_{ab}\big)\\ &+G^{ab}\big(\nabla_i\nabla_a h_{bj} - \tfrac{1}{2}\nabla_i\nabla_j h_{ab} + \nabla_j\nabla_a h_{bi} - \tfrac{1}{2}\nabla_j\nabla_i h_{ab}\big).\end{aligned}$$

Their sum is

$$G^{ab}\nabla_a\nabla_b h_{ij} + \mathcal{U}_G(h)_{ij}. \tag{D.2}$$

For the trace term one uses the exact identity $C^k_{ik} = \frac{1}{2}G^{ab}\nabla_i h_{ab}$.

Every remaining term is quadratic in ∇h. Explicitly, their sum is

$$\begin{aligned}\mathcal{Q}_{ij} = &-(\nabla_k G^{k\ell})\Theta_{ij\ell} + (\nabla_j G^{k\ell})\nabla_i h_{k\ell} - 2C^k_{k\ell}C^\ell_{ij} + 2C^k_{j\ell}C^\ell_{ik} + \mathcal{W}^k\nabla_k h_{ij}\\ &+ (g+h)_{kj}(\nabla_i G^{k\ell})G^{ab}T_{ab\ell} + (\nabla_i G^{ab})T_{abj}\\ &+ (g+h)_{ki}(\nabla_j G^{k\ell})G^{ab}T_{ab\ell} + (\nabla_j G^{ab})T_{abi}.\end{aligned} \tag{D.3}$$

The derivative of G in each term is replaced by (26.8). Equations (D.2)–(D.3) give

$$\mathcal{F}^{\mathrm{RD}}_g(h)_{ij} = G^{ab}\nabla_a\nabla_b h_{ij} + \mathcal{U}_G(h)_{ij} + \mathcal{Q}_{ij}.$$

Consequently one can take

$$\mathcal{C}(h)(\mathrm{Rm}_g) = \mathcal{U}_{G-g^{-1}}(h), \qquad \mathcal{B}(h)(\nabla h, \nabla h) = \mathcal{Q}.$$

Since $G - g^{-1} = O(h)$ and $\mathcal{U}_A(h)$ is linear in both A and h, one has $\mathcal{C}(h) = O(h^2)$. The coefficients of $\mathcal{B}$ are finite products of the inverse metric and $g + h$, and hence are smooth and uniformly bounded on $|h| \leq 1/2$. All contractions in this identity are with the displayed inverse metric; the calculation is covariant and does not require coordinates normal for $g + h$.

Appendix E. Component formulas and elliptic coordinate estimates

E.1. **The vector equation in a warped product.** We give the covariant calculation used in Proposition 5.2. Let $g = g_B + r^2\gamma_n$ be Ricci-flat, with $r > 0$ on the regular part. Write D and $\widehat{\nabla}$ for the base and unit-sphere connections, and put $\ell_a = D_a \log r$. Base indices are raised with g_B and fibre indices in this subsection with γ_n. For a coclosed one-form ω, put

$$\widehat{\nabla}^*\widehat{\nabla}\omega = k_V^2\omega, \qquad S_{ij} = \widehat{\nabla}_{(i}\omega_{j)}.$$

Subtracting the Lie derivative of $C\omega^\sharp$ reduces the calculation to

$$h_{ab} = h_{ij} = 0, \qquad h_{ai} = r^2\mathcal{A}_a\omega_i, \qquad \mathcal{A} = B - dC.$$

In the Killing case take $C = 0$. Naturality and $\mathrm{Ric}_g = 0$ justify this subtraction on every relatively compact subset of the regular part.

Lemma E.1. *Set $F_{ab} = D_a\mathcal{A}_b - D_b\mathcal{A}_a$. The Ricci variation of this tensor has components*

$$\begin{aligned}
\mathrm{Ric}_g'(h)_{ab} &= 0,\\
2r^n\,\mathrm{Ric}_g'(h)_{ai} &= \left\{\delta_B(r^{n+2}F)_a + (k_V^2 - n + 1)r^n\mathcal{A}_a\right\}\omega_i,\\
\mathrm{Ric}_g'(h)_{ij} &= r^{2-n}D_a(r^n\mathcal{A}^a)S_{ij}.
\end{aligned}\tag{E.1}$$

Proof. Let C^A_{BC} be the variation of the full Levi–Civita connection. The nonzero first derivatives of h, before taking their symmetric combination, are

$$\begin{aligned}
\nabla_a h_{bi} &= r^2(D_a\mathcal{A}_b + \ell_a\mathcal{A}_b)\omega_i,\\
\nabla_i h_{ab} &= -r^2(\ell_a\mathcal{A}_b + \ell_b\mathcal{A}_a)\omega_i,\\
\nabla_i h_{aj} &= r^2\mathcal{A}_a\widehat{\nabla}_i\omega_j,\\
\nabla_i h_{jk} &= r^4\ell^a\mathcal{A}_a(\gamma_{ij}\omega_k + \gamma_{ik}\omega_j).
\end{aligned}$$

The other derivative types, $\nabla_a h_{bc}$ and $\nabla_a h_{ij}$, are zero. Consequently the complete list of connection variations is

$$\begin{aligned}
C^a_{bc} &= 0,\\
C^i_{ab} &= P_{ab}\omega^i, \quad P_{ab} = D_{(a}\mathcal{A}_{b)} + \ell_a\mathcal{A}_b + \ell_b\mathcal{A}_a,\\
C^a_{bi} &= \frac{r^2}{2}(F_b{}^a - 2\ell^a\mathcal{A}_b)\omega_i,\\
C^i_{aj} &= \frac{\mathcal{A}_a}{2}(\widehat{\nabla}_j\omega^i - \widehat{\nabla}^i\omega_j),\\
C^a_{ij} &= r^2\mathcal{A}^aS_{ij}, \qquad C^i_{jk} = r^2\ell^a\mathcal{A}_a\gamma_{jk}\omega^i.
\end{aligned}\tag{E.2}$$

Lower connection indices are symmetric. These formulas give $C^A_{aA} = C^A_{iA} = 0$, in agreement with $\mathrm{tr}_g h = 0$. Hence $\mathrm{Ric}_g'(h)_{AB} = \nabla_C C^C_{AB}$.

At a point choose normal coordinates separately on the base and the unit sphere. The cross connection coefficients are $\Gamma^i_{aj} = \ell_a\delta^i_j$ and $\Gamma^a_{ij} = -r^2\ell^a\gamma_{ij}$. The divergence of a $(1,2)$-tensor is

$$\nabla_C C^C_{AB} = \partial_C C^C_{AB} + \Gamma^C_{CD}C^D_{AB} - \Gamma^D_{CA}C^C_{DB} - \Gamma^D_{CB}C^C_{AD}.$$

Separating its base and fibre contractions gives the six identities

$$\begin{aligned}
\nabla_c C^c_{ab} &= 0,\\
\nabla_i C^i_{ab} &= \widehat{\nabla}_i C^i_{ab} - \ell_a C^i_{ib} - \ell_b C^i_{ai} = 0,\\
\nabla_b C^b_{ai} &= D_b C^b_{ai} - \ell_b C^b_{ai},\\
\nabla_j C^j_{ai} &= \widehat{\nabla}_j C^j_{ai} + n\ell_b C^b_{ai} - \ell_a C^j_{ji} + r^2\ell^b \gamma_{ji} C^j_{ab},\\
\nabla_a C^a_{ij} &= D_a C^a_{ij} - 2\ell_a C^a_{ij},\\
\nabla_k C^k_{ij} &= \widehat{\nabla}_k C^k_{ij} + n\ell_a C^a_{ij} + r^2\ell^a(\gamma_{ki} C^k_{aj} + \gamma_{kj} C^k_{ia}).
\end{aligned} \tag{E.3}$$

In these formulas D differentiates only the base coefficient, and $\widehat{\nabla}$ only the displayed sphere tensors. The equalities in the second line use $\widehat{\nabla}_i\omega^i = 0$ and $C^i_{ai} = 0$. In the fourth line,

$$\widehat{\nabla}_j C^j_{ai} = \tfrac{1}{2}(k_V^2 + n - 1)\mathcal{A}_a\omega_i, \quad C^j_{ji} = r^2\ell^b\mathcal{A}_b\omega_i, \quad \gamma_{ji}C^j_{ab} = P_{ab}\omega_i.$$

The two terms in parentheses in the last line cancel, by the antisymmetry of C^k_{aj} after lowering its fibre index. Its first term is zero by coclosedness. These formulas account for both covariant slots as well as the contravariant slot.

The base and fibre contributions to the mixed component, with the common factor ω_i suppressed, are respectively

$$\begin{aligned}
\nabla_b C^b_{ai} &= \frac{r^2}{2}\left[D_b F_a{}^b + \ell_b F_a{}^b - 2(D_b\ell^b)\mathcal{A}_a - 2\ell^b D_b\mathcal{A}_a - 2|\ell|^2\mathcal{A}_a\right],\\
\nabla_j C^j_{ai} &= \frac{k_V^2 + n - 1}{2}\mathcal{A}_a + r^2\left[\tfrac{n}{2}\ell_b F_a{}^b + \tfrac{1}{2}\ell^b(D_a\mathcal{A}_b + D_b\mathcal{A}_a) - (n-1)|\ell|^2\mathcal{A}_a\right].
\end{aligned}$$

Here the fibre commutator is $\widehat{\nabla}_j\widehat{\nabla}_i\omega^j = (n-1)\omega_i$. Their sum is

$$\frac{1}{2}\left\{r^2 D_b F_a{}^b + (n+2)r^2\ell_b F_a{}^b - 2r^2(D_b\ell^b + n|\ell|^2)\mathcal{A}_a + (k_V^2 + n - 1)\mathcal{A}_a\right\}.$$

The vertical Ricci-flat equation is

$$D_b\ell^b + n|\ell|^2 = (n-1)r^{-2}.$$

Also $\delta_B(r^{n+2}F)_a = D^b(r^{n+2}F_{ab})$. These identities give the middle equation in (E.1). For the ij component the base contribution is $r^2(D_a\mathcal{A}^a)S_{ij}$ and the remaining contribution is $nr^2\ell_a\mathcal{A}^a S_{ij}$. In the ab component all angular contractions are either $\widehat{\nabla}_i\omega^i$ or the trace of an antisymmetric fibre derivative; both vanish. This proves the other two equations. □

If $k_V^2 > n-1$, applying δ_B to the middle equation gives $\delta_B(r^n\mathcal{A}) = 0$, because $\delta_B^2 = 0$ on two-forms. Thus the last equation is then a consequence of the middle one. When $k_V^2 = n-1$, integration on the sphere gives $S = 0$, so the last equation is identically zero. These statements include every vector component, not only the mixed projection.

The endpoint estimates in Proposition 5.2 can be obtained without choosing a smooth gauge potential at the collapsing fibre. In a fixed angular type, the coefficients of a smooth mixed tensor in Cartesian normal coordinates give $B = O(t^{-1})$, $dB = O(t^{-2})$. If $S \neq 0$, projection of the fibre block gives $C = O(1)$ and $dC = O(t^{-1})$. Therefore $\mathcal{A} = O(t^{-1})$ and $d\mathcal{A} = dB = O(t^{-2})$ on the regular part. The inner boundary integral is bounded by $Ct^{n+2}t^{-1}t^{-2} = Ct^{n-1}$ times a bounded base-boundary area. It tends to zero for $n \geq 2$. If the chosen fibre does not collapse, all projected coefficients are smooth on the base across its centre; there is no inner boundary. This proves the integration statements used in (5.3).

E.2. **All scalar components of the Ricci variation.** At a fixed regular radius put $\gamma_P = a^2\gamma_p$ and $\gamma_Q = b^2\gamma_q$. We use the ordered covariant basis

$$YZ\gamma_p, \quad YZ\gamma_q, \quad Z\nabla^2 Y, \quad Y\nabla^2 Z, \quad dY \otimes dZ.$$

The last entry denotes its mixed component; the full tensor is symmetric. Its coefficient vector is $(2a^2A, 2b^2B, 2a^2C, 2b^2D, 2e)$. Define

$$\Xi = pA + qB - \lambda C - \mu D, \qquad d_P = -2A + 2(\lambda - p + 1)C + 2M_0 e,$$
$$d_Q = -2B + 2(\mu - q + 1)D + 2L_0 e.$$

Then

$$\operatorname{tr}_\gamma h = 2\Xi YZ, \qquad (\delta_\gamma h)_P = d_P Z\, dY, \qquad (\delta_\gamma h)_Q = d_Q Y\, dZ. \tag{E.4}$$

Lemma E.2. *In this basis the intrinsic Ricci variation has coefficients*

component	$\operatorname{Ric}'_\gamma(h)$
PP trace	$a^2(L_0 + M_0)A$
QQ trace	$b^2(L_0 + M_0)B$
PP Hessian	$(2-p)A - qB - M_0 c + 2(p-1)C$
QQ Hessian	$-pA + (2-q)B - L_0 c + 2(q-1)D$
PQ	$-(p-1)(A-C) - (q-1)(B-D)$

(E.5)

Together with the normal-coordinate variation, these coefficients give all five equations in (7.7)*, as well as the single-factor equations* (6.4)–(6.6).

Proof. For a scalar eigenfunction Y on the unit sphere, commuting derivatives with its curvature-one tensor gives

$$\nabla^*\nabla dY = (\lambda - p + 1)dY,$$
$$\operatorname{div} \nabla^2 Y = -(\lambda - p + 1)dY,$$
$$\nabla^*\nabla(\nabla^2 Y) = (\lambda - 2p)\nabla^2 Y - 2\lambda Y\gamma_p.$$

Writing $H_{ij} = \widehat{\nabla}_i\widehat{\nabla}_j Y$, the two successive commutations give, since the curvature tensor is parallel,

$$(\widehat{\nabla}^*\widehat{\nabla} H)_{ij} = \widehat{\nabla}_i\widehat{\nabla}_j(\Delta Y) - 2(p-1)H_{ij} + 2\big((\operatorname{tr} H)\gamma_{p,ij} - H_{ij}\big).$$

Here $\operatorname{tr} H = -\lambda Y$. Substitution yields the third identity, including its scalar-metric term. After scaling and taking products, the orbit Lichnerowicz expression

$$\Delta_{L,\gamma} h = \nabla^*_\gamma\nabla_\gamma h + \operatorname{Ric}_\gamma \circ h + h \circ \operatorname{Ric}_\gamma - 2\mathring{R}_\gamma h$$

is multiplication by $L_0 + M_0$ on each of the five displayed generators. For the mixed generator the rough Laplacian has coefficient $L_0 + M_0 - r_P - r_Q$; the two Ricci contractions restore $r_P + r_Q$, and the product curvature action on the mixed block is zero. The factor traces have cancellation of their Ricci and curvature terms. For a Hessian block the third identity above gives the stated coefficient.

Equation (E.4) follows by contracting each block. The coefficients of $\delta^*_\gamma\delta_\gamma h$ are

$$(0, 0, d_P, d_Q, (d_P + d_Q)/2),$$

and those of $\nabla^2_\gamma \operatorname{tr}_\gamma h$ are $(0, 0, 2\Xi, 2\Xi, 2\Xi)$. Substitution in

$$2\operatorname{Ric}'_\gamma h = \Delta_{L,\gamma} h - 2\delta^*_\gamma\delta_\gamma h - \nabla^2_\gamma \operatorname{tr}_\gamma h$$

gives the following coefficient vector before collecting:

$$\begin{pmatrix} a^2(L_0 + M_0)A \\ b^2(L_0 + M_0)B \\ a^2(L_0 + M_0)C - d_P - \Xi \\ b^2(L_0 + M_0)D - d_Q - \Xi \\ (L_0 + M_0)e - (d_P + d_Q)/2 - \Xi \end{pmatrix}. \tag{E.6}$$

Equivalently, its matrix on $(A, B, C, D, e)^T$ is

$$\begin{pmatrix} a^2(L_0+M_0) & 0 & 0 & 0 & 0 \\ 0 & b^2(L_0+M_0) & 0 & 0 & 0 \\ 2-p & -q & a^2M_0+2(p-1) & b^2M_0 & -2M_0 \\ -p & 2-q & a^2L_0 & b^2L_0+2(q-1) & -2L_0 \\ -(p-1) & -(q-1) & p-1 & q-1 & 0 \end{pmatrix}. \tag{E.7}$$

For the third row use $a^2L_0 = \lambda$, $b^2M_0 = \mu$; for the fourth use the same two identities in the opposite order. The last row follows from

$$\tfrac{1}{2}(d_P + d_Q) + \Xi = (p-1)A + (q-1)B - (p-1)C - (q-1)D + (L_0 + M_0)e.$$

Replacing $2e - a^2C - b^2D$ by c gives every row of (E.5).

Write k_i for the five coefficients of h. Differentiating the normal-coordinate Ricci formula, the extrinsic coefficient in a PP position is

$$-\tfrac{1}{2}k_i'' + (2u - H/2)k_i' - 2u^2k_i.$$

The PP trace has in addition $-ua^2\Xi'$. In a QQ position replace u by v; its trace has in addition $-vb^2\Xi'$. The mixed coefficient is

$$-\tfrac{1}{2}k_5'' + (u + v - H/2)k_5' - 2uvk_5 = -e'' - (H - 2u - 2v)e' - 4uve.$$

For any scalar coefficient f,

$$-\tfrac{1}{2}(2a^2f)'' + (2u - H/2)(2a^2f)' - 2u^2(2a^2f) = -a^2\big(f'' + Hf' + 2(u' + Hu)f\big).$$

Use $u' + Hu = r_P$ and its exchanged identity. Adding the five intrinsic rows therefore gives the full coefficient vector

$$\begin{pmatrix} -a^2[A'' + HA' + u\Xi' - (L_0 + M_0 - 2r_P)A] \\ -b^2[B'' + HB' + v\Xi' - (L_0 + M_0 - 2r_Q)B] \\ -a^2(C'' + HC') + (2-p)A - qB - M_0c \\ -b^2(D'' + HD') - pA + (2-q)B - L_0c \\ -e'' - (H - 2u - 2v)e' - 4uve - (p-1)(A-C) - (q-1)(B-D) \end{pmatrix}. \tag{E.8}$$

The third and fourth rows use respectively $2a^2r_P = 2(p-1)$ and $2b^2r_Q = 2(q-1)$. Setting this vector to zero gives (7.7), with all five nonzero leading coefficients specified.

Here are the remaining constraints without a coefficient projection left implicit. Let $\dot{A}$ denote the variation of the shape endomorphism. Its diagonal blocks have coefficients (A', C') and (B', D'); the two mixed blocks have coefficients $(e' - 2ve)/a^2$ and $(e' - 2ue)/b^2$. In the variation of $\operatorname{div} A - dH$, the connection contribution in the first factor is $(u - v)(qB - \mu D)Z\,dY$, and in the second it is $(v - u)(pA - \lambda C)Y\,dZ$. Indeed, if $\dot{\Gamma}^a_{bc}$ is the orbit connection variation, then

$$\dot{\Gamma}^b_{ab} = \tfrac{1}{2}\nabla_a \operatorname{tr}_\gamma h = \Xi\nabla_a(YZ), \qquad \sum_b A^b{}_b\dot{\Gamma}^b_{ba} = \tfrac{1}{2}\sum_b A^b{}_b\nabla_a h_b{}^b.$$

Here the diagonal coefficients of the background shape operator are parallel on the orbit, whereas Ξ is a radial coefficient. It follows that the momentum coefficients are

$$\begin{aligned} M_P &= A' - (\lambda - p + 1)C' - M_0(e' - 2ue) - \Xi' + (u - v)(qB - \mu D), \\ M_Q &= B' - (\mu - q + 1)D' - L_0(e' - 2ve) - \Xi' + (v - u)(pA - \lambda C). \end{aligned} \tag{E.9}$$

These are the two rows in (7.5). Furthermore,

$$\begin{aligned} \mathfrak{r} &= L_0d_P + M_0d_Q + 2(L_0 + M_0)\Xi - 2r_P(pA - \lambda C) - 2r_Q(qB - \mu D), \\ \mathfrak{g} &= s_P(pA' - \lambda C') + s_Q(qB' - \mu D'). \end{aligned}$$

The full scalar identities are

$$DR_\gamma(h) = \mathfrak{r}\,YZ, \qquad H\dot{H} - \operatorname{tr}(A\dot{A}) = \mathfrak{g}\,YZ.$$

The linearized Gauss identity therefore gives $\mathfrak{r} = 2\mathfrak{g}$ by equality of the coefficients of the nonzero function YZ. There is no pointwise division at its zeros. Expanding this coefficient identity gives (7.6). Finally set $Y = 1$, $\lambda = 0$, and remove the vanishing first-factor Hessian and mixed components. Renaming the second Hessian coefficient D as C gives (6.4)–(6.6). □

The remaining elimination in Lemma 6.1 can be recorded in a form with no division by a possibly vanishing coefficient. Use the notation of that lemma and put $y = A/u$. The momentum equation gives

$$(q-1)C' = pA' + (q-1)B' + p(u-v)A.$$

Insert $A = uy$, $B = Z_0 + vy$ into the Hamiltonian equation, substitute this expression for C', and use $D_0 = H^2 - pu^2 - qv^2$. The coefficient of y is zero, and the result is

$$Fy' + (q-1)(\mu - q)(s_0 Z_0' + r_Q Z_0) = 0.$$

Since $F > 0$, this is (6.7). The PP equation becomes

$$y'' + \left(\frac{2r_P}{u} - \frac{\mu s_0}{q-1}\right) y' - (\mu - q) Z_0' = 0. \tag{E.10}$$

For

$$b_1 = -\frac{(q-1)(\mu-q)s_0}{F}, \qquad b_0 = -\frac{(q-1)(\mu-q)r_Q}{F}, \qquad a_1 = \frac{2r_P}{u} - \frac{\mu s_0}{q-1},$$

substitution of $y' = b_1 Z_0' + b_0 Z_0$ in (E.10) has coefficients

$$b_1, \qquad b_1' + b_0 + a_1 b_1 - (\mu - q), \qquad b_0' + a_1 b_0.$$

Direct differentiation using (3.3) gives

$$\frac{b_1' + b_0 + a_1 b_1 - (\mu-q)}{b_1} = H - \frac{J_0'}{J_0}, \qquad \frac{b_0' + a_1 b_0}{b_1} = -V_\mu. \tag{E.11}$$

Here $b_1 < 0$, so division is legitimate. This proves the scalar equation with exactly the weight $\varrho_0 = W/J_0$.

E.3. **The normal component of the boundary Hessian.** We verify the covariant normal calculation in Proposition 12.2. Let $e_0 = \nu$ and let e_i be an orthonormal frame on a product orbit, extended by radial parallel transport. At the point of the calculation choose the tangential connection coefficients to vanish. The full connection is

$$\nabla_0 e_i = 0, \qquad \nabla_i e_0 = A_i{}^j e_j, \qquad \nabla_i e_j = \nabla_i^\Sigma e_j - A_{ij} e_0.$$

For $s = S_{00}$, $\eta_i = S_{0i}$ and $K_{ij} = S_{ij}$, this gives

$$\begin{aligned} (\nabla_i S)_{00} &= D_i s - 2A_i{}^j \eta_j, \\ (\nabla_i S)_{0j} &= D_i \eta_j - A_i{}^k K_{kj} + A_{ij} s, \\ (\nabla_i S)_{jk} &= D_i K_{jk} + A_{ij}\eta_k + A_{ik}\eta_j. \end{aligned} \tag{E.12}$$

Contracting the derivative index once more, including its connection term, yields

$$\begin{aligned} (\nabla^*\nabla S)_{00} = &-s'' - Hs' + \Delta_\Sigma s + 4A : D\eta + 2(\operatorname{div}_\Sigma A)\cdot \eta \\ &- 2A^2 : K + 2|A|^2 s. \end{aligned} \tag{E.13}$$

More explicitly, at the chosen point the tangential contraction is

$$\begin{aligned} &\sum_i \left(\nabla_{e_i}\nabla_{e_i} S - \nabla_{\nabla_{e_i} e_i} S\right)_{00} \\ &= \sum_i D_i(D_i s - 2A_i{}^j \eta_j) - 2\sum_{i,j} A_i{}^j (D_i \eta_j - A_i{}^k K_{kj} + A_{ij} s) + Hs' \\ &= -\Delta_\Sigma s - 2(\operatorname{div}_\Sigma A)\cdot\eta - 4A : D\eta + 2A^2 : K - 2|A|^2 s + Hs'. \end{aligned} \tag{E.14}$$

The last Hs' comes from $\sum_i \nabla_{e_i} e_i = -H\nu$ at this point. Negating this contraction and subtracting s'' proves (E.13). The normal curvature endomorphism is $\mathcal{R}_\nu = -A' - A^2$, so

$$-2(\mathring{R}S)_{00} = 2(A' + A^2) : K.$$

Since A is parallel on the product orbit, the divergence term in (E.13) vanishes. Therefore

$$(LS)_{00} = -s'' - Hs' + \Delta_\Sigma s + 4A : D\eta + 2|A|^2 s + 2A' : K. \tag{E.15}$$

The equations $\delta S = 0$ are

$$s' + Hs + \operatorname{div}_\Sigma \eta - A : K = 0, \qquad \eta' + (H\gamma + A)\eta + \operatorname{div}_\Sigma K = 0.$$

In radial parallel identification the product connection gives

$$(\operatorname{div}_\Sigma \eta)' = \operatorname{div}_\Sigma \eta' - A : D\eta.$$

At Σ_R one has $s = 0$, $K = -\phi A_0$ and $\operatorname{tr} K' = -s'$. Thus

$$\begin{aligned}
s' &= -\operatorname{div}_\Sigma \eta - \phi|A_0|^2,\\
\eta' &= A_0 d\phi - (H\gamma + A)\eta,\\
(LS)_{00} &= \operatorname{div}_\Sigma \eta' + 3A : D\eta + A' : K - A : K'\\
&= A_0 : D^2\phi - \frac{(m-1)H}{m}\operatorname{div}_\Sigma \eta + 2A_0 : D\eta - \mathcal{V}_R\phi - A_0 : K'.
\end{aligned}$$

This is (12.11).

For the Green identity, put $J = LS - (m-1)\nabla^2 F = 2E'(h)$. It is TT. Since $h = S + \delta^* X^\flat + Fg$, $X^\top = 0$ and $S_{00} = 0$ on the boundary, integration gives

$$\int_M \langle J, h\rangle = Q(S) - \int_\Sigma \langle S', S\rangle - (m-1)\int_\Sigma \langle \eta, dF\rangle + \int_\Sigma \phi\big((LS)_{00} - (m-1)F_{00}\big).$$

The terms $\phi A_0 : K'$ cancel. The other terms, after tangential integration by parts, are respectively

$$\begin{aligned}
-2\langle \eta', \eta\rangle &= 2(H\gamma + A)(\eta,\eta) - 2A_0(\eta, d\phi),\\
-(m-1)\langle \eta, dF\rangle &= \frac{(m-1)H}{m}\langle \eta, d\phi\rangle,\\
\int_\Sigma \phi\big((LS)_{00} + A_0 : K'\big) &= \int_\Sigma \left[-A_0(d\phi, d\phi) + \frac{(m-1)H}{m}\langle \eta, d\phi\rangle - 2A_0(\eta, d\phi) - \mathcal{V}_R\phi^2\right],\\
-(m-1)\int_\Sigma \phi F_{00} &= \frac{(m-1)H}{m}\int_\Sigma |d\phi|^2 - \frac{(m-1)H^2}{m}\langle \phi, \mathcal{N}_R\phi\rangle.
\end{aligned}$$

For the last line use $F|_\Sigma = -H\phi/m$ and $F_{00} = \Delta_\Sigma F - H\mathcal{N}_R F$. Summation gives (12.8), since $((m-1)H/m)\gamma - A_0 = H\gamma - A$.

For two compatible pairs (S_1, ϕ_1) and (S_2, ϕ_2), put $T_i = S_i + \mathcal{K}X_i$, $\eta_i = S_i(\nu, \cdot)^\top$, and $\mathsf{C}_R = ((m-1)H/m)\gamma - 2A_0$. Polarization of the preceding calculation gives

$$\begin{aligned}
q_R(T_1, T_2) &= Q_R(S_1, S_2)\\
&\quad + \int_{\Sigma_R} \big[2(H\gamma + A)(\eta_1, \eta_2) + \mathsf{C}_R(\eta_1, d\phi_2) + \mathsf{C}_R(\eta_2, d\phi_1)\big]\, dA_\gamma\\
&\quad + \int_{\Sigma_R} \big[(H\gamma - A)(d\phi_1, d\phi_2) - \mathcal{V}_R\phi_1\phi_2\big]\, dA_\gamma\\
&\quad - \frac{(m-1)H^2}{m}\langle \phi_1, \mathcal{N}_R\phi_2\rangle.
\end{aligned} \tag{E.16}$$

Indeed each term is obtained by subtracting the two diagonal values from its value on the sum and dividing by two. The last term is symmetric because

$$\langle \phi_1, \mathcal{N}_R\phi_2\rangle = \int_{M_R} \langle d\mathcal{H}_R\phi_1, d\mathcal{H}_R\phi_2\rangle\, dv_g.$$

Thus (E.16) contains all mixed terms required in the form-norm comparison.

E.4. **Norm convergence in one fixed representation.** We detail the estimates used at the singular limit in Theorem 20.1. All operators below are conjugated by the bundle identification in (20.3); the volume is normalized as in (20.2). Fix a representation ϖ before taking $R \downarrow 0$. We use the reference product Sobolev norms, denoted $\|\cdot\|_j$.

The finite-dimensional vertical angular spaces give, for every fixed j, k and each of the tensor bundles involved,

$$\|\nabla^j_{S^q} U\|_{H^k_B} \leq C_{\varpi,j,k}\|U\|_{H^k_B}. \tag{E.17}$$

Indeed the multiplicity of a fixed compact-group representation in sections of a homogeneous finite-rank bundle is bounded by the dimension of the intertwining maps into its isotropy fibre. Covariant vertical differentiation is equivariant and maps into another such finite-rank bundle. The resulting linear maps on these finite-dimensional spaces are bounded. The assertion applies pointwise in the normal variable and then after each normal derivative. No estimate uniform in ϖ follows or is used.

The Cartesian estimates follow from the smooth even expansions at the singular orbit. Set $f_R(x) = a(R|x|)/(R|x|)$ and $b_R(x) = b(R|x|)/b_0$. There are smooth functions a_1, b_1 on a fixed interval such that

$$f_R(x) = 1 + R^2|x|^2 a_1(R^2|x|^2), \qquad b_R(x) = 1 + R^2|x|^2 b_1(R^2|x|^2).$$

The normal metric is

$$(\widehat{g}_R)_{AB} = \delta_{AB} + R^2 c_R(x)(|x|^2\delta_{AB} - x_A x_B), \quad c_R = 2a_1(R^2|x|^2) + R^2|x|^2 a_1(R^2|x|^2)^2. \tag{E.18}$$

Every fixed number of derivatives of c_R is uniformly bounded on the closed unit ball. The metric inverse, density, and positive square root have the same $O(R^2)$ difference from their limiting coefficients. The vertical frame factor is $R/b(R|x|) = R/b_0 + O(R^3)$, and $d_x \log b(R|x|) = O(R^2)$ in every fixed coefficient norm.

