SPECTRAL IDENTITIES AND CLOSED-GEODESIC PERIODS

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Abstract. We construct closed real-analytic locally symmetric nine-manifolds with equal scalar Laplace spectra and explicitly determine the infinite difference of their closed-geodesic period sets. Their common half-wave trace is real analytic near every period in this difference, although both geodesic flows are clean. In dimension eight, all finite-rank unitary flat connection spectra agree under a fixed $\mathbb{Z}/7\mathbb{Z}$-marking, while the period sets differ. On a fixed closed simply connected eighteen-manifold, one of two real-analytic scalar-isospectral metrics has infinitely many primitive periods absent from the period set of the other. These spectral equalities follow from intrinsic equivariant identities for $\mathrm{U}(2n+1)$ and $\mathrm{Sp}(n)\times\mathrm{SO}(2n+2)$, $n\geq 1$. For every central extension of a finite group with finitely generated kernel, prescribed finite families of central quotients occur as fundamental groups of closed homogeneous manifolds. Their canonical rough connection spectra agree for every finite-dimensional unitary representation of a common compact isotropy group. Simple connectivity varies in dimension eight, and rational formality in dimension seven. For a closed connected Riemannian manifold, marked finite-holonomy spectra determine the profinite conjugacy envelope of least lengths. Among compact connected Lie groups with bi-invariant metrics and a $\mathrm{U}(n)\times\mathrm{U}(1)$ marking, prescribed finite jets of the rate and logarithmic amplitude of the normalized equivariant heat trace, dimension and a volume lower bound characterize exactly these two models. The ambient weighted identity gives scalar-isospectral left-invariant normal homogeneous metrics on every $\mathrm{SU}(m)$, $m\geq 6$, which are not locally isometric, answering Gordon–Sutton's uniqueness question negatively.

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1. Introduction

Zelditch [118, Problem 4.1], Colin de Verdière [28, §13], and Sutton [102, arXiv v1, Problem 1] ask whether the scalar Laplace spectrum determines the set of lengths of smoothly closed geodesics. The answer is negative, even for closed real-analytic locally symmetric manifolds of nonnegative sectional curvature.

A closed manifold is compact and has no boundary. All operator spectra include multiplicities. For a closed Riemannian manifold X, Δ_X denotes the nonnegative scalar Laplace–Beltrami operator, and $\operatorname{Spec}(\Delta_X)$ its spectrum. For a group J, let $Z(J)$ be its center. For a Lie group J, write $\mathfrak{j} = \operatorname{Lie}(J)$, let J^0 be its identity component, and put $\pi_0(J) = J/J^0$. Denote the center of $\mathfrak{j}$ by $\mathfrak{z}(\mathfrak{j})$. For compact J, let $R(J)$ be the representation ring of finite-dimensional continuous complex representations of J. Write $\operatorname{Lsp}(X)$ for the set of lengths of nonconstant smoothly closed geodesics, including iterates and without length multiplicities. A nonconstant closed geodesic is primitive if it is not an iterate of a shorter closed geodesic. For a closed connected Riemannian manifold X, let $\ell_X(c)$ be the minimum length of a piecewise smooth loop in the free homotopy class of $c \in \Pi = \pi_1(X)$, with $\ell_X(1) = 0$. The minimum is attained (Lemma 5.3); conjugation and inversion preserve it. The half-wave trace is the tempered distribution $w_X(t) = \operatorname{Tr}(e^{it\sqrt{\Delta_X}})$; $\operatorname{singsupp} w_X$ denotes its distributional singular support. Let SX be the unit tangent bundle and Φ_t its geodesic flow. The flow is clean if, for every $t > 0$, its fixed set is a finite disjoint union of closed embedded submanifolds F satisfying $T_vF = \ker(d\Phi_t(v) - \mathrm{Id})$ for every $v \in F$.

1.1. Closed-geodesic periods and the Poisson relation.

Put $h = 2\pi/7$ and $\mathcal{S}_3 = \{a^2 + b^2 + c^2 : a, b, c \in \mathbb{Z}\}$.

Theorem 1.1 (Closed-geodesic periods). *There are closed connected real-analytic nine-dimensional manifolds M_9, N_9, locally symmetric and of nonnegative sectional curvature, such that*

$$\begin{aligned} \operatorname{Spec}(\Delta_{M_9}) = \operatorname{Spec}(\Delta_{N_9}), \qquad \operatorname{Lsp}(M_9) \subsetneq \operatorname{Lsp}(N_9), \\ \operatorname{Lsp}(N_9) \setminus \operatorname{Lsp}(M_9) = \{h\sqrt{n} : n \in \mathcal{S}_3,\ 7 \mid n,\ 49 \nmid n\}. \end{aligned} \tag{1.1}$$

Both geodesic flows are clean. The common half-wave trace is real analytic near every length in $\operatorname{Lsp}(N_9) \setminus \operatorname{Lsp}(M_9)$. *Infinitely many of these lengths are attained by embedded primitive closed geodesics on N_9.*

The manifolds are the order-seven quotients (6.4) (Theorem 7.1, Corollary 7.4). By (7.3), as $T \to \infty$,

$$\#\big((\operatorname{Lsp}(N_9) \setminus \operatorname{Lsp}(M_9)) \cap (0, T]\big) = \frac{5}{4\pi^2} T^2 + O(\log(2 + T)).$$

The Poisson relation [102, arXiv v1, §1.2, equation (1.1)] is

$$\operatorname{singsupp} w_X \subseteq \{0\} \cup \operatorname{Lsp}(X) \cup (-\operatorname{Lsp}(X)).$$

Zelditch [118, Problem 4.1] asks whether w_X can be smooth at an actual closed-geodesic length; Sutton [102, arXiv v1, Problem 2] asks whether this inclusion is always an equality. Theorem 1.1 gives a negative answer to the latter question. For $\ell = h\sqrt{14}$, the fixed set of the time-ℓ geodesic flow on the unit tangent bundle of N_9 consists of twelve clean three-dimensional tori. Their vectors are the initial velocities of embedded primitive closed geodesics, and exactly two components correspond to each nonidentity deck transformation (Proposition 7.5). In each pair, the complete singular expansions cancel coefficient by coefficient (Proposition 7.6).

Let $\widetilde{X}$ be the universal Riemannian cover of a closed connected manifold X, with the marked left deck action of $\Pi = \pi_1(X)$. For a finite-dimensional Hermitian space V and a unitary representation $\rho : \Pi \to \mathrm{U}(V)$, put

$$E_\rho = (\widetilde{X} \times V)/\Pi, \qquad c \cdot (x, v) = (cx, \rho(c)v).$$

The trivial connection descends to a flat Hermitian connection ∇_ρ on E_ρ; write $\Delta_{X,\rho} = \nabla_\rho^* \nabla_\rho$. For an integer $q \geq 1$, put $C_q = \mathbb{Z}/q\mathbb{Z}$. There are closed connected eight-dimensional manifolds M_8, N_8 such that

$$\pi_1(M_8) \simeq \pi_1(N_8) \simeq C_7, \qquad \operatorname{Spec}(\Delta_{M_8,\rho}) = \operatorname{Spec}(\Delta_{N_8,\rho}) \quad (\rho : C_7 \to \mathrm{U}(V)),$$

where each marking sends $1 \in C_7$ to the corresponding deck generator, and ρ ranges over all finite-dimensional unitary representations (Theorem 7.7). Theorem 7.7 and Corollary 7.9 give

$$3h \in \mathrm{Lsp}(N_8) \setminus \mathrm{Lsp}(M_8), \qquad \ell_{M_8}(3) = \ell_{N_8}(3) = h\sqrt{2}. \tag{1.2}$$

In fact $\ell_{M_8}(c) = \ell_{N_8}(c)$ for every $c \in C_7$, by Theorem 5.4 with the identity homomorphism of C_7.

On one closed simply connected eighteen-dimensional manifold there are scalar-isospectral analytic metrics for which infinitely many lengths of embedded primitive closed geodesics for one metric are absent from the length set of the other. Such pairs with positive Ricci curvature occur in every C^∞ neighborhood of a fixed product metric (Theorem 8.8). Theorem 8.2 gives pairs of metrics on one closed seventeen-dimensional manifold with equal marked unitary flat spectra on every corresponding connected finite cover and a primitive closed-geodesic length for one metric absent from the length set of the other. Theorem 9.2 gives an exact period-set difference and counting asymptotics for two locally symmetric metrics on one nineteen-dimensional manifold. Their flat spectra agree for representations pulled back from the base circle (Proposition 9.1).