It follows that each first-order difference is a finite sum of terms

$$a^A_R\partial_A + Rb^I_R\widehat{\nabla}_I + c^0_R, \qquad \|a^A_R\|_{C^j} + \|c^0_R\|_{C^j} \leq C_j R^2, \quad \|b^I_R\|_{C^j} \leq C_j.$$

For second-order differences the normal–normal, normal–vertical, and vertical–vertical coefficients have orders R^2, R, and R^2, respectively. Coefficient differentiation preserves these bounds. The product rule and (E.17) bound every term from H^{j+1}_ϖ to H^j_ϖ, or from H^{j+2}_ϖ to H^j_ϖ, by $C_{\varpi,j}R$. Restriction to the fixed outer collar and the trace estimate $H^2 \to H^{1/2}$ for first derivatives give the same bound for the first-order boundary operators. No constant in these estimates is asserted to be independent of ϖ.

The expansions (20.2)–(20.4) and (E.17) imply, on the fixed type,

$$\|\nabla_R - \nabla_0\|_{H^{j+1}\to H^j} + \|\delta_R - \delta_0\|_{H^{j+1}\to H^j} + \|\mathcal{K}_R - \mathcal{K}_0\|_{H^{j+1}\to H^j} \leq C_{\varpi,j}R, \tag{E.19}$$

$$\|\delta_R\mathcal{K}_R - \delta_0\mathcal{K}_0\|_{H^{j+2}\to H^j} \leq C_{\varpi,j}R. \tag{E.20}$$

Here ∇_0 differentiates only in the normal ball. To verify the orders, a second-order expression has a normal-normal part whose coefficient error is $O(R^2)$, mixed normal-vertical parts with a factor R, and vertical-vertical parts with a factor R^2. Every vertical derivative in the latter two is bounded by (E.17). First and zeroth order connection terms have the same or higher powers of R. Differentiating coefficients in the fixed charts preserves their displayed orders. The metric densities differ by $O(R^2)$ in every fixed differentiability norm. The tangential derivatives on the outer boundary satisfy the corresponding estimate

$$\|D^\Sigma_R - D^\Sigma_0\|_{H^{j+1}(\partial)\to H^j(\partial)} \leq C_{\varpi,j}R. \tag{E.21}$$

Boundary traces of the first-order operators differ by $O_\varpi(R)$ as maps $H^2 \to H^{1/2}$; the tangential Dirichlet trace is fixed by the bundle identification.

For the mixed vector operator include the boundary operators in the map

$$\mathfrak{L}_R : H^2_\varpi \longrightarrow L^2_\varpi \oplus H^{3/2}_\varpi(\omega^\top|_\partial) \oplus H^{1/2}_\varpi((\mathcal{K}_R\omega)_{\nu\nu}|_\partial).$$

The $R=0$ problem is the flat-ball problem with a fixed vertical vector space, fibrewise over S^q. Its solution is independent of the choice of orthonormal vertical frame, because the equations are $O(q)$-equivariant. Thus it defines a global inverse on the vertical bundle, not merely in one trivialization. Its H^2 estimate and the finite-type bound give $\|\mathfrak{L}_0^{-1}\| \le C_\varpi$. The preceding estimates give

$$\|\mathfrak{L}_R - \mathfrak{L}_0\| \le C_\varpi R, \qquad \|\mathfrak{L}_R^{-1} - \mathfrak{L}_0^{-1}\| \le \frac{\|\mathfrak{L}_0^{-1}\|^2 C_\varpi R}{1 - C_\varpi R\|\mathfrak{L}_0^{-1}\|}.$$

This follows by factoring $\mathfrak{L}_R = (I + (\mathfrak{L}_R - \mathfrak{L}_0)\mathfrak{L}_0^{-1})\mathfrak{L}_0$. For the full vector Dirichlet problem use instead $\mathfrak{L}_R\omega = (\delta_R \boldsymbol{\mathcal{K}}_R\omega, \omega|_\partial)$. Its domain is H^2_ϖ and its target is $L^2_\varpi \oplus H^{3/2}_\varpi(\partial)$. The coefficient estimate and the limiting Dirichlet coercivity give the same factorization and inverse bound.

We also need their weak versions. For the zero-Dirichlet scalar and vector spaces the limiting normal-ball Poincaré inequalities give uniform coercivity. The difference of their bilinear forms is bounded by $C_\varpi R\|u\|_1\|v\|_1$. Given boundary data in $H^{1/2}_\varpi$, subtract one fixed trace extension and solve the zero-boundary problem. Let a_R be either of the zero-Dirichlet bilinear forms, and let u_R, u_0 be its solutions with the same prescribed boundary value f. Their difference $w_R = u_R - u_0$ has zero trace and satisfies

$$a_R(w_R, v) = -(a_R - a_0)(u_0, v) \qquad (v \in H^1_{0,\varpi}).$$

For a constant $c_0 > 0$ independent of small R, coercivity and the coefficient estimate give

$$c_0\|w_R\|^2_{H^1} \le C_\varpi R\|u_0\|_{H^1}\|w_R\|_{H^1}.$$

Division when $w_R \neq 0$, and the bounded trace extension followed by the zero-Dirichlet inverse, give $\|w_R\|_{H^1} \le C'_\varpi R\|f\|_{H^{1/2}}$. The inequality is also true when $w_R = 0$. Consequently

$$\|\mathcal{H}_R - \mathcal{H}_0\|_{H^{1/2}\to H^1} + \|X^D_R - X^D_0\|_{H^{1/2}\to H^1} \le C_\varpi R. \tag{E.22}$$

All the stated maps preserve the specified representation.

Here is the precise common space for the constrained forms. In the reference bundle choose a bounded trace extension $\boldsymbol{\mathcal{T}}$ into trace-free H^1 tensors, and average it over $\mathfrak{G}$. With

$$\mathscr{V} = H^1_{0,\varpi}(S^2_0E^*) \oplus H^{1/2}_\varpi(T^*\partial) \oplus H^1_\varpi(\partial),$$

set

$$J_R(S_0, \eta, \phi) = \Big(S_0 + \boldsymbol{\mathcal{T}}\big(\nu^\flat \otimes \eta + \eta \otimes \nu^\flat - \phi(\widehat{\mathrm{II}}_R)_0\big), \phi\Big).$$

Its inverse subtracts the same extension and reads off $S(\nu,\cdot)^\top$. Thus J_R and J_R^{-1} are bounded, and $J_R - J_0 = O_\varpi(R^2)$ in their operator norms. Define

$$\widetilde{C}_R = C_R J_R, \qquad \widetilde{B}_R = J_R^{-1}B_R, \qquad \widetilde{P}_R = I - \widetilde{B}_R\widetilde{C}_R.$$

The construction of B_R in Section 20 gives $\widetilde{C}_R\widetilde{B}_R = I$ exactly. Hence $\widetilde{P}_R^2 = \widetilde{P}_R$ and $\widetilde{P}_R - \widetilde{P}_0 = O_\varpi(R)$. The operator

$$V_R = \widetilde{P}_R\widetilde{P}_0 + (I - \widetilde{P}_R)(I - \widetilde{P}_0)$$

is invertible for small R, and $\widetilde{P}_R V_R = V_R\widetilde{P}_0$. The actual map of constrained pairs is $J_R V_R J_0^{-1}$. This specifies the identifications implicit in Section 20. Smoothing the free data in $\mathscr{V}$, then applying $\widetilde{P}_R$, proves the required density without violating a boundary condition or the divergence constraint.

Lemma E.3. *On the fixed compatible-pair space $\mathscr{P}_{0,\varpi}$, the pulled-back forms and masses satisfy*

$$|(b_R - b_0)(v, w)| + |(\mathfrak{m}_R - \mathfrak{m}_0)(v, w)| \le C_\varpi R\|v\|_{\mathscr{P}_{0,\varpi}}\|w\|_{\mathscr{P}_{0,\varpi}}. \tag{E.23}$$

Proof. First compare forms before applying $V_R = I + O_\varpi(R)$; this last change adds another error of the stated size because the forms below are uniformly bounded. Denote the normalized interior and boundary densities by μ_R and ν_R. They converge in every fixed coefficient norm. The identity (E.16) consists of the following terms:

$$\begin{aligned}
I_R(S,T) &= \int \langle \nabla_R S, \nabla_R T\rangle \mu_R - 2\int \langle \mathring{R}_R S, T\rangle \mu_R,\\
E_R(\eta,\xi) &= 2\int_\partial (H_R\gamma_R + \mathrm{II}_R)(\eta,\xi)\nu_R,\\
J_R^\partial((\eta,\phi),(\xi,\psi)) &= \int_\partial \left\{ \mathsf{C}_R(\eta, D_R^\Sigma\psi) + \mathsf{C}_R(\xi, D_R^\Sigma\phi)\right\} \nu_R,\\
P_R^\partial(\phi,\psi) &= \int_\partial \Pi_R(D_R^\Sigma\phi, D_R^\Sigma\psi)\nu_R,\\
Z_R(\phi,\psi) &= -\int_\partial \mathcal{V}_R\phi\psi\nu_R,\\
N_R^\partial(\phi,\psi) &= -\frac{(m-1)H_R^2}{m}\langle \phi, \mathcal{N}_R\psi\rangle_{\nu_R},
\end{aligned}$$

where $\mathsf{C}_R = ((m-1)H_R/m)\gamma_R - 2(\mathrm{II}_R)_0$. This coefficient is distinct from the divergence constraint C_R.

For I_R, subtract the two gradient products one factor at a time. Equation (E.19), $\|\mathrm{Rm}_R\|_{C^0} = O(R^2)$ and the density estimate give $|(I_R - I_0)(S,T)| \le C_\varpi R\|S\|_1\|T\|_1$. The shape coefficients in $E_R, J_R^\partial, P_R^\partial, Z_R$ differ from their limiting values by $O(R^2)$. Their arguments are bounded respectively by

$$\|\eta\|_{L^2(\partial)}, \quad \|\phi\|_{H^1(\partial)}, \quad \|S\|_{H^1(M)},$$

using the trace theorem and (E.21). Replacing one coefficient, one derivative, or the density at a time therefore bounds each difference by $C_\varpi R\|(S,\phi)\|\|(T,\psi)\|$. For the limiting shape operator $A^{\mathrm{lim}} = \mathrm{diag}(I_p, 0)$, its radial derivative is $\mathrm{diag}(-I_p,0)$ and its trace-free part is $\mathrm{diag}((q/m)I_p, -(p/m)I_q)$. Thus the zeroth-order coefficient has limit

$$\mathcal{V}_0 = -\frac{pq}{m} + \frac{p^2q}{m^2} = -\frac{pq^2}{m^2}.$$

For the last term use the weak definition

$$\langle \phi, \mathcal{N}_R\psi\rangle_{\nu_R} = \int \langle d_R\mathcal{H}_R\phi, d_R\mathcal{H}_R\psi\rangle \mu_R.$$

Let $a_R^D(u,v) = \int \langle d_R u, d_R v\rangle \mu_R$ on the fixed reference space. The exact difference is

$$\begin{aligned}
&a_R^D(\mathcal{H}_R\phi, \mathcal{H}_R\psi) - a_0^D(\mathcal{H}_0\phi, \mathcal{H}_0\psi)\\
&\quad = (a_R^D - a_0^D)(\mathcal{H}_R\phi, \mathcal{H}_R\psi) + a_0^D((\mathcal{H}_R - \mathcal{H}_0)\phi, \mathcal{H}_R\psi)\\
&\qquad + a_0^D(\mathcal{H}_0\phi, (\mathcal{H}_R - \mathcal{H}_0)\psi).
\end{aligned}$$

The coefficient estimate bounds the first term by $C_\varpi R\|\mathcal{H}_R\phi\|_{H^1}\|\mathcal{H}_R\psi\|_{H^1}$. The other two are bounded using (E.22) and the boundedness of a_0^D. The uniform harmonic-extension bounds therefore give

$$|\langle \phi, \mathcal{N}_R\psi\rangle_{\nu_R} - \langle \phi, \mathcal{N}_0\psi\rangle_{\nu_0}| \le C_\varpi R\|\phi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}.$$

The coefficient H_R^2 also converges. This treats every term of the boundary form.

Finally the mass is

$$\mathfrak{m}_R((S,\phi),(T,\psi)) = \langle S,T\rangle_{\mu_R} + \langle \mathcal{K}_R X_R^D(\phi), \mathcal{K}_R X_R^D(\psi)\rangle_{\mu_R}.$$

Equations (E.19) and (E.22) give its asserted bound. Applying $J_R V_R J_0^{-1}$ changes all these bounded forms by $O_\varpi(R)$, proving (E.23). □

The flat normal-model coercivity in (18.6) and Theorem 18.1 gives $b_0(v,v) \ge c_\varpi\|v\|^2_{\mathscr{P}_{0,\varpi}}$. Consequently the explicit choice

$$0 < R < \min\{R_{\varpi,0}, c_\varpi/(2C_\varpi)\}$$

in (E.23) proves $b_R(v,v) \ge (c_\varpi/2)\|v\|^2$. Rescaling multiplies the form and its mass by positive constants. This proves the initial-index assertion in the fixed representation.

E.5. **Proof of the smooth boundary-coordinate estimates.** We use the full map and the graded spaces defined in Section 9.2. The constants below are allowed to depend on the fixed compact metric; no uniform estimate along the normalized sequence of truncations is used.

Proof of Proposition 9.4. Write $\mathscr{G}$ for the bordered map and $\mathscr{L}_q = D\mathscr{G}_q$. Finite-dimensional source and target variables are included in the norms. Fix finite interior and boundary atlases and a partition of unity. All constants below are uniform when $q - g_0$ is sufficiently small in $C^{r_0,\alpha}$.

The product estimates used in the proof follow from the Leibniz rule and interpolation. For $r > r_0$ and $0 \le j \le r - r_0$,

$$\|u\|_{r_0+j} \le C_r \|u\|_{r_0}^{1-j/(r-r_0)} \|u\|_r^{j/(r-r_0)}. \tag{E.24}$$

For nonnegative L_i, H_i and θ_i with $\sum_i \theta_i \le 1$, weighted arithmetic–geometric mean gives

$$\prod_i L_i^{1-\theta_i} H_i^{\theta_i} \le \sum_i \theta_i H_i \prod_{j \ne i} L_j + \Big(1 - \sum_i \theta_i\Big) \prod_i L_i. \tag{E.25}$$

For positive L_i this follows after division by $\prod_i L_i$; the general case follows by replacing each L_i by $L_i + \varepsilon$ and letting $\varepsilon \downarrow 0$. Applying (E.24) to each differentiated factor, then (E.25), bounds each product by a sum with only one high norm. The Hölder seminorm satisfies $[uv]_\alpha \le \|u\|_\infty [v]_\alpha + \|v\|_\infty [u]_\alpha$; the same estimates therefore hold in the stated Hölder norms.

In each chart the interior expression is a finite sum of terms $a(q)\partial^2 q$, $b(q)(\partial q, \partial q)$, and terms with smooth fixed coefficients of lower differential order. The boundary map B_0 is algebraic in q; the Bianchi boundary operator is of first order. The scalar boundary map $B_1 = H_q^2 - |A_q|^2$ is *quadratic in the first jet*, with coefficients smooth in q. Its second differential may therefore contain $\partial v_1\, \partial v_2$; it is not an expression with only one derivative in total. After $r - 1$ boundary differentiations, at most r derivatives fall on any single metric argument. The other factors are estimated in the fixed low norm. Since $r_0 \ge 4$, (E.25) applies to all these terms. The target orders in (9.9) are consequently the interior order $r - 2$, the Bianchi and scalar boundary order $r - 1$, and the conformal boundary order r.

The derivatives of an inverse metric are given, for $n \ge 1$, by

$$D^n(q^{-1})[v_1, \dots, v_n] = (-1)^n \sum_{\sigma \in \mathfrak{S}_n} q^{-1} v_{\sigma(1)} q^{-1} \cdots v_{\sigma(n)} q^{-1}. \tag{E.26}$$

The case $n = 1$ is the derivative of $q^{-1}q = I$. Differentiation of each inverse factor inserts the next argument in each possible position, proving the formula by induction. The inverse factors are uniformly bounded in the low neighborhood. The product estimates just proved, applied to this identity and to the local expressions of $\mathscr{G}$, give

$$\begin{aligned} \|D^n \mathscr{G}_q[v_1, \dots, v_n]\|_r \le C_{r,n} \Bigg[& \sum_i \|v_i\|_r \prod_{j \ne i} \|v_j\|_{r_0} \\ & + (1 + \|q - g_0\|_r) \prod_i \|v_i\|_{r_0} \Bigg]. \end{aligned} \tag{E.27}$$

Smooth finite-rank bordering maps satisfy this bound because their coefficients are bounded functionals in the low norm and their images are spanned by fixed smooth vectors.

We next prove the corresponding inverse estimate. The localized complementing-boundary estimates are those of [1]. For a local coefficient a and a multi-index γ with $|\gamma| \le 2$, the commutator is

$$[D^\beta, aD^\gamma]v = \sum_{0 < \zeta \le \beta} \binom{\beta}{\zeta} (D^\zeta a) D^{\beta - \zeta + \gamma} v.$$

For a first-order boundary term the same identity holds with $|\gamma| \le 1$. Apply the fixed-order estimate to tangential difference quotients, and then to tangential derivatives. The normal principal coefficient of the interior equation is invertible, so the equation determines the two additional normal derivatives. The commutator terms with an intermediate derivative are estimated by (E.24) and Young's inequality. For every $\delta > 0$ their total bound has the form

$$\delta \|v\|_r + C_{r,\delta}(1 + \|q - g_0\|_r)\|v\|_{r_0}.$$

Terms in which all high derivatives fall on a coefficient already have the second form, by (E.25). Derivatives of the fixed partition of unity have lower order and satisfy the same bound. Choose δ so that its contribution to the localized estimate is at most one half of the left side. Absorption gives

$$\|v\|_r \leq C_r\left(|\mathscr{L}_q v|_r + (1+\|q-g_0\|_r)\|v\|_{r_0}\right). \tag{E.28}$$

All coefficient derivatives used here have order at most r in q. In particular the first-jet dependence of the boundary principal coefficients does not require $\|q\|_{r+1}$.

At the low order, continuity in the operator norm permits the choice

$$\|(\mathscr{L}_q - \mathscr{L}_{g_0})\mathscr{L}_{g_0}^{-1}\|_{\mathcal{L}(Y_{r_0})} < \tfrac{1}{2}.$$

The factorization

$$\mathscr{L}_q^{-1} = \mathscr{L}_{g_0}^{-1}[I + (\mathscr{L}_q - \mathscr{L}_{g_0})\mathscr{L}_{g_0}^{-1}]^{-1}$$

gives $\|\mathscr{L}_q^{-1}\|_{Y_{r_0}\to X_{r_0}} \leq 2\|\mathscr{L}_{g_0}^{-1}\|$. Boundary regularity applied successively to the linear equation upgrades this inverse on smooth data to every order. Substitution in (E.28) gives

$$\|\mathscr{L}_q^{-1}v\|_r \leq C_r\left(|v|_r + (1+\|q-g_0\|_r)|v|_{r_0}\right). \tag{E.29}$$

The Banach inverse function theorem gives $\mathscr{S}$ on one fixed low-order neighborhood. Smooth target data give smooth solutions: tangential difference quotients of the nonlinear equation satisfy the linearized complementing system, with lower-order terms controlled by the preceding product estimates. The boundary estimates give one additional tangential derivative at each step; the interior equation then determines the remaining normal derivatives. Induction proves regularity at every order. Uniqueness in the low-order chart identifies all higher-regularity inverses on their common domains.

Make the low target neighborhood convex and put $q_t = \mathscr{S}(y_0 + t(y-y_0))$ for $0 \leq t \leq 1$. Then $\partial_t q_t = \mathscr{L}_{q_t}^{-1}(y-y_0)$. Writing $A = |y-y_0|_r$, $B = |y-y_0|_{r_0}$ and $F(t) = \|q_t - g_0\|_r$, (E.29) gives

$$F(t) \leq C_r \int_0^t [A + B + BF(s)]\,ds.$$

The continuous inclusion of the high target space in the low one implies $B \leq C_r A$. Since B is bounded in the fixed neighborhood, Grönwall's inequality yields

$$F(t) \leq C_r' A \exp(C_r B t) \leq C_r'' A \qquad (0 \leq t \leq 1).$$

At $t = 1$ this is (9.11). No smallness of A is used.

For $n \geq 2$, differentiate $\mathscr{G}(\mathscr{S}(y)) = y$. If Π_n is the set of partitions of $\{1,\dots,n\}$, then

$$\mathscr{L}_q D^n\mathscr{S}[y_1,\dots,y_n] = -\sum_{\substack{\pi\in\Pi_n\\ |\pi|\geq 2}} D^{|\pi|}\mathscr{G}_q\left[D^{|B|}\mathscr{S}[y_i : i\in B] : B\in\pi\right]. \tag{E.30}$$

Each block on the right has cardinality less than n. For $n = 1$, (E.29) and the zero-order bound prove (9.10). Assume its bounds through order $n-1$. In every term of (E.30), (E.27) places a high norm on at most one block or on $q - g_0$. The induction hypothesis on that block places it on at most one target argument or on $y - y_0$; all other blocks use the low-order bounds. Apply (E.29) to the sum. The finite number of partitions is absorbed in $C_{r,n}$. The resulting bound is exactly (9.10). This proves the estimate at every order. □

We specify the bordering and the use of the lemma in the two cases. At an umbilic cap let k be the smooth Bianchi representative spanning K_R, and choose

$$\pi_K(v) = \frac{\langle v,k\rangle_{L^2(g_0)}}{\|k\|^2_{L^2(g_0)}}.$$

Choose a smooth boundary vector e whose class spans the cokernel of $D\mathcal{B}_\partial$. Such a vector is supplied by the nonzero shape second derivative in (9.12); the averaging argument there proves that its class is nonzero in the full cokernel, not only the invariant cokernel. View e as a target vector of $\mathscr{F}$ with zero interior and Bianchi components. It spans the cokernel of $D\mathscr{F}$: an equation with zero interior and

Bianchi components has an infinitesimal Einstein solution, and the remaining boundary equation is precisely the differential of $\mathcal{B}_\partial$. The full index is zero and its kernel is $\mathbb{R}k$. The required bordered map is

$$(q,z) \longmapsto \big(\Phi(q), \beta_{g_0} q|_\Sigma, (B_0(q), B_1(q)) + ze, \pi_K(q-g_0)\big)\,. \tag{E.31}$$

Its derivative is an isomorphism: a vector in its kernel has $z=0$ after projection to the cokernel, is then a multiple of k, and is zero by π_K. Its Fredholm index is zero, which proves surjectivity. A bounded cokernel functional at the low order, normalized to be one on e, defines the target projection. Its image is the smooth line $\mathbb{R}e$, so for every r its norm is bounded by C_r times the low norm. The same assertion holds for the source projection. Thus both projections satisfy the estimates required in the proposition.

Set the interior and Bianchi target data in (E.31) equal to zero. The resulting metrics are Ricci-flat: for $V = \beta_{g_0} q$, the contracted Bianchi identity gives

$$\beta_q \delta_q^* V = 0, \qquad V|_\Sigma = 0.$$

Near g_0, this vector Dirichlet operator is an invertible perturbation of $\frac{1}{2}\nabla_{g_0}^* \nabla_{g_0}$, so $V=0$. The local boundary-fixing gauge theorem [4, Lemma 2.9] gives a unique nearby representative with $\beta_{g_0} q = 0$. We use it to identify quotient classes with these representatives. The smooth-scale regularity of the coordinates below comes from the displayed inverse estimates; it is not an assertion that the diffeomorphism action defines a smooth slice on unchanged finite Hölder spaces. For target boundary coordinates y in the complementary hyperplane, the bordered inverse has metric component $q(y,s)$ and border component $z(y,s)$. Hence

$$\mathcal{B}_\partial(q(y,s)) = (y, -z(y,s))$$

in the associated target splitting. Equations (9.11)–(E.30) prove that this is a smooth tame chart. Equation (9.12) and its averaging argument give $z_s = 0$ and $z_{ss} \neq 0$ at the basepoint. The scalar implicit equation for its critical point and the subsequent square-root coordinate change have nonzero denominators. Differentiating these scalar equations gives finite sums of products of the already estimated derivatives and reciprocal powers of those denominators. Their reciprocals are uniformly bounded in a low neighborhood. The same tame estimates therefore hold for the fold coordinate changes.

For a sufficiently late normalized cap, the kernel and cokernel are zero; no border variables are needed. Apply the proposition directly to $\mathscr{F}$, and set its interior and Bianchi target data to zero. For smooth prescribed boundary metrics use $y = \Theta(\gamma)$. At these normalized metrics the boundary metric is γ_* and $R_{\gamma_*} = m$. The derivative of its scalar curvature in the conformal direction $2f\gamma_*$ is

$$DR_{\gamma_*}(2f\gamma_*) = 2\big((m-1)\Delta_{\gamma_*} - m\big)f.$$

This operator is invertible by (22.4); its eigenvalue on constants is $-2m$. At an umbilic metric the corresponding operator is $2(m-1)(\Delta_\gamma - m\kappa^2)$, as in Lemma 9.2. Applying the product, inverse, and partition formulas to either scalar equation gives a smooth tame inverse, with the grading

$$\|\bar{\gamma}\|_{r,\alpha} + \|R\|_{r-2,\alpha} \quad \longmapsto \quad \|f\|_{r,\alpha}.$$

Finally the map $\gamma \mapsto \Theta(\gamma)$ sends $C^{r+1,\alpha}$ into the boundary part of (9.9). Thus the actual filling map $\mathcal{B}$ satisfies, on a fixed low-order neighborhood,

$$\begin{aligned}\|D^n\mathcal{B}_\gamma[\gamma_1,\dots,\gamma_n]\|_{r,\alpha} \le C_{r,n}\Bigg[\sum_i \|\gamma_i\|_{r+1,\alpha}\prod_{j\neq i}\|\gamma_j\|_{r_0+1,\alpha} \\ + (1+\|\gamma-\gamma_*\|_{r+1,\alpha})\prod_i \|\gamma_i\|_{r_0+1,\alpha}\Bigg].\end{aligned} \tag{E.32}$$

The zero-order estimate is

$$\|\mathcal{B}(\gamma) - \mathcal{B}(\gamma_*)\|_{r,\alpha} \le C_r\|\gamma-\gamma_*\|_{r+1,\alpha}.$$

Indeed the scalar-curvature formula and the product estimates give $|\Theta(\gamma)-\Theta(\gamma_*)|_r \le C_r\|\gamma-\gamma_*\|_{r+1,\alpha}$ in the target grading; apply (9.11). The one-derivative shift in this displayed filling estimate is retained.

No inverse on unchanged finite-regularity Dirichlet spaces is asserted. These estimates complete the smooth-scale compatibility statements in Proposition 9.5 and Lemma 22.3.

Appendix F. Higher-mode coefficient identities

Put $a = p - 2$, $b = q - 2$. The polynomials $\Pi_i(a, b)$ are defined completely in Appendix G, and $\Pi_i^{\vee}(a, b) := \Pi_i(b, a)$. A row $(i, j; c)$ in the table for (H, k, ℓ, D) means

$$c^{H}_{k\ell ij}(a, b) = c.$$

All omitted (i, j) have coefficient zero. The denominator and the excess-parameter powers are those in (8.22). In a superscript $\vee n$, the meaning is $(\Pi_i^{\vee})^n$.