Miatello–Rossetti [80, arXiv v1, Example 2.3 and Theorem 4.1(ii)] construct compact flat manifolds with different full length sets and equal spectra on certain p-forms, $p > 0$, whereas the scalar spectrum determines their full length sets. Their scalar and Sunada examples with different length spectra differ in multiplicities [80, arXiv v1, Examples 3.4 and 3.6].

Gordon–Rossetti [55, Corollary 1.6] show that, on a closed nonorientable $2m$-manifold, an isolated nondegenerate simple closed geodesic with orientation-reversing holonomy has zero contribution to the complexified heat trace on m-forms. For scalar-isospectral Heisenberg manifolds, Gornet [56, Example 2.15 and §5] computes the leading wave coefficients and explains their equality despite different length multiplicities.

Colin de Verdière [28, §13] describes cancellation of complete semiclassical expansions as a possible obstruction to length recovery. Koval–Vig [67, Theorem 1.1] obtain finite-order cancellation for planar billiards: for each finite N there are smooth strictly convex domains, arbitrarily C^∞-close to an ellipse, whose Dirichlet cosine trace is C^N near a billiard length. The analytic regularity in Theorem 1.1 follows from the analytic Poisson inclusion of Proposition A.3.

1.2. **Equivariant identities and descent.** For $n \geq 1$, put

$$G_n = \mathrm{U}(2n+1), \qquad H_n = \mathrm{Sp}(n) \times \mathrm{SO}(2n+2), \qquad P_n = \mathrm{U}(n) \times \mathrm{U}(1).$$

The embeddings $i_n : P_n \hookrightarrow G_n$ and $j_n : P_n \hookrightarrow H_n$ are

$$i_n(A, z) = \mathrm{diag}(A, \bar{A}, z), \qquad j_n(A, z) = (\mathrm{diag}(A, \bar{A}), r(A \oplus z)),$$

where $\bar{A}$ is the complex conjugate matrix and $r : \mathrm{U}(n+1) \to \mathrm{SO}(2n+2)$ is realification. Give the groups the bi-invariant metrics

$$Q_G = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}, \qquad Q_H = -\tfrac{1}{2}\operatorname{Re}\operatorname{Tr}_{\mathbb{C}} \oplus -\tfrac{1}{2}\operatorname{Tr}_{\mathbb{R}}.$$

The traces are evaluated on products of Lie algebra elements in the defining representations. The pullbacks of these forms to $\mathfrak{p}_n$ agree. Put $\iota_{G_n} = i_n$ and $\iota_{H_n} = j_n$. For $J = G_n, H_n$, the right P_n action on $L^2(J)$ is $R_p f(x) = f(x\iota_J(p))$. There is a unitary equivalence of these actions and Laplacians,

$$(L^2(G_n), \Delta_{G_n}, P_n) \simeq (L^2(H_n), \Delta_{H_n}, P_n), \tag{1.3}$$

which preserves smooth functions and the self-adjoint domains (Theorem 3.3). It also intertwines the left actions of $L_n = \mathrm{U}(n)$ through $i_n(A, 1)$ and $j_n(A, 1)$. For $n = 1$, restriction to the invariant subspaces of the order-seven actions in (6.4) gives the scalar spectral equality for M_9, N_9. On the subspaces invariant under the common right circle $\{1\} \times \mathrm{U}(1) \leq P_1$, the same unitary intertwines the residual C_7-actions; their character subspaces give the marked unitary flat spectral equalities for M_8, N_8 (Corollary 4.14, Theorem 7.7).

Let J be a compact connected Lie group with a positive bi-invariant metric Q, and let $K \leq J$ be closed. Give J/K the quotient metric. The horizontal space at $g \in J$ of the canonical connection on

$J \to J/K$ is $dL_g(\mathfrak{k}^{\perp_Q})$, where L_g is left translation. For a finite-dimensional unitary K-module (V, ρ), let $\Delta_{J/K,V}$ be the rough Laplacian of the induced connection on $J \times_K V$, with $(gk, v) \sim (g, \rho(k)v)$. The trivial module gives the scalar Laplacian. Let $C_J = \Delta_J$ act componentwise, and choose a Q-orthonormal basis (Y_a) of $\mathfrak{k}$. Put $C_{K,V} = -\sum_a d\rho(Y_a)^2$. On smooth functions satisfying $f(gk) = \rho(k)^{-1} f(g)$, one has

$$\Delta_{J/K,V} = C_J - C_{K,V}$$

by (4.3). For integers $r, n_1, \dots, n_r \geq 1$ and a closed subgroup $K \leq \prod_{j=1}^{r} P_{n_j}$, the metric on $\mathfrak{k}$ is the same for every choice of ambient factors G_{n_j} or H_{n_j}. Tensor products of (1.3) therefore give equal canonical rough spectra for every common K-module (Theorem 4.3).

For the bilateral action $((p_1, p_2) \cdot f)(x) = f(\iota_J(p_1)^{-1} x \iota_J(p_2))$, the difference of the eigenspace classes at $2n + 1$, the first positive eigenvalue, generates the ideal of all eigenspace differences in $R(P_n \times P_n)$. After restriction to any closed subgroup, all these differences vanish if and only if the first one vanishes (Corollary 3.10). The $L_1 \times P_1$ equivalence gives the scalar spectral equalities for the connection metrics of Theorem 8.8. More generally, closed Riemannian fibers with equivariantly equivalent scalar Laplacians and isometric structure-group actions give isospectral connection metrics on associated bundles over a closed Riemannian base, for a fixed principal bundle with compact structure group and a fixed connection (Proposition 4.7).

Sutton [100, Theorems 2.3 and 3.6] obtained simply connected, locally non-isometric normal homogeneous spaces from representation-equivalent connected subgroups. The groups G_n and H_n occur in the dimension-data identity of An–Yu–Yu [6, Theorem 1.5(1)]; their construction gives simply connected isospectral homogeneous spaces of different homotopy types [6, Theorem 1.7]. Yu obtained nontrivial weighted dimension-data identities and vector bundles with isospectral ambient Casimir operators [116, Theorem 2.1 and Corollary 2.6] [117, arXiv version 2, §2.3]. Yu's weighted identities imply the ambient Casimir identity of Theorem 20.1. The intrinsic equivalence (1.3) identifies the right-P_n representations in each scalar eigenspace of G_n and H_n, with the metrics above.

1.3. **Realization and topology.** With K fixed, the boundary maps $\pi_1(J/K) \twoheadrightarrow \pi_0(K)$ for the ambient products J have the same finite target. Their kernels are the finitely generated central groups $\operatorname{coker}(\pi_1(K^0) \to \pi_1(J))$ (Lemma 10.2). By Wilking's theorem [114, Theorem 2.2(a),(d)], a group is the fundamental group of a compact connected Riemannian homogeneous manifold if and only if it contains a finitely generated central subgroup of finite index.

Let

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} \Gamma \longrightarrow 1$$

be central, with Γ finite and A finitely generated. Choose generators $c_1, \dots, c_r$ of A, where $r \geq 1$; identity entries are allowed. Put $[r] = \{1, \dots, r\}$ and $A_I = \langle c_j : j \in I \rangle$ for $I \subseteq [r]$. For a finitely generated abelian group B, write $\operatorname{rank} B = \dim_{\mathbb{Q}}(B \otimes_{\mathbb{Z}} \mathbb{Q})$. For a homogeneous Riemannian manifold M, let Ric_M denote its Ricci endomorphism at any point.