$H = A$, $k = 0$, $\ell = 0$, $D^{A}_{0,0} = 5$.

i	j	$c^{A}_{0,0,i,j}$
1	0	$128\,\Pi_1^{3}\,\Pi_1^{\vee 6}\,\Pi_2^{6}\,\Pi_2^{\vee 9}\,\Pi_3$
1	1	$128\,\Pi_1^{3}\,\Pi_1^{\vee 6}\,\Pi_2^{5}\,\Pi_2^{\vee 9}\,\Pi_3\,\Pi_4$
1	2	$256\,\Pi_1^{3}\,\Pi_1^{\vee 6}\,\Pi_2^{5}\,\Pi_2^{\vee 9}\,\Pi_3\,\Pi_5$
1	3	$256\,\Pi_1^{3}\,\Pi_1^{\vee 6}\,\Pi_2^{5}\,\Pi_2^{\vee 9}\,\Pi_3\,\Pi_6$
1	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee 6}\,\Pi_2^{5}\,\Pi_2^{\vee 9}\,\Pi_3\,\Pi_7$
1	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee 6}\,\Pi_2^{5}\,\Pi_2^{\vee 9}\,\Pi_3$
2	0	$128\,\Pi_1^{2}\,\Pi_1^{\vee 5}\,\Pi_2^{6}\,\Pi_2^{\vee 7}\,\Pi_8^{\vee}$
2	1	$128\,\Pi_1^{2}\,\Pi_1^{\vee 5}\,\Pi_2^{5}\,\Pi_2^{\vee 7}\,\Pi_9^{\vee}$
2	2	$256\,\Pi_1^{2}\,\Pi_1^{\vee 5}\,\Pi_2^{4}\,\Pi_2^{\vee 7}\,\Pi_{10}$
2	3	$256\,\Pi_1^{2}\,\Pi_1^{\vee 5}\,\Pi_2^{4}\,\Pi_2^{\vee 7}\,\Pi_{11}$
2	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee 5}\,\Pi_2^{4}\,\Pi_2^{\vee 7}\,\Pi_{12}^{\vee}$
2	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee 5}\,\Pi_2^{4}\,\Pi_2^{\vee 8}\,\Pi_{13}^{\vee}$
3	0	$128\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^{6}\,\Pi_2^{\vee 6}\,\Pi_{14}^{\vee}$
3	1	$128\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^{5}\,\Pi_2^{\vee 6}\,\Pi_{15}^{\vee}$
3	2	$128\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^{4}\,\Pi_2^{\vee 6}\,\Pi_{16}$
3	3	$128\,\Pi_1^{2}\,\Pi_1^{\vee 4}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{17}$
3	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee 4}\,\Pi_2^{3}\,\Pi_2^{\vee 7}\,\Pi_{18}$
3	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee 4}\,\Pi_2^{3}\,\Pi_2^{\vee 7}\,\Pi_{19}^{\vee}$
4	0	$128\,\Pi_1^{\vee 4}\,\Pi_2^{6}\,\Pi_2^{\vee 5}\,\Pi_{20}^{\vee}$
4	1	$128\,\Pi_1^{\vee 3}\,\Pi_2^{5}\,\Pi_2^{\vee 5}\,\Pi_{21}^{\vee}$
4	2	$128\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{22}^{\vee}$
4	3	$128\,\Pi_1^{2}\,\Pi_1^{\vee 3}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{23}$
4	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee 3}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{24}^{\vee}$
4	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee 3}\,\Pi_2^{3}\,\Pi_2^{\vee 7}\,\Pi_{25}^{\vee}$
5	0	$128\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^{6}\,\Pi_2^{\vee 4}\,\Pi_{26}^{\vee}$
5	1	$128\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{27}^{\vee}$
5	2	$128\,\Pi_1\,\Pi_1^{\vee 2}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{28}$
5	3	$128\,\Pi_1^{2}\,\Pi_1^{\vee 2}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{29}^{\vee}$
5	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee 2}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{30}$
5	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee 2}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{31}^{\vee}$
6	0	$128\,\Pi_1^{2}\,\Pi_1^{\vee 4}\,\Pi_2^{6}\,\Pi_2^{\vee 3}\,\Pi_{32}$
6	1	$128\,\Pi_1^{2}\,\Pi_1^{\vee 3}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{33}$
6	2	$128\,\Pi_1^{2}\,\Pi_1^{\vee 2}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{34}$
6	3	$128\,\Pi_1^{2}\,\Pi_1^{\vee}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{35}$
6	4	$128\,\Pi_1^{3}\,\Pi_1^{\vee}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{36}$
6	5	$128\,\Pi_1^{4}\,\Pi_1^{\vee}\,\Pi_2^{3}\,\Pi_2^{\vee 6}\,\Pi_{37}$
7	0	$128\,\Pi_1^{3}\,\Pi_1^{\vee 4}\,\Pi_2^{6}\,\Pi_2^{\vee 3}\,\Pi_{38}$
7	1	$128\,\Pi_1^{3}\,\Pi_1^{\vee 3}\,\Pi_2^{6}\,\Pi_2^{\vee 3}\,\Pi_{39}$
7	2	$128\,\Pi_1^{3}\,\Pi_1^{\vee 2}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{40}$
7	3	$128\,\Pi_1^{3}\,\Pi_1^{\vee}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{41}$
7	4	$128\,\Pi_1^{3}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{42}$
7	5	$128\,\Pi_1^{4}\,\Pi_2^{4}\,\Pi_2^{\vee 5}\,\Pi_{43}$
8	0	$128\,\Pi_1^{4}\,\Pi_1^{\vee 4}\,\Pi_2^{7}\,\Pi_2^{\vee 3}\,\Pi_{44}$
8	1	$128\,\Pi_1^{4}\,\Pi_1^{\vee 3}\,\Pi_2^{6}\,\Pi_2^{\vee 3}\,\Pi_{45}$
8	2	$128\,\Pi_1^{4}\,\Pi_1^{\vee 2}\,\Pi_2^{6}\,\Pi_2^{\vee 3}\,\Pi_{46}^{\vee}$
8	3	$128\,\Pi_1^{4}\,\Pi_1^{\vee}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{47}$
8	4	$128\,\Pi_1^{4}\,\Pi_1^{\vee}\,\Pi_2^{5}\,\Pi_2^{\vee 4}\,\Pi_{48}$

i	j	$c^A_{0,0,i,j}$
8	5	$128\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee 5}\,\Pi_{49}$
9	0	$128\,\Pi_1^5\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 3}\,\Pi_{50}^{\vee}$
9	1	$128\,\Pi_1^5\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 3}\,\Pi_{51}^{\vee}$
9	2	$128\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{52}^{\vee}$
9	3	$128\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{53}^{\vee}$
9	4	$128\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 4}\,\Pi_{54}$
9	5	$128\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 5}\,\Pi_{55}$
10	0	$128\,\Pi_1^6\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 4}\,\Pi_{56}^{\vee}$
10	1	$128\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 4}\,\Pi_{57}$
10	2	$256\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 4}\,\Pi_{58}$
10	3	$256\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 4}\,\Pi_{59}$
10	4	$128\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 4}\,\Pi_{60}$
10	5	$128\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^7\,\Pi_2^{\vee 5}\,\Pi_{61}$
11	0	$128\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$
11	1	$640\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$
11	2	$1280\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$
11	3	$1280\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$
11	4	$640\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$
11	5	$128\,\Pi_1^7\,\Pi_1^{\vee 4}\,\Pi_2^8\,\Pi_2^{\vee 5}\,\Pi_{62}$

$H = A$, $k = 0$, $\ell = 1$, $D^A_{0,1} = 4$.

i	j	$c^A_{0,1,i,j}$
0	0	$32\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	1	$128\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	2	$192\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	3	$128\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	4	$32\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
1	0	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^4\,\Pi_2^{\vee 6}\,\Pi_{63}^{\vee}$
1	1	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{64}^{\vee}$
1	2	$192\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{65}^{\vee}$
1	3	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{66}^{\vee}$
1	4	$64\,\Pi_1^3\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{67}^{\vee}$
2	0	$32\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^4\,\Pi_2^{\vee 5}\,\Pi_{68}^{\vee}$
2	1	$32\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^3\,\Pi_2^{\vee 5}\,\Pi_{69}^{\vee}$
2	2	$32\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{70}$
2	3	$32\,\Pi_1^2\,\Pi_1^{\vee 4}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{71}^{\vee}$
2	4	$32\,\Pi_1^3\,\Pi_1^{\vee 4}\,\Pi_2^2\,\Pi_2^{\vee 6}\,\Pi_{72}^{\vee}$
3	0	$32\,\Pi_1^{\vee 4}\,\Pi_2^4\,\Pi_2^{\vee 4}\,\Pi_{73}^{\vee}$
3	1	$32\,\Pi_1^{\vee 3}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{74}^{\vee}$
3	2	$32\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{75}^{\vee}$
3	3	$32\,\Pi_1^2\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{76}^{\vee}$
3	4	$32\,\Pi_1^3\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{77}^{\vee}$
4	0	$32\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{78}$
4	1	$32\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{79}$
4	2	$32\,\Pi_1\,\Pi_1^{\vee 2}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{80}^{\vee}$
4	3	$32\,\Pi_1^2\,\Pi_1^{\vee 2}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{81}^{\vee}$
4	4	$32\,\Pi_1^3\,\Pi_1^{\vee 2}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{82}^{\vee}$
5	0	$32\,\Pi_1^2\,\Pi_1^{\vee 4}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{83}$
5	1	$32\,\Pi_1^2\,\Pi_1^{\vee 3}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{84}$
5	2	$32\,\Pi_1^2\,\Pi_1^{\vee 2}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{85}$
5	3	$32\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{86}$
5	4	$32\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{87}$
6	0	$32\,\Pi_1^3\,\Pi_1^{\vee 4}\,\Pi_2^5\,\Pi_2^{\vee 2}\,\Pi_{88}$
6	1	$32\,\Pi_1^3\,\Pi_1^{\vee 3}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{89}$
6	2	$32\,\Pi_1^3\,\Pi_1^{\vee 2}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{90}$
6	3	$32\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{91}$
6	4	$32\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{92}$
7	0	$32\,\Pi_1^4\,\Pi_1^{\vee 4}\,\Pi_2^6\,\Pi_2^{\vee 2}\,\Pi_{93}^{\vee}$
7	1	$64\,\Pi_1^4\,\Pi_1^{\vee 3}\,\Pi_2^5\,\Pi_2^{\vee 2}\,\Pi_{94}^{\vee}$
7	2	$32\,\Pi_1^4\,\Pi_1^{\vee 2}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{95}^{\vee}$

i	j	$c^A_{0,1,i,j}$
7	3	$64\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{96}$
7	4	$32\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,4}\,\Pi_{97}$
8	0	$64\,\Pi_1^5\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{98}^\vee$
8	1	$64\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{99}$
8	2	$192\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{100}$
8	3	$64\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{101}$
8	4	$64\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,4}\,\Pi_{102}$
9	0	$32\,\Pi_1^6\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{103}$
9	1	$128\,\Pi_1^6\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{103}$
9	2	$192\,\Pi_1^6\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{103}$
9	3	$128\,\Pi_1^6\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{103}$
9	4	$32\,\Pi_1^6\,\Pi_1^{\vee\,4}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{103}$

$H = A$, $k = 0$, $\ell = 2$, $D^A_{0,2} = 3$.

i	j	$c^A_{0,2,i,j}$
0	0	$16\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{104}^\vee$
0	1	$48\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{104}^\vee$
0	2	$48\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{104}^\vee$
0	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{104}^\vee$
1	0	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{105}^\vee$
1	1	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{106}^\vee$
1	2	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{107}^\vee$
1	3	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{108}^\vee$
2	0	$8\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{109}^\vee$
2	1	$8\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{110}^\vee$
2	2	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{111}^\vee$
2	3	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{112}^\vee$
3	0	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{113}$
3	1	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{114}$
3	2	$8\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{115}$
3	3	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{116}^\vee$
4	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{117}$
4	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{118}$
4	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{119}^\vee$
4	3	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{120}$
5	0	$8\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee}\,\Pi_{121}^\vee$
5	1	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{122}^\vee$
5	2	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{123}$
5	3	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{124}$
6	0	$16\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{125}$
6	1	$16\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{126}$
6	2	$16\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{127}$
6	3	$16\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{128}$
7	0	$16\,\Pi_1^5\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{129}$
7	1	$48\,\Pi_1^5\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{129}$
7	2	$48\,\Pi_1^5\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{129}$
7	3	$16\,\Pi_1^5\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{129}$

$H = A$, $k = 0$, $\ell = 3$, $D^A_{0,3} = 2$.

i	j	$c^A_{0,3,i,j}$
0	0	$4\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{130}^\vee$
0	1	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{130}^\vee$
0	2	$4\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{130}^\vee$
1	0	$4\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{131}^\vee$
1	1	$4\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{132}$
1	2	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{133}^\vee$
2	0	$4\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{134}$

i	j	$c^A_{0,3,i,j}$
2	1	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{135}$
2	2	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee\,3}\,\Pi_{136}$
3	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{137}$
3	1	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{138}$
3	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee\,2}\,\Pi_{139}$
4	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{140}$
4	1	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee}\,\Pi_{141}$
4	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{142}$
5	0	$8\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_5$
5	1	$16\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_5$
5	2	$8\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_5$

$H = A$, $k = 0$, $\ell = 4$, $D^A_{0,4} = 0$.

i	j	$c^A_{0,4,i,j}$
0	0	$\Pi_1^{\vee\,4}\,\Pi_2^{\vee\,3}$
1	0	$4\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^{\vee\,2}$
2	0	$5\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^{\vee}$
3	0	$2\,\Pi_1^3\,\Pi_1^{\vee\,4}$

$H = A$, $k = 1$, $\ell = 0$, $D^A_{1,0} = 4$.

i	j	$c^A_{1,0,i,j}$
1	0	$32\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^5\,\Pi_2^{\vee\,7}\,\Pi_{143}^{\vee}$
1	1	$32\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^4\,\Pi_2^{\vee\,7}\,\Pi_{144}$
1	2	$96\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^4\,\Pi_2^{\vee\,7}\,\Pi_{145}$
1	3	$32\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^4\,\Pi_2^{\vee\,7}\,\Pi_{146}$
1	4	$32\,\Pi_1^4\,\Pi_1^{\vee\,5}\,\Pi_2^4\,\Pi_2^{\vee\,7}\,\Pi_{147}^{\vee}$
2	0	$32\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^5\,\Pi_2^{\vee\,5}\,\Pi_{148}^{\vee}$
2	1	$64\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,5}\,\Pi_{149}^{\vee}$
2	2	$32\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{150}$
2	3	$64\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{151}^{\vee}$
2	4	$32\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{152}^{\vee}$
3	0	$32\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,4}\,\Pi_{153}^{\vee}$
3	1	$32\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,4}\,\Pi_{154}^{\vee}$
3	2	$32\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{155}^{\vee}$
3	3	$32\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{156}$
3	4	$32\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{157}^{\vee}$
4	0	$32\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{158}^{\vee}$
4	1	$32\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{159}^{\vee}$
4	2	$32\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{160}$
4	3	$32\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{161}$
4	4	$32\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{162}$
5	0	$32\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{163}$
5	1	$32\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{164}$
5	2	$32\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{165}$
5	3	$32\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{166}$
5	4	$32\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{167}^{\vee}$
6	0	$32\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{168}$
6	1	$32\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{169}$
6	2	$32\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{170}$
6	3	$32\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{171}$
6	4	$32\,\Pi_1^4\,\Pi_2^4\,\Pi_2^{\vee\,4}\,\Pi_{172}$
7	0	$32\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{173}$
7	1	$64\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{174}$
7	2	$32\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{175}^{\vee}$
7	3	$64\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{176}$
7	4	$32\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,4}\,\Pi_{177}$
8	0	$32\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{178}$

i	j	$c^A_{1,0,i,j}$
8	1	$32\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{179}$
8	2	$96\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{180}$
8	3	$32\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{181}$
8	4	$32\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,4}\,\Pi_{182}$
9	0	$64\,\Pi_1^6\,\Pi_1^{\vee\,3}\,\Pi_2^7\,\Pi_2^{\vee\,4}\,\Pi_{183}$
9	1	$256\,\Pi_1^6\,\Pi_1^{\vee\,3}\,\Pi_2^7\,\Pi_2^{\vee\,4}\,\Pi_{183}$
9	2	$384\,\Pi_1^6\,\Pi_1^{\vee\,3}\,\Pi_2^7\,\Pi_2^{\vee\,4}\,\Pi_{183}$
9	3	$256\,\Pi_1^6\,\Pi_1^{\vee\,3}\,\Pi_2^7\,\Pi_2^{\vee\,4}\,\Pi_{183}$
9	4	$64\,\Pi_1^6\,\Pi_1^{\vee\,3}\,\Pi_2^7\,\Pi_2^{\vee\,4}\,\Pi_{183}$

$H = A$, $k = 1$, $\ell = 1$, $D^A_{1,1} = 3$.

i	j	$c^A_{1,1,i,j}$
0	0	$40\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	1	$120\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	2	$120\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	3	$40\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
1	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi^\vee_{184}$
1	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{185}$
1	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{186}$
1	3	$8\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{187}$
2	0	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi^\vee_{188}$
2	1	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{189}$
2	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{190}$
2	3	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{191}$
3	0	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi^\vee_{192}$
3	1	$8\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{193}$
3	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{194}$
3	3	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{195}$
4	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{196}$
4	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{197}$
4	2	$8\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{198}$
4	3	$8\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{199}$
5	0	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{200}$
5	1	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{201}$
5	2	$8\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{202}$
5	3	$8\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{203}$
6	0	$8\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{204}$
6	1	$8\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{205}$
6	2	$8\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{206}$
6	3	$8\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{207}$
7	0	$16\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{208}$
7	1	$48\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{208}$
7	2	$48\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{208}$
7	3	$16\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{208}$

$H = A$, $k = 1$, $\ell = 2$, $D^A_{1,2} = 2$.

i	j	$c^A_{1,2,i,j}$
0	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{209}$
0	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{209}$
0	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^\vee_{209}$
1	0	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{210}$
1	1	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{211}$
1	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi^\vee_{212}$
2	0	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{213}$
2	1	$4\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{214}$
2	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{215}$
3	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi^\vee_{216}$

i	j	$c^A_{1,2,i,j}$
3	1	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee}\,\Pi_{217}$
3	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{218}$
4	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{219}$
4	1	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{220}$
4	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{221}$
5	0	$4\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{222}$
5	1	$8\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{222}$
5	2	$4\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{222}$

$H = A$, $k = 1$, $\ell = 3$, $D^A_{1,3} = 0$.

i	j	$c^A_{1,3,i,j}$
0	0	$\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{130}$
1	0	$\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee}\,\Pi^{\vee}_{223}$
2	0	$\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi^{\vee}_{224}$
3	0	$\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_{225}$

$H = A$, $k = 2$, $\ell = 0$, $D^A_{2,0} = 3$.

i	j	$c^A_{2,0,i,j}$
1	0	$8\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^4\,\Pi_2^{\vee\,5}\,\Pi^{\vee}_{226}$
1	1	$8\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{227}$
1	2	$8\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{228}$
1	3	$8\,\Pi_1^4\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi^{\vee}_{229}$
2	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi^{\vee}_{230}$
2	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi^{\vee}_{231}$
2	2	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi^{\vee}_{232}$
2	3	$8\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi^{\vee}_{233}$
3	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{234}$
3	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{235}$
3	2	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{236}$
3	3	$8\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi^{\vee}_{237}$
4	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee}\,\Pi^{\vee}_{238}$
4	1	$8\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{239}$
4	2	$8\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{240}$
4	3	$8\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi^{\vee}_{241}$
5	0	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee}\,\Pi_{242}$
5	1	$8\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^4\,\Pi_2^{\vee}\,\Pi_{243}$
5	2	$8\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{244}$
5	3	$8\,\Pi_1^4\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{245}$
6	0	$16\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{246}$
6	1	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{247}$
6	2	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{248}$
6	3	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{249}$
7	0	$16\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{250}$
7	1	$48\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{250}$
7	2	$48\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{250}$
7	3	$16\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{250}$

$H = A$, $k = 2$, $\ell = 1$, $D^A_{2,1} = 2$.

i	j	$c^A_{2,1,i,j}$
0	0	$16\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
0	1	$32\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
0	2	$16\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
1	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{251}$
1	1	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{252}$
1	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^{\vee}_{253}$

i	j	$c^A_{2,1,i,j}$
2	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^\vee\,\Pi_{254}^\vee$
2	1	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^\vee\,\Pi_{255}^\vee$
2	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{256}^\vee$
3	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_{257}^\vee$
3	1	$4\,\Pi_1^2\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^\vee\,\Pi_{258}^\vee$
3	2	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{259}$
4	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^\vee\,\Pi_{260}$
4	1	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^\vee\,\Pi_{261}$
4	2	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{262}$
5	0	$28\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$
5	1	$56\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$
5	2	$28\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$

$H = A$, $k = 2$, $\ell = 2$, $D^A_{2,2} = 0$.

i	j	$c^A_{2,2,i,j}$
0	0	$\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^\vee\,\Pi_{263}^\vee$
1	0	$\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_{264}^\vee$
2	0	$\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_{265}$
3	0	$\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_{266}$

$H = A$, $k = 3$, $\ell = 0$, $D^A_{3,0} = 2$.

i	j	$c^A_{3,0,i,j}$
1	0	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{267}^\vee$
1	1	$8\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{268}$
1	2	$8\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{269}^\vee$
2	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^\vee\,\Pi_{270}^\vee$
2	1	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^\vee\,\Pi_{271}^\vee$
2	2	$4\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{272}^\vee$
3	0	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^4\,\Pi_{273}^\vee$
3	1	$8\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^\vee\,\Pi_{274}^\vee$
3	2	$4\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{275}^\vee$
4	0	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^4\,\Pi_2^\vee\,\Pi_{276}$
4	1	$4\,\Pi_1^3\,\Pi_2^4\,\Pi_2^\vee\,\Pi_{277}^\vee$
4	2	$4\,\Pi_1^4\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{278}$
5	0	$8\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{130}$
5	1	$16\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{130}$
5	2	$8\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{130}$

$H = A$, $k = 3$, $\ell = 1$, $D^A_{3,1} = 0$.

i	j	$c^A_{3,1,i,j}$
0	0	$\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^\vee$
1	0	$\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_{279}$
2	0	$\Pi_1^3\,\Pi_1^\vee\,\Pi_2\,\Pi_{280}$
3	0	$\Pi_1^3\,\Pi_1^\vee\,\Pi_2^2\,\Pi_7$

$H = A$, $k = 4$, $\ell = 0$, $D^A_{4,0} = 0$.

i	j	$c^A_{4,0,i,j}$
1	0	$\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2$
2	0	$2\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^2$
3	0	$\Pi_1^4\,\Pi_2^3$

$H = C$, $k = 0$, $\ell = 0$, $D^C_{0,0} = 4$.

i	j	$c^C_{0,0,i,j}$
0	0	$64\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	1	$256\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	2	$384\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	3	$256\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
0	4	$64\,\Pi_1^3\,\Pi_1^{\vee 6}\,\Pi_2^4\,\Pi_2^{\vee 8}$
1	0	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^4\,\Pi_2^{\vee 6}\,\Pi_{281}^{\vee}$
1	1	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{282}^{\vee}$
1	2	$192\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{283}^{\vee}$
1	3	$64\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{284}^{\vee}$
1	4	$64\,\Pi_1^3\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{285}^{\vee}$
2	0	$64\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^4\,\Pi_2^{\vee 5}\,\Pi_{286}^{\vee}$
2	1	$64\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^3\,\Pi_2^{\vee 5}\,\Pi_{287}^{\vee}$
2	2	$64\,\Pi_1\,\Pi_1^{\vee 4}\,\Pi_2^3\,\Pi_2^{\vee 5}\,\Pi_{288}^{\vee}$
2	3	$64\,\Pi_1^2\,\Pi_1^{\vee 4}\,\Pi_2^3\,\Pi_2^{\vee 5}\,\Pi_{289}^{\vee}$
2	4	$64\,\Pi_1^3\,\Pi_1^{\vee 4}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{290}^{\vee}$
3	0	$64\,\Pi_1^{\vee 3}\,\Pi_2^4\,\Pi_2^{\vee 4}\,\Pi_{291}^{\vee}$
3	1	$64\,\Pi_1^{\vee 3}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{292}^{\vee}$
3	2	$64\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{293}^{\vee}$
3	3	$64\,\Pi_1^2\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{294}^{\vee}$
3	4	$64\,\Pi_1^3\,\Pi_1^{\vee 3}\,\Pi_2^2\,\Pi_2^{\vee 6}\,\Pi_{295}^{\vee}$
4	0	$64\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{296}^{\vee}$
4	1	$64\,\Pi_1\,\Pi_1^{\vee 2}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{297}^{\vee}$
4	2	$64\,\Pi_1\,\Pi_1^{\vee 2}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{298}^{\vee}$
4	3	$64\,\Pi_1^2\,\Pi_1^{\vee 2}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{299}^{\vee}$
4	4	$64\,\Pi_1^3\,\Pi_1^{\vee 2}\,\Pi_2^2\,\Pi_2^{\vee 5}\,\Pi_{300}^{\vee}$
5	0	$64\,\Pi_1\,\Pi_1^{\vee 3}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{301}$
5	1	$64\,\Pi_1\,\Pi_1^{\vee 2}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{302}$
5	2	$64\,\Pi_1\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{303}$
5	3	$64\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{302}^{\vee}$
5	4	$64\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee 4}\,\Pi_{301}^{\vee}$
6	0	$64\,\Pi_1^2\,\Pi_1^{\vee 3}\,\Pi_2^5\,\Pi_2^{\vee 2}\,\Pi_{300}$
6	1	$64\,\Pi_1^2\,\Pi_1^{\vee 2}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{299}$
6	2	$64\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{298}$
6	3	$64\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 3}\,\Pi_{297}$
6	4	$64\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee 4}\,\Pi_{296}$
7	0	$64\,\Pi_1^3\,\Pi_1^{\vee 3}\,\Pi_2^6\,\Pi_2^{\vee 2}\,\Pi_{295}$
7	1	$64\,\Pi_1^3\,\Pi_1^{\vee 2}\,\Pi_2^5\,\Pi_2^{\vee 2}\,\Pi_{294}$
7	2	$64\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^4\,\Pi_2^{\vee 2}\,\Pi_{293}$
7	3	$64\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee 3}\,\Pi_{292}$
7	4	$64\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee 4}\,\Pi_{291}$
8	0	$64\,\Pi_1^4\,\Pi_1^{\vee 3}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{290}$
8	1	$64\,\Pi_1^4\,\Pi_1^{\vee 2}\,\Pi_2^5\,\Pi_2^{\vee 3}\,\Pi_{289}$
8	2	$64\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee 3}\,\Pi_{288}$
8	3	$64\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee 3}\,\Pi_{287}$
8	4	$64\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee 4}\,\Pi_{286}$
9	0	$64\,\Pi_1^5\,\Pi_1^{\vee 3}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{285}$
9	1	$64\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{284}$
9	2	$192\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{283}$
9	3	$64\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 3}\,\Pi_{282}$
9	4	$64\,\Pi_1^5\,\Pi_1^{\vee 2}\,\Pi_2^6\,\Pi_2^{\vee 4}\,\Pi_{281}$
10	0	$64\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^8\,\Pi_2^{\vee 4}$
10	1	$256\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^8\,\Pi_2^{\vee 4}$
10	2	$384\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^8\,\Pi_2^{\vee 4}$
10	3	$256\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^8\,\Pi_2^{\vee 4}$
10	4	$64\,\Pi_1^6\,\Pi_1^{\vee 3}\,\Pi_2^8\,\Pi_2^{\vee 4}$

$H = C$, $k = 0$, $\ell = 1$, $D^C_{0,1} = 3$.

i	j	$c^C_{0,1,i,j}$
0	0	$16\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{263}^{\vee}$
0	1	$48\,\Pi_1^2\,\Pi_1^{\vee 5}\,\Pi_2^3\,\Pi_2^{\vee 6}\,\Pi_{263}^{\vee}$

i	j	$c^C_{0,1,i,j}$
0	2	$48\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{263}^{\vee}$
0	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}\,\Pi_{263}^{\vee}$
1	0	$16\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,5}\,\Pi_{304}^{\vee}$
1	1	$16\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{305}$
1	2	$16\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{306}^{\vee}$
1	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{307}^{\vee}$
2	0	$16\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{308}^{\vee}$
2	1	$16\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{309}$
2	2	$16\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{310}^{\vee}$
2	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{311}^{\vee}$
3	0	$16\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{312}^{\vee}$
3	1	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{313}^{\vee}$
3	2	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{314}$
3	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{315}^{\vee}$
4	0	$16\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{316}$
4	1	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{317}$
4	2	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{318}^{\vee}$
4	3	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{319}$
5	0	$16\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee}\,\Pi_{320}^{\vee}$
5	1	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{321}^{\vee}$
5	2	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{322}$
5	3	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{323}$
6	0	$16\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{324}$
6	1	$16\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{325}$
6	2	$16\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{326}$
6	3	$16\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{327}$
7	0	$16\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{328}$
7	1	$16\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{329}$
7	2	$16\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{330}$
7	3	$16\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{331}$
8	0	$64\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^6\,\Pi_2^{\vee\,3}$
8	1	$192\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^6\,\Pi_2^{\vee\,3}$
8	2	$192\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^6\,\Pi_2^{\vee\,3}$
8	3	$64\,\Pi_1^5\,\Pi_1^{\vee\,3}\,\Pi_2^6\,\Pi_2^{\vee\,3}$

$H = C$, $k = 0$, $\ell = 2$, $D^C_{0,2} = 2$.

i	j	$c^C_{0,2,i,j}$
0	0	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{332}^{\vee}$
0	1	$16\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{332}^{\vee}$
0	2	$8\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,5}\,\Pi_{332}^{\vee}$
1	0	$4\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{333}^{\vee}$
1	1	$4\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{334}$
1	2	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{335}^{\vee}$
2	0	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{336}^{\vee}$
2	1	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{337}$
2	2	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee\,3}\,\Pi_{338}$
3	0	$8\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{339}$
3	1	$8\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{340}$
3	2	$8\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^{\vee\,2}\,\Pi_{341}$
4	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{342}$
4	1	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee}\,\Pi_{343}$
4	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{344}$
5	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{345}$
5	1	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee}\,\Pi_{346}$
5	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{347}$
6	0	$20\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}$
6	1	$40\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}$
6	2	$20\,\Pi_1^4\,\Pi_1^{\vee\,3}\,\Pi_2^4\,\Pi_2^{\vee\,2}$

$H = C$, $k = 0$, $\ell = 3$, $D^C_{0,3} = 0$.

i	j	$c^C_{0,3,i,j}$
0	0	$\Pi_1^{\vee\,4}\,\Pi_2^{\vee\,3}$
1	0	$4\,\Pi_1\,\Pi_1^{\vee\,4}\,\Pi_2^{\vee\,2}$
2	0	$\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee}\,\Pi_{348}$
3	0	$2\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_{349}$
4	0	$\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2$

$H = C$, $k = 1$, $\ell = 0$, $D^C_{1,0} = 3$.

i	j	$c^C_{1,0,i,j}$
0	0	$64\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	1	$192\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	2	$192\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
0	3	$64\,\Pi_1^3\,\Pi_1^{\vee\,5}\,\Pi_2^3\,\Pi_2^{\vee\,6}$
1	0	$16\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^3\,\Pi_2^{\vee\,4}\,\Pi_{331}^{\vee}$
1	1	$16\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{330}^{\vee}$
1	2	$16\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{329}^{\vee}$
1	3	$16\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{328}^{\vee}$
2	0	$16\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{327}^{\vee}$
2	1	$16\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{326}^{\vee}$
2	2	$16\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{325}^{\vee}$
2	3	$16\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{324}^{\vee}$
3	0	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{323}^{\vee}$
3	1	$16\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{322}^{\vee}$
3	2	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{321}$
3	3	$16\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{320}$
4	0	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{319}^{\vee}$
4	1	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{318}$
4	2	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{317}^{\vee}$
4	3	$16\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{316}^{\vee}$
5	0	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{315}$
5	1	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee}\,\Pi_{314}^{\vee}$
5	2	$16\,\Pi_1^2\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{313}$
5	3	$16\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^3\,\Pi_2^{\vee\,3}\,\Pi_{312}$
6	0	$16\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{311}$
6	1	$16\,\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{310}$
6	2	$16\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{309}^{\vee}$
6	3	$16\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee\,3}\,\Pi_{308}$
7	0	$16\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{307}$
7	1	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{306}$
7	2	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{305}^{\vee}$
7	3	$16\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^5\,\Pi_2^{\vee\,3}\,\Pi_{304}$
8	0	$16\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{263}$
8	1	$48\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{263}$
8	2	$48\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{263}$
8	3	$16\,\Pi_1^5\,\Pi_1^{\vee\,2}\,\Pi_2^6\,\Pi_2^{\vee\,3}\,\Pi_{263}$

$H = C$, $k = 1$, $\ell = 1$, $D^C_{1,1} = 2$.

i	j	$c^C_{1,1,i,j}$
0	0	$12\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{263}^{\vee}$
0	1	$24\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{263}^{\vee}$
0	2	$12\,\Pi_1^2\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}\,\Pi_{263}^{\vee}$
1	0	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,3}\,\Pi_{350}^{\vee}$
1	1	$4\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,3}\,\Pi_{351}^{\vee}$
1	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,4}\,\Pi_{352}^{\vee}$
2	0	$4\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{353}^{\vee}$
2	1	$8\,\Pi_1\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{354}^{\vee}$

i	j	$c^C_{1,1,i,j}$
2	2	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{355}$
3	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^\vee\,\Pi^\vee_{356}$
3	1	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^\vee\,\Pi_{357}$
3	2	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi_{356}$
4	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^\vee\,\Pi_{355}$
4	1	$8\,\Pi_1^2\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^\vee\,\Pi_{354}$
4	2	$4\,\Pi_1^2\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi_{353}$
5	0	$4\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^\vee\,\Pi_{352}$
5	1	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^\vee\,\Pi_{351}$
5	2	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{350}$
6	0	$12\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$
6	1	$24\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$
6	2	$12\,\Pi_1^4\,\Pi_1^{\vee\,2}\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{263}$

$H = C$, $k = 1$, $\ell = 2$, $D^C_{1,2} = 0$.

i	j	$c^C_{1,2,i,j}$
0	0	$\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee\,2}\,\Pi^\vee_{130}$
1	0	$2\,\Pi_1\,\Pi_1^{\vee\,3}\,\Pi_2^\vee\,\Pi^\vee_{358}$
2	0	$\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi^\vee_{359}$
3	0	$2\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_{360}$
4	0	$\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_{263}$

$H = C$, $k = 2$, $\ell = 0$, $D^C_{2,0} = 2$.

i	j	$c^C_{2,0,i,j}$
0	0	$20\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
0	1	$40\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
0	2	$20\,\Pi_1^3\,\Pi_1^{\vee\,4}\,\Pi_2^2\,\Pi_2^{\vee\,4}$
1	0	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{347}$
1	1	$4\,\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{346}$
1	2	$4\,\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{345}$
2	0	$8\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2^2\,\Pi_2^\vee\,\Pi^\vee_{344}$
2	1	$16\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^\vee\,\Pi^\vee_{343}$
2	2	$8\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{342}$
3	0	$8\,\Pi_1^2\,\Pi_1^\vee\,\Pi_2^2\,\Pi^\vee_{341}$
3	1	$8\,\Pi_1^2\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^\vee\,\Pi^\vee_{340}$
3	2	$8\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^2\,\Pi_2^{\vee\,2}\,\Pi^\vee_{339}$
4	0	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi^\vee_{338}$
4	1	$8\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^\vee\,\Pi^\vee_{337}$
4	2	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^3\,\Pi_2^{\vee\,2}\,\Pi_{336}$
5	0	$4\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2^4\,\Pi_2^\vee\,\Pi_{335}$
5	1	$4\,\Pi_1^3\,\Pi_2^4\,\Pi_2^\vee\,\Pi^\vee_{334}$
5	2	$4\,\Pi_1^3\,\Pi_2^4\,\Pi_2^{\vee\,2}\,\Pi_{333}$
6	0	$8\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{332}$
6	1	$16\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{332}$
6	2	$8\,\Pi_1^4\,\Pi_1^\vee\,\Pi_2^5\,\Pi_2^{\vee\,2}\,\Pi_{332}$

$H = C$, $k = 2$, $\ell = 1$, $D^C_{2,1} = 0$.

i	j	$c^C_{2,1,i,j}$
0	0	$\Pi_1^2\,\Pi_1^{\vee\,3}\,\Pi_2^\vee\,\Pi^\vee_{263}$
1	0	$2\,\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi^\vee_{360}$
2	0	$\Pi_1^2\,\Pi_1^{\vee\,2}\,\Pi_{359}$
3	0	$2\,\Pi_1^3\,\Pi_1^\vee\,\Pi_2\,\Pi_{358}$
4	0	$\Pi_1^3\,\Pi_1^\vee\,\Pi_2^2\,\Pi_{130}$

$H = C$, $k = 3$, $\ell = 0$, $D^C_{3,0} = 0$.

i	j	$c^C_{3,0,i,j}$
0	0	$\Pi_1^3\,\Pi_1^{\vee\,3}\,\Pi_2^{\vee}$
1	0	$2\,\Pi_1^3\,\Pi_1^{\vee\,2}\,\Pi_{349}$
2	0	$\Pi_1^3\,\Pi_1^{\vee}\,\Pi_2\,\Pi_{348}$
3	0	$4\,\Pi_1^4\,\Pi_1^{\vee}\,\Pi_2^2$
4	0	$\Pi_1^4\,\Pi_2^3$

Appendix G. Higher-mode dimension polynomials

Put $a = p - 2$ and $b = q - 2$. For a finite list write

$$[c_0, \dots, c_n]_b = \sum_{j=0}^{n} c_j b^j.$$

Each row below specifies the entire polynomial as a sum of terms $a^u b^v [c_0, \dots, c_n]_b$. Factors with exponent zero are omitted. Every coefficient is an exact nonnegative integer. Every polynomial has a positive coefficient. A polynomial may vanish on the boundary $a = 0$ or $b = 0$; the strict inequalities required in the scalar form follow from (8.26)–(8.27).

i	$\Pi_i(a, b)$
1	$[1, 1]_b$
2	$[2, 1]_b$
3	$[4, 1]_b + a^1[1]_b$
4	$[9, 5]_b$
5	$[8, 5]_b$
6	$[7, 5]_b$
7	$[6, 5]_b$
8	$[576, 783, 391, 82, 6]_b + a^1[560, 685, 297, 48, 2]_b + a^2[184, 203, 75, 8]_b + a^3[20, 19, 5]_b$
9	$[4856, 6576, 3249, 671, 48]_b + a^1[7870, 10000, 4545, 818, 46]_b + a^2[4616, 5414, 2190, 313, 10]_b + a^3[1170, 1244, 427, 40]_b + a^4[108, 100, 25]_b$
10	$[8080, 18528, 16391, 7006, 1449, 116]_b + a^1[10892, 23862, 19923, 7916, 1491, 105]_b + a^2[5315, 10968, 8445, 3000, 479, 25]_b + a^3[1078, 2027, 1358, 386, 40]_b + a^4[75, 121, 62, 10]_b$
11	$[6624, 16516, 15627, 7030, 1509, 124]_b + a^1[8880, 21218, 18960, 7907, 1536, 110]_b + a^2[4271, 9647, 7955, 2953, 481, 25]_b + a^3[848, 1760, 1272, 379, 40]_b + a^4[57, 103, 58, 10]_b$
12	$[5344, 7116, 3366, 652, 42]_b + a^1[9216, 11528, 5010, 854, 44]_b + a^2[5594, 6411, 2445, 332, 10]_b + a^3[1434, 1464, 458, 40]_b + a^4[132, 115, 25]_b$
13	$[424, 348, 86, 6]_b + a^1[418, 285, 53, 2]_b + a^2[134, 71, 8]_b + a^3[14, 5]_b$
14	$[8534, 20233, 18844, 8907, 2253, 290, 15]_b + a^1[16774, 37718, 32440, 13744, 2972, 303, 11]_b + a^2[13474, 28762, 22786, 8592, 1548, 114, 2]_b + a^3[5288, 10617, 7612, 2484, 350, 15]_b + a^4[988, 1833, 1145, 301, 27]_b + a^5[70, 117, 59, 10]_b$
15	$[80212, 177268, 156435, 70595, 17112, 2113, 105]_b + a^1[216856, 459672, 383328, 160471, 35067, 3740, 152]_b + a^2[238982, 484587, 379436, 145594, 27948, 2418, 69]_b + a^3[136908, 264190, 192326, 66552, 10852, 694, 10]_b + a^4[42676, 77558, 51478, 15539, 2006, 75]_b + a^5[6812, 11458, 6682, 1631, 135]_b + a^6[434, 659, 315, 50]_b$
16	$[288016, 990848, 1418050, 1095114, 492378, 128584, 18008, 1042]_b + a^1[603864, 2011804, 2776997, 2057750, 881158, 216737, 28121, 1473]_b + a^2[508958, 1626424, 2139467, 1498179, 598772, 134690, 15430, 670]_b + a^3[219843, 665296, 820275, 531363, 192572, 37883, 3514, 100]_b + a^4[51001, 143313, 161065, 92865, 28872, 4524, 270]_b + a^5[6013, 15234, 14863, 7042, 1636, 150]_b + a^6[285, 629, 496, 166, 20]_b$
17	$[495328, 1582464, 2117220, 1537576, 653938, 162606, 21846, 1222]_b + a^1[998240, 3085000, 3974804, 2763534, 1116211, 260611, 32349, 1635]_b + a^2[809316, 2397928, 2942643, 1931263, 726696, 154816, 16938, 710]_b + a^3[335676, 941468, 1082991, 657307, 223929, 41589, 3666, 100]_b + a^4[74511, 193857, 203461, 110107, 32218, 4766, 270]_b + a^5[8354, 19512, 17762, 7914, 1734, 150]_b + a^6[375, 755, 552, 174, 20]_b$
18	$[102296, 253676, 256032, 134546, 38776, 5802, 352]_b + a^1[148984, 349100, 329416, 159525, 41493, 5431, 275]_b + a^2[82550, 179885, 155316, 67145, 14943, 1539, 50]_b + a^3[21492, 42453, 32419, 11917, 2084, 135]_b + a^4[2598, 4415, 2746, 746, 75]_b + a^5[120, 163, 71, 10]_b$
19	$[16360, 23092, 12302, 3059, 348, 15]_b + a^1[29312, 38236, 18436, 4013, 368, 11]_b + a^2[20656, 24520, 10434, 1914, 129, 2]_b + a^3[7140, 7538, 2707, 387, 15]_b + a^4[1208, 1095, 303, 27]_b + a^5[80, 59, 10]_b$

i	$\Pi_i(a,b)$
20	$[79444, 188894, 186269, 98606, 30127, 5282, 488, 18]_b + a^1[225298, 512378, 476531, 234148, 64941, 10000, 769, 21]_b + a^2[277740, 603053, 526074, 237733, 58924, 7748, 470, 8]_b + a^3[191370, 395391, 320770, 131437, 28380, 3028, 131, 1]_b + a^4[78768, 154040, 114831, 41858, 7591, 598, 14]_b + a^5[18978, 34706, 23216, 7263, 1035, 48]_b + a^6[2408, 4009, 2274, 553, 50]_b + a^7[122, 177, 75, 10]_b$
21	$[667496, 2124460, 2824174, 2053559, 893129, 237211, 37377, 3166, 108]_b + a^1[2509776, 7763950, 9953159, 6918145, 2845324, 704361, 101351, 7602, 216]_b + a^2[4077030, 12233221, 15065789, 9948925, 3833253, 871349, 111708, 7114, 153]_b + a^3[3732182, 10838630, 12771884, 7960624, 2843623, 583182, 64505, 3290, 46]_b + a^4[2096624, 5875976, 6591057, 3845602, 1256992, 227171, 20623, 758, 5]_b + a^5[735016, 1978464, 2095612, 1130225, 331558, 50971, 3468, 70]_b + a^6[155490, 398533, 392803, 191205, 48494, 5921, 240]_b + a^7[17930, 43096, 38413, 16004, 3185, 250]_b + a^8[856, 1886, 1445, 455, 50]_b$
22	$[2348640, 6961904, 8711740, 6008516, 2492030, 633452, 95698, 7768, 252]_b + a^1[8558256, 24610680, 29648256, 19512227, 7636057, 1803670, 247763, 17709, 474]_b + a^2[13445388, 37448432, 43303560, 27064948, 9918541, 2149521, 262570, 15900, 322]_b + a^3[11864372, 31930543, 35305141, 20832569, 7084150, 1386517, 146089, 7079, 94]_b + a^4[6407730, 16601978, 17452113, 9641088, 3004088, 519497, 45008, 1572, 10]_b + a^5[2158436, 5351645, 5299019, 2703216, 756351, 111601, 7272, 140]_b + a^6[440156, 1034195, 949158, 435846, 105249, 12354, 480]_b + a^7[49344, 108164, 89388, 35020, 6610, 500]_b + a^8[2318, 4635, 3281, 970, 100]_b$
23	$[2079712, 7244864, 10878900, 9177236, 4744066, 1533750, 301656, 32862, 1514]_b + a^1[4732544, 15896432, 22948156, 18530863, 9111526, 2775336, 506895, 50163, 2033]_b + a^2[4440956, 14236668, 19525222, 14881850, 6839057, 1916599, 313536, 26456, 830]_b + a^3[2223980, 6721045, 8631478, 6104237, 2567205, 643278, 89981, 5850, 100]_b + a^4[638326, 1787530, 2104017, 1344271, 500131, 107041, 11906, 500]_b + a^5[104088, 263269, 274062, 150315, 45780, 7328, 480]_b + a^6[8826, 19468, 17025, 7405, 1608, 140]_b + a^7[288, 516, 338, 96, 10]_b$
24	$[897616, 1915360, 1692168, 799264, 216427, 33249, 2655, 81]_b + a^1[2525904, 5162244, 4322378, 1907960, 472537, 64082, 4294, 99]_b + a^2[2993764, 5828656, 4592045, 1876411, 418966, 48706, 2585, 39]_b + a^3[1932562, 3556851, 2608129, 972406, 191642, 18237, 692, 5]_b + a^4[731640, 1258670, 844247, 280258, 47033, 3352, 70]_b + a^5[161894, 255991, 152508, 43002, 5709, 240]_b + a^6[19322, 27381, 13728, 2945, 250]_b + a^7[958, 1173, 445, 50]_b$
25	$[74664, 114388, 70278, 22019, 3635, 299, 9]_b + a^1[160840, 228694, 128527, 35952, 5002, 317, 6]_b + a^2[142054, 184983, 93489, 22878, 2569, 114, 1]_b + a^3[65632, 76707, 33856, 6988, 580, 14]_b + a^4[16674, 16945, 6181, 997, 48]_b + a^5[2202, 1848, 493, 50]_b + a^6[118, 75, 10]_b$
26	$[283176, 691496, 711974, 399912, 131931, 25466, 2651, 114]_b + a^1[693636, 1604286, 1551456, 810358, 245185, 42540, 3860, 137]_b + a^2[722910, 1574071, 1415776, 677987, 184256, 27814, 2082, 54]_b + a^3[415066, 845262, 698399, 300741, 71130, 8835, 492, 7]_b + a^4[141278, 267194, 199857, 75540, 14864, 1368, 43]_b + a^5[28050, 48614, 32108, 10266, 1571, 85]_b + a^6[2896, 4425, 2395, 575, 55]_b + a^7[116, 144, 51, 5]_b$
27	$[2629456, 8344000, 11302680, 8545112, 3941613, 1133759, 197981, 19105, 774]_b + a^1[8447024, 25935796, 33795578, 24432090, 10702951, 2898213, 470812, 41555, 1497]_b + a^2[11707332, 34626412, 43114739, 29531634, 12133661, 3040269, 448299, 34893, 1049]_b + a^3[9138486, 25922089, 30626916, 19672882, 7469420, 1693629, 218849, 14188, 317]_b + a^4[4379396, 11857008, 13189605, 7851951, 2705843, 538785, 57701, 2800, 35]_b + a^5[1309824, 3362578, 3485173, 1893542, 578672, 97031, 7793, 215]_b + a^6[235676, 566872, 537211, 259145, 67528, 8895, 425]_b + a^7[22866, 50281, 41727, 16628, 3295, 275]_b + a^8[900, 1732, 1143, 300, 25]_b$
28	$[4765088, 19314960, 34049832, 34274498, 21683082, 8913152, 2368214, 389112, 35504, 1358]_b + a^1[16260400, 64220688, 109946418, 107121480, 65350491, 25780612, 6526891, 1009833, 85012, 2903]_b + a^2[23432936, 89695818, 148088324, 138422786, 80544533, 30081404, 7129707, 1012685, 75312, 2103]_b + a^3[18691226, 68976988, 109028347, 96832341, 53073768, 18467743, 4011841, 506650, 31218, 590]_b + a^4[9037996, 31978360, 48004835, 40055489, 20365410, 6469403, 1253741, 135282, 6330, 50]_b + a^5[2710166, 9133251, 12890317, 9953047, 4589855, 1287932, 212200, 18170, 550]_b + a^6[490710, 1561707, 2044353, 1429099, 576377, 134048, 16632, 850]_b + a^7[48816, 145073, 172964, 106207, 35506, 6140, 430]_b + a^8[2022, 5507, 5790, 2947, 728, 70]_b$
29	$[4384416, 13759248, 18492792, 13926056, 6423853, 1853910, 325490, 31594, 1281]_b + a^1[14885968, 45237184, 58501888, 42113379, 18434392, 5003165, 816602, 72618, 2632]_b + a^2[21592968, 63286236, 78285253, 53422241, 21929119, 5500766, 813220, 63770, 1941]_b + a^3[17458804, 49126145, 57747618, 37014164, 14059390, 3191791, 412396, 26872, 612]_b + a^4[8582016, 23045767, 25526235, 15190868, 5252346, 1050336, 112364, 5460, 70]_b + a^5[2614100, 6639737, 6841008, 3714083, 1141384, 193312, 15558, 430]_b + a^6[478702, 1133608, 1062223, 509133, 132948, 17750, 850]_b + a^7[47804, 103044, 84062, 32966, 6470, 550]_b + a^8[1982, 3747, 2441, 630, 50]_b$
30	$[2006800, 6476224, 8966572, 6945016, 3282798, 966198, 172130, 16886, 696]_b + a^1[4809408, 14913492, 19730530, 14501410, 6442372, 1757106, 283780, 24360, 830]_b + a^2[4887120, 14445620, 18080840, 12455088, 5118354, 1264685, 178296, 12390, 285]_b + a^3[2731174, 7628717, 8931520, 5685136, 2123215, 464234, 54972, 2805, 25]_b + a^4[902922, 2356447, 2538658, 1459836, 480729, 89302, 8365, 275]_b + a^5[175398, 421249, 407182, 202977, 55013, 7662, 425]_b + a^6[18426, 40061, 33723, 13789, 2750, 215]_b + a^7[792, 1518, 1057, 318, 35]_b$
31	$[355088, 785984, 745268, 392020, 122816, 22748, 2300, 96]_b + a^1[910928, 1917460, 1715862, 842982, 242677, 40319, 3563, 125]_b + a^2[982200, 1949356, 1628978, 737579, 191261, 27480, 1992, 52]_b + a^3[575598, 1064861, 819151, 335986, 76550, 9031, 482, 7]_b + a^4[197334, 334663, 231695, 83744, 16136, 1425, 43]_b + a^5[39420, 59728, 35749, 10787, 1631, 85]_b + a^6[4232, 5498, 2622, 575, 55]_b + a^7[188, 196, 61, 5]_b$

i	$\Pi_i(a,b)$
32	$[504784, 1128696, 1045104, 518382, 148408, 24192, 1976, 58]_b + a^1[1277288, 2703932, 2346342, 1077072, 281418, 41072, 2840, 60]_b + a^2[1372684, 2737932, 2209102, 926173, 216298, 27434, 1508, 17]_b + a^3[810544, 1514029, 1124788, 423418, 85864, 9080, 366, 1]_b + a^4[283222, 491225, 331363, 109165, 18275, 1476, 36]_b + a^5[58366, 92820, 55648, 15380, 1918, 90]_b + a^6[6542, 9357, 4811, 1033, 75]_b + a^7[306, 381, 156, 21]_b$
33	$[2629936, 7586792, 9339268, 6393340, 2651604, 676928, 102180, 8068, 244]_b + a^1[7150408, 19808940, 23232540, 15014381, 5815829, 1367575, 185787, 12510, 282]_b + a^2[8191832, 21700730, 24081012, 14533864, 5174010, 1095409, 129504, 6914, 85]_b + a^3[5125622, 12929256, 13480272, 7510708, 2412258, 447303, 44302, 1734, 5]_b + a^4[1887802, 4510068, 4378738, 2217370, 624715, 96556, 7399, 180]_b + a^5[407906, 915914, 817555, 367668, 86657, 10010, 450]_b + a^6[47666, 99442, 80035, 30836, 5610, 375]_b + a^7[2308, 4386, 3066, 935, 105]_b$
34	$[5187872, 16154368, 21332688, 15582770, 6860488, 1849886, 294150, 24612, 806]_b + a^1[17754496, 53534992, 68005458, 47448043, 19790276, 5000279, 731401, 54174, 1449]_b + a^2[26116976, 75979174, 92299627, 60981344, 23810061, 5547099, 729170, 45630, 865]_b + a^3[21585046, 60379117, 69731622, 43219481, 15579322, 3282842, 377452, 18846, 190]_b + a^4[10954799, 29356690, 32018609, 18405904, 6012794, 1114273, 107749, 3976, 10]_b + a^5[3488934, 8916498, 9107877, 4782043, 1376241, 213151, 15916, 360]_b + a^6[678782, 1644269, 1555413, 729872, 176758, 20560, 900]_b + a^7[73420, 167075, 143982, 58445, 11030, 750]_b + a^8[3355, 7061, 5374, 1760, 210]_b$
35	$[5011936, 19516896, 32738032, 31033380, 18287202, 6921826, 1672540, 246384, 19794, 650]_b + a^1[18132032, 68824560, 112005412, 102480716, 57941008, 20880021, 4749860, 646692, 46428, 1279]_b + a^2[28081680, 103531932, 162625268, 142610576, 76642163, 25976171, 5471341, 671008, 40906, 823]_b + a^3[24382892, 87044560, 131321152, 109572985, 55409083, 17422182, 3335417, 358257, 17406, 190]_b + a^4[12983428, 44743704, 64497821, 50780669, 23858434, 6833370, 1159357, 105255, 3784, 10]_b + a^5[4332552, 14364582, 19663955, 14454156, 6196794, 1568928, 224695, 15954, 360]_b + a^6[881608, 2800510, 3613635, 2446104, 932263, 198032, 21400, 900]_b + a^7[99476, 301086, 362731, 222188, 72669, 11920, 750]_b + a^8[4716, 13458, 14862, 7956, 2070, 210]_b$
36	$[2423056, 7730840, 10454156, 7813338, 3518396, 972182, 159696, 14108, 508]_b + a^1[7990504, 24670124, 32078282, 22884774, 9743307, 2510142, 375821, 29070, 852]_b + a^2[11506864, 34236460, 42550695, 28745092, 11451620, 2711851, 361634, 23297, 479]_b + a^3[9440720, 26961247, 31823876, 20172455, 7428833, 1591320, 184505, 9348, 100]_b + a^4[4811117, 13131357, 14606644, 8581458, 2869450, 542444, 52844, 1961, 5]_b + a^5[1552790, 4030667, 4184059, 2238330, 659383, 104681, 7906, 180]_b + a^6[308470, 757388, 725107, 344687, 85033, 10145, 450]_b + a^7[34270, 79084, 68863, 28187, 5385, 375]_b + a^8[1609, 3445, 2654, 875, 105]_b$
37	$[458480, 1376056, 1765784, 1262150, 547674, 147030, 23706, 2084, 76]_b + a^1[1154360, 3320380, 4049638, 2722540, 1095194, 266702, 37646, 2720, 72]_b + a^2[1279700, 3513972, 4052148, 2545536, 941387, 205395, 24767, 1370, 19]_b + a^3[800264, 2086117, 2255811, 1309589, 439086, 84352, 8474, 336, 1]_b + a^4[301252, 740024, 740959, 389702, 114998, 18624, 1459, 36]_b + a^5[67676, 155452, 141820, 65512, 16044, 1948, 90]_b + a^6[8352, 17840, 14617, 5730, 1070, 75]_b + a^7[428, 839, 599, 185, 21]_b$
38	$[249232, 519008, 434488, 187720, 44830, 5822, 350, 6]_b + a^1[534800, 1048640, 813432, 317953, 66499, 7253, 325, 2]_b + a^2[475856, 875900, 625909, 218612, 39046, 3449, 108]_b + a^3[223916, 385003, 251203, 76892, 11191, 732, 13]_b + a^4[58518, 93231, 54760, 14235, 1539, 57]_b + a^5[8012, 11667, 6015, 1252, 80]_b + a^6[446, 579, 248, 35]_b$
39	$[1420144, 3060424, 2678584, 1222528, 310666, 42982, 2778, 54]_b + a^1[4012760, 8263268, 6820180, 2883819, 663831, 80785, 4245, 48]_b + a^2[4809356, 9434158, 7298039, 2825622, 577319, 60093, 2435, 10]_b + a^3[3165358, 5893836, 4243016, 1482063, 260794, 22170, 637]_b + a^4[1232947, 2168874, 1439815, 444527, 64077, 4033, 65]_b + a^5[283337, 467614, 282494, 74759, 8041, 285]_b + a^6[35413, 54212, 29142, 6255, 400]_b + a^7[1845, 2566, 1170, 175]_b$
40	$[3055584, 8522160, 9996684, 6404318, 2430186, 551900, 71442, 4550, 96]_b + a^1[9235248, 24848160, 27858526, 16863163, 5957472, 1234579, 141138, 7318, 92]_b + a^2[11802084, 30570862, 32619562, 18514459, 6013898, 1116107, 109816, 4394, 20]_b + a^3[8258150, 20550701, 20774030, 10957102, 3217800, 519567, 42198, 1208]_b + a^4[3409546, 8130843, 7745610, 3754490, 973280, 129921, 7932, 130]_b + a^5[827758, 1884665, 1680246, 737769, 163026, 16314, 570]_b + a^6[108826, 235023, 193866, 75379, 13400, 800]_b + a^7[5924, 11970, 8904, 2900, 350]_b$
41	$[3172768, 9420256, 11714716, 7934342, 3177202, 760638, 103982, 7088, 168]_b + a^1[11789984, 33982152, 40699346, 26303265, 9934319, 2207045, 272495, 15758, 260]_b + a^2[18965548, 52976994, 60889716, 37325124, 13176419, 2682431, 294002, 13908, 128]_b + a^3[17238438, 46586893, 51184176, 29543465, 9630357, 1763578, 167373, 6186, 20]_b + a^4[9669630, 25237786, 26391602, 14216746, 4207299, 672507, 52816, 1402]_b + a^5[3420285, 8604454, 8523283, 4239373, 1112370, 147327, 8678, 130]_b + a^6[742529, 1795683, 1675118, 759118, 170782, 17006, 570]_b + a^7[89965, 208135, 181149, 73295, 13390, 800]_b + a^8[4613, 10099, 8026, 2760, 350]_b$
42	$[1612432, 5945520, 9377488, 8286236, 4502584, 1549924, 334420, 42856, 2868, 72]_b + a^1[6280704, 22620920, 34652644, 29544063, 15363678, 5006078, 1005756, 116651, 6650, 120]_b + a^2[10603296, 37237452, 55235521, 45230265, 22366565, 6839511, 1264726, 130446, 6076, 62]_b + a^3[10125820, 34614847, 49555386, 38764700, 18080347, 5128450, 859046, 77125, 2813, 10]_b + a^4[5971646, 19838174, 27314325, 20279352, 8823064, 2279969, 336389, 25237, 668]_b + a^5[2222412, 7164608, 9454746, 6614706, 2646897, 606390, 74816, 4272, 65]_b + a^6[508092, 1587418, 2001146, 1309358, 473799, 92483, 8619, 285]_b + a^7[64832, 195825, 234722, 142255, 45546, 7140, 400]_b + a^8[3486, 10100, 11354, 6209, 1660, 175]_b$

i	$\Pi_i(a,b)$
43	$[324336, 952168, 1180000, 802938, 325950, 80030, 11434, 848, 24]_b + a^1[1152408, 3279756, 3907754, 2531110, 964952, 217962, 27642, 1688, 32]_b + a^2[1820460, 5012120, 5722736, 3513072, 1251241, 258414, 28837, 1410, 14]_b + a^3[1651844, 4387963, 4777543, 2758452, 908048, 169046, 16305, 617, 2]_b + a^4[935884, 2392639, 2469613, 1326681, 396465, 64676, 5174, 140]_b + a^5[337880, 829816, 807135, 398023, 105098, 14248, 859, 13]_b + a^6[75680, 178368, 162608, 72500, 16283, 1656, 57]_b + a^7[9556, 21577, 18318, 7249, 1312, 80]_b + a^8[512, 1097, 849, 283, 35]_b$
44	$[34488, 50616, 26414, 6040, 638, 24]_b + a^1[61972, 83930, 38974, 7372, 594, 14]_b + a^2[44330, 55115, 22503, 3368, 187, 2]_b + a^3[15708, 17769, 6257, 683, 20]_b + a^4[2738, 2769, 808, 52]_b + a^5[186, 162, 35]_b$
45	$[416368, 824040, 642816, 249480, 50392, 5072, 192]_b + a^1[1042136, 1976032, 1454082, 518854, 92638, 7898, 232]_b + a^2[1069680, 1936382, 1335473, 431828, 65765, 4440, 86]_b + a^3[576556, 992986, 637827, 184011, 22773, 1091, 10]_b + a^4[171748, 280028, 166089, 41848, 3869, 100]_b + a^5[26678, 40805, 21990, 4649, 260]_b + a^6[1674, 2357, 1110, 175]_b$
46	$[951584, 3080848, 4202900, 3135542, 1381027, 358265, 50445, 2949]_b + a^1[1940560, 6060904, 7958786, 5703984, 2407798, 596374, 79576, 4338]_b + a^2[1571268, 4673478, 5814373, 3928819, 1555291, 358523, 43888, 2120]_b + a^3[640010, 1778394, 2043416, 1259492, 448310, 91096, 9476, 350]_b + a^4[137188, 346476, 352786, 186344, 54136, 8246, 520]_b + a^5[14730, 32870, 28344, 11898, 2454, 200]_b + a^6[600, 1118, 738, 204, 20]_b$
47	$[1051680, 2767264, 3019388, 1762484, 590870, 113104, 11394, 456]_b + a^1[3655520, 9349680, 9832412, 5470501, 1721712, 303179, 27406, 938]_b + a^2[5334612, 13237924, 13362534, 7030358, 2046726, 322766, 25056, 670]_b + a^3[4249124, 10219151, 9869688, 4874788, 1291834, 176155, 11014, 196]_b + a^4[1995782, 4648184, 4283926, 1972852, 467404, 52414, 2366, 20]_b + a^5[551292, 1241491, 1088178, 463551, 95950, 8114, 200]_b + a^6[82394, 178654, 147850, 57442, 9994, 520]_b + a^7[5076, 10472, 8012, 2720, 350]_b$
48	$[569200, 1563328, 1776956, 1080508, 377874, 75588, 7970, 336]_b + a^1[1825952, 4865972, 5312850, 3065060, 1000535, 182714, 17085, 604]_b + a^2[2504704, 6473804, 6770276, 3680858, 1106622, 179917, 14303, 386]_b + a^3[1906370, 4780363, 4779050, 2433537, 663942, 92967, 5915, 104]_b + a^4[867716, 2111996, 2015450, 955609, 232569, 26679, 1219, 10]_b + a^5[235012, 555112, 504656, 221337, 47028, 4051, 100]_b + a^6[34756, 79444, 68418, 27421, 4911, 260]_b + a^7[2130, 4661, 3728, 1310, 175]_b$
49	$[60856, 175200, 207792, 131498, 47814, 9942, 1090, 48]_b + a^1[147956, 411242, 464026, 274493, 91171, 16753, 1547, 52]_b + a^2[154908, 416393, 446354, 245228, 73296, 11531, 844, 18]_b + a^3[88546, 230586, 234569, 118851, 31350, 4017, 211, 2]_b + a^4[28844, 72907, 70361, 32660, 7419, 705, 20]_b + a^5[4998, 12258, 11195, 4718, 895, 52]_b + a^6[348, 818, 691, 252, 35]_b$
50	$[2524, 3730, 2039, 486, 42]_b + a^1[2298, 3056, 1478, 303, 21]_b + a^2[462, 477, 163, 19]_b + a^3[28, 19, 3]_b$
51	$[31544, 70516, 61058, 25672, 5231, 409]_b + a^1[42824, 91870, 75639, 29962, 5676, 401]_b + a^2[19618, 39336, 29808, 10698, 1793, 105]_b + a^3[3338, 5783, 3564, 948, 95]_b + a^4[196, 273, 116, 15]_b$
52	$[148832, 453296, 556648, 353726, 122752, 21992, 1574]_b + a^1[275024, 817328, 975966, 600832, 201082, 34485, 2325]_b + a^2[194088, 557230, 639240, 375917, 119388, 19219, 1184]_b + a^3[63790, 173324, 185876, 100818, 29072, 4142, 210]_b + a^4[9494, 23403, 21946, 9867, 2162, 190]_b + a^5[532, 1145, 869, 274, 30]_b$
53	$[168928, 486288, 571976, 352178, 119540, 21096, 1494]_b + a^1[314800, 880496, 1004786, 599264, 196492, 33291, 2229]_b + a^2[225544, 605978, 661848, 376493, 117214, 18679, 1146]_b + a^3[76186, 192356, 195400, 102304, 28964, 4108, 210]_b + a^4[11842, 26865, 23638, 10177, 2178, 190]_b + a^5[700, 1371, 963, 286, 30]_b$
54	$[46616, 88428, 64622, 22444, 3626, 224]_b + a^1[103496, 188886, 130551, 41650, 5813, 292]_b + a^2[91322, 160468, 104579, 30237, 3425, 119]_b + a^3[39908, 67556, 41449, 10761, 907, 15]_b + a^4[8540, 13869, 7955, 1827, 95]_b + a^5[698, 1067, 554, 105]_b$
55	$[5036, 9794, 7323, 2614, 437, 28]_b + a^1[7866, 14532, 10018, 3154, 421, 19]_b + a^2[4814, 8533, 5450, 1503, 145, 3]_b + a^3[1320, 2247, 1328, 320, 19]_b + a^4[128, 204, 107, 21]_b$
56	$[78, 52, 8]_b + a^1[81, 50, 7]_b + a^2[9, 3]_b$
57	$[1012, 1406, 568, 54]_b + a^1[1494, 2063, 804, 63]_b + a^2[650, 870, 315, 15]_b + a^3[84, 107, 35]_b$
58	$[1244, 1612, 641, 63]_b + a^1[1688, 2166, 833, 66]_b + a^2[700, 870, 315, 15]_b + a^3[88, 104, 35]_b$
59	$[1476, 1818, 714, 72]_b + a^1[1882, 2269, 862, 69]_b + a^2[750, 870, 315, 15]_b + a^3[92, 101, 35]_b$
60	$[1708, 2024, 787, 81]_b + a^1[2076, 2372, 891, 72]_b + a^2[800, 870, 315, 15]_b + a^3[96, 98, 35]_b$
61	$[194, 223, 86, 9]_b + a^1[130, 136, 49, 3]_b + a^2[20, 19, 7]_b$
62	$b^1[1]_b$
63	$[151, 171, 63, 7]_b + a^1[110, 115, 38, 3]_b + a^2[19, 18, 5]_b$
64	$[1084, 1225, 448, 49]_b + a^1[1421, 1536, 529, 49]_b + a^2[596, 607, 191, 12]_b + a^3[79, 74, 20]_b$
65	$[480, 541, 196, 21]_b + a^1[679, 734, 251, 23]_b + a^2[300, 305, 95, 6]_b + a^3[41, 38, 10]_b$
66	$[836, 939, 336, 35]_b + a^1[1295, 1400, 475, 43]_b + a^2[604, 613, 189, 12]_b + a^3[85, 78, 20]_b$
67	$[178, 199, 70, 7]_b + a^1[130, 134, 42, 3]_b + a^2[22, 20, 5]_b$
68	$[6274, 13569, 11041, 4246, 769, 53]_b + a^1[10356, 21363, 16100, 5526, 837, 44]_b + a^2[6634, 13060, 9054, 2710, 313, 9]_b + a^3[1872, 3495, 2198, 556, 41]_b + a^4[184, 317, 171, 32]_b$
69	$[47616, 96556, 74722, 27516, 4788, 318]_b + a^1[112244, 219170, 160606, 54707, 8463, 476]_b + a^2[104164, 195642, 134907, 41789, 5466, 230]_b + a^3[47380, 85364, 54942, 15157, 1547, 36]_b + a^4[10444, 17902, 10573, 2499, 164]_b + a^5[872, 1390, 718, 128]_b$