Theorem 1.2 (Realization). *There are compact connected Lie groups J_I, a compact Lie group K with $K^0 \subseteq Z(K)$ and embeddings into every J_I, and normal homogeneous metrics on $M_I = J_I/K$, $I \subseteq [r]$, such that*

$$\pi_1(M_I) \simeq E/A_I, \qquad \pi_0(K) \simeq \Gamma, \qquad \dim \ker \mathrm{Ric}_{M_I} = \operatorname{rank}(A/A_I). \tag{1.4}$$

The boundary homomorphism $\pi_1(M_I) \to \pi_0(K)$ is the prescribed map $E/A_I \to \Gamma$. For every finite-dimensional unitary K-module V, the spectrum of $\Delta_{M_I,V}$ is independent of I. The tangent spaces at the identity cosets are isomorphic as metric K-modules. The metrics are real analytic and have nonnegative sectional curvature; their Ricci tensors are positive definite exactly when E/A_I is finite.

Theorem 10.4 gives this realization, including the prescribed boundary homomorphisms. There are connected covering groups $\widehat{J}_I \to J_I$ for which the inverse images $\widehat{K}_I$ of K satisfy $\pi_0(\widehat{K}_I) \simeq E/A_I$. Under these identifications, the quotient maps $E \to E/A_I$ are induced on component groups by homomorphisms $\widehat{K}_\emptyset \to \widehat{K}_I$ (Theorem 10.4(1) and its proof). Let $\mathfrak{m}$ be the common real metric tangent

K-module. Its complexified tensor powers and their invariant summands give associated bundles with equal canonical rough spectra. There are compatible homogeneous spin structures for which the squares of the cubic Dirac operators have equal spectra after every common homogeneous unitary twist. For $X \in \mathfrak{m}$, let $X^\flat$ be its metric dual and let ι_X denote contraction. On $\Lambda^\bullet(\mathfrak{m}^* \otimes_{\mathbb{R}} \mathbb{C})$, with its parity grading and Clifford multiplication $c(X)\alpha = X^\flat \wedge \alpha - \iota_X \alpha$, the cubic Dirac operators themselves have equal spectra, also after every common unitary twist (Theorem 4.9 and Proposition 4.11). For $L \leq \Gamma$, the inverse image of L under $K \to \pi_0(K) = \Gamma$ defines degree-$[\Gamma : L]$ covers with fundamental groups $p^{-1}(L)/A_I$. These equalities hold on the covers, including all flat twists pulled back from finite-dimensional unitary representations of L.

A nonempty finite list of groups $E_1, \dots, E_m$ admits a realization $E_i \simeq \pi_1(J_i/K)$, with compact connected Lie groups J_i and one compact isotropy group K, if and only if there are surjections

$$p_i : E_i \twoheadrightarrow \Gamma \tag{1.5}$$

onto one finite group, with finitely generated central kernels. For prescribed maps p_i, the realization may be chosen with these boundary homomorphisms and the canonical spectral equalities above (Theorem 11.1).

Let A be a nontrivial finitely generated abelian group, and let $d(A)$ be its least number of generators. Among pairs $(\prod_{j=1}^r J_j)/K$ and $(\prod_{j=1}^r J_j')/K$, where $r, n_j \geq 1$, $J_j, J_j' \in \{G_{n_j}, H_{n_j}\}$ and $K \leq \prod_{j=1}^r P_{n_j}$ is one common closed subgroup, the least dimension for fundamental groups A and 1 is

$$8d(A) \tag{1.6}$$

(Theorem 11.5). Bérard asks whether a closed manifold scalar-isospectral to a simply connected manifold must itself be simply connected [12, VIII.4, p. 141]. For every $q \geq 2$, Corollary 11.12 gives eight-dimensional manifolds with fundamental groups C_q and 1, equal scalar and canonical connection spectra, positive Ricci curvature and nonnegative sectional curvature, answering this question negatively.

Let $S^1_{(1,-1,0)} = \{\mathrm{diag}(e^{it}, e^{-it}, 1) : t \in \mathbb{R}\} \leq \mathrm{SU}(3)$. On the standard maximal torus $\mathrm{U}(1)^3 \leq \mathrm{SU}(2)^3$, put $\chi(u, v, w) = uv^{-1}w^{-1}$. The seven-dimensional quotients

$$A_7 = G_1/P_1 \cong \mathrm{SU}(3)/S^1_{(1,-1,0)},$$
$$B_7 = H_1/P_1 \cong \mathrm{SU}(2)^3/\ker\chi,$$

are simply connected and carry the normal metrics induced by Q_G and Q_H. For every common finite-dimensional unitary P_1-module, the associated bundles have equal canonical rough spectra. Their Ricci tensors are positive, and their second Betti numbers are 1 and 2 (Theorem 12.1). The Aloff–Wallach space $A_7 = N_{1,-1}$ is rationally formal [66, arXiv v2, Theorem 6 and Remark 2] [44, Theorem 5.3]; the space $B_7 = Q^{1,1,1}$ has a triple Massey product which does not contain zero [17, proof of Theorem 3.2, case $n = 3$]. Their Hodge two-form spectra differ in the multiplicity of zero. Since every simply connected closed manifold of dimension at most six is formal [86, Proposition 4.6, p. 574], seven is the least dimension for this formality contrast.

A group is large if a finite-index subgroup surjects onto a nonabelian free group. Cornelissen–Peyerimhoff ask whether closed connected oriented Riemannian manifolds with large fundamental groups must be isometric if a bijection between all finite-dimensional unitary representations of their fundamental groups preserves the corresponding flat spectra, without assuming a common finite orbifold quotient [30, Chapter 1, Open Problem, p. 6]. For a closed oriented surface Σ_g, $g \geq 2$, products with the simply connected pair of [6, Theorem 1.7] already give a negative answer: every marked unitary flat spectrum agrees, while the second homotopy groups differ. For a hyperbolic metric on Σ_g, there is a positive real-analytic function with nondegenerate Hessian at some point whose warped product metrics on $\Sigma_g \times A_7$ and $\Sigma_g \times B_7$ have restricted holonomy $\mathrm{SO}(9)$ and equal marked unitary flat spectra on every corresponding finite cover (Corollary 13.3). Their second Betti numbers are 2 and 3; the first product is rationally formal and the second is not. Their universal covers have second Betti numbers 1 and 2, so they have no common connected Riemannian cover.

1.4. **Finite holonomy and least lengths.** For elements of a group Q, write $a \sim_Q b$ for conjugacy and $\mathrm{Conj}_Q(a)$ for the conjugacy class of a. A group is conjugacy separable if each pair of nonconjugate elements has nonconjugate images in some finite quotient. Let $p_I : E/A_I \twoheadrightarrow \Gamma$ be induced by p. For the manifolds of Theorem 1.2,

$$\ell_{M_{[r]}}(\gamma) = \min_{p_I(c)=\gamma} \ell_{M_I}(c), \qquad \gamma \in \Gamma.$$

The minimum is attained for every $I \subseteq [r]$ (Corollary 14.10).

Let X be an arbitrary closed connected smooth Riemannian manifold of positive dimension, with marked fundamental group Π. For a finite quotient $q : \Pi \twoheadrightarrow Q$, let $Y_q \to X$ be the corresponding regular Riemannian cover, with its marked left Q-action, and let d_{Y_q} be its distance. For $g \in Q$ and $t > 0$, put

$$\ell_q(g) = \min_{y \in Y_q} d_{Y_q}(y, gy), \qquad R_g f(y) = f(g^{-1}y), \qquad H_{q,g}(t) = \mathrm{Tr}(R_g e^{-t\Delta_{Y_q}}).$$

These traces are positive, and character inversion recovers them from the spectra of $\Delta_{X,\rho\circ q}$, with ρ ranging over the irreducible unitary representations of Q. One has

$$\lim_{t\downarrow 0} -4t \log H_{q,g}(t) = \ell_q(g)^2, \qquad \ell_q(g) = \min_{q(c)\sim_Q g} \ell_X(c). \tag{1.7}$$

Every minimum is attained (Lemma 5.3). Let $\widehat{\Pi}$ be the profinite completion of Π, with canonical map $\iota : \Pi \to \widehat{\Pi}$, and let $\widehat{q}$ be the continuous extension of q. For $z \in \widehat{\Pi}$, put $\widehat{\ell}_X(z) = \sup_q \ell_q(\widehat{q}(z)) \in [0, \infty]$, where q ranges over the finite quotients of Π. Then

$$\{z \in \widehat{\Pi} : \widehat{\ell}_X(z) \le R\} = \bigcup_{\substack{c \in \Pi \\ \ell_X(c) \le R}} \mathrm{Conj}_{\widehat{\Pi}}(\iota(c)), \qquad 0 \le R < \infty. \tag{1.8}$$

Only finitely many distinct orbits occur. For each $0 \le R < \infty$, one finite quotient attains the supremum at every $\iota(c)$ with $\ell_X(c) \le R$ (Theorem 14.2). If Π is conjugacy separable, then $\widehat{\ell}_X(\iota(c)) = \ell_X(c)$ for every $c \in \Pi$. In this case, the spectra indexed by the finite-image unitary representations of Π determine every $\ell_X(c)$.