i	$\Pi_i(a,b)$
70	$[131952, 404088, 501682, 323580, 114186, 20800, 1512]_b + a^1[253040, 753708, 908023, 566511, 192380, 33407, 2271]_b + a^2[186470, 534272, 615856, 365176, 116680, 18718, 1128]_b + a^3[65523, 177561, 191322, 104468, 30059, 4161, 192]_b + a^4[10874, 27087, 26057, 12157, 2763, 246]_b + a^5[689, 1526, 1227, 426, 54]_b$
71	$[78592, 143848, 101264, 33886, 5324, 318]_b + a^1[179352, 317792, 213734, 66798, 9404, 476]_b + a^2[159232, 271936, 173346, 49751, 6001, 230]_b + a^3[68676, 112308, 67065, 17236, 1645, 36]_b + a^4[14324, 22180, 12151, 2665, 164]_b + a^5[1144, 1640, 786, 128]_b$
72	$[8388, 10628, 4723, 844, 53]_b + a^1[12944, 15434, 6257, 943, 44]_b + a^2[7344, 8143, 2937, 344, 9]_b + a^3[1804, 1822, 557, 41]_b + a^4[160, 141, 32]_b$
73	$[75360, 162586, 141071, 62727, 14967, 1801, 84]_b + a^1[188622, 390331, 319931, 131767, 28278, 2919, 106]_b + a^2[199810, 395690, 304258, 114577, 21576, 1817, 44]_b + a^3[114048, 215330, 153656, 51811, 8202, 515, 6]_b + a^4[36792, 65910, 43001, 12574, 1539, 56]_b + a^5[6250, 10515, 6115, 1468, 114]_b + a^6[414, 630, 300, 48]_b$
74	$[499712, 1495304, 1824226, 1177587, 433523, 90529, 9835, 420]_b + a^1[1691064, 4930700, 5814630, 3593086, 1250143, 242258, 23745, 866]_b + a^2[2426482, 6879053, 7808177, 4585159, 1489750, 262574, 22413, 644]_b + a^3[1914730, 5266726, 5728161, 3170202, 947816, 147886, 10441, 206]_b + a^4[895216, 2383280, 2470699, 1275579, 343979, 45533, 2418, 24]_b + a^5[246568, 632990, 620913, 294768, 69553, 7220, 224]_b + a^6[36614, 89963, 82360, 35015, 6820, 456]_b + a^7[2206, 5088, 4190, 1476, 192]_b$
75	$[1292352, 3615296, 4163608, 2554688, 898196, 179632, 18680, 756]_b + a^1[4208128, 11450000, 12718792, 7450398, 2467334, 456064, 42578, 1458]_b + a^2[5809816, 15355272, 16400824, 9118852, 2815349, 472331, 38330, 1032]_b + a^3[4400128, 11276406, 11537556, 6045688, 1717146, 254858, 17096, 318]_b + a^4[1968108, 4877494, 4756039, 2326427, 596591, 75201, 3792, 36]_b + a^5[517300, 1234096, 1137592, 511850, 115000, 11406, 336]_b + a^6[73372, 167046, 143367, 57685, 10680, 684]_b + a^7[4268, 9102, 7018, 2340, 288]_b$
76	$[379520, 801416, 671706, 285033, 64108, 7147, 294]_b + a^1[982840, 1997948, 1586336, 624461, 126487, 12158, 392]_b + a^2[1044874, 2038447, 1523120, 549066, 97574, 7677, 170]_b + a^3[581816, 1084103, 755085, 244679, 36615, 2142, 24]_b + a^4[178112, 314479, 201056, 56793, 6628, 224]_b + a^5[28208, 46525, 26513, 6124, 456]_b + a^6[1782, 2666, 1260, 192]_b$
77	$[82336, 161772, 126636, 50215, 10554, 1101, 42]_b + a^1[157588, 296012, 217271, 78511, 14388, 1220, 32]_b + a^2[118800, 212046, 144393, 46647, 7165, 451, 6]_b + a^3[43876, 73694, 45746, 12785, 1525, 56]_b + a^4[7880, 12244, 6700, 1510, 114]_b + a^5[544, 756, 336, 48]_b$
78	$[270864, 594016, 536860, 255590, 67530, 9290, 490]_b + a^1[598808, 1247920, 1063068, 472350, 115208, 14410, 652]_b + a^2[541496, 1063138, 842448, 342044, 74421, 8012, 273]_b + a^3[255666, 467684, 338940, 122280, 22589, 1916, 37]_b + a^4[66198, 111023, 71702, 21959, 3132, 166]_b + a^5[8864, 13287, 7304, 1733, 148]_b + a^6[476, 610, 258, 36]_b$
79	$[2034944, 5943104, 7324744, 4939116, 1964672, 458724, 57488, 2888]_b + a^1[6022976, 17073024, 20319784, 13158528, 4996184, 1105276, 129640, 5900]_b + a^2[7436760, 20345324, 23193186, 14258107, 5085130, 1042599, 110882, 4296]_b + a^3[4968976, 13041072, 14112362, 8126094, 2667846, 491778, 45208, 1328]_b + a^4[1938470, 4845631, 4923328, 2609898, 766784, 121143, 8834, 148]_b + a^5[440276, 1038106, 975234, 464002, 116514, 14368, 664]_b + a^6[53630, 117531, 99594, 40691, 7970, 592]_b + a^7[2680, 5324, 3880, 1232, 144]_b$
80	$[1384928, 4461504, 5924880, 4217044, 1739272, 414870, 52664, 2710]_b + a^1[4465264, 14007032, 18000468, 12318245, 4851090, 1095475, 129984, 6110]_b + a^2[5991872, 18209328, 22467458, 14615096, 5409329, 1131381, 121554, 4948]_b + a^3[4345060, 12727157, 14948358, 9118985, 3107445, 583313, 53906, 1716]_b + a^4[1837588, 5158608, 5710481, 3211869, 979389, 156957, 11304, 216]_b + a^5[451340, 1205458, 1241317, 628814, 164355, 20544, 888]_b + a^6[59012, 148120, 138779, 60777, 12570, 996]_b + a^7[3096, 7097, 5711, 1902, 222]_b$
81	$[1686784, 5009600, 6209720, 4175588, 1645682, 379016, 46834, 2352]_b + a^1[5133632, 14820832, 17730584, 11423284, 4280759, 929074, 106815, 4872]_b + a^2[6551112, 18303956, 20987186, 12821542, 4500678, 901086, 93424, 3672]_b + a^3[4541664, 12221852, 13325915, 7624952, 2459418, 440642, 39169, 1200]_b + a^4[1843444, 4749808, 4876736, 2571962, 742138, 113406, 7838, 144]_b + a^5[436196, 1067102, 1017239, 482920, 119551, 14280, 592]_b + a^6[55276, 126608, 109314, 44700, 8738, 664]_b + a^7[2852, 5946, 4386, 1356, 148]_b$
82	$[191760, 429384, 392228, 188024, 49852, 6868, 372]_b + a^1[453008, 973920, 841398, 375903, 91412, 11325, 530]_b + a^2[436240, 893106, 719882, 293448, 63268, 6666, 242]_b + a^3[218782, 422409, 312363, 113035, 20570, 1667, 36]_b + a^4[60016, 107784, 71285, 21931, 3088, 148]_b + a^5[8478, 13845, 7809, 1868, 166]_b + a^6[476, 676, 287, 37]_b$
83	$[450848, 914232, 741276, 305560, 67164, 7392, 296]_b + a^1[1030168, 1985156, 1510648, 574036, 113514, 10880, 342]_b + a^2[971104, 1770362, 1254433, 432889, 74797, 5923, 124]_b + a^3[482294, 826899, 540025, 165890, 23984, 1438, 14]_b + a^4[132642, 211975, 125583, 33238, 3700, 134]_b + a^5[19064, 27984, 14617, 3119, 212]_b + a^6[1112, 1462, 635, 91]_b$
84	$[956192, 2057984, 1776958, 783756, 185020, 21872, 954]_b + a^1[2378768, 4887948, 3975609, 1622375, 346059, 35759, 1238]_b + a^2[2417790, 4718282, 3578664, 1325795, 246780, 20975, 480]_b + a^3[1285197, 2369269, 1658386, 545937, 84709, 5418, 56]_b + a^4[375780, 649477, 413406, 117083, 13918, 536]_b + a^5[56991, 91142, 51375, 11740, 848]_b + a^6[3474, 5016, 2364, 364]_b$
85	$[2875840, 8186096, 9675696, 6141876, 2251932, 473616, 52208, 2248]_b + a^1[9225520, 25495320, 29037628, 17604622, 6097018, 1193410, 119634, 4432]_b + a^2[12439032, 33267376, 36303946, 20826850, 6713046, 1194837, 104891, 3052]_b + a^3[9149664, 23614828, 24560578, 13204545, 3893942, 611634, 44615, 864]_b + a^4[3962356, 9839822, 9697614, 4832247, 1275358, 169097, 9402, 84]_b + a^5[1007194, 2396421, 2220648, 1009923, 230612, 23706, 804]_b + a^6[138340, 313060, 269181, 108811, 20142, 1272]_b + a^7[7846, 16649, 12927, 4374, 546]_b$

i	$\Pi_i(a,b)$
86	$[938464, 2897088, 3678498, 2489460, 967332, 214608, 24914, 1140]_b + a^1[3211504, 9655556, 11851875, 7688548, 2834288, 588210, 62485, 2494]_b + a^2[4595794, 13400958, 15791023, 9712955, 3341295, 632567, 59022, 1870]_b + a^3[3580463, 10091152, 11341495, 6539830, 2059581, 344398, 26657, 560]_b + a^4[1641701, 4459399, 4751116, 2537138, 714437, 100803, 5934, 56]_b + a^5[441631, 1152298, 1155406, 562540, 136747, 14922, 536]_b + a^6[64033, 159561, 148865, 64461, 12692, 848]_b + a^7[3802, 8928, 7550, 2740, 364]_b$
87	$[452896, 1293400, 1540148, 989372, 368952, 79420, 9060, 416]_b + a^1[1419768, 3939764, 4525668, 2781328, 981806, 197064, 20466, 816]_b + a^2[1891480, 5071398, 5573179, 3237821, 1063433, 193919, 17546, 548]_b + a^3[1393578, 3596211, 3754671, 2038781, 611855, 98408, 7350, 150]_b + a^4[612816, 1517465, 1495181, 749156, 200655, 27253, 1548, 14]_b + a^5[160132, 379343, 350102, 159201, 36688, 3856, 134]_b + a^6[22808, 51441, 43990, 17709, 3292, 212]_b + a^7[1338, 2834, 2185, 732, 91]_b$
88	$[96824, 136852, 68592, 14740, 1356, 36]_b + a^1[182688, 239296, 107496, 19362, 1356, 18]_b + a^2[137218, 165769, 65969, 9590, 472, 2]_b + a^3[51054, 56389, 19465, 2109, 57]_b + a^4[9354, 9294, 2667, 173]_b + a^5[670, 580, 124]_b$
89	$[896000, 1760912, 1355376, 512576, 98104, 8780, 252]_b + a^1[2315568, 4364904, 3175028, 1107116, 187534, 13972, 270]_b + a^2[2465960, 4449036, 3044084, 968047, 141273, 8318, 86]_b + a^3[1383088, 2381670, 1523860, 435622, 52523, 2231, 8]_b + a^4[429658, 703001, 416862, 104710, 9611, 228]_b + a^5[69778, 107617, 58197, 12302, 692]_b + a^6[4596, 6578, 3132, 496]_b$
90	$[735632, 1520360, 1234970, 496916, 102242, 9956, 324]_b + a^1[2062320, 4088524, 3135207, 1162384, 211767, 17164, 378]_b + a^2[2373430, 4507568, 3247658, 1096339, 171852, 10971, 126]_b + a^3[1433567, 2603944, 1753289, 531015, 68584, 3153, 12]_b + a^4[477624, 827156, 516347, 137204, 13419, 342]_b + a^5[82735, 135712, 77419, 17340, 1038]_b + a^6[5760, 8808, 4428, 744]_b$
91	$[1039360, 2811760, 3125920, 1838820, 611568, 113244, 10568, 360]_b + a^1[3654608, 9618232, 10309408, 5778232, 1799906, 303924, 24730, 648]_b + a^2[5448488, 13941312, 14372820, 7635833, 2205333, 333025, 22775, 398]_b + a^3[4465292, 11103582, 10986034, 5502517, 1456064, 191635, 10506, 98]_b + a^4[2168110, 5235699, 4958640, 2326072, 554783, 60939, 2445, 8]_b + a^5[621356, 1454994, 1314190, 572279, 120080, 10121, 228]_b + a^6[96718, 218819, 187190, 74547, 13194, 692]_b + a^7[6236, 13502, 10746, 3764, 496]_b$
92	$[135224, 388484, 456076, 282596, 99036, 19384, 1928, 72]_b + a^1[366776, 1018868, 1141072, 662898, 212414, 36482, 2954, 72]_b + a^2[423382, 1139513, 1217763, 661702, 192534, 28545, 1815, 22]_b + a^3[262512, 684830, 696935, 351799, 91419, 11245, 522, 2]_b + a^4[91370, 231051, 223349, 103794, 23460, 2191, 57]_b + a^5[16732, 40948, 37427, 15811, 2989, 173]_b + a^6[1228, 2878, 2438, 896, 124]_b$
93	$[10432, 15984, 9124, 2290, 211]_b + a^1[8792, 12000, 6008, 1290, 96]_b + a^2[1708, 1796, 631, 76]_b + a^3[96, 64, 10]_b$
94	$[50688, 115024, 102180, 44508, 9491, 787]_b + a^1[67536, 146064, 122872, 50360, 10011, 759]_b + a^2[30476, 61068, 46965, 17348, 3039, 192]_b + a^3[5120, 8760, 5428, 1474, 152]_b + a^4[288, 384, 158, 20]_b$
95	$[351744, 1072704, 1331216, 863104, 308572, 57550, 4346]_b + a^1[652288, 1933024, 2325632, 1458764, 503262, 90157, 6475]_b + a^2[461840, 1315072, 1513104, 903734, 295391, 49707, 3288]_b + a^3[152320, 407636, 435706, 238845, 70627, 10494, 576]_b + a^4[22692, 54528, 50425, 22701, 5058, 456]_b + a^5[1248, 2560, 1858, 564, 60]_b$
96	$[65536, 124400, 90620, 31094, 4894, 288]_b + a^1[153840, 280672, 193226, 60771, 8217, 384]_b + a^2[143172, 251162, 162884, 46314, 5063, 158]_b + a^3[65990, 111385, 67920, 17268, 1402, 20]_b + a^4[14954, 24223, 13801, 3077, 152]_b + a^5[1312, 2016, 1049, 192]_b$
97	$[17856, 34984, 26296, 9370, 1546, 96]_b + a^1[29440, 54656, 37808, 11827, 1539, 64]_b + a^2[19064, 33902, 21728, 5943, 559, 10]_b + a^3[5542, 9457, 5607, 1328, 76]_b + a^4[582, 939, 501, 96]_b$
98	$[238, 171, 30]_b + a^1[208, 132, 20]_b + a^2[22, 7]_b$
99	$[2336, 3036, 1174, 110]_b + a^1[3564, 4508, 1649, 123]_b + a^2[1654, 1987, 661, 28]_b + a^3[240, 270, 80]_b$
100	$[1384, 1728, 670, 66]_b + a^1[1928, 2330, 853, 65]_b + a^2[850, 977, 329, 14]_b + a^3[120, 130, 40]_b$
101	$[3200, 3876, 1506, 154]_b + a^1[4148, 4812, 1763, 137]_b + a^2[1746, 1921, 655, 28]_b + a^3[240, 250, 80]_b$
102	$[454, 537, 209, 22]_b + a^1[328, 352, 123, 7]_b + a^2[60, 60, 20]_b$
103	$[4, 7]_b$
104	$[5, 2]_b$
105	$[388, 565, 244, 31]_b + a^1[324, 439, 158, 13]_b + a^2[80, 102, 28]_b$
106	$[2248, 3070, 1265, 155]_b + a^1[3328, 4310, 1596, 158]_b + a^2[1640, 2016, 643, 39]_b + a^3[272, 320, 84]_b$
107	$[2168, 2750, 1066, 124]_b + a^1[3548, 4291, 1512, 145]_b + a^2[1828, 2103, 644, 39]_b + a^3[304, 334, 84]_b$
108	$[696, 810, 289, 31]_b + a^1[560, 614, 187, 13]_b + a^2[112, 116, 28]_b$
109	$[6762, 9723, 4926, 1026, 73]_b + a^1[11352, 15340, 6999, 1219, 62]_b + a^2[7388, 9353, 3752, 506, 13]_b + a^3[2188, 2572, 859, 73]_b + a^4[254, 276, 68]_b$
110	$[33660, 77750, 66164, 25886, 4608, 292]_b + a^1[83918, 187941, 152685, 55692, 8889, 467]_b + a^2[82672, 178912, 137455, 45750, 6243, 238]_b + a^3[40440, 84285, 60551, 17834, 1917, 39]_b + a^4[9848, 19706, 13047, 3222, 219]_b + a^5[958, 1838, 1102, 204]_b$
111	$[56984, 122428, 98022, 36349, 6152, 365]_b + a^1[133264, 276980, 210740, 72425, 10922, 529]_b + a^2[123736, 248312, 178131, 55641, 7161, 251]_b + a^3[56980, 110166, 73780, 20333, 2058, 39]_b + a^4[13004, 24166, 14915, 3444, 219]_b + a^5[1176, 2096, 1172, 204]_b$

i	$\Pi_i(a,b)$
112	$[16712, 32868, 24290, 8385, 1332, 73]_b + a^1[27784, 52748, 36600, 11414, 1544, 62]_b + a^2[17180, 31380, 20211, 5517, 587, 13]_b + a^3[4676, 8184, 4815, 1086, 73]_b + a^4[472, 788, 414, 68]_b$
113	$[24680, 35820, 19190, 4478, 388]_b + a^1[36454, 48894, 23676, 4834, 354]_b + a^2[19573, 23713, 9931, 1607, 76]_b + a^3[4496, 4747, 1590, 163]_b + a^4[369, 316, 67]_b$
114	$[141200, 302008, 252444, 103144, 20612, 1616]_b + a^1[318516, 656608, 524022, 202000, 37534, 2696]_b + a^2[274252, 538898, 403218, 142258, 23330, 1404]_b + a^3[112895, 208728, 143106, 44238, 5821, 228]_b + a^4[22160, 37859, 22983, 5781, 489]_b + a^5[1641, 2501, 1241, 201]_b$
115	$[259680, 750352, 878080, 533788, 177924, 30836, 2172]_b + a^1[790728, 2238648, 2553754, 1505152, 483428, 80160, 5362]_b + a^2[960964, 2650080, 2921347, 1648175, 500653, 77285, 4720]_b + a^3[597768, 1594964, 1681044, 892547, 249322, 34137, 1746]_b + a^4[200804, 514410, 511788, 250109, 61592, 6821, 228]_b + a^5[34400, 83728, 77240, 33615, 6792, 489]_b + a^6[2312, 5234, 4303, 1534, 201]_b$
116	$[79328, 223576, 253688, 149102, 47972, 7966, 520]_b + a^1[172736, 474940, 520416, 292307, 88931, 13777, 813]_b + a^2[146256, 389572, 407454, 214639, 60010, 8309, 408]_b + a^3[60280, 154424, 152046, 73191, 17951, 2045, 67]_b + a^4[12088, 29532, 26872, 11337, 2216, 163]_b + a^5[944, 2180, 1792, 620, 76]_b$
117	$[38016, 50832, 23668, 4440, 284]_b + a^1[58244, 71882, 29778, 4622, 218]_b + a^2[32998, 37217, 13367, 1574, 40]_b + a^3[8154, 8275, 2468, 181]_b + a^4[734, 647, 141]_b$
118	$[245376, 475520, 354536, 125600, 20868, 1300]_b + a^1[555000, 1033256, 729962, 239626, 35542, 1878]_b + a^2[489852, 871476, 577155, 171593, 21516, 848]_b + a^3[211168, 357096, 219218, 57431, 5587, 120]_b + a^4[44356, 70742, 39590, 8741, 543]_b + a^5[3600, 5322, 2607, 423]_b$
119	$[251008, 756280, 926780, 593030, 209083, 38384, 2839]_b + a^1[524320, 1532904, 1815480, 1118708, 378379, 66228, 4605]_b + a^2[419056, 1174820, 1323096, 768893, 243041, 39195, 2427]_b + a^3[159136, 420888, 439736, 232538, 65311, 8992, 423]_b + a^4[28408, 69108, 64078, 28551, 6212, 543]_b + a^5[1912, 4152, 3206, 1042, 120]_b$
120	$[84224, 233408, 259680, 148356, 45744, 7180, 448]_b + a^1[189384, 510524, 546452, 296224, 85012, 12068, 656]_b + a^2[169888, 443352, 452396, 228801, 59180, 7107, 292]_b + a^3[76118, 192005, 185903, 86699, 19611, 1816, 40]_b + a^4[16860, 41006, 37456, 15885, 3020, 181]_b + a^5[1438, 3333, 2809, 1031, 141]_b$
121	$[13592, 21956, 13254, 3529, 347]_b + a^1[10384, 14996, 8002, 1847, 151]_b + a^2[1832, 2044, 772, 100]_b + a^3[80, 52, 8]_b$
122	$[94640, 221664, 205120, 93712, 21089, 1861]_b + a^1[123216, 274480, 241024, 104174, 22054, 1810]_b + a^2[53720, 110408, 88692, 34674, 6507, 453]_b + a^3[8456, 14660, 9464, 2730, 300]_b + a^4[400, 500, 196, 24]_b$
123	$[105424, 196768, 139036, 45196, 6464, 320]_b + a^1[273984, 492912, 330448, 98784, 12088, 448]_b + a^2[279180, 483900, 306763, 83327, 8280, 188]_b + a^3[139736, 233290, 139441, 34001, 2544, 24]_b + a^4[34256, 54954, 30753, 6567, 300]_b + a^5[3268, 5004, 2579, 453]_b$
124	$[18984, 37092, 27468, 9448, 1456, 80]_b + a^1[35352, 65536, 44784, 13514, 1614, 52]_b + a^2[25354, 45057, 28676, 7625, 670, 8]_b + a^3[8048, 13712, 8090, 1858, 100]_b + a^4[934, 1515, 814, 151]_b$
125	$[500, 370, 36]_b + a^1[384, 246, 11]_b + a^2[74, 41]_b$
126	$[3496, 4284, 1580, 144]_b + a^1[5492, 6450, 2204, 152]_b + a^2[2700, 2994, 918, 33]_b + a^3[428, 448, 123]_b$
127	$[3992, 4848, 1834, 180]_b + a^1[5680, 6552, 2278, 163]_b + a^2[2652, 2892, 909, 33]_b + a^3[412, 428, 123]_b$
128	$[748, 902, 348, 36]_b + a^1[604, 658, 218, 11]_b + a^2[132, 136, 41]_b$
129	$[10, 9]_b$
130	$[4, 3]_b$
131	$[135, 97, 14]_b + a^1[115, 73, 6]_b + a^2[28, 16]_b$
132	$[464, 723, 369, 62]_b + a^1[761, 1146, 565, 92]_b + a^2[351, 491, 220, 32]_b + a^3[42, 46, 12]_b$
133	$[194, 297, 129, 14]_b + a^1[164, 241, 93, 6]_b + a^2[34, 48, 16]_b$
134	$[508, 366, 64]_b + a^1[376, 229, 29]_b + a^2[63, 27]_b$
135	$[988, 1337, 580, 79]_b + a^1[1557, 2013, 819, 101]_b + a^2[731, 874, 311, 29]_b + a^3[102, 106, 27]_b$
136	$[1888, 3812, 2764, 854, 94]_b + a^1[2644, 5230, 3664, 1075, 109]_b + a^2[1150, 2174, 1413, 366, 29]_b + a^3[154, 268, 149, 27]_b$
137	$[714, 472, 60]_b + a^1[551, 306, 22]_b + a^2[103, 46]_b$
138	$[3072, 3786, 1464, 164]_b + a^1[4790, 5593, 1982, 182]_b + a^2[2331, 2535, 782, 44]_b + a^3[361, 362, 92]_b$
139	$[3224, 5868, 3868, 1080, 104]_b + a^1[4424, 7764, 4848, 1240, 100]_b + a^2[2010, 3369, 1949, 430, 22]_b + a^3[306, 489, 259, 46]_b$
140	$[400, 252, 20]_b + a^1[332, 184, 6]_b + a^2[70, 35]_b$
141	$[896, 1040, 360, 30]_b + a^1[1456, 1608, 506, 29]_b + a^2[764, 804, 228, 6]_b + a^3[132, 134, 35]_b$
142	$[496, 588, 214, 20]_b + a^1[476, 512, 155, 6]_b + a^2[124, 128, 35]_b$
143	$[21, 5]_b + a^1[6]_b$
144	$[148, 127, 24]_b + a^1[35, 20]_b$
145	$[64, 61, 12]_b + a^1[15, 10]_b$
146	$[108, 117, 24]_b + a^1[25, 20]_b$
147	$[22, 5]_b + a^1[6]_b$

i	$\Pi_i(a,b)$
148	$[2042, 2827, 1418, 294, 21]_b + a^1[2234, 2792, 1207, 188, 7]_b + a^2[832, 945, 349, 36]_b + a^3[100, 98, 26]_b$
149	$[6736, 9214, 4530, 915, 63]_b + a^1[12022, 15440, 6946, 1205, 63]_b + a^2[7780, 9259, 3702, 504, 14]_b + a^3[2172, 2358, 804, 72]_b + a^4[218, 207, 52]_b$
150	$[32816, 82264, 79254, 36794, 8236, 708]_b + a^1[44256, 106112, 96711, 41916, 8597, 654]_b + a^2[21246, 47876, 40189, 15599, 2730, 156]_b + a^3[4157, 8474, 6139, 1896, 216]_b + a^4[273, 469, 252, 42]_b$
151	$[8736, 11588, 5410, 1019, 63]_b + a^1[15396, 19194, 8200, 1336, 63]_b + a^2[9724, 11193, 4223, 544, 14]_b + a^3[2622, 2726, 860, 72]_b + a^4[254, 229, 52]_b$
152	$[1716, 1376, 323, 21]_b + a^1[1796, 1216, 214, 7]_b + a^2[614, 333, 36]_b + a^3[68, 26]_b$
153	$[22774, 53201, 48597, 22283, 5375, 644, 30]_b + a^1[49056, 109288, 92862, 38396, 7922, 738, 22]_b + a^2[42854, 91147, 71943, 26705, 4629, 312, 4]_b + a^3[18292, 36834, 26578, 8616, 1176, 46]_b + a^4[3732, 7008, 4480, 1192, 106]_b + a^5[284, 482, 252, 44]_b$
154	$[161040, 350728, 302942, 132059, 30298, 3435, 150]_b + a^1[476328, 999336, 820214, 333400, 69128, 6716, 230]_b + a^2[572654, 1153599, 893950, 334948, 61264, 4801, 108]_b + a^3[357160, 687338, 498284, 169434, 26476, 1532, 16]_b + a^4[121050, 220445, 146946, 43965, 5478, 184]_b + a^5[20960, 35542, 21092, 5190, 424]_b + a^6[1432, 2196, 1080, 176]_b$
155	$[413552, 850808, 697830, 289047, 62824, 6685, 270]_b + a^1[1141040, 2258244, 1759906, 679944, 133569, 12129, 378]_b + a^2[1283690, 2434701, 1791512, 638822, 111087, 8166, 168]_b + a^3[751592, 1357598, 933533, 302039, 44992, 2448, 24]_b + a^4[240490, 409487, 258435, 73434, 8742, 276]_b + a^5[39648, 62592, 35082, 8154, 636]_b + a^6[2612, 3718, 1728, 264]_b$
156	$[228832, 588400, 619696, 341358, 103394, 16260, 1032]_b + a^1[335328, 817728, 810732, 415775, 115267, 16148, 872]_b + a^2[184376, 418658, 381794, 176531, 42585, 4834, 176]_b + a^3[46560, 95435, 76969, 30376, 5810, 424]_b + a^4[5250, 9099, 5897, 1700, 184]_b + a^5[210, 274, 116, 16]_b$
157	$[46264, 64456, 33478, 7948, 828, 30]_b + a^1[87772, 114080, 54075, 11283, 938, 22]_b + a^2[65604, 78560, 33279, 5899, 358, 4]_b + a^3[24056, 26030, 9484, 1328, 46]_b + a^4[4308, 4082, 1182, 106]_b + a^5[300, 236, 44]_b$
158	$[107792, 246342, 228335, 109839, 28837, 3909, 214]_b + a^1[275522, 606003, 534246, 241286, 58440, 7132, 341]_b + a^2[286398, 601063, 496520, 205659, 44178, 4514, 164]_b + a^3[155560, 309413, 236349, 87837, 16046, 1238, 25]_b + a^4[45890, 85465, 59043, 19000, 2751, 127]_b + a^5[6786, 11532, 6857, 1751, 164]_b + a^6[380, 558, 250, 36]_b$
159	$[781632, 2396600, 3035018, 2063367, 813563, 185747, 22695, 1146]_b + a^1[2641256, 7903852, 9705734, 6353255, 2391260, 515170, 58462, 2691]_b + a^2[3705738, 10777205, 12749681, 7959126, 2819374, 560587, 56941, 2248]_b + a^3[2800038, 7878191, 8915222, 5250515, 1721228, 306841, 26334, 791]_b + a^4[1227250, 3321519, 3563311, 1952879, 579823, 89074, 5860, 100]_b + a^5[309678, 798899, 800722, 399136, 103460, 12757, 508]_b + a^6[41124, 99492, 90586, 39118, 8108, 656]_b + a^7[2180, 4802, 3742, 1228, 144]_b$
160	$[1079072, 3435712, 4568740, 3285728, 1376186, 333892, 43098, 2260]_b + a^1[2526416, 7785744, 9968710, 6858821, 2724109, 618173, 72923, 3356]_b + a^2[2382036, 7040672, 8575082, 5555391, 2047847, 421057, 42947, 1536]_b + a^3[1158742, 3251513, 3713211, 2221030, 739317, 131933, 10650, 216]_b + a^4[305698, 803019, 841575, 450202, 129079, 18495, 984]_b + a^5[41254, 99295, 91936, 41109, 8940, 762]_b + a^6[2230, 4797, 3724, 1239, 150]_b$
161	$[1296000, 3875600, 4868348, 3324474, 1329274, 309960, 38812, 2000]_b + a^1[2793712, 8071312, 9738650, 6340699, 2392691, 518592, 58916, 2646]_b + a^2[2457812, 6806178, 7807000, 4780521, 1670208, 326645, 31908, 1108]_b + a^3[1126970, 2961973, 3187401, 1803109, 568651, 96292, 7404, 144]_b + a^4[282398, 694265, 686358, 348116, 94824, 12919, 656]_b + a^5[36334, 81563, 71159, 30214, 6260, 508]_b + a^6[1878, 3741, 2724, 861, 100]_b$
162	$[140608, 325068, 307070, 151302, 40856, 5704, 320]_b + a^1[213124, 464220, 407663, 183311, 43917, 5189, 226]_b + a^2[127734, 259377, 208404, 83411, 16947, 1527, 36]_b + a^3[37638, 70035, 50215, 17247, 2795, 164]_b + a^4[5318, 8693, 5174, 1336, 127]_b + a^5[290, 395, 174, 25]_b$
163	$[231648, 514368, 456696, 207900, 50396, 5920, 248]_b + a^1[534592, 1137664, 956074, 407122, 90704, 9392, 312]_b + a^2[508560, 1030956, 810468, 316834, 63044, 5446, 120]_b + a^3[254906, 488826, 354244, 123961, 21158, 1407, 14]_b + a^4[70820, 127279, 83198, 24948, 3362, 141]_b + a^5[10312, 17147, 9764, 2293, 188]_b + a^6[614, 930, 438, 66]_b$
164	$[913280, 2695728, 3292428, 2151828, 808172, 172312, 18876, 792]_b + a^1[2307504, 6595512, 7739816, 4816303, 1701936, 334937, 32596, 1116]_b + a^2[2372204, 6528000, 7292770, 4261949, 1389187, 244751, 19931, 464]_b + a^3[1272948, 3354269, 3533159, 1909054, 560330, 85081, 5351, 56]_b + a^4[375578, 941210, 922563, 449927, 113995, 13835, 564]_b + a^5[57604, 135993, 121564, 51223, 10144, 752]_b + a^6[3586, 7892, 6274, 2136, 264]_b$
165	$[2674560, 9935008, 15622600, 13565916, 7096194, 2278836, 435222, 44612, 1844]_b + a^1[8683808, 31583936, 48421932, 40785750, 20559849, 6304932, 1133617, 106708, 3864]_b + a^2[11790968, 41838108, 62216970, 50476788, 24287921, 7018315, 1164073, 96971, 2818]_b + a^3[8706588, 30034930, 43095925, 33420409, 15183393, 4069771, 607436, 42588, 840]_b + a^4[3778084, 12621990, 17366254, 12749116, 5389535, 1310795, 169919, 9303, 84]_b + a^5[961760, 3096254, 4050259, 2774669, 1065402, 225665, 23589, 846]_b + a^6[132588, 408598, 502006, 313610, 104504, 17466, 1128]_b + a^7[7628, 22344, 25426, 14058, 3780, 396]_b$
166	$[873600, 2604240, 3215180, 2125778, 809510, 176370, 20154, 920]_b + a^1[2149712, 6202552, 7351506, 4617423, 1645915, 328047, 32945, 1224]_b + a^2[2190828, 6084474, 6866982, 4049273, 1327546, 235029, 19492, 488]_b + a^3[1182610, 3143685, 3348068, 1828224, 539975, 82061, 5193, 56]_b + a^4[354260, 894346, 886552, 438023, 112130, 13669, 564]_b + a^5[55314, 131061, 118138, 50387, 10096, 752]_b + a^6[3508, 7710, 6150, 2112, 264]_b$
167	$[207456, 461552, 435764, 222010, 63582, 9574, 590]_b + a^1[470032, 1000664, 899466, 433163, 115866, 15991, 886]_b + a^2[429324, 862818, 725984, 323370, 78266, 9376, 426]_b + a^3[201486, 375309, 288752, 115430, 24128, 2269, 66]_b + a^4[50934, 85517, 57871, 19725, 3279, 188]_b + a^5[6524, 9414, 5169, 1328, 141]_b + a^6[328, 372, 132, 14]_b$

i	$\Pi_i(a,b)$
168	$[124112, 245940, 190386, 73032, 14254, 1244, 32]_b + a^1[234812, 442456, 319177, 111294, 19015, 1320, 18]_b + a^2[177342, 316933, 211172, 65658, 9489, 488, 2]_b + a^3[66598, 112403, 68222, 18260, 2061, 64]_b + a^4[12386, 19615, 10612, 2282, 157]_b + a^5[910, 1341, 627, 94]_b$
169	$[1054720, 2776624, 2997220, 1705482, 545044, 95406, 8040, 224]_b + a^1[2734160, 6976496, 7231350, 3904027, 1163926, 184741, 13242, 254]_b + a^2[2926092, 7222386, 7156882, 3637001, 998623, 140788, 8222, 86]_b + a^3[1653278, 3938357, 3710324, 1755334, 435000, 52567, 2324, 8]_b + a^4[519186, 1190073, 1058033, 458293, 98857, 9454, 256]_b + a^5[85722, 188347, 156460, 60481, 10574, 628]_b + a^6[5802, 12165, 9327, 3098, 376]_b$
170	$[825920, 2320256, 2663016, 1606062, 542986, 100774, 9138, 288]_b + a^1[2307760, 6290360, 6941066, 3975587, 1253800, 210551, 16178, 354]_b + a^2[2654440, 7006062, 7399366, 3993574, 1160214, 172678, 10736, 126]_b + a^3[1607358, 4097747, 4118220, 2071276, 543564, 69315, 3252, 12]_b + a^4[539044, 1323058, 1255260, 578224, 132150, 13344, 384]_b + a^5[94654, 222811, 197522, 81149, 15018, 942]_b + a^6[6784, 15254, 12486, 4404, 564]_b$
171	$[569600, 1713944, 2092198, 1335206, 476398, 93490, 9084, 320]_b + a^1[1983592, 5831876, 6902605, 4226723, 1424929, 257589, 21930, 596]_b + a^2[2937062, 8422322, 9632550, 5627515, 1776624, 291433, 21024, 382]_b + a^3[2395819, 6687309, 7359719, 4072490, 1189686, 173617, 10190, 98]_b + a^4[1160000, 3143669, 3311076, 1716990, 455060, 56977, 2524, 8]_b + a^5[332462, 872183, 872941, 417543, 96925, 9594, 256]_b + a^6[52106, 131905, 124380, 53627, 10202, 628]_b + a^7[3439, 8376, 7371, 2774, 376]_b$
172	$[73464, 163538, 143680, 63150, 14432, 1592, 64]_b + a^1[194738, 417823, 347727, 141235, 28625, 2580, 68]_b + a^2[222031, 457834, 359241, 133886, 23772, 1690, 22]_b + a^3[136330, 268748, 196790, 66104, 9972, 526, 2]_b + a^4[46916, 87846, 59064, 17248, 2038, 64]_b + a^5[8524, 15065, 9099, 2159, 157]_b + a^6[637, 1058, 559, 94]_b$
173	$[33448, 59592, 40082, 12590, 1816, 92]_b + a^1[51812, 87848, 54809, 15357, 1852, 68]_b + a^2[30052, 48454, 27801, 6744, 621, 12]_b + a^3[7728, 11846, 6191, 1243, 70]_b + a^4[740, 1076, 504, 76]_b$
174	$[73040, 136928, 97440, 32526, 5010, 276]_b + a^1[167816, 302880, 203912, 62773, 8569, 388]_b + a^2[151356, 262348, 165631, 45986, 5279, 172]_b + a^3[67292, 111886, 65687, 15985, 1408, 24]_b + a^4[14776, 23556, 12746, 2606, 140]_b + a^5[1280, 1954, 962, 152]_b$
175	$[237584, 726168, 906074, 593635, 216078, 41483, 3278]_b + a^1[468608, 1389964, 1679040, 1063349, 373880, 69329, 5290]_b + a^2[351430, 998529, 1147300, 686249, 226300, 39064, 2748]_b + a^3[123870, 330877, 351804, 190939, 55676, 8178, 456]_b + a^4[20212, 49324, 46346, 21128, 4722, 420]_b + a^5[1196, 2540, 1932, 624, 72]_b$
176	$[42680, 88504, 69637, 25736, 4407, 276]_b + a^1[99168, 198050, 147391, 50159, 7574, 388]_b + a^2[91819, 176395, 123141, 37732, 4755, 172]_b + a^3[42580, 78728, 51179, 13735, 1320, 24]_b + a^4[9895, 17635, 10598, 2390, 140]_b + a^5[918, 1580, 870, 152]_b$
177	$[11442, 24918, 20479, 7884, 1405, 92]_b + a^1[18275, 38039, 29031, 9939, 1460, 68]_b + a^2[11554, 23070, 16257, 4818, 533, 12]_b + a^3[3371, 6475, 4181, 1027, 70]_b + a^4[378, 702, 412, 76]_b$
178	$[1080, 1182, 391, 37]_b + a^1[826, 833, 230, 13]_b + a^2[158, 149, 33]_b$
179	$[4560, 5192, 1813, 185]_b + a^1[7400, 8042, 2561, 213]_b + a^2[3693, 3789, 1044, 52]_b + a^3[589, 575, 132]_b$
180	$[2400, 2828, 1031, 111]_b + a^1[3588, 4012, 1319, 113]_b + a^2[1725, 1825, 518, 26]_b + a^3[273, 277, 66]_b$
181	$[5040, 6120, 2311, 259]_b + a^1[6952, 8006, 2715, 239]_b + a^2[3207, 3511, 1028, 52]_b + a^3[503, 533, 132]_b$
182	$[660, 823, 320, 37]_b + a^1[511, 587, 189, 13]_b + a^2[115, 128, 33]_b$
183	$[7, 3]_b$
184	$[990, 1165, 440, 49]_b + a^1[790, 858, 285, 21]_b + a^2[160, 161, 45]_b$
185	$[5212, 6084, 2263, 245]_b + a^1[7360, 8212, 2845, 252]_b + a^2[3368, 3560, 1117, 63]_b + a^3[500, 496, 135]_b$
186	$[4484, 5178, 1886, 196]_b + a^1[7010, 7781, 2660, 231]_b + a^2[3406, 3580, 1109, 63]_b + a^3[520, 509, 135]_b$
187	$[1252, 1424, 503, 49]_b + a^1[968, 1026, 322, 21]_b + a^2[180, 174, 45]_b$
188	$[15352, 32804, 26246, 9792, 1680, 106]_b + a^1[27538, 56373, 42003, 14000, 1976, 88]_b + a^2[19148, 37656, 26083, 7665, 828, 18]_b + a^3[5882, 11053, 7029, 1766, 124]_b + a^4[648, 1142, 639, 121]_b$
189	$[81008, 162896, 123856, 43998, 7158, 424]_b + a^1[207644, 404246, 292810, 96575, 13885, 670]_b + a^2[209094, 393547, 270050, 81465, 9876, 336]_b + a^3[103272, 187448, 120981, 32756, 3113, 54]_b + a^4[24838, 43195, 25836, 6043, 372]_b + a^5[2296, 3768, 2015, 363]_b$
190	$[139808, 265296, 191032, 64084, 9750, 530]_b + a^1[327576, 602652, 414386, 129285, 17357, 758]_b + a^2[302872, 539364, 352468, 100985, 11489, 354]_b + a^3[137732, 236604, 145768, 37586, 3370, 54]_b + a^4[30652, 50440, 28847, 6425, 372]_b + a^5[2648, 4116, 2113, 363]_b$
191	$[19600, 25560, 11372, 1942, 106]_b + a^1[32124, 39852, 16239, 2319, 88]_b + a^2[19508, 22856, 8357, 925, 18]_b + a^3[5172, 5647, 1779, 124]_b + a^4[500, 495, 121]_b$
192	$[78808, 162296, 130050, 50538, 9476, 684]_b + a^1[179312, 357932, 274794, 100894, 17525, 1141]_b + a^2[158456, 303298, 218888, 73568, 11188, 590]_b + a^3[68706, 125044, 83239, 24649, 2997, 97]_b + a^4[14484, 24762, 14792, 3632, 290]_b + a^5[1138, 1764, 873, 143]_b$
193	$[428688, 1232864, 1421252, 842088, 270286, 44448, 2918]_b + a^1[1308376, 3688468, 4142910, 2374571, 730578, 113723, 6946]_b + a^2[1613092, 4439500, 4825729, 2647316, 767679, 110021, 5961]_b + a^3[1030048, 2755114, 2875764, 1489999, 397824, 50187, 2188]_b + a^4[359080, 928388, 921561, 443091, 105226, 10779, 291]_b + a^5[64408, 159606, 148376, 64286, 12734, 870]_b + a^6[4564, 10644, 8948, 3237, 429]_b$

i	$\Pi_i(a,b)$
194	$[393168, 1110144, 1277624, 766662, 252454, 43042, 2938]_b + a^1[857240, 2359784, 2634740, 1524957, 480639, 77507, 4893]_b + a^2[715660, 1900122, 2027904, 1109378, 325328, 47544, 2568]_b + a^3[286118, 723345, 723729, 363362, 94644, 11533, 429]_b + a^4[54624, 129236, 117716, 51538, 10824, 870]_b + a^5[3966, 8565, 6759, 2315, 291]_b$
195	$[118256, 237192, 184276, 69230, 12532, 866]_b + a^1[249248, 485676, 361164, 127527, 21168, 1289]_b + a^2[205604, 385968, 270779, 87658, 12751, 620]_b + a^3[82788, 148218, 96238, 27524, 3237, 97]_b + a^4[16196, 27254, 15878, 3738, 290]_b + a^5[1220, 1880, 919, 143]_b$
196	$[161040, 322816, 246636, 89796, 15368, 928]_b + a^1[331880, 642292, 466492, 158576, 24686, 1262]_b + a^2[270732, 503276, 343425, 106450, 14401, 552]_b + a^3[109102, 193707, 122403, 33441, 3633, 78]_b + a^4[21656, 36444, 20876, 4728, 336]_b + a^5[1686, 2657, 1328, 213]_b$
197	$[483312, 1302248, 1413484, 788406, 236910, 35922, 2086]_b + a^1[1114544, 2921280, 3058846, 1628445, 460190, 64049, 3234]_b + a^2[1001128, 2539108, 2540682, 1270643, 328804, 40021, 1558]_b + a^3[439286, 1073219, 1016630, 468927, 107154, 10534, 234]_b + a^4[94120, 220354, 195291, 80894, 15309, 1008]_b + a^5[7834, 17419, 14141, 4971, 639]_b$
198	$[477808, 1390072, 1616064, 958572, 304448, 48644, 3000]_b + a^1[1461440, 4167100, 4718362, 2703704, 820298, 123032, 6888]_b + a^2[1811544, 5041794, 5522347, 3024005, 861323, 117705, 5618]_b + a^3[1170590, 3167537, 3332198, 1721707, 449966, 53558, 1928]_b + a^4[417040, 1092958, 1095658, 525803, 122034, 11717, 234]_b + a^5[77638, 196165, 185500, 80773, 15696, 1008]_b + a^6[5868, 14186, 12431, 4680, 639]_b$
199	$[78024, 166852, 136876, 53560, 9908, 684]_b + a^1[180396, 374466, 294130, 108244, 18326, 1102]_b + a^2[163424, 325332, 239956, 80396, 11728, 532]_b + a^3[73348, 138883, 94556, 27833, 3214, 78]_b + a^4[16304, 29134, 17928, 4386, 336]_b + a^5[1416, 2353, 1250, 213]_b$
200	$[74016, 133840, 89888, 27388, 3612, 144]_b + a^1[120280, 208656, 131298, 36040, 3954, 100]_b + a^2[73372, 122136, 71641, 17363, 1478, 16]_b + a^3[19840, 31662, 17163, 3538, 193]_b + a^4[1992, 3034, 1491, 237]_b$
201	$[238304, 459024, 330416, 108572, 15628, 720]_b + a^1[561064, 1043896, 713338, 216236, 27142, 932]_b + a^2[524356, 942184, 608455, 167771, 17628, 380]_b + a^3[243168, 421912, 256138, 62975, 5149, 48]_b + a^4[55848, 93484, 52924, 11195, 579]_b + a^5[5060, 8148, 4239, 711]_b$
202	$[126960, 260888, 200444, 70560, 11004, 576]_b + a^1[328060, 651476, 475536, 154524, 21012, 832]_b + a^2[335112, 642658, 443151, 130804, 14807, 364]_b + a^3[169791, 314437, 203691, 53543, 4706, 48]_b + a^4[42574, 76080, 45901, 10354, 579]_b + a^5[4203, 7233, 4005, 711]_b$
203	$[22408, 49148, 40090, 14984, 2498, 144]_b + a^1[40598, 85166, 64619, 21420, 2861, 100]_b + a^2[29003, 58547, 41401, 12073, 1228, 16]_b + a^3[9356, 18168, 11865, 2934, 193]_b + a^4[1135, 2119, 1257, 237]_b$
204	$[3644, 4142, 1402, 128]_b + a^1[2850, 3001, 855, 44]_b + a^2[566, 563, 133]_b$
205	$[11708, 13940, 5048, 512]_b + a^1[18808, 21346, 7026, 560]_b + a^2[9507, 10228, 2937, 132]_b + a^3[1567, 1618, 399]_b$
206	$[12484, 15454, 5890, 640]_b + a^1[18134, 21263, 7281, 604]_b + a^2[8766, 9764, 2910, 132]_b + a^3[1436, 1547, 399]_b$
207	$[2210, 2828, 1122, 128]_b + a^1[1805, 2116, 695, 44]_b + a^2[435, 492, 133]_b$
208	$[31, 15]_b$
209	$[15, 7]_b$
210	$[432, 623, 264, 31]_b + a^1[397, 537, 192, 13]_b + a^2[109, 142, 42]_b$
211	$[1544, 2106, 847, 93]_b + a^1[2490, 3266, 1197, 101]_b + a^2[1327, 1681, 548, 26]_b + a^3[237, 293, 84]_b$
212	$[680, 860, 319, 31]_b + a^1[584, 709, 230, 13]_b + a^2[128, 151, 42]_b$
213	$[2452, 3276, 1385, 181]_b + a^1[3710, 4806, 1914, 224]_b + a^2[1750, 2127, 736, 61]_b + a^3[276, 309, 81]_b$
214	$[9056, 20004, 15572, 5050, 570]_b + a^1[20004, 43388, 32841, 10193, 1072]_b + a^2[15900, 33647, 24416, 7035, 642]_b + a^3[5402, 11073, 7555, 1910, 122]_b + a^4[666, 1314, 822, 162]_b$
215	$[4216, 8644, 6286, 1922, 208]_b + a^1[6012, 12120, 8498, 2433, 235]_b + a^2[2714, 5301, 3480, 870, 61]_b + a^3[390, 729, 432, 81]_b$
216	$[4704, 6152, 2634, 368]_b + a^1[6184, 7776, 3165, 416]_b + a^2[2454, 2829, 1007, 108]_b + a^3[290, 279, 64]_b$
217	$[19072, 37088, 25920, 7630, 782]_b + a^1[40288, 76544, 51642, 14397, 1347]_b + a^2[30880, 57018, 36650, 9385, 737]_b + a^3[10238, 18285, 11029, 2470, 128]_b + a^4[1246, 2147, 1197, 216]_b$
218	$[4864, 10088, 7448, 2302, 246]_b + a^1[6888, 13948, 9875, 2841, 266]_b + a^2[3204, 6231, 4088, 1011, 64]_b + a^3[508, 947, 565, 108]_b$
219	$[1740, 2038, 686, 54]_b + a^1[1450, 1599, 464, 18]_b + a^2[312, 332, 85]_b$
220	$[3728, 4642, 1714, 162]_b + a^1[6070, 7195, 2398, 162]_b + a^2[3227, 3663, 1086, 36]_b + a^3[573, 634, 170]_b$
221	$[994, 1302, 514, 54]_b + a^1[943, 1128, 367, 18]_b + a^2[261, 302, 85]_b$
222	$[46, 25]_b$
223	$[26, 16]_b + a^1[20, 13]_b$
224	$[46, 25]_b + a^1[30, 17]_b$
225	$[12, 7]_b$
226	$[37, 8]_b + a^1[13]_b$
227	$[190, 181, 39]_b + a^1[40, 24]_b$

i	$\Pi_i(a,b)$
228	$[158,173,39]_b + a^1[32,24]_b$
229	$[42,8]_b + a^1[13]_b$
230	$[2114,2947,1444,281,18]_b + a^1[2844,3601,1517,216,6]_b + a^2[1266,1467,532,51]_b + a^3[176,177,47]_b$
231	$[10388,14040,6605,1219,72]_b + a^1[21632,27584,11878,1859,78]_b + a^2[16172,19262,7420,909,18]_b + a^3[5148,5644,1876,153]_b + a^4[580,562,141]_b$
232	$[16760,21884,9816,1698,90]_b + a^1[31016,38204,15662,2292,84]_b + a^2[20894,24045,8816,1017,18]_b + a^3[6058,6408,2015,153]_b + a^4[632,593,141]_b$
233	$[2220,1686,348,18]_b + a^1[2538,1675,261,6]_b + a^2[948,522,51]_b + a^3[114,47]_b$
234	$[14408,31720,26530,10558,1984,140]_b + a^1[24726,52439,41408,15182,2521,146]_b + a^2[15568,31418,22889,7407,984,34]_b + a^3[4198,7953,5206,1410,127]_b + a^4[388,662,355,63]_b$
235	$[77968,159880,126700,48198,8704,590]_b + a^1[193316,384022,291758,104856,17456,1044]_b + a^2[184022,351355,252548,83828,12309,584]_b + a^3[84048,152608,101885,30193,3618,102]_b + a^4[18306,31089,18698,4648,381]_b + a^5[1492,2286,1143,189]_b$
236	$[68208,153892,134916,57316,11736,916]_b + a^1[97544,209912,173490,68335,12628,843]_b + a^2[50008,100883,76531,26760,4115,189]_b + a^3[10710,19671,13041,3699,381]_b + a^4[790,1240,628,102]_b$
237	$[19312,25744,12316,2478,170]_b + a^1[30020,37436,16377,2897,156]_b + a^2[17032,19458,7482,1074,34]_b + a^3[4156,4211,1322,127]_b + a^4[364,307,63]_b$
238	$[38288,80624,66572,26926,5312,406]_b + a^1[87932,180472,144227,56032,10501,748]_b + a^2[75716,149432,113362,41157,7025,432]_b + a^3[30494,56856,39812,12917,1847,78]_b + a^4[5648,9672,5958,1587,155]_b + a^5[362,528,245,37]_b$
239	$[112336,340636,410232,250404,81208,13104,800]_b + a^1[266016,789480,924212,543068,167082,24944,1334]_b + a^2[242220,699193,787975,439062,125040,16503,687]_b + a^3[106598,297524,319497,165820,42296,4594,111]_b + a^4[22628,60565,61024,28632,6142,465]_b + a^5[1834,4630,4230,1656,234]_b$
240	$[110320,235640,198968,82932,16956,1336]_b + a^1[249596,516722,420214,167440,32344,2344]_b + a^2[214046,424045,326257,121217,21297,1314]_b + a^3[86490,160831,113726,37701,5582,234]_b + a^4[16302,27553,16953,4559,465]_b + a^5[1126,1621,750,111]_b$
241	$[17368,25276,14048,3502,314]_b + a^1[26362,35981,18574,4227,324]_b + a^2[14222,17555,7965,1518,78]_b + a^3[3198,3359,1198,155]_b + a^4[250,197,37]_b$
242	$[21672,43200,31774,10518,1486,62]_b + a^1[35068,67616,47125,14274,1711,46]_b + a^2[21224,39506,25900,7036,674,8]_b + a^3[5664,10130,6159,1438,93]_b + a^4[556,944,514,90]_b$
243	$[68344,147440,117094,41806,6422,310]_b + a^1[159948,335404,255127,84952,11505,416]_b + a^2[148532,302408,219273,67264,7757,178]_b + a^3[68304,134742,92475,25630,2365,24]_b + a^4[15484,29464,18891,4542,279]_b + a^5[1372,2494,1452,270]_b$
244	$[36076,83960,71504,27328,4532,248]_b + a^1[91868,208250,170258,60748,8874,370]_b + a^2[92793,204512,159624,52436,6477,170]_b + a^3[46534,99462,73586,21848,2150,24]_b + a^4[11501,23716,16406,4230,279]_b + a^5[1104,2172,1362,270]_b$
245	$[6370,9550,4872,970,62]_b + a^1[11113,15875,7321,1181,46]_b + a^2[7796,10586,4370,544,8]_b + a^3[2477,3155,1123,93]_b + a^4[288,334,90]_b$
246	$[612,768,287,29]_b + a^1[470,551,177,11]_b + a^2[90,99,26]_b$
247	$[1956,2610,1046,116]_b + a^1[3104,3963,1460,131]_b + a^2[1537,1864,606,33]_b + a^3[245,283,78]_b$
248	$[2076,2916,1231,145]_b + a^1[2962,3969,1528,142]_b + a^2[1394,1778,603,33]_b + a^3[220,269,78]_b$
249	$[366,537,236,29]_b + a^1[287,394,148,11]_b + a^2[65,85,26]_b$
250	$[11,6]_b$
251	$[426,525,200,21]_b + a^1[412,467,152,9]_b + a^2[106,114,32]_b$
252	$[1472,1766,647,63]_b + a^1[2402,2759,931,69]_b + a^2[1270,1398,428,18]_b + a^3[220,233,64]_b$
253	$[620,716,247,21]_b + a^1[532,584,180,9]_b + a^2[114,119,32]_b$
254	$[3488,6960,4966,1494,156]_b + a^1[4944,9668,6642,1875,173]_b + a^2[2284,4309,2765,686,45]_b + a^3[348,625,361,69]_b$
255	$[12736,24160,16564,4810,486]_b + a^1[26816,49952,33274,9227,857]_b + a^2[20500,37306,23864,6159,489]_b + a^3[6746,11903,7181,1644,90]_b + a^4[806,1365,755,138]_b$
256	$[2848,3680,1474,174]_b + a^1[3808,4792,1799,181]_b + a^2[1644,1978,661,45]_b + a^3[228,256,69]_b$
257	$[9296,17280,11836,3534,386]_b + a^1[19200,35216,23666,6889,727]_b + a^2[13972,24976,16172,4469,437]_b + a^3[4216,7192,4328,1064,84]_b + a^4[436,680,346,58]_b$
258	$[9136,19976,15682,5264,638]_b + a^1[20432,43916,33675,10966,1279]_b + a^2[16082,33665,24810,7634,821]_b + a^3[5216,10450,7162,1952,168]_b + a^4[574,1061,621,116]_b$
259	$[2260,3164,1369,177]_b + a^1[3408,4664,1905,217]_b + a^2[1616,2088,746,58]_b + a^3[252,300,84]_b$
260	$[1104,1456,543,47]_b + a^1[916,1151,376,17]_b + a^2[196,239,69]_b$
261	$[2360,3354,1375,141]_b + a^1[3786,5162,1933,145]_b + a^2[1995,2621,882,34]_b + a^3[353,453,138]_b$
262	$[628,949,416,47]_b + a^1[569,803,299,17]_b + a^2[157,214,69]_b$

i	$\Pi_i(a,b)$
263	$[5,3]_b$
264	$[54,70,22]_b + a^1[34,46,15]_b$
265	$[72,98,32]_b + a^1[44,62,21]_b$
266	$[13,9]_b$
267	$[6,1]_b + a^1[3]_b$
268	$[20,23,6]_b + a^1[3,2]_b$
269	$[8,1]_b + a^1[3]_b$
270	$[350,441,176,21]_b + a^1[306,356,125,11]_b + a^2[64,65,17]_b$
271	$[1216,1482,569,65]_b + a^1[1886,2193,783,77]_b + a^2[918,998,316,22]_b + a^3[140,137,34]_b$
272	$[258,171,23]_b + a^1[204,118,11]_b + a^2[38,17]_b$
273	$[700,1366,965,293,32]_b + a^1[610,1160,784,223,22]_b + a^2[102,170,91,16]_b$
274	$[672,889,381,52]_b + a^1[1045,1330,542,69]_b + a^2[486,576,210,22]_b + a^3[65,65,16]_b$
275	$[314,224,40]_b + a^1[239,149,22]_b + a^2[39,16]_b$
276	$[218,333,140,13]_b + a^1[180,266,101,5]_b + a^2[38,55,19]_b$
277	$[472,741,384,67]_b + a^1[779,1188,600,103]_b + a^2[358,510,238,38]_b + a^3[39,41,10]_b$
278	$[127,96,13]_b + a^1[108,75,5]_b + a^2[29,19]_b$
279	$[20,11]_b + a^1[12,7]_b$
280	$[22,14]_b + a^1[16,11]_b$
281	$[98,122,47,5]_b + a^1[90,103,34,2]_b + a^2[22,24,7]_b$
282	$[760,918,343,35]_b + a^1[1146,1313,448,34]_b + a^2[570,623,193,8]_b + a^3[94,99,28]_b$
283	$[368,430,155,15]_b + a^1[590,657,218,16]_b + a^2[302,321,97,4]_b + a^3[50,51,14]_b$
284	$[712,802,277,25]_b + a^1[1214,1315,424,30]_b + a^2[638,661,195,8]_b + a^3[106,105,28]_b$
285	$[172,186,61,5]_b + a^1[140,143,42,2]_b + a^2[28,27,7]_b$
286	$[1150,2943,2602,1027,182,12]_b + a^1[2480,5786,4603,1570,219,10]_b + a^2[2026,4387,3171,929,93,2]_b + a^3[714,1453,960,239,14]_b + a^4[90,173,104,21]_b$
287	$[9912,22478,18401,6838,1147,72]_b + a^1[27506,58963,44985,15122,2158,108]_b + a^2[30042,61317,43690,13160,1511,52]_b + a^3[16062,31381,20917,5590,472,8]_b + a^4[4178,7831,4873,1134,56]_b + a^5[420,754,434,84]_b$
288	$[15584,31926,24221,8421,1322,78]_b + a^1[45170,88947,63857,20404,2777,134]_b + a^2[50720,96190,65201,18886,2103,72]_b + a^3[27558,50407,32186,8325,690,12]_b + a^4[7222,12729,7613,1716,84]_b + a^5[726,1227,678,126]_b$
289	$[10632,19962,14055,4544,659,36]_b + a^1[21630,39103,25936,7566,901,36]_b + a^2[15812,27566,17176,4438,392,8]_b + a^3[4934,8277,4809,1070,56]_b + a^4[552,884,470,84]_b$
290	$[1330,1649,691,108,6]_b + a^1[1584,1863,693,77,2]_b + a^2[618,685,221,14]_b + a^3[78,80,21]_b$
291	$[13176,44472,58743,39629,14761,3023,312,12]_b + a^1[37372,123607,157802,101053,34881,6396,561,16]_b + a^2[45882,147701,180979,108928,34265,5457,387,7]_b + a^3[31002,96581,112841,63126,17679,2316,120,1]_b + a^4[12022,36131,40085,20649,5011,490,14]_b + a^5[2478,7168,7520,3530,716,42]_b + a^6[204,564,552,229,35]_b$
292	$[89856,291616,369948,239976,86004,16932,1672,60]_b + a^1[340352,1073656,1314548,813974,274078,49570,4358,128]_b + a^2[552220,1688740,1987544,1166633,364428,59190,4474,99]_b + a^3[495992,1468460,1656332,915387,261299,36876,2267,33]_b + a^4[265302,760187,819996,423933,108591,12640,567,4]_b + a^5[84046,233045,239925,115247,25893,2268,56]_b + a^6[14474,38781,37949,16753,3183,168]_b + a^7[1030,2651,2436,961,140]_b$
293	$[254208,760416,902532,552436,187656,35076,3280,108]_b + a^1[927744,2690688,3073836,1790601,569283,97299,8073,216]_b + a^2[1433964,4035372,4432522,2447134,720809,110312,7860,159]_b + a^3[1214956,3319811,3502473,1824311,491571,65284,3780,51]_b + a^4[608374,1614684,1634087,798872,193777,21249,897,6]_b + a^5[179578,462756,448214,204273,43629,3612,84]_b + a^6[28806,71916,66337,27852,5043,252]_b + a^7[1922,4621,4011,1509,210]_b$
294	$[85440,188640,158648,64712,13390,1324,42]_b + a^1[247328,528240,422074,159198,29267,2470,59]_b + a^2[292600,604544,457474,157776,24957,1699,27]_b + a^3[181022,361448,257871,80118,10360,509,4]_b + a^4[61702,118800,79391,21765,2100,56]_b + a^5[10954,20244,12529,2925,168]_b + a^6[786,1380,771,140]_b$
295	$[10920,15992,8360,1910,200,6]_b + a^1[22948,32505,15701,3062,255,5]_b + a^2[18967,25787,11300,1779,105,1]_b + a^3[7704,9974,3876,448,14]_b + a^4[1534,1871,622,42]_b + a^5[119,134,35]_b$
296	$[70200,219568,279306,188138,72604,16046,1874,88]_b + a^1[160716,492736,605579,388095,139972,28269,2923,114]_b + a^2[156380,467732,551080,331783,109591,19587,1704,49]_b + a^3[83548,242330,271157,150850,44266,6617,439,7]_b + a^4[25920,72494,76388,38589,9656,1085,42]_b + a^5[4352,11666,11478,5170,1046,70]_b + a^6[292,738,660,251,35]_b$