For every smooth metric and every $g \neq 1$, the equivariant cosine trace $\mathrm{Tr}(R_g \cos(t\sqrt{\Delta_{Y_q}}))$ is singular at $t = \ell_q(g)$ (Theorem 15.3). For each $t > 0$, there is a unique Borel probability measure $\nu_{X,t}$ on $\widehat{\Pi}$ such that

$$(\widehat{q})_* \nu_{X,t}(\{g\}) = \frac{H_{q,g}(t)}{|Q|\,\mathrm{Tr}(e^{-t\Delta_X})} \qquad (q : \Pi \twoheadrightarrow Q,\ g \in Q).$$

As $t \downarrow 0$, these measures satisfy a large-deviation principle in the profinite topology, with speed $1/t$ and good rate function $z \mapsto \widehat{\ell}_X(z)^2/4$ (Theorems 14.4 and 14.8).

For Brownian motion on a closed Riemannian manifold, with a positive constant killing rate and flat connections on trivial bundles with compact Lie structure group, Le Jan [71, §3, proof of Proposition 5] identifies Brownian-loop integrals of normalized holonomy characters minus 1 with differences of normalized twisted and scalar zeta derivatives at 0. His Remark 2 relates conjugacy separability to the separation of free homotopy classes by holonomy and notes that every finite-dimensional unitary representation of Higman's group is trivial. Wang–Xie–Yu prove finite-quotient approximation of delocalized traces for equivariant bounded smooth ℓ^1-summable kernels, along sequences of finite-index normal subgroups distinguishing the chosen nontrivial conjugacy class [112, Lemmas 2.4 and 2.6]. For a smooth closed connected Riemannian three-manifold X, Li–Wu–Zhou prove that the distributional trace of $\sin(t\sqrt{\Delta_X})/\sqrt{\Delta_X}$ is not C^1 on any neighborhood of the shortest positive closed-geodesic length when that length is below the conjugate radius [72, Theorem 4.1]. Here the operator acts as t on $\ker \Delta_X$.

1.5. **Marked recognition.** Fix $n \geq 1$. Let J be a compact connected Lie group with a positive bi-invariant metric Q and a Lie group homomorphism $\iota : P_n \to J$. Write $d\iota : \mathfrak{p}_n \to \mathfrak{j}$ for its differential at the identity. Two such triples (J, Q, ι) and (J', Q', ι') are isomorphic as marked metric Lie groups if there is a Lie group isomorphism $F : J \to J'$ with $F^*Q' = Q$ and $F \circ \iota = \iota'$. Let j_J be the Riemannian Jacobian of the Lie exponential, normalized by $j_J(0) = 1$. For $X \in \mathfrak{p}_n$ near zero, put

$$I_J(X) = Q(d\iota X, d\iota X), \qquad \Phi_J(X) = -\tfrac{1}{2} \log j_J(d\iota X).$$

For $p \in P_n$, put $R_p f(x) = f(x\iota(p))$. Then

$$\frac{\operatorname{Tr}(R_{\exp_{P_n} X} e^{-t\Delta_J})}{\operatorname{Tr}(e^{-t\Delta_J})} = e^{-I_J(X)/(4t)} \big(e^{\Phi_J(X)} + O(t) \big) \qquad (t \downarrow 0),$$

locally uniformly for X near zero (Lemma 16.1). Together with $\dim J$, the germs of I_J and Φ_J determine the orthogonal P_n-module $(\mathfrak{j}, Q, \operatorname{Ad} \circ \iota)$ and the map $d\iota$, up to an orthogonal P_n-intertwiner (proof of Theorem 16.2). Write $\operatorname{jet}_0^k f$ for the Taylor polynomial of f at zero through total degree k. If

$$\begin{gathered} \dim J = (2n+1)^2, \qquad \operatorname{jet}_0^2 I_J = \operatorname{jet}_0^2 I_{G_n}, \qquad \operatorname{Vol}(J) \geq \operatorname{Vol}(G_n), \\ \operatorname{jet}_0^6(\Phi_J|_{\mathfrak{z}(\mathfrak{p}_n)}) = \operatorname{jet}_0^6(\Phi_{G_n}|_{\mathfrak{z}(\mathfrak{p}_n)}), \end{gathered} \tag{1.9}$$

then (J, Q, ι) is isomorphic as a marked metric Lie group to (G_n, Q_G, i_n) or (H_n, Q_H, j_n) (Theorem 19.1). Let I_n be the $n \times n$ identity matrix. In central coordinates $(a, b) \mapsto (iaI_n, ib) \in \mathfrak{p}_n$, it suffices to require equality with the reference coefficients of

$$b^2,\ b^4,\ b^6,\ a^2,\ a^4,\ a^6,\ a^2b^2,\ a^4b^2,\ ab$$

in Φ_J, and those of a^2, ab, b^2 in I_J. For $n \geq 2$, fix a nonzero $H_0 \in \mathfrak{su}(n)$ and also require equality of the coefficient of s^2 in $I_J(s(H_0, 0))$. There are thirteen real coefficients, or twelve when $n = 1$. Dimension and the volume bound are separate hypotheses.

Fix integers $r, n_1, \dots, n_r \geq 1$, put $P = \prod_{j=1}^r P_{n_j}$, and choose positive scales c_j. The scalar eigenspaces of $\prod_{j=1}^r (G_{n_j}, c_j Q_G)$, with their eigenvalues and right-P actions, have exactly the 2^r marked realizations

$$\prod_{j=1}^r (J_j, c_j Q_{J_j}), \qquad J_j \in \{G_{n_j}, H_{n_j}\}, \tag{1.10}$$

among compact connected Lie groups with positive bi-invariant metrics (Theorem 19.6). Here Q_{J_j} is the corresponding reference form. These marked classes remain distinct when ranks or scales repeat.

1.6. **Metrics on compact simple groups.** Schueth constructed continuous isospectral families of left-invariant metrics on $\mathrm{SU}(m)$, $m \geq 6$, arbitrarily close to a bi-invariant metric, with sectional curvatures of both signs [94, Example 3.9 and Remarks 3.12(i)–(ii), author version].

For $m \geq 6$, equip $\mathrm{SU}(m)$ with the trace metric $Q(X, Y) = -\operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY)$ for $X, Y \in \mathfrak{su}(m)$. The nonisomorphic reference groups $G_1 = \mathrm{U}(3)$ and $H_1 = \mathrm{Sp}(1) \times \mathrm{SO}(4)$ embed in a six-dimensional complex block by (22.1)–(22.2). Their Casimirs for the restricted trace form have identical spectra on every finite-dimensional complex $\mathrm{SU}(m)$-module (Theorem 20.1 and Lemma 22.2). For either image subgroup K, put $\mathfrak{k} = \operatorname{Lie}(K)$ and let D_K be its positive Casimir for $Q|_{\mathfrak{k}}$. The left-invariant metric and its scalar Laplacian are

$$g_{K,s} = Q|_{\mathfrak{k}^{\perp_Q}} \oplus sQ|_{\mathfrak{k}}, \qquad \Delta_{g_{K,s}} = C_{\mathrm{SU}(m)} + (s^{-1} - 1)D_K, \qquad 0 < s < 1,$$

where the Casimirs act in the right regular representation.

Gordon–Sutton [50, Introduction] ask whether spectral isolation among naturally reductive left-invariant metrics on a fixed compact simple group can be strengthened to uniqueness or finiteness. Theorem 22.1 gives a negative answer to uniqueness on every $\mathrm{SU}(m)$, $m \geq 6$. For each $0 < s < 1$, the two metrics above are scalar-isospectral, left-invariant, and normal homogeneous. Both are locally irreducible, with positive Ricci curvature and nonnegative sectional curvature. Their Ricci endomorphisms have different least eigenvalues, so they are not locally isometric.

For fixed integers $r \geq 1$ and $m \geq 6r$, the block metrics of Theorem 22.5 form a family $(g_{\omega,\varepsilon})$, with $\omega \in \{A, B\}^r$ and $0 < \varepsilon < 1$, where $\omega_j = A$ selects G_1 and $\omega_j = B$ selects H_1 in the jth block. For $\omega, \omega' \in \{A, B\}^r$ and $\varepsilon, \varepsilon' \in (0, 1)$, one has

$$\operatorname{Spec}(\Delta_{g_{\omega,\varepsilon}}) = \operatorname{Spec}(\Delta_{g_{\omega',\varepsilon'}}) \quad \Longleftrightarrow \quad \varepsilon = \varepsilon'.$$

For each fixed ε, the 2^r metrics are pairwise not locally isometric. Thus within this family each spectrum has exactly 2^r isometry classes. Every C^∞ neighborhood of the bi-invariant trace metric on $\mathrm{SU}(m)$ contains one of these 2^r-member sets.

Part I. Spectral identities and descent

2. Reference groups, metrics, and Weyl lattices

All L^2 spaces are complex. On compact groups they are defined with Haar probability measure; volumes refer to Riemannian measure. For an integer $q \geq 1$, let I_q be the identity matrix. The groups $\mathrm{U}(q)$ and $\mathrm{SO}(q)$ consist of the unitary complex $q \times q$ matrices and the real orthogonal $q \times q$ matrices of determinant one, respectively. For an integer $m \geq 1$, put

$$\Omega_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \qquad \mathrm{Sp}(m) = \{B \in \mathrm{U}(2m) \,:\, B^t \Omega_m B = \Omega_m\},$$

where B^t denotes transpose. For an integer $n \geq 1$, set

$$N = 2n + 1, \qquad G_n = \mathrm{U}(N), \qquad H_n = \mathrm{Sp}(n) \times \mathrm{SO}(2n + 2), \qquad P_n = \mathrm{U}(n) \times \mathrm{U}(1).$$

For a $q \times q$ complex matrix A, let $\bar{A}$ denote entrywise conjugation and let $r(A)$ be its real matrix in the ordered coordinates $(\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_q, \operatorname{Im} z_q)$. Put $c(A) = \operatorname{diag}(A, \bar{A})$. Define $i_n : P_n \to G_n$ and $j_n : P_n \to H_n$ by

$$i_n(A, z) = \operatorname{diag}(A, \bar{A}, z), \qquad j_n(A, z) = \big(c(A), r(A \oplus z)\big). \tag{2.1}$$

For $A \in \mathrm{U}(n)$, the matrix $c(A)$ is unitary, and $A^t \bar{A} = I_n$ gives $c(A)^t \Omega_n c(A) = \Omega_n$. Moreover, $r(A \oplus z)$ is orthogonal and $\det_{\mathbb{R}} r(A \oplus z) = |\det_{\mathbb{C}}(A \oplus z)|^2 = 1$. Both maps are homomorphisms; their matrix blocks recover A, z smoothly, so they are embeddings.

On $\mathfrak{u}(N)$, $\mathfrak{sp}(m)$, and $\mathfrak{so}(2m)$, respectively, define

$$\begin{aligned} Q_G(X, Y) &= -\operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY), \\ Q_C(X, Y) &= -\tfrac{1}{2} \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY), \\ Q_D(X, Y) &= -\tfrac{1}{2} \operatorname{Tr}_{\mathbb{R}}(XY). \end{aligned} \tag{2.2}$$

The traces are in the defining representations, of dimensions N, $2m$ over $\mathbb{C}$, and $2m$ over $\mathbb{R}$, respectively. These positive adjoint-invariant forms define bi-invariant metrics. On H_n put $Q_H = Q_C \oplus Q_D$.

Lemma 2.1. *The pullbacks of Q_G and Q_H by i_n and j_n coincide. Their common value Q_P on $\mathfrak{p}_n = \mathfrak{u}(n) \oplus i\mathbb{R}$ is*

$$Q_P((X, it), (Y, is)) = -2 \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY) + ts, \qquad X, Y \in \mathfrak{u}(n), \quad t, s \in \mathbb{R}. \tag{2.3}$$

Proof. Let di_n, dj_n be the differentials at the identity. For $u = (X, it)$ and $v = (Y, is)$ in $\mathfrak{p}_n$,

$$\begin{aligned} Q_G(di_n u, di_n v) &= -\operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY + \bar{X}\bar{Y}) + ts = -2 \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY) + ts, \\ Q_H(dj_n u, dj_n v) &= -\tfrac{1}{2} \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY + \bar{X}\bar{Y}) - \tfrac{1}{2} \operatorname{Tr}_{\mathbb{R}} r(XY) + ts \\ &= -2 \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY) + ts. \end{aligned}$$

Here $\operatorname{Tr}_{\mathbb{R}} r(XY) = 2 \operatorname{Re} \operatorname{Tr}_{\mathbb{C}}(XY)$. □

Write $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, identified with U(1) by $[t] \mapsto e^{it}$; the identification $\mathbb{R}/\mathbb{Z} \simeq \mathbb{T}$ is $[x] \mapsto [2\pi x]$. For $\theta \in \mathbb{R}$, put

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For $m \geq 1$ and $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, put

$$c_m(x) = \operatorname{diag}(e^{ix_1}, \dots, e^{ix_m}, e^{-ix_1}, \dots, e^{-ix_m}),$$
$$d_m(x) = \operatorname{diag}(R(x_1), \dots, R(x_m)).$$

These maps parametrize maximal tori in $\mathrm{Sp}(m)$ and $\mathrm{SO}(2m)$. Their differentials at zero have squared norm $\sum_j x_j^2$ on a vector $x \in \mathbb{R}^m$, for Q_C and Q_D, respectively. For $x \in \mathbb{R}^N$, the diagonal torus of $\mathrm{U}(N)$ is parametrized by $x \mapsto \operatorname{diag}(e^{ix_1}, \dots, e^{ix_N})$, and $Q_G(i \operatorname{diag} x, i \operatorname{diag} x) = \sum_j x_j^2$. The kernels of c_m and d_m are $(2\pi\mathbb{Z})^m$; the unitary parametrization has kernel $(2\pi\mathbb{Z})^N$. In rank d, their characters are $x \mapsto e^{i\sum_{j=1}^d k_j x_j}$, $k \in \mathbb{Z}^d$.

2.1. **Highest weights and the scalar Casimir.** Let e_j be the coordinate covectors dual to the angular coordinates just defined; they are orthonormal for the dual torus metrics. Write $|\cdot|$ for this Euclidean norm. The positive roots and dominant integral weights are

J	positive roots	dominant weights
$\mathrm{U}(N)$	$e_i - e_j$ $(i < j)$	$\lambda_1 \geq \cdots \geq \lambda_N, \quad \lambda_i \in \mathbb{Z}$
$\mathrm{Sp}(m)$	$e_i \pm e_j$ $(i < j)$, $2e_i$	$\lambda_1 \geq \cdots \geq \lambda_m \geq 0, \quad \lambda_i \in \mathbb{Z}$
$\mathrm{SO}(2m)$	$e_i \pm e_j$ $(i < j)$	$\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq \lvert\lambda_m\rvert, \quad \lambda_i \in \mathbb{Z}.$

(2.4)

The last row has $m \geq 2$; for $\mathrm{SO}(2)$ the root system is empty and the irreducible characters are $R(x) \mapsto e^{ikx}$, $k \in \mathbb{Z}$. Half-integral spin weights do not belong to the character lattice of $\mathrm{SO}(2m)$. Define

$$\rho_{\mathrm{U}(N)} = \rho_A = (n, n-1, \dots, -n),$$
$$\rho_{\mathrm{Sp}(m)} = \rho_{C,m} = (m, m-1, \dots, 1),$$
$$\rho_{\mathrm{SO}(2m)} = \rho_{D,m} = (m-1, m-2, \dots, 0).$$

These are the half-sums of the positive roots; in particular $\rho_{D,1} = 0$. The weight parametrizations and Weyl character formula are those of [20, Chapter V, §6, pp. 218–222; Chapter VI, Theorem 1.7, p. 242, and Proposition 2.6, p. 252]. For finite products, weights and ρ-vectors are concatenated, and the squared norms are added.