i	$\Pi_i(a,b)$
297	$[498528, 1976104, 3267832, 2949250, 1590476, 524816, 103308, 11030, 480]_b + a^1[1549560, 6041732, 9763130, 8543977, 4427420, 1388658, 256374, 25201, 972]_b + a^2[2061524, 7879288, 12382883, 10435171, 5142953, 1510253, 255830, 22432, 722]_b + a^3[1525548, 5697654, 8667983, 6982206, 3233213, 871039, 130964, 9731, 234]_b + a^4[678344, 2469064, 3621111, 2767825, 1187633, 285561, 36072, 2054, 28]_b + a^5[180568, 639012, 900113, 648299, 253883, 52505, 5040, 168]_b + a^6[26356, 90368, 121622, 81784, 28658, 4824, 280]_b + a^7[1588, 5226, 6622, 4052, 1204, 140]_b$
298	$[336768, 1102252, 1450910, 1001840, 393354, 87928, 10308, 472]_b + a^1[1171588, 3748060, 4784203, 3172763, 1182979, 247914, 26836, 1093]_b + a^2[1707614, 5320737, 6547399, 4129837, 1439384, 276057, 26679, 909]_b + a^3[1355872, 4102184, 4838697, 2872004, 917288, 155426, 12705, 324]_b + a^4[634134, 1857753, 2088810, 1153088, 328954, 46545, 2898, 42]_b + a^5[174424, 493340, 525651, 266352, 65619, 7014, 252]_b + a^6[25964, 70564, 70628, 32244, 6516, 420]_b + a^7[1588, 4098, 3768, 1476, 210]_b$
299	$[433824, 1304040, 1591988, 1028848, 381520, 81344, 9172, 400]_b + a^1[1427800, 4199076, 4970196, 3080510, 1081438, 215234, 22318, 860]_b + a^2[1969432, 5650670, 6452867, 3803132, 1245674, 226082, 20865, 670]_b + a^3[1478526, 4127265, 4524462, 2511917, 753803, 120608, 9395, 226]_b + a^4[652840, 1767850, 1850296, 957246, 257008, 34342, 2034, 28]_b + a^5[169366, 443303, 440065, 209323, 48659, 4928, 168]_b + a^6[23816, 59914, 55840, 23926, 4564, 280]_b + a^7[1388, 3318, 2840, 1044, 140]_b$
300	$[55368, 122644, 106844, 47090, 11218, 1382, 62]_b + a^1[140408, 304536, 254720, 105168, 22816, 2502, 94]_b + a^2[144946, 305407, 242203, 91687, 17426, 1604, 45]_b + a^3[78128, 158807, 117944, 39937, 6285, 436, 7]_b + a^4[23192, 45141, 30945, 9069, 1071, 42]_b + a^5[3586, 6618, 4100, 986, 70]_b + a^6[224, 385, 206, 35]_b$
301	$[170240, 505760, 621808, 414362, 163404, 38460, 4972, 258]_b + a^1[515616, 1510672, 1816810, 1173954, 444746, 99692, 12168, 582]_b + a^2[649616, 1863330, 2170574, 1341893, 479550, 99954, 11184, 471]_b + a^3[442306, 1233538, 1377967, 802979, 264512, 49493, 4831, 164]_b + a^4[175926, 473674, 501861, 270765, 79653, 12621, 971, 21]_b + a^5[40854, 105346, 104473, 51005, 12798, 1540, 70]_b + a^6[5114, 12496, 11387, 4865, 968, 70]_b + a^7[264, 600, 484, 167, 21]_b$
302	$[323136, 1056576, 1417172, 1021058, 430080, 106340, 14204, 762]_b + a^1[1068384, 3429176, 4485112, 3127952, 1265682, 298510, 37734, 1874]_b + a^2[1454084, 4554618, 5759062, 3839767, 1466868, 322504, 37486, 1647]_b + a^3[1059844, 3219928, 3898460, 2447697, 862165, 170536, 17347, 617]_b + a^4[447882, 1311433, 1504298, 873289, 274397, 45990, 3699, 84]_b + a^5[109790, 307665, 330323, 173359, 46347, 5880, 280]_b + a^6[14406, 38277, 37845, 17415, 3693, 280]_b + a^7[770, 1903, 1670, 623, 84]_b$
303	$[959744, 3875776, 6615792, 6250724, 3583610, 1278400, 276952, 33132, 1646]_b + a^1[3875776, 15449600, 25934900, 24000428, 13420119, 4649004, 973482, 111852, 5263]_b + a^2[6615792, 25934900, 42602652, 38360719, 20740441, 6900628, 1377655, 149457, 6492]_b + a^3[6250724, 24000428, 38360719, 33340530, 17235502, 5423297, 1011316, 100836, 3870]_b + a^4[3583610, 13420119, 20740441, 17235502, 8396544, 2444052, 411967, 36010, 1119]_b + a^5[1278400, 4649004, 6900628, 5423297, 2444052, 637046, 91656, 6363, 126]_b + a^6[276952, 973482, 1377655, 1011316, 411967, 91656, 10080, 420]_b + a^7[33132, 111852, 149457, 100836, 36010, 6363, 420]_b + a^8[1646, 5263, 6492, 3870, 1119, 126]_b$
304	$[95, 208, 107, 14]_b + a^1[107, 196, 81, 6]_b + a^2[36, 56, 18]_b$
305	$[614, 1062, 616, 120]_b + a^1[1167, 1820, 963, 174]_b + a^2[563, 772, 351, 54]_b + a^3[70, 72, 18]_b$
306	$[658, 1086, 484, 56]_b + a^1[1197, 1840, 737, 66]_b + a^2[695, 1002, 351, 18]_b + a^3[132, 180, 54]_b$
307	$[234, 335, 135, 14]_b + a^1[210, 285, 99, 6]_b + a^2[48, 62, 18]_b$
308	$[1906, 5114, 4803, 1968, 345, 20]_b + a^1[3405, 9127, 8299, 3133, 462, 18]_b + a^2[2423, 6417, 5580, 1893, 219, 4]_b + a^3[827, 2115, 1711, 491, 36]_b + a^4[119, 287, 209, 45]_b$
309	$[8996, 23962, 25711, 13921, 3821, 429]_b + a^1[24134, 63045, 65746, 34304, 9010, 965]_b + a^2[22205, 55966, 55719, 27407, 6688, 657]_b + a^3[8834, 20888, 19137, 8415, 1749, 135]_b + a^4[1487, 3129, 2441, 841, 108]_b + a^5[80, 132, 70, 12]_b$
310	$[7660, 19496, 17077, 6473, 1030, 50]_b + a^1[15804, 38436, 31701, 10952, 1483, 50]_b + a^2[12062, 27976, 21503, 6580, 701, 12]_b + a^3[4044, 8924, 6302, 1620, 108]_b + a^4[510, 1072, 687, 135]_b$
311	$[2476, 5590, 4478, 1575, 233, 10]_b + a^1[3266, 7075, 5257, 1621, 187, 4]_b + a^2[1398, 2887, 1952, 497, 36]_b + a^3[200, 394, 239, 45]_b$
312	$[10464, 25468, 22839, 9470, 1809, 126]_b + a^1[15312, 36795, 31615, 12106, 2020, 110]_b + a^2[8454, 19958, 16179, 5528, 745, 24]_b + a^3[2156, 4915, 3633, 1024, 90]_b + a^4[230, 492, 314, 60]_b$
313	$[55616, 184016, 238404, 154184, 52258, 8724, 550]_b + a^1[122016, 399412, 506810, 316641, 101732, 15633, 856]_b + a^2[105812, 340802, 419944, 250016, 74436, 10104, 434]_b + a^3[45688, 143730, 169960, 94443, 25074, 2777, 72]_b + a^4[9956, 30322, 33874, 17010, 3732, 270]_b + a^5[896, 2622, 2724, 1182, 180]_b$
314	$[96512, 290528, 354528, 225630, 79392, 14742, 1140]_b + a^1[306928, 911356, 1092608, 680199, 232990, 41907, 3128]_b + a^2[381428, 1109530, 1295065, 779094, 255483, 43506, 3042]_b + a^3[237296, 670312, 752457, 429745, 131331, 20289, 1242]_b + a^4[77700, 210666, 223599, 118092, 32149, 4122, 180]_b + a^5[12560, 32096, 31333, 14626, 3247, 270]_b + a^6[760, 1760, 1478, 538, 72]_b$
315	$[29824, 88224, 102704, 60382, 18860, 2926, 168]_b + a^1[68608, 199760, 226572, 128230, 37978, 5458, 274]_b + a^2[61032, 173822, 190204, 101991, 27926, 3565, 142]_b + a^3[26460, 73284, 76520, 38065, 9256, 969, 24]_b + a^4[5632, 15082, 14832, 6634, 1324, 90]_b + a^5[476, 1228, 1124, 434, 60]_b$
316	$[24984, 55296, 45712, 17814, 3344, 258]_b + a^1[57864, 127246, 103086, 38678, 6800, 474]_b + a^2[51000, 109798, 85300, 29799, 4632, 263]_b + a^3[21460, 44611, 32485, 10143, 1272, 45]_b + a^4[4308, 8506, 5624, 1466, 120]_b + a^5[328, 599, 337, 60]_b$

i	$\Pi_i(a,b)$
317	$[143824, 425064, 496340, 295328, 95020, 15840, 1096]_b + a^1[450112, 1315236, 1509942, 876586, 272240, 43190, 2806]_b + a^2[563084, 1619016, 1814432, 1016657, 299545, 43935, 2559]_b + a^3[360572, 1014378, 1100123, 586771, 160115, 20715, 984]_b + a^4[124582, 340559, 353527, 176225, 42988, 4506, 135]_b + a^5[21948, 57732, 56428, 25520, 5218, 360]_b + a^6[1526, 3795, 3378, 1291, 180]_b$
318	$[141776, 411680, 484500, 295848, 98672, 16940, 1152]_b + a^1[449448, 1288588, 1490388, 890192, 288824, 47838, 3086]_b + a^2[562308, 1581246, 1781342, 1028757, 319901, 50085, 2961]_b + a^3[356472, 974476, 1055562, 578955, 168321, 24013, 1211]_b + a^4[121292, 318632, 325721, 164760, 42767, 5138, 180]_b + a^5[21152, 52636, 49402, 21870, 4566, 360]_b + a^6[1504, 3502, 2925, 1044, 135]_b$
319	$[47152, 136320, 157004, 92748, 29754, 4936, 334]_b + a^1[102280, 290788, 326186, 185616, 56593, 8778, 547]_b + a^2[86852, 240810, 259778, 139565, 39108, 5349, 278]_b + a^3[36250, 97364, 99923, 49702, 12297, 1347, 45]_b + a^4[7368, 18994, 18278, 8198, 1686, 120]_b + a^5[562, 1356, 1173, 437, 60]_b$
320	$[16384, 36996, 32576, 14002, 2935, 239]_b + a^1[24048, 53656, 46133, 19152, 3827, 291]_b + a^2[11780, 25060, 20076, 7566, 1318, 80]_b + a^3[2228, 4238, 2869, 836, 90]_b + a^4[152, 238, 116, 18]_b$
321	$[99328, 302168, 373976, 241304, 85606, 15802, 1180]_b + a^1[216976, 648712, 786906, 496085, 171252, 30568, 2183]_b + a^2[174516, 505876, 591646, 357286, 117121, 19574, 1273]_b + a^3[64552, 177710, 194982, 108797, 32245, 4684, 240]_b + a^4[10932, 27584, 26878, 12718, 2946, 270]_b + a^5[704, 1566, 1256, 432, 54]_b$
322	$[102464, 296960, 336404, 191458, 57700, 8722, 524]_b + a^1[337784, 962408, 1063754, 584497, 167378, 23469, 1274]_b + a^2[452120, 1263676, 1357384, 714608, 191480, 24092, 1108]_b + a^3[315030, 862072, 896423, 448560, 110412, 11867, 408]_b + a^4[120546, 322196, 322853, 152091, 33580, 2844, 54]_b + a^5[23938, 62224, 59609, 26009, 4934, 270]_b + a^6[1910, 4784, 4305, 1673, 240]_b$
323	$[18336, 56200, 67000, 39958, 12588, 1986, 124]_b + a^1[46180, 138998, 160995, 91902, 27123, 3882, 212]_b + a^2[45606, 134286, 150059, 80875, 21736, 2646, 110]_b + a^3[22394, 64384, 69144, 34853, 8313, 788, 18]_b + a^4[5426, 15166, 15539, 7217, 1468, 90]_b + a^5[506, 1358, 1299, 531, 80]_b$
324	$[3032, 3770, 1537, 216, 9]_b + a^1[3780, 4546, 1688, 175, 3]_b + a^2[1542, 1776, 583, 36]_b + a^3[204, 221, 60]_b$
325	$[19168, 35976, 24976, 7767, 1024, 45]_b + a^1[38860, 70968, 46931, 13326, 1443, 42]_b + a^2[28856, 51351, 32295, 8272, 681, 9]_b + a^3[9256, 16000, 9485, 2129, 108]_b + a^4[1076, 1793, 983, 180]_b$
326	$[20240, 40844, 30250, 10078, 1452, 72]_b + a^1[56560, 111332, 78952, 24379, 3025, 114]_b + a^2[62458, 119975, 81322, 23063, 2352, 57]_b + a^3[33946, 63605, 41087, 10573, 819, 9]_b + a^4[9032, 16466, 10066, 2293, 108]_b + a^5[936, 1652, 943, 180]_b$
327	$[3568, 7720, 6059, 2140, 331, 18]_b + a^1[7326, 15342, 11298, 3561, 438, 15]_b + a^2[5842, 11862, 8214, 2302, 211, 3]_b + a^3[2064, 4049, 2620, 639, 36]_b + a^4[268, 505, 301, 60]_b$
328	$[532, 578, 189, 15]_b + a^1[444, 459, 135, 6]_b + a^2[92, 91, 24]_b$
329	$[1684, 1920, 668, 60]_b + a^1[2854, 3128, 1011, 69]_b + a^2[1520, 1602, 475, 18]_b + a^3[260, 265, 72]_b$
330	$[1772, 2106, 769, 75]_b + a^1[2780, 3145, 1050, 75]_b + a^2[1432, 1554, 473, 18]_b + a^3[244, 257, 72]_b$
331	$[310, 382, 145, 15]_b + a^1[296, 336, 109, 6]_b + a^2[76, 83, 24]_b$
332	$[1, 2]_b$
333	$[101, 191, 97, 11]_b + a^1[83, 165, 81, 5]_b + a^2[18, 40, 20]_b$
334	$[312, 475, 237, 38]_b + a^1[621, 971, 500, 84]_b + a^2[314, 479, 241, 40]_b + a^3[33, 37, 10]_b$
335	$[110, 239, 120, 11]_b + a^1[98, 211, 100, 5]_b + a^2[20, 44, 20]_b$
336	$[514, 883, 430, 57]_b + a^1[360, 621, 276, 25]_b + a^2[58, 104, 40]_b$
337	$[940, 1244, 524, 68]_b + a^1[1536, 2041, 861, 112]_b + a^2[720, 910, 354, 40]_b + a^3[92, 97, 25]_b$
338	$[1608, 3280, 2376, 726, 78]_b + a^1[2532, 5176, 3737, 1133, 120]_b + a^2[1138, 2250, 1542, 432, 40]_b + a^3[138, 244, 137, 25]_b$
339	$[594, 884, 381, 45]_b + a^1[927, 1393, 588, 65]_b + a^2[437, 638, 244, 20]_b + a^3[62, 84, 25]_b$
340	$[2328, 5028, 3747, 1144, 117]_b + a^1[5656, 12129, 8913, 2654, 258]_b + a^2[4848, 10255, 7341, 2083, 182]_b + a^3[1718, 3538, 2405, 615, 40]_b + a^4[210, 412, 253, 50]_b$
341	$[2280, 6424, 6828, 3436, 816, 72]_b + a^1[5228, 14634, 15388, 7614, 1760, 148]_b + a^2[4240, 11727, 12096, 5805, 1275, 97]_b + a^3[1440, 3898, 3882, 1757, 346, 20]_b + a^4[172, 449, 420, 169, 25]_b$
342	$[748, 938, 352, 34]_b + a^1[1072, 1347, 486, 39]_b + a^2[493, 608, 201, 10]_b + a^3[73, 87, 25]_b$
343	$[1560, 2936, 1956, 536, 48]_b + a^1[3520, 6558, 4269, 1116, 89]_b + a^2[2880, 5297, 3344, 817, 53]_b + a^3[1014, 1833, 1110, 245, 10]_b + a^4[130, 230, 132, 25]_b$
344	$[812, 1624, 1135, 326, 31]_b + a^1[1294, 2551, 1721, 461, 37]_b + a^2[668, 1284, 819, 194, 10]_b + a^3[114, 213, 127, 25]_b$
345	$[440, 480, 152, 10]_b + a^1[392, 416, 121, 4]_b + a^2[88, 92, 25]_b$
346	$[944, 1088, 372, 30]_b + a^1[1624, 1810, 573, 32]_b + a^2[906, 987, 291, 8]_b + a^3[166, 179, 50]_b$
347	$[252, 304, 110, 10]_b + a^1[270, 305, 95, 4]_b + a^2[78, 87, 25]_b$
348	$[9, 7]_b + a^1[7, 6]_b$
349	$[5, 3]_b + a^1[3, 2]_b$
350	$[320, 475, 200, 21]_b + a^1[321, 461, 169, 9]_b + a^2[97, 138, 45]_b$
351	$[1184, 1664, 663, 63]_b + a^1[2000, 2750, 1015, 69]_b + a^2[1123, 1523, 518, 18]_b + a^3[211, 285, 90]_b$

i	$\Pi_i(a,b)$
352	$[272,221,21]_b + a^1[247,196,9]_b + a^2[57,45]_b$
353	$[1784,4236,3443,1130,123]_b + a^1[2924,6866,5428,1678,160]_b + a^2[1496,3440,2581,702,45]_b + a^3[260,586,412,90]_b$
354	$[3280,7476,5839,1844,193]_b + a^1[7644,17206,13117,3955,376]_b + a^2[6417,14208,10468,2935,232]_b + a^3[2308,5011,3528,882,45]_b + a^4[303,645,430,90]_b$
355	$[3088,6560,4814,1430,140]_b + a^1[4744,10008,7149,1996,169]_b + a^2[2276,4712,3203,792,45]_b + a^3[348,704,448,90]_b$
356	$[2304,5054,3897,1258,143]_b + a^1[3170,6892,5186,1601,169]_b + a^2[1363,2873,2024,551,45]_b + a^3[187,374,232,45]_b$
357	$[9216,18704,13650,4242,472]_b + a^1[18704,37596,26874,8077,855]_b + a^2[13650,26874,18462,5191,491]_b + a^3[4242,8077,5191,1282,90]_b + a^4[472,855,491,90]_b$
358	$[11,7]_b + a^1[9,6]_b$
359	$[54,79,27]_b + a^1[34,51,18]_b$
360	$[19,25,8]_b + a^1[13,18,6]_b$

Appendix H. Exceptional-mode coefficient identity

Put $a = p - 2$ and $b = q - 2$. The columns N_{ijk} and D_{ijk} specify $c_{ijk} = N_{ijk}/D_{ijk}$. An omitted coefficient is zero. Every denominator is strictly positive for $a, b \geq 0$. The polynomials F_i are defined in Section I.

i	j	k	N_{ijk}	D_{ijk}
2	0	0	$(a+2)^2\mathsf{F}_1$	$2(b+1)^2$
2	0	1	$(a+3)(a+2)^2\mathsf{F}_2$	$2(b+1)^2$
2	0	2	$(a+3)^2(a+2)^3$	$2(b+1)^2$
2	1	0	$(a+2)\mathsf{F}_3$	$(b+2)(b+1)^2$
2	1	1	$(a+2)\mathsf{F}_4$	$(b+2)(b+1)^2$
2	1	2	$(a+3)(2ab+3a+5b+8)(a+2)^3$	$(b+2)(b+1)^2$
2	2	0	$(a+2)\mathsf{F}_5$	$(b+1)^2(b+2)^2$
2	2	1	$(a+2)\mathsf{F}_6$	$2(b+1)^2(b+2)^2$
2	2	2	$(a+2)^3\mathsf{F}_7$	$2(b+1)^2(b+2)^2$
2	3	0	$(a+2)\mathsf{F}_8$	$(b+1)(b+2)^2$
2	3	1	$(a+2)\mathsf{F}_9$	$(b+1)(b+2)^2$
2	3	2	$(2ab+3a+5b+8)(a+2)^4$	$(b+1)(b+2)^2$
2	4	0	$(a+2)^2\mathsf{F}_{10}$	$2(b+2)^2$
2	4	1	$(a+2)^2\mathsf{F}_{11}$	$2(b+2)^2$
2	4	2	$(a+2)^5$	$2(b+2)^2$
3	0	0	F_{12}	$2(b+1)^3$
3	0	1	F_{13}	$2(b+1)^3$
3	0	2	$(a+2)(a+3)\mathsf{F}_{14}$	$2(b+1)^3$
3	0	3	$(a+2)^2(a+3)^3$	$2(b+1)^3$
3	1	0	F_{15}	$2(b+2)(b+1)^3(a+1)$
3	1	1	F_{16}	$2(b+2)(b+1)^3(a+1)$
3	1	2	$(a+2)\mathsf{F}_{17}$	$2(b+2)(b+1)^3(a+1)$
3	1	3	$(4ab+5a+9b+12)(a+2)^2(a+3)^2$	$2(b+2)(b+1)^3$
3	2	0	F_{18}	$2(b+1)^2(b+2)^2(a+1)$
3	2	1	F_{19}	$2(b+1)^2(b+2)^2(a+1)$
3	2	2	$(a+2)\mathsf{F}_{20}$	$2(b+1)^2(b+2)^2(a+1)$
3	2	3	$3(a+3)(2ab+3a+5b+8)(a+2)^3$	$2(b+1)^2(b+2)^2$
3	3	0	F_{21}	$2(b+1)(b+2)^3(a+1)$

i	j	k	N_{ijk}	D_{ijk}
3	3	1	F_{22}	$2(b+1)(b+2)^3(a+1)$
3	3	2	$(a+2)^2\mathsf{F}_{23}$	$2(b+1)(b+2)^3(a+1)$
3	3	3	$(4ab+7a+11b+20)(a+2)^4$	$2(b+1)(b+2)^3$
3	4	0	$(a+2)\mathsf{F}_{24}$	$2(b+2)^3(a+1)$
3	4	1	$(a+2)\mathsf{F}_{25}$	$2(b+2)^3(a+1)$
3	4	2	$(a+2)^3\mathsf{F}_{26}$	$2(b+2)^3(a+1)$
3	4	3	$(a+2)^5$	$2(b+2)^3$
4	0	0	F_{27}	$2(b+1)^2(a+2)$
4	0	1	F_{28}	$2(b+1)^2(a+2)$
4	0	2	F_{29}	$2(b+1)^2$
4	0	3	$(a+2)(5a+12)(a+3)^2$	$2(b+1)^2$
4	1	0	F_{30}	$(b+2)(b+1)^2(a+1)(a+2)$
4	1	1	F_{31}	$(b+2)(b+1)^2(a+1)(a+2)$
4	1	2	F_{32}	$(b+2)(b+1)^2(a+1)$
4	1	3	$(a+2)(a+3)\mathsf{F}_{33}$	$(b+2)(b+1)^2$
4	2	0	F_{34}	$2(b+1)^2(b+2)^2(a+2)(a+1)^2$
4	2	1	F_{35}	$2(b+1)^2(b+2)^2(a+2)(a+1)^2$
4	2	2	F_{36}	$2(b+1)^2(b+2)^2(a+1)$
4	2	3	$3(a+2)^2\mathsf{F}_{37}$	$2(b+1)^2(b+2)^2$
4	3	0	F_{38}	$(b+1)(b+2)^3(a+1)^2$
4	3	1	F_{39}	$(b+1)(b+2)^3(a+1)^2$
4	3	2	$(a+2)\mathsf{F}_{40}$	$(b+1)(b+2)^3(a+1)$
4	3	3	$\mathsf{F}_{41}(a+2)^3$	$(b+1)(b+2)^3$
4	4	0	F_{42}	$2(b+2)^4(a+1)^2$
4	4	1	F_{43}	$2(b+2)^4(a+1)^2$
4	4	2	$(a+2)^2\mathsf{F}_{44}$	$2(b+2)^4(a+1)$
4	4	3	$(5b+8)(a+2)^4$	$2(b+2)^3$
5	0	0	F_{45}	$2(b+1)(a+2)^2$
5	0	1	F_{46}	$2(b+1)(a+2)^2$
5	0	2	F_{47}	$2(b+1)(a+2)$
5	0	3	$(a+3)(10a^2+48a+57)$	$2(b+1)$
5	1	0	F_{48}	$2(b+1)(b+2)(a+1)(a+2)^2$
5	1	1	F_{49}	$2(b+1)(b+2)(a+1)(a+2)^2$
5	1	2	F_{50}	$2(b+1)(b+2)(a+1)(a+2)$
5	1	3	F_{51}	$2(b+1)(b+2)$
5	2	0	F_{52}	$2(b+1)(b+2)^2(a+2)(a+1)^2$
5	2	1	F_{53}	$2(b+1)(b+2)^2(a+2)(a+1)^2$
5	2	2	F_{54}	$2(b+1)(b+2)^2(a+1)$
5	2	3	$3(a+2)\mathsf{F}_{55}$	$2(b+1)(b+2)^2$
5	3	0	F_{56}	$2(b+1)(b+2)^3(a+2)(a+1)^3$
5	3	1	F_{57}	$2(b+1)(b+2)^3(a+2)(a+1)^2$
5	3	2	F_{58}	$2(b+1)(b+2)^2(a+1)$
5	3	3	$\mathsf{F}_{59}(a+2)^2$	$2(b+1)(b+2)^3$
5	4	0	F_{60}	$2(b+2)^3(a+1)^3$

i	j	k	N_{ijk}	D_{ijk}
5	4	1	F_{61}	$2(b+2)^3(a+1)^2$
5	4	2	$(a+2)\mathsf{F}_{62}$	$2(b+2)^2(a+1)$
5	4	3	$(10b^2+32b+25)(a+2)^3$	$2(b+2)^3$
6	0	0	F_{63}	$2(a+2)^2$
6	0	1	F_{64}	$2(a+2)^2$
6	0	2	F_{65}	$2(a+2)$
6	0	3	$(10a^2+52a+67)$	2
6	1	0	F_{66}	$(b+2)(a+1)(a+2)^2$
6	1	1	F_{67}	$(b+2)(a+1)(a+2)^2$
6	1	2	F_{68}	$(b+2)(a+1)(a+2)$
6	1	3	F_{69}	$(b+2)$
6	2	0	F_{70}	$2(b+2)(a+1)^2(a+2)^2$
6	2	1	F_{71}	$2(b+2)^2(a+1)^2(a+2)^2$
6	2	2	F_{72}	$(b+2)^2(a+1)(a+2)$
6	2	3	$3\mathsf{F}_{73}$	$2(b+2)^2$
6	3	0	F_{74}	$(b+2)(a+2)(a+1)^2$
6	3	1	F_{75}	$(b+2)^2(a+2)(a+1)^2$
6	3	2	F_{76}	$(b+2)^2(a+1)$
6	3	3	$(a+2)\mathsf{F}_{77}$	$(b+2)^2$
6	4	0	$(b+4)\mathsf{F}_{78}$	$2(b+2)(a+1)^2$
6	4	1	F_{79}	$2(b+2)^2(a+1)^2$
6	4	2	F_{80}	$2(b+2)^2(a+1)$
6	4	3	$(10b^2+28b+19)(a+2)^2$	$2(b+2)^2$
7	0	0	$(b+1)(b+4)\mathsf{F}_{81}$	$(a+2)$
7	0	1	$(b+1)\mathsf{F}_{82}$	$2(a+2)$
7	0	2	$(b+1)\mathsf{F}_{83}$	$2(a+2)$
7	0	3	$(b+1)(5a+13)$	2
7	1	0	$(b+1)(b+4)\mathsf{F}_{84}$	$(b+2)(a+1)(a+2)$
7	1	1	$(b+1)\mathsf{F}_{85}$	$2(b+2)(a+1)(a+2)$
7	1	2	$(b+1)\mathsf{F}_{86}$	$2(b+2)(a+1)(a+2)$
7	1	3	$(b+1)(20ab+37a+49b+92)$	$2(b+2)$
7	2	0	$3(b+1)(b+4)\mathsf{F}_{87}$	$(b+2)(a+1)(a+2)$
7	2	1	$3(b+1)\mathsf{F}_{88}$	$2(b+2)(a+1)(a+2)$
7	2	2	$3(b+1)\mathsf{F}_{89}$	$2(b+2)(a+1)(a+2)$
7	2	3	$3(b+1)(10ab+17a+23b+40)$	$2(b+2)$
7	3	0	$(b+1)(b+4)\mathsf{F}_{90}$	$(b+2)(a+1)(a+2)$
7	3	1	$(b+1)\mathsf{F}_{91}$	$2(b+2)(a+1)(a+2)$
7	3	2	$(b+1)\mathsf{F}_{92}$	$2(b+2)(a+1)(a+2)$
7	3	3	$(b+1)(20ab+31a+43b+68)$	$2(b+2)$
7	4	0	$(b+1)(b+3)(b+4)\mathsf{F}_{93}$	$(b+2)(a+1)$
7	4	1	$(b+1)\mathsf{F}_{94}$	$2(b+2)(a+1)$
7	4	2	$(b+1)\mathsf{F}_{95}$	$2(b+2)(a+1)$
7	4	3	$(b+1)(5b+7)(a+2)$	$2(b+2)$
8	0	0	$(b+3)(b+1)^2(b+4)^2$	1
8	0	1	$(b+4)(5b+16)(b+1)^2$	2
8	0	2	$(2b+7)(b+1)^2$	1

i	j	k	N_{ijk}	D_{ijk}
8	0	3	$(b+1)^2$	2
8	1	0	$4(b+3)(b+1)^2(b+4)^2$	1
8	1	1	$2(b+4)(5b+16)(b+1)^2$	1
8	1	2	$4(2b+7)(b+1)^2$	1
8	1	3	$2(b+1)^2$	1
8	2	0	$6(b+3)(b+1)^2(b+4)^2$	1
8	2	1	$3(b+4)(5b+16)(b+1)^2$	1
8	2	2	$6(2b+7)(b+1)^2$	1
8	2	3	$3(b+1)^2$	1
8	3	0	$4(b+3)(b+1)^2(b+4)^2$	1
8	3	1	$2(b+4)(5b+16)(b+1)^2$	1
8	3	2	$4(2b+7)(b+1)^2$	1
8	3	3	$2(b+1)^2$	1
8	4	0	$(b+3)(b+1)^2(b+4)^2$	1
8	4	1	$(b+4)(5b+16)(b+1)^2$	2
8	4	2	$(2b+7)(b+1)^2$	1
8	4	3	$(b+1)^2$	2