Let J be one of these groups or a finite product of them. For a dominant weight λ of J, let $\pi_\lambda : J \to \mathrm{U}(V_\lambda)$ be the irreducible representation on the finite-dimensional Hermitian space V_λ with highest weight λ. Put $d_\lambda = \dim V_\lambda$ and $\chi_\lambda(g) = \operatorname{Tr} \pi_\lambda(g)$. Let $C_\lambda(J)$ be the span of the matrix entries of $\pi_\lambda(g)$ in an orthonormal basis of V_λ. The Peter–Weyl theorem [41, Theorem 35.9] gives

$$L^2(J) = \widehat{\bigoplus}_{\lambda \text{ dominant}} C_\lambda(J), \qquad \dim C_\lambda(J) = d_\lambda^2.$$

Let Δ_J be the nonnegative self-adjoint scalar Laplacian on $L^2(J)$.

Lemma 2.2. *Let $J = \mathrm{U}(N)$, $\mathrm{Sp}(m)$, or $\mathrm{SO}(2m)$, with the metric in* (2.2)*, or a finite product of these groups with the product metric. On $C_\lambda(J)$ the scalar Laplacian acts by*

$$c_J(\lambda) = |\lambda + \rho_J|^2 - |\rho_J|^2. \tag{2.5}$$

Proof. For an orthonormal basis (Z_a) of left-invariant vector fields, the nonnegative scalar Laplacian is $-\sum_a Z_a^2$. Apply [103, Propositions 9.A.1–9.A.2 and equation (9.A.21)], with the opposite sign convention for the Laplacian. The torus norm computations above identify the dual metric with the displayed Euclidean norm, including the central direction of $\mathrm{U}(N)$. For $\mathrm{SO}(2)$ the formula is $c_{\mathrm{SO}(2)}(k) = k^2$; on a finite product the factor Laplacians and their eigenvalues add. □

For $\mu \in \mathbb{R}$, write $E_\mu(J) = \ker(\Delta_J - \mu)$, with $E_\mu(J) = 0$ when μ is not an eigenvalue. The Weyl dimension formula [20, Chapter VI, Theorem 1.7(iv), p. 242] gives constants $C_J > 0$ and an integer $a_J \geq 0$ such that

$$d_\lambda \leq C_J(1 + |\lambda + \rho_J|)^{a_J}.$$

For $0 < \varepsilon \leq T < \infty$, there is $C_{J,\varepsilon,T} > 0$ such that

$$d_\lambda^2 e^{-tc_J(\lambda)} \leq C_{J,\varepsilon,T} e^{-\varepsilon|\lambda+\rho_J|^2/2}, \qquad \varepsilon \leq t \leq T.$$

The sum of the right-hand side over the integer weight lattice is finite. Thus $e^{-t\Delta_J}$ is trace class for $t > 0$. For $x, y \in J$, define $F_{x,y} : J \to J$ by $F_{x,y}(g) = xgy^{-1}$ and $F_{x,y}^* f = f \circ F_{x,y}$. This pullback is unitary, since $F_{x,y}$ preserves Haar measure. Put

$$B_J(t; x, y) = \mathrm{Tr}_{L^2(J)}(F_{x,y}^* e^{-t\Delta_J}), \qquad t > 0.$$

For $f_{ab}(g) = \pi_\lambda(g)_{ab}$, $1 \leq a, b \leq d_\lambda$,

$$F_{x,y}^* f_{ab} = \sum_{c,d=1}^{d_\lambda} \pi_\lambda(x)_{ac} \left(\pi_\lambda(y)^{-1}\right)_{db} f_{cd}.$$

Thus the trace of $F_{x,y}^*$ on $C_\lambda(J)$ is $\chi_\lambda(x)\overline{\chi_\lambda(y)}$. Consequently

$$B_J(t; x, y) = \sum_{\lambda \text{ dominant}} \chi_\lambda(x)\overline{\chi_\lambda(y)} e^{-tc_J(\lambda)}. \tag{2.6}$$

Unitarity gives $|\chi_\lambda(x)| \leq d_\lambda$, so the series converges absolutely and uniformly in x, y and $\varepsilon \leq t \leq T$. At $x = 1$ the first character is d_λ; at $x = y = 1$ the corresponding summand is $d_\lambda^2 e^{-tc_J(\lambda)}$. Bounded eigenvalue intervals contain only finitely many highest weights, since $c_J(\lambda) = |\lambda + \rho_J|^2 - |\rho_J|^2$.

Lemma 2.3. *Let $\mathcal{E} \subset [0, \infty)$ meet every bounded interval in a finite set. Let $a_E \in \mathbb{C}$ for $E \in \mathcal{E}$ and $t_0 \in \mathbb{R}$, and suppose $\sum_{E\in\mathcal{E}} |a_E| e^{-t_0 E} < \infty$. If $\sum_{E\in\mathcal{E}} a_E e^{-tE} = 0$ for every $t \geq t_0$, then every a_E is zero.*

Proof. For $\mathcal{E} \neq \emptyset$, apply the uniqueness theorem for general Dirichlet series [52, Chapter II, §5, Theorem 6] to $\sum_{E\in\mathcal{E}} (a_E e^{-t_0 E}) e^{-sE}$, where $s = t - t_0$. If $\mathcal{E}$ is infinite, its increasing enumeration tends to infinity. The series converges absolutely at $s = 0$ and vanishes for every real $s \geq 0$, hence at all positive integers. A finite nonempty set is extended by adjoining an increasing unbounded sequence of larger exponents with zero coefficients. The empty set gives no coefficients. □

Lemma 2.4. *Let D be a Hausdorff topological group and $\mathcal{A}$ an index set. For $i = 1, 2$ and $\eta \in \mathcal{A}$, let $U_{i,\eta} : D \to \mathrm{U}(E_{i,\eta})$ be continuous unitary representations on finite-dimensional Hermitian spaces. Put*

$$\mathcal{H}_i = \widehat{\bigoplus}_{\eta\in\mathcal{A}} E_{i,\eta}, \qquad U_i(g) = \widehat{\bigoplus}_{\eta\in\mathcal{A}} U_{i,\eta}(g).$$

Suppose $S_\eta : E_{1,\eta} \to E_{2,\eta}$ are equivariant unitaries. Then $S = \widehat{\bigoplus}_\eta S_\eta$ is an equivariant unitary $\mathcal{H}_1 \to \mathcal{H}_2$. For $u \in \mathcal{H}_i$, write $u_\eta \in E_{i,\eta}$ for its orthogonal components. For every $f : \mathcal{A} \to \mathbb{C}$, S intertwines the diagonal operators $f_i u = \sum_\eta f(\eta) u_\eta$, with their maximal domains

$$\mathcal{D}_{i,f} = \left\{ u = \sum_\eta u_\eta \in \mathcal{H}_i \,:\, \sum_\eta |f(\eta)|^2 \|u_\eta\|^2 < \infty \right\}.$$

It intertwines averaging over every compact subgroup of D and the self-adjoint generators of $U_i \circ \gamma$, with their domains, for every continuous homomorphism $\gamma : \mathbb{R} \to D$.

Suppose in addition that $\mathcal{H}_i = L^2(M_i)$ for closed Riemannian manifolds M_i and that there is $\nu : \mathcal{A} \to [0, \infty)$ such that $E_{i,\eta} \subset \ker(\Delta_{M_i} - \nu(\eta))$. Write $H^s(M_i)$ for the complex L^2 Sobolev space of order $s \in \mathbb{R}$. Then S induces a topological isomorphism $H^s(M_1) \to H^s(M_2)$ for every $s \in \mathbb{R}$, and it restricts to a topological isomorphism $C^\infty(M_1) \to C^\infty(M_2)$. Finite Hilbert tensor products of these unitaries intertwine the product group actions, their restrictions, and arbitrary diagonal functions of the tuples of factor indices. They also induce Sobolev isomorphisms on the corresponding product manifolds with product metrics.