Appendix I. Exceptional-mode dimension polynomials

Put $a = p - 2$ and $b = q - 2$. For a finite list write

$$[c_0, \dots, c_n]_b = \sum_{j=0}^{n} c_j b^j.$$

Each row below specifies the entire polynomial as a sum of terms $a^u b^v [c_0, \dots, c_n]_b$. Factors with exponent zero are omitted. Every coefficient is an exact nonnegative integer. Every polynomial has a strictly positive constant term.

i	$\mathsf{F}_i(a,b)$
1	$[708,557,155,19,1]_b + a^1[740,497,98,5]_b + a^2[258,152,22]_b + a^3[30,16,2]_b$
2	$[75,29,2]_b + a^1[58,18]_b + a^2[11,3]_b$
3	$[4176,6144,3255,803,97,5]_b + a^1[6372,8753,4074,795,64,2]_b + a^2[3658,4737,1950,289,10]_b + a^3[936,1158,432,50]_b + a^4[90,108,38,4]_b$
4	$[1260,1309,362,25]_b + a^1[2037,2032,495,20]_b + a^2[1232,1192,260,4]_b + a^3[330,313,63]_b + a^4[33,31,6]_b$
5	$[8736,19256,16212,6714,1468,166,8]_b + a^1[13192,27602,21324,7656,1321,102,3]_b + a^2[7598,15247,10935,3414,439,15]_b + a^3[1975,3849,2615,733,72]_b + a^4[195,374,247,66,6]_b$
6	$[4960,8708,5090,1130,76]_b + a^1[8100,13830,7609,1459,60]_b + a^2[5000,8380,4405,741,12]_b + a^3[1379,2289,1170,180]_b + a^4[143,237,120,18]_b$
7	$[88,116,37]_b + a^1[68,92,30]_b + a^2[13,18,6]_b$
8	$[3744,5616,3070,787,98,5]_b + a^1[5656,7826,3711,751,64,2]_b + a^2[3296,4258,1757,265,10]_b + a^3[878,1080,400,46]_b + a^4[90,108,38,4]_b$
9	$[1000,1066,313,25]_b + a^1[1662,1673,420,20]_b + a^2[1056,1020,223,4]_b + a^3[303,286,57]_b + a^4[33,31,6]_b$
10	$[528,452,140,19,1]_b + a^1[568,390,82,5]_b + a^2[218,126,18]_b + a^3[30,16,2]_b$
11	$[140,62,6]_b + a^1[170,58,2]_b + a^2[73,21]_b + a^3[11,3]_b$
12	$[20136,22798,13657,5000,1018,106,5]_b + a^1[37480,38798,20833,6717,1097,73,2]_b + a^2[27440,25592,12100,3436,444,12]_b + a^3[9868,8100,3244,812,88]_b + a^4[1740,1204,364,76,8]_b + a^5[120,64,8]_b$
13	$[10674,8099,3222,651,38]_b + a^1[20382,14270,5155,932,33]_b + a^2[15334,9749,3119,503,7]_b + a^3[5678,3200,862,124]_b + a^4[1034,496,98,12]_b + a^5[74,28,2]_b$
14	$[311,118,23]_b + a^1[349,116,19]_b + a^2[127,35,4]_b + a^3[15,3]_b$

i	$\mathsf{F}_i(a,b)$
15	$[109344, 230032, 207334, 108397, 34452, 6406, 646, 29]_b + a^1[298200, 593450, 496769, 239805, 69375, 11243, 924, 34]_b + a^2[327000, 613556, 471795, 207119, 53351, 7073, 378, 8]_b + a^3[185544, 326842, 228046, 89998, 20626, 2192, 48]_b + a^4[57564, 94432, 58416, 20216, 4164, 376]_b + a^5[9252, 13908, 7220, 1996, 360, 32]_b + a^6[600, 800, 296, 32]_b$
16	$[52040, 85882, 54791, 18812, 3235, 188]_b + a^1[148574, 235503, 140920, 45242, 7122, 319]_b + a^2[169996, 258032, 143100, 42303, 5982, 167]_b + a^3[100460, 145411, 73581, 19669, 2491, 28]_b + a^4[32454, 44494, 20006, 4658, 532]_b + a^5[5442, 6986, 2646, 478, 48]_b + a^6[370, 436, 122, 8]_b$
17	$[4108, 4917, 1722, 265]_b + a^1[10161, 11850, 3873, 564]_b + a^2[9456, 10710, 3189, 423]_b + a^3[4206, 4607, 1209, 136]_b + a^4[902, 949, 207, 16]_b + a^5[75, 75, 12]_b$
18	$[221760, 471984, 435568, 227230, 70929, 13009, 1303, 57]_b + a^1[570160, 1129004, 957516, 455709, 127340, 20021, 1614, 56]_b + a^2[602884, 1107574, 855906, 367473, 90176, 11439, 604, 12]_b + a^3[335536, 569896, 397622, 153128, 33102, 3308, 72]_b + a^4[103356, 160922, 99320, 33782, 6564, 552]_b + a^5[16616, 23244, 11984, 3324, 576, 48]_b + a^6[1080, 1296, 456, 48]_b$
19	$[91360, 148348, 96718, 32800, 5494, 328]_b + a^1[255044, 392220, 238145, 74899, 11317, 523]_b + a^2[289814, 420741, 235375, 67781, 9050, 261]_b + a^3[172018, 234706, 119283, 31048, 3663, 42]_b + a^4[56264, 71583, 32199, 7356, 780]_b + a^5[9602, 11226, 4218, 762, 72]_b + a^6[666, 696, 186, 12]_b$
20	$[6272, 7026, 2526, 368]_b + a^1[15718, 16959, 5682, 787]_b + a^2[15043, 15549, 4731, 601]_b + a^3[6943, 6835, 1820, 198]_b + a^4[1553, 1444, 315, 24]_b + a^5[135, 117, 18]_b$
21	$[196992, 418176, 382728, 195088, 59068, 10521, 1024, 43]_b + a^1[480864, 945656, 794664, 368192, 99123, 15041, 1172, 38]_b + a^2[490208, 888548, 679618, 283196, 66491, 8111, 422, 8]_b + a^3[266644, 443166, 305984, 114224, 23410, 2216, 48]_b + a^4[81164, 122406, 74942, 24868, 4572, 360]_b + a^5[12984, 17320, 8876, 2452, 408, 32]_b + a^6[840, 928, 312, 32]_b$
22	$[69600, 112376, 73412, 24452, 4006, 242]_b + a^1[190408, 288796, 175200, 53810, 7879, 373]_b + a^2[214800, 304606, 169827, 47505, 6060, 181]_b + a^3[127870, 168567, 85277, 21525, 2389, 28]_b + a^4[42276, 51289, 22968, 5122, 508]_b + a^5[7332, 8028, 2994, 538, 48]_b + a^6[518, 492, 126, 8]_b$
23	$[2024, 2190, 792, 110]_b + a^1[4170, 4263, 1416, 185]_b + a^2[3035, 2882, 835, 96]_b + a^3[940, 815, 189, 16]_b + a^4[105, 81, 12]_b$
24	$[14880, 23560, 15872, 5684, 1123, 116, 5]_b + a^1[28768, 40264, 23780, 7242, 1133, 81, 2]_b + a^2[22156, 26862, 13630, 3434, 388, 12]_b + a^3[8564, 8826, 3830, 828, 72]_b + a^4[1640, 1360, 476, 92, 8]_b + a^5[120, 64, 8]_b$
25	$[4824, 5332, 2356, 464, 32]_b + a^1[10672, 10444, 4086, 685, 33]_b + a^2[9378, 7961, 2696, 369, 7]_b + a^3[4100, 2947, 852, 100]_b + a^4[884, 506, 114, 12]_b + a^5[74, 28, 2]_b$
26	$[118, 66, 12]_b + a^1[193, 92, 15]_b + a^2[96, 34, 4]_b + a^3[15, 3]_b$
27	$[72624, 69556, 29114, 7306, 1138, 98, 4]_b + a^1[142304, 125608, 46592, 10017, 1219, 59, 1]_b + a^2[110564, 89688, 29219, 5404, 535, 10]_b + a^3[42566, 31574, 8845, 1380, 119]_b + a^4[8114, 5452, 1250, 148, 12]_b + a^5[612, 366, 60, 2]_b$
28	$[39916, 26042, 6924, 976, 50]_b + a^1[79680, 47964, 11226, 1368, 42]_b + a^2[63118, 34941, 7060, 734, 9]_b + a^3[24791, 12561, 2120, 182]_b + a^4[4825, 2219, 292, 18]_b + a^5[372, 153, 13]_b$
29	$[3614, 1198, 122]_b + a^1[5558, 1657, 131]_b + a^2[3179, 854, 48]_b + a^3[801, 193, 6]_b + a^4[75, 16]_b$
30	$[218112, 376208, 264308, 103952, 25467, 3851, 329, 13]_b + a^1[612544, 1006136, 654222, 232251, 50162, 6363, 410, 12]_b + a^2[700908, 1098850, 660176, 210071, 39584, 4059, 154, 2]_b + a^3[419692, 628988, 348210, 98554, 16222, 1370, 20]_b + a^4[138936, 199082, 100856, 24880, 3536, 262]_b + a^5[24124, 32988, 15064, 3064, 344, 24]_b + a^6[1716, 2228, 888, 124, 4]_b$
31	$[110288, 153232, 77518, 19826, 2668, 136]_b + a^1[321504, 430036, 202532, 46989, 5623, 218]_b + a^2[380178, 490254, 213918, 44304, 4604, 110]_b + a^3[234564, 291949, 117393, 21287, 1889, 18]_b + a^4[79868, 95996, 35294, 5422, 400]_b + a^5[14252, 16533, 5485, 664, 36]_b + a^6[1042, 1164, 340, 26]_b$
32	$[9172, 9515, 2656, 261]_b + a^1[23215, 23454, 6027, 514]_b + a^2[22426, 22107, 5176, 359]_b + a^3[10461, 10082, 2131, 108]_b + a^4[2372, 2238, 422, 12]_b + a^5[210, 194, 32]_b$
33	$[84, 57]_b + a^1[69, 48]_b + a^2[14, 10]_b$
34	$[903552, 2348192, 2520640, 1497876, 551746, 130174, 19122, 1598, 60]_b + a^1[3319040, 8325952, 8486800, 4707244, 1593782, 339923, 43877, 3085, 97]_b + a^2[5128848, 12428644, 12015962, 6189328, 1906718, 361055, 39236, 2055, 46]_b + a^3[4328064, 10145536, 9303300, 4436433, 1235883, 206151, 18319, 572, 6]_b + a^4[2155916, 4894944, 4256650, 1872089, 468642, 69021, 5082, 60]_b + a^5[634068, 1395238, 1148466, 461706, 102020, 13204, 858]_b + a^6[101916, 217220, 168358, 60572, 11150, 1176, 72]_b + a^7[6900, 14210, 10252, 3138, 384, 12]_b$
35	$[424800, 940304, 798556, 342994, 81962, 10450, 526]_b + a^1[1636256, 3534264, 2883616, 1167776, 259108, 30120, 1276]_b + a^2[2639396, 5563014, 4351212, 1649332, 335035, 34677, 1108]_b + a^3[2315532, 4764584, 3567682, 1257103, 230363, 20770, 408]_b + a^4[1195190, 2402580, 1720359, 559235, 90795, 7015, 54]_b + a^5[363408, 714099, 488219, 144851, 20199, 1308]_b + a^6[60314, 115876, 75429, 20055, 2232, 108]_b + a^7[4216, 7915, 4879, 1122, 78]_b$
36	$[32544, 59196, 35934, 8670, 792]_b + a^1[82740, 147594, 85970, 19051, 1535]_b + a^2[81600, 143058, 79983, 16055, 1073]_b + a^3[39250, 67771, 36404, 6522, 324]_b + a^4[9239, 15741, 8133, 1278, 36]_b + a^5[853, 1436, 714, 96]_b$
37	$[136, 194, 67]_b + a^1[102, 148, 52]_b + a^2[19, 28, 10]_b$
38	$[211968, 536736, 577448, 349568, 131340, 31360, 4622, 384, 14]_b + a^1[630336, 1508192, 1504396, 828522, 277829, 57717, 7081, 455, 12]_b + a^2[780920, 1770572, 1637320, 816439, 241808, 42805, 4120, 166, 2]_b + a^3[515160, 1111194, 954652, 430589, 112434, 16995, 1260, 20]_b + a^4[190200, 391438, 312510, 126878, 29054, 3826, 238]_b + a^5[37112, 72894, 53738, 19174, 3658, 404, 24]_b + a^6[2976, 5558, 3712, 1078, 128, 4]_b$

i	$\mathsf{F}_i(a,b)$
39	$[87648, 184224, 153728, 66622, 16159, 2080, 107]_b + a^1[287664, 582000, 459090, 184425, 40738, 4653, 194]_b + a^2[386296, 751946, 558251, 205387, 40393, 3920, 108]_b + a^3[272258, 510447, 355966, 118822, 20418, 1634, 18]_b + a^4[106310, 192255, 125722, 37647, 5519, 364]_b + a^5[21806, 38073, 23256, 6104, 710, 36]_b + a^6[1834, 3091, 1749, 382, 26]_b$
40	$[6008, 10042, 5808, 1398, 128]_b + a^1[12486, 20179, 10971, 2366, 186]_b + a^2[9435, 14777, 7524, 1407, 84]_b + a^3[3096, 4719, 2253, 354, 12]_b + a^4[373, 556, 250, 32]_b$
41	$[68, 85, 26]_b + a^1[25, 32, 10]_b$
42	$[143232, 354080, 376624, 226408, 84244, 19800, 2854, 230, 8]_b + a^1[410432, 952048, 935864, 511828, 170114, 34752, 4151, 255, 6]_b + a^2[491616, 1069920, 968568, 478324, 140360, 24429, 2298, 91, 1]_b + a^3[317040, 649096, 541084, 240470, 62008, 9145, 651, 10]_b + a^4[116192, 224540, 172428, 68748, 15592, 2008, 119]_b + a^5[22848, 41672, 29232, 10184, 1936, 212, 12]_b + a^6[1872, 3204, 2000, 554, 64, 2]_b$
43	$[52704, 106960, 88056, 38164, 9276, 1194, 62]_b + a^1[170000, 329224, 254556, 101934, 22518, 2565, 109]_b + a^2[226352, 417584, 301308, 109898, 21509, 2060, 58]_b + a^3[159968, 281372, 188962, 62186, 10596, 825, 9]_b + a^4[63372, 106430, 66380, 19501, 2842, 182]_b + a^5[13336, 21392, 12314, 3142, 364, 18]_b + a^6[1164, 1782, 936, 195, 13]_b$
44	$[1576, 2476, 1382, 330, 30]_b + a^1[2580, 3834, 1954, 403, 30]_b + a^2[1362, 1908, 869, 141, 6]_b + a^3[238, 318, 131, 16]_b$
45	$[108480, 101488, 37208, 7034, 747, 42, 1]_b + a^1[221792, 192664, 62992, 10064, 815, 23]_b + a^2[181012, 146254, 42560, 5734, 368, 4]_b + a^3[73620, 55394, 14262, 1582, 82]_b + a^4[14904, 10452, 2358, 202, 8]_b + a^5[1200, 784, 152, 8]_b$
46	$[60496, 38368, 8428, 816, 32]_b + a^1[126232, 74306, 14213, 1100, 25]_b + a^2[105002, 57453, 9480, 563, 5]_b + a^3[43488, 22159, 3121, 131]_b + a^4[8960, 4260, 506, 12]_b + a^5[734, 326, 32]_b$
47	$[5594, 1807, 119]_b + a^1[9120, 2671, 118]_b + a^2[5547, 1484, 38]_b + a^3[1490, 367, 4]_b + a^4[149, 34]_b$
48	$[724224, 1150336, 718608, 234088, 43878, 4781, 278, 7]_b + a^1[2098560, 3191488, 1861824, 549110, 89635, 7955, 314, 4]_b + a^2[2494240, 3644268, 1989050, 530298, 75025, 5357, 116]_b + a^3[1559652, 2195666, 1122514, 269538, 32808, 1906, 16]_b + a^4[541844, 736870, 352982, 75716, 7740, 368]_b + a^5[99240, 130672, 58630, 11070, 888, 32]_b + a^6[7488, 9564, 4012, 648, 32]_b$
49	$[376448, 481344, 216816, 45060, 4568, 184]_b + a^1[1130336, 1394128, 586358, 108751, 9312, 275]_b + a^2[1384640, 1652284, 649102, 106286, 7414, 131]_b + a^3[889098, 1029459, 378051, 54020, 2928, 20]_b + a^4[316392, 356489, 122545, 15077, 584]_b + a^5[59252, 65150, 21004, 2188, 48]_b + a^6[4566, 4912, 1488, 128]_b$
50	$[32440, 31006, 7769, 545]_b + a^1[84786, 78879, 18196, 1006]_b + a^2[85214, 77430, 16414, 646]_b + a^3[41579, 37046, 7235, 172]_b + a^4[9906, 8690, 1574, 16]_b + a^5[925, 802, 136]_b$
51	$[924, 573]_b + a^1[1137, 714]_b + a^2[462, 294]_b + a^3[62, 40]_b$
52	$[845184, 1936768, 1812712, 912776, 273872, 50679, 5655, 345, 9]_b + a^1[2764416, 6116912, 5432820, 2537232, 685967, 110068, 9973, 426, 6]_b + a^2[3696936, 7921668, 6691364, 2899178, 704023, 97132, 6941, 166]_b + a^3[2589072, 5388788, 4340236, 1745720, 379690, 44908, 2546, 24]_b + a^4[1002264, 2032224, 1564332, 583656, 112712, 11248, 516]_b + a^5[203520, 403148, 297228, 102598, 17226, 1356, 48]_b + a^6[16944, 32876, 23260, 7392, 1032, 48]_b$
53	$[416960, 821248, 617568, 228472, 44934, 4588, 190]_b + a^1[1430144, 2743352, 1973444, 679648, 119985, 10500, 335]_b + a^2[1993288, 3729988, 2567260, 819059, 127454, 9122, 183]_b + a^3[1447504, 2648906, 1747582, 515278, 69476, 3854, 30]_b + a^4[578824, 1038878, 658926, 179517, 20601, 822]_b + a^5[121064, 213774, 130852, 32988, 3144, 72]_b + a^6[10360, 18054, 10712, 2508, 192]_b$
54	$[33664, 54898, 29975, 6302, 441]_b + a^1[70902, 112829, 58447, 10882, 566]_b + a^2[53756, 84045, 41627, 6836, 216]_b + a^3[17576, 27174, 13006, 1908, 24]_b + a^4[2102, 3232, 1512, 204]_b$
55	$[296, 379, 119]_b + a^1[237, 308, 98]_b + a^2[47, 62, 20]_b$
56	$[834048, 2568576, 3318752, 2375192, 1044996, 295124, 53682, 6042, 378, 10]_b + a^1[3435392, 10294208, 12799160, 8693616, 3569344, 921779, 149312, 14348, 708, 13]_b + a^2[6009216, 17553056, 21033376, 13564810, 5187619, 1217931, 173395, 13751, 474, 4]_b + a^3[5782768, 16500812, 19095514, 11709278, 4169761, 887209, 110255, 7059, 140]_b + a^4[3303248, 9226808, 10334096, 6033878, 1999734, 384190, 41492, 2142, 16]_b + a^5[1119360, 3066860, 3331070, 1853798, 570388, 97988, 9060, 384]_b + a^6[208288, 560864, 591892, 314060, 89170, 13346, 984, 32]_b + a^7[16416, 43536, 44740, 22648, 5884, 728, 32]_b$
57	$[388352, 1059456, 1157792, 659208, 212884, 39566, 3976, 166]_b + a^1[1306048, 3476992, 3660372, 1970824, 586487, 97088, 8295, 266]_b + a^2[1808256, 4709112, 4786764, 2439180, 665998, 96547, 6690, 133]_b + a^3[1319564, 3370676, 3318441, 1604091, 401097, 50163, 2688, 20]_b + a^4[535588, 1345746, 1287765, 592527, 135615, 14379, 560]_b + a^5[114696, 284316, 265478, 116774, 24468, 2132, 48]_b + a^6[10128, 24846, 22742, 9624, 1856, 128]_b$
58	$[14664, 27350, 16784, 3898, 304]_b + a^1[31002, 56325, 32672, 6655, 386]_b + a^2[24117, 42959, 23755, 4223, 148]_b + a^3[8209, 14433, 7698, 1210, 16]_b + a^4[1033, 1805, 942, 136]_b$
59	$[356, 711, 462, 98]_b + a^1[139, 282, 186, 40]_b$
60	$[79104, 191968, 194712, 108988, 37088, 7862, 1006, 70, 2]_b + a^1[267456, 615776, 580380, 294872, 88621, 15986, 1630, 78, 1]_b + a^2[382000, 838844, 737738, 340754, 90259, 13759, 1091, 31]_b + a^3[293808, 618696, 510092, 214620, 49982, 6394, 384, 4]_b + a^4[127872, 259572, 201610, 77458, 15832, 1706, 82]_b + a^5[29728, 58384, 42796, 14916, 2594, 218, 8]_b + a^6[2880, 5496, 3816, 1200, 168, 8]_b$

i	$\mathsf{F}_i(a,b)$
61	$[32672, 66896, 53376, 21592, 4766, 552, 26]_b + a^1[91152, 177852, 131536, 47525, 8934, 823, 26]_b + a^2[102012, 190688, 131223, 42196, 6596, 446, 5]_b + a^3[57188, 103070, 66465, 19089, 2457, 119]_b + a^4[16032, 28020, 17038, 4366, 442, 12]_b + a^5[1800, 3074, 1784, 414, 32]_b$
62	$[1126, 1312, 404, 38]_b + a^1[1853, 2014, 509, 30]_b + a^2[1011, 1040, 215, 4]_b + a^3[185, 185, 34]_b$
63	$[38112, 35000, 11896, 1854, 134, 4]_b + a^1[63360, 53968, 16244, 2058, 99, 1]_b + a^2[39544, 31404, 8404, 856, 25]_b + a^3[10972, 8184, 1968, 160, 2]_b + a^4[1140, 806, 178, 12]_b$
64	$[22072, 13712, 2658, 174, 4]_b + a^1[37520, 21596, 3562, 151, 1]_b + a^2[23888, 12816, 1805, 41]_b + a^3[6746, 3402, 417, 3]_b + a^4[712, 341, 38]_b$
65	$[2116, 670, 30]_b + a^1[2620, 744, 14]_b + a^2[1078, 281, 1]_b + a^3[147, 36]_b$
66	$[138240, 207648, 120024, 34332, 5154, 390, 12]_b + a^1[352032, 507560, 275304, 71436, 9220, 547, 11]_b + a^2[349888, 485580, 247452, 57970, 6279, 266, 2]_b + a^3[170800, 229086, 110132, 23312, 2081, 56]_b + a^4[41112, 53564, 24464, 4714, 342, 4]_b + a^5[3912, 4982, 2186, 392, 24]_b$
67	$[76032, 91512, 37824, 6606, 474, 12]_b + a^1[199992, 232348, 89796, 13764, 747, 11]_b + a^2[204364, 229976, 83321, 11110, 409, 2]_b + a^3[102186, 111867, 38215, 4432, 91]_b + a^4[25116, 26886, 8741, 895, 6]_b + a^5[2434, 2563, 804, 76]_b$
68	$[6920, 6198, 1426, 72]_b + a^1[15362, 13367, 2816, 93]_b + a^2[12097, 10270, 1986, 31]_b + a^3[4078, 3402, 615, 2]_b + a^4[501, 414, 72]_b$
69	$[208, 119]_b + a^1[169, 98]_b + a^2[34, 20]_b$
70	$[364416, 583712, 359040, 110212, 18050, 1518, 52]_b + a^1[1253184, 1937872, 1127760, 318852, 46328, 3269, 85]_b + a^2[1757824, 2627852, 1445538, 373852, 47170, 2633, 42]_b + a^3[1291064, 1871284, 974516, 229749, 24669, 1010, 6]_b + a^4[524992, 740670, 366590, 78821, 7060, 186]_b + a^5[112296, 154972, 73382, 14492, 1068, 12]_b + a^6[9888, 13430, 6150, 1140, 72]_b$
71	$[383552, 692384, 473512, 154392, 24938, 1870, 52]_b + a^1[1365600, 2411280, 1590216, 487604, 70718, 4351, 85]_b + a^2[1978600, 3423208, 2179946, 627870, 80739, 3821, 42]_b + a^3[1496408, 2543264, 1568504, 425177, 48051, 1582, 6]_b + a^4[624572, 1046014, 627425, 160921, 15951, 300]_b + a^5[136712, 226424, 132836, 32550, 2868, 18]_b + a^6[12284, 20202, 11677, 2778, 228]_b$
72	$[16544, 24740, 12388, 2317, 123]_b + a^1[36444, 53300, 25479, 4249, 147]_b + a^2[29000, 41772, 19263, 2896, 48]_b + a^3[9984, 14257, 6419, 894, 3]_b + a^4[1262, 1798, 801, 108]_b$
73	$[312, 364, 105]_b + a^1[268, 316, 92]_b + a^2[57, 68, 20]_b$
74	$[51072, 88496, 58276, 19212, 3416, 316, 12]_b + a^1[144816, 239396, 145790, 42736, 6395, 456, 11]_b + a^2[162376, 257574, 145656, 37889, 4679, 238, 2]_b + a^3[90152, 138196, 73054, 16887, 1687, 56]_b + a^4[24808, 37046, 18488, 3822, 298, 4]_b + a^5[2712, 3988, 1916, 368, 24]_b$
75	$[51968, 100744, 73568, 25682, 4508, 378, 12]_b + a^1[155400, 292388, 202930, 64949, 9797, 621, 11]_b + a^2[182532, 334784, 221899, 65299, 8371, 368, 2]_b + a^3[105456, 189574, 120848, 32939, 3564, 91]_b + a^4[30020, 53220, 32935, 8433, 777, 6]_b + a^5[3376, 5946, 3620, 896, 76]_b$
76	$[4248, 6910, 3729, 752, 45]_b + a^1[7178, 11301, 5684, 960, 29]_b + a^2[3912, 6058, 2916, 429, 2]_b + a^3[694, 1073, 510, 72]_b$
77	$[112, 145, 46]_b + a^1[47, 62, 20]_b$
78	$[2508, 4214, 2404, 630, 80, 4]_b + a^1[5592, 8736, 4269, 857, 69, 1]_b + a^2[4806, 7108, 3047, 478, 25]_b + a^3[1848, 2610, 984, 112, 2]_b + a^4[270, 372, 130, 12]_b$
79	$[10040, 21392, 16882, 6342, 1208, 112, 4]_b + a^1[24304, 49736, 36514, 12095, 1842, 110, 1]_b + a^2[22124, 43874, 30365, 9024, 1114, 41]_b + a^3[8928, 17288, 11406, 3074, 299, 3]_b + a^4[1356, 2598, 1677, 433, 38]_b$
80	$[1560, 2824, 1662, 362, 24]_b + a^1[2632, 4596, 2502, 449, 15]_b + a^2[1484, 2538, 1315, 203, 1]_b + a^3[280, 476, 243, 36]_b$
81	$[390, 247, 45, 2]_b + a^1[318, 176, 23]_b + a^2[66, 34, 4]_b$
82	$[1924, 1152, 195, 7]_b + a^1[1612, 876, 116]_b + a^2[340, 176, 22]_b$
83	$[390, 119, 3]_b + a^1[336, 93]_b + a^2[72, 19]_b$
84	$[3000, 3502, 1383, 213, 10]_b + a^1[5250, 5863, 2137, 279, 8]_b + a^2[2832, 3020, 1005, 104]_b + a^3[492, 512, 164, 16]_b$
85	$[14416, 16364, 6184, 887, 35]_b + a^1[25900, 28452, 10097, 1257, 28]_b + a^2[14312, 15234, 5072, 534]_b + a^3[2528, 2650, 862, 88]_b$
86	$[2840, 2390, 499, 15]_b + a^1[5250, 4317, 838, 12]_b + a^2[2974, 2397, 435]_b + a^3[534, 427, 76]_b$
87	$[1440, 1734, 709, 115, 6]_b + a^1[2418, 2755, 1019, 135, 4]_b + a^2[1296, 1412, 477, 50]_b + a^3[228, 244, 80, 8]_b$
88	$[6720, 7908, 3100, 469, 21]_b + a^1[11756, 13268, 4797, 607, 14]_b + a^2[6504, 7122, 2416, 258]_b + a^3[1168, 1266, 422, 44]_b$
89	$[1280, 1134, 249, 9]_b + a^1[2346, 2017, 402, 6]_b + a^2[1342, 1133, 211]_b + a^3[246, 207, 38]_b$
90	$[2760, 3434, 1453, 247, 14]_b + a^1[4422, 5157, 1939, 261, 8]_b + a^2[2352, 2628, 903, 96]_b + a^3[420, 464, 156, 16]_b$
91	$[12464, 15268, 6216, 989, 49]_b + a^1[21124, 24620, 9091, 1171, 28]_b + a^2[11704, 13254, 4592, 498]_b + a^3[2144, 2414, 826, 88]_b$
92	$[2280, 2146, 497, 21]_b + a^1[4134, 3751, 770, 12]_b + a^2[2394, 2135, 409]_b + a^3[450, 401, 76]_b$
93	$[110, 105, 27, 2]_b + a^1[112, 92, 15]_b + a^2[32, 26, 4]_b$
94	$[1436, 1840, 779, 130, 7]_b + a^1[1624, 1918, 684, 76]_b + a^2[488, 574, 202, 22]_b$

i	$\mathsf{F}_i(a,b)$
95	$[250, 253, 62, 3]_b + a^1[322, 307, 61]_b + a^2[102, 97, 19]_b$

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206 route de Bayonne, 31300 Toulouse, France
Email address: nifa.anass@gmail.com