Proof. The representations U_i are strongly continuous: continuity on finite sums and approximation in $\mathcal{H}_i$, together with $\|U_i(g)\| \le 1$, give continuity on each vector. Let $P_{i,\eta}$ be the orthogonal projection onto $E_{i,\eta}$. The blockwise isometries give a unitary S with inverse $\widehat{\bigoplus}_\eta S_\eta^{-1}$ and $SP_{1,\eta} = P_{2,\eta}S$. The equality

$$\sum_\eta |f(\eta)|^2 \|S_\eta u_\eta\|^2 = \sum_\eta |f(\eta)|^2 \|u_\eta\|^2$$

gives $S\mathcal{D}_{1,f} = \mathcal{D}_{2,f}$ and $Sf_1 = f_2 S$ on $\mathcal{D}_{1,f}$. For a compact subgroup $K \le D$, Haar probability measure gives

$$S \int_K U_1(k)u \, dk = \int_K U_2(k)Su \, dk, \qquad u \in \mathcal{H}_1.$$

For a continuous homomorphism $\gamma : \mathbb{R} \to D$, let $A_{i,\gamma}$ be the self-adjoint generator, with convention $U_i(\gamma(t)) = e^{itA_{i,\gamma}}$. Write $\mathcal{D}(A)$ for the domain of an operator A. Stone's theorem [106, Theorems 5.1–5.2] and $SU_1(\gamma(t))S^{-1} = U_2(\gamma(t))$ give

$$S\mathcal{D}(A_{1,\gamma}) = \mathcal{D}(A_{2,\gamma}), \qquad SA_{1,\gamma}S^{-1} = A_{2,\gamma}.$$

Every vector has countable support in $\mathcal{A}$; its invariant subsum is separable, so the cited theorem applies on that subsum.

In the manifold case, put $\Delta_i = \Delta_{M_i}$ and let $\mathcal{H}_i^{\mathrm{fin}} = \bigoplus_{\eta\in\mathcal{A}} E_{i,\eta}$ be the algebraic direct sum. For $u \in \mathcal{H}_i^{\mathrm{fin}}$ and $s \in \mathbb{R}$, put

$$\|u\|_{s,\Delta_i}^2 = \sum_{\eta\in\mathcal{A}} (1 + \nu(\eta))^s \|u_\eta\|^2,$$

and let $\mathcal{H}_i^s$ be its completion. For an empty or zero-dimensional M_i, one has $\Delta_i = 0$ and $\mathcal{H}_i^s = H^s(M_i) = L^2(M_i)$. In positive dimension, the Laplace eigenspaces are finite dimensional and smooth [38, Theorem 17.1]; each is the orthogonal sum of the blocks with its eigenvalue, so every finite spectral sum belongs to $\mathcal{H}_i^{\mathrm{fin}}$. For integers $k \ge 0$, $\|u\|_{2k,\Delta_i} = \|(1 + \Delta_i)^k u\|_{L^2}$. For $k \ge 1$, the self-adjoint domain of $(1 + \Delta_i)^k$ is $H^{2k}(M_i)$ [38, §17.3], and the finite spectral sums form a core by the spectral domain formula [106, §3.1, equation (3.25)]. The elliptic estimate [38, Theorem 16.7], applied to $(1 + \Delta_i)^k$ for $k \ge 1$, gives equivalence with the $H^{2k}(M_i)$ norm; the lower-order L^2 term is bounded by $\|(1 + \Delta_i)^k u\|_{L^2}$, and the reverse bound follows from the order $2k$ of this differential operator. Thus $\mathcal{H}_i^{2k} = H^{2k}(M_i)$ with equivalent norms. For $k \ge 1$, interpolation with $A = (1 + \Delta_i)^k$ and $\theta = s/(2k)$ for $0 \le s \le 2k$, and then L^2 duality, give $\mathcal{H}_i^s = H^s(M_i)$ for all real s [104, §2, Proposition 2.2; §3, Propositions 3.1–3.2]. The map S preserves these spectral norms, so it extends to each H^s; the extensions agree on intersections by density of the common finite sums. Sobolev embedding [104, §3, Proposition 3.3] identifies $\bigcap_{k\ge0} H^{2k}(M_i)$ with $C^\infty(M_i)$, with its usual Fréchet topology.

For an integer $q \ge 1$, let $\mathcal{A}_j$ be the index set of the jth decomposition, $1 \le j \le q$. The tensor products of the summands are the summands of the Hilbert tensor product; their projections and unitaries are the corresponding tensor products, with the product group action. The same norm identity applies to every function of $(\eta_1, \dots, \eta_q)$, where $\eta_j \in \mathcal{A}_j$. On a product manifold the scalar Laplacian is the sum of the factor Laplacians. Let ν_j be the eigenvalue function of the jth factor; the product eigenvalue on the summand with indices $(\eta_1, \dots, \eta_q)$ is $\sum_{j=1}^q \nu_j(\eta_j)$. The corresponding spectral norms give the Sobolev isomorphisms on the product. □

2.2. **Global periods of the markings.** The fundamental groups and the maps from the maximal tori are given by [20, Chapter V, Theorem 7.1, p. 223]. For a loop $(A(t), z(t))$ in P_n, $0 \le t \le 1$, based at the identity, let $m, l \in \mathbb{Z}$ be the winding numbers of $\det A(t)$ and $z(t)$, respectively. These coordinates identify $\pi_1(P_n)$ with $\mathbb{Z}^2$. The maps induced by (2.1) are

$$(m, l) \longmapsto l \quad \text{in } \pi_1(G_n) = \mathbb{Z}, \qquad (m, l) \longmapsto m + l \pmod 2 \quad \text{in } \pi_1(H_n) = \mathbb{Z}/2. \tag{2.7}$$

The determinant of $i_n(A, z)$ is z, and $\mathrm{Sp}(n)$ is simply connected. Under $r(A \oplus z)$, the determinant generators of $\mathrm{U}(n)$ and $\mathrm{U}(1)$ give, respectively, the loops

$$t \longmapsto \mathrm{diag}(R(2\pi t), I_{2n}), \qquad t \longmapsto \mathrm{diag}(I_{2n}, R(2\pi t)), \qquad 0 \le t \le 1.$$

Each represents the nonzero element of $\pi_1(\mathrm{SO}(2n+2))$, including $\mathrm{SO}(4)$. Both maps in (2.7) are therefore onto.

For positive integers $r, n_1, \dots, n_r$ and positive real numbers $s_1, \dots, s_r$, write

$$P = \prod_{j=1}^{r} P_{n_j}, \qquad J_I = \prod_{j=1}^{r} J_{I,j}, \qquad J_{I,j} = \begin{cases} G_{n_j}, & j \notin I, \\ H_{n_j}, & j \in I, \end{cases} \qquad I \subseteq [r].$$

Let $Q_{P_{n_j}}$ be (2.3) with $n = n_j$. Equip J_I with the product metric whose jth factor has s_j times its reference metric. Under the product of the embeddings i_{n_j} or j_{n_j} specified by I, the pullback of this metric to P is $\bigoplus_{j=1}^{r} s_j Q_{P_{n_j}}$, independent of I.

3. Intrinsic spectral identity, bilateral defect, and maximal equivariance

Fix an integer $n \geq 1$ and put $N = 2n+1$, $G = G_n$, $H = H_n$, $P = P_n$, $i = i_n$, and $j = j_n$. For a finite-dimensional representation V, write χ_V for its character and $[V]$ for its class in the representation ring. For continuous homomorphisms $\iota_-, \iota_+ : P \to R$ into a compact connected Lie group R, define a unitary action on $L^2(R)$ by

$$T_{p,q} f(g) = f\big(\iota_-(p)^{-1} g \iota_+(q)\big), \qquad g \in R, \quad (p,q) \in P \times P.$$

On a matrix-coefficient space $C_\lambda(R)$, with irreducible character χ_λ, its character is

$$\overline{\chi_\lambda(\iota_-(p))}\, \chi_\lambda(\iota_+(q)).$$

On G both markings are i; on H both are j. Put

$$J_n' = \mathrm{SO}(2n) \times \mathrm{Sp}(n+1),$$

with metric $Q_{J_n'} = Q_D \oplus Q_C$. Its dimension is $N^2 + 2$, and $\pi_1(J_n') = \pi_1(\mathrm{SO}(2n))$, equal to $\mathbb{Z}$ for $n = 1$ and $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. In the real coordinates of the rotation two-planes, let $s = \mathrm{diag}(1, -1, 1, \dots, 1)$ be the real orthogonal matrix of order $2n$. Define $\sigma : \mathrm{SO}(2n) \to \mathrm{SO}(2n)$ by $\sigma(g) = sgs^{-1}$. Define $k_-, k_+ : P \to J_n'$ by

$$k_-(A, z) = \big(\sigma(r(A)), c(A \oplus z)\big), \qquad k_+(A, z) = \big(r(A), c(A \oplus z)\big).$$

The differentials $dk_\pm$ at the identity satisfy, for $u = (X, it), v = (Y, it') \in \mathfrak{u}(n) \oplus i\mathbb{R}$,

$$Q_{J_n'}(dk_\pm u, dk_\pm v) = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(XY) - \operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(XY) + tt' = Q_P(u, v).$$

For $n = 1$, σ is inversion on $\mathrm{SO}(2)$. The isometry $(g, h) \mapsto (\sigma(g), h)$ exchanges k_- and k_+. On J_n' the action T has $\iota_- = k_-$ and $\iota_+ = k_+$.

For $R = G, H, J_n'$ and a dominant weight λ of R, let $M_R(\lambda)$ be the multiset of the absolute coordinates of $\lambda + \rho_R$, with both factors concatenated on a product. For a multiset M of N nonnegative integers, set

$$\mathcal{V}_R(M) = \bigoplus_{M_R(\lambda) = M} C_\lambda(R) \subset L^2(R). \tag{3.1}$$

This is a finite-dimensional $P \times P$ module: a fixed absolute multiset admits only finitely many signed ordered tuples. An empty sum is zero. Each summand has dimension $(\dim V_\lambda)^2$.

3.1. **Finite Weyl determinants.** Choose $D \in \mathbb{Z}_{\geq 0}$ and work over the polynomial ring $\mathbb{C}[a_0, \dots, a_D]$ in independent variables. Put

$$\theta(t) = \sum_{k=-D}^{D} a_{|k|} e^{ikt}, \qquad E(t,s) = \theta(t-s) + \theta(t+s), \qquad O(t,s) = \theta(t-s) - \theta(t+s), \qquad t, s \in \mathbb{R}.$$

For any of the reference groups R or their products, let d be its rank and let $x, y \in \mathbb{R}^d$ be maximal-torus angle vectors; $\chi_\lambda(x)$ denotes evaluation at the corresponding torus element. Put

$$E[x, y] = (E(x_i, y_j))_{1 \leq i,j \leq d}, \qquad O[x, y] = (O(x_i, y_j))_{1 \leq i,j \leq d}.$$

Define the finite character polynomial

$$F_R(x,y)=\sum_{\substack{\lambda\ \text{dominant}\\ |\lambda_j+\rho_{R,j}|\le D\ \text{for all } j}}\chi_\lambda(x)\overline{\chi_\lambda(y)}\prod_j a_{|\lambda_j+\rho_{R,j}|}.$$

The bar conjugates the character value and leaves the formal variables fixed. The same notation applies to group arguments. For a product group, F_R is the product of the corresponding factor polynomials. For $x\in\mathbb{R}^N$ in type A and $x\in\mathbb{R}^m$ in types C,D, with an integer $m\ge 1$, define

$$\delta_{A,N}(x)=\prod_{i<j}2i\sin((x_i-x_j)/2), \tag{3.2}$$

$$\delta_{D,m}(x)=\prod_{i<j}2(\cos x_i-\cos x_j), \tag{3.3}$$

$$\delta_{C,m}(x)=(2i)^m\prod_j \sin x_j\,\delta_{D,m}(x). \tag{3.4}$$

In particular, $\delta_{D,1}=1$. These are the Weyl denominators $\prod_{\alpha>0}2i\sin(\alpha(x)/2)$ for the positive roots in (2.4) [41, Corollary 25.5, p. 112]. The Weyl character formula [20, Chapter VI, Theorem 1.7, p. 242] gives

$$\begin{aligned}F_{\mathrm{U}(N)}(x,y)&=\frac{\det[\theta(x_i-y_j)]}{\delta_{A,N}(x)\overline{\delta_{A,N}(y)}},\\ F_{\mathrm{Sp}(m)}(x,y)&=\frac{\det O[x,y]}{\delta_{C,m}(x)\overline{\delta_{C,m}(y)}},\\ F_{\mathrm{SO}(2m)}(x,y)&=\frac{\det E[x,y]+\det O[x,y]}{2\delta_{D,m}(x)\delta_{D,m}(y)}.\end{aligned} \tag{3.5}$$

The displayed quotients are taken where their denominators are nonzero.

For the unitary formula, finite Cauchy–Binet gives a sum over $k_1>\cdots>k_N$ in $[-D,D]\cap\mathbb{Z}$, with terms

$$\Big(\prod_j a_{|k_j|}\Big)\det[e^{ik_ix_j}]\overline{\det[e^{ik_iy_j}]}.$$

Reversing the selected columns changes both determinants by the same sign. Each decreasing tuple therefore occurs once. For the symplectic formula,

$$O(t,s)=4\sum_{k=1}^{D}a_k\sin(kt)\sin(ks).$$

Cauchy–Binet sums over $k_1>\cdots>k_m>0$, with alternants $\det[2i\sin(k_ix_j)]$ and their conjugates at y. Division by the corresponding Weyl denominators gives the first two formulas in (3.5).

For $m\ge 2$, the orthogonal Weyl group consists of signed permutations with an even number of sign changes. Its alternant vanishes when two coordinates have the same absolute value. A regular orbit has a unique representative $k_1>\cdots>k_{m-1}>|k_m|$ and trivial stabilizer, also when $k_m=0$: changing the sign of the zero coordinate alone is not in the Weyl group. For $k=(k_1,\dots,k_m)\in\mathbb{Z}^m$ with $k_1>\cdots>k_m\ge 0$, put

$$C_k(x)=\det[2\cos(k_ix_j)],\qquad S_k(x)=\det[2i\sin(k_ix_j)].$$

If $k_m>0$, the alternants for the two signs of the last coordinate are $(C_k+S_k)/2$ and $(C_k-S_k)/2$. The sum of their bilateral products is

$$\tfrac12\big(C_k(x)C_k(y)+S_k(x)\overline{S_k(y)}\big).$$

If $k_m=0$, there is one alternant, $C_k/2$, and its product is $C_k(x)C_k(y)/4$. On the other hand,

$$E(t,s)=2a_0+\sum_{k=1}^{D}a_k(2\cos kt)(2\cos ks),\qquad O(t,s)=\sum_{k=1}^{D}a_k(2i\sin kt)\overline{(2i\sin ks)}.$$

In Cauchy–Binet for $E[x,y]$, the constant row 2 has weight $a_0/2$. Thus $(\det E[x,y]+\det O[x,y])/2$ has precisely the preceding coefficients, including those with a zero coordinate. Division by the two Weyl denominators gives the orthogonal formula. For $m=1$, its numerator divided by two is $\frac{1}{2}(E(x,y)+O(x,y))=\theta(x-y)$, the full character sum of SO(2).

Lemma 3.1. *For complex matrices A,B of order n, column vectors $c,d,e,f\in\mathbb{C}^n$, and scalars $g,h\in\mathbb{C}$, put*

$$\mathcal{M}=\begin{pmatrix} A & B & c\\ B & A & d\\ e^t & f^t & g\end{pmatrix},\quad \mathcal{E}=\begin{pmatrix} A+B & c+d\\ (e+f)^t & g+h\end{pmatrix},\quad \mathcal{O}=\begin{pmatrix} A-B & c-d\\ (e-f)^t & g-h\end{pmatrix}.$$

Then

$$2\det\mathcal{M}=\det(A-B)\det\mathcal{E}+\det(A+B)\det\mathcal{O}. \tag{3.6}$$

The same identity holds for matrices over $\mathbb{C}[a_0,\dots,a_D]$.

Proof. For

$$Q=\begin{pmatrix} I_n/\sqrt{2} & I_n/\sqrt{2} & 0\\ I_n/\sqrt{2} & -I_n/\sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix},\qquad Q^tQ=I_{2n+1},$$

one has

$$Q^t\mathcal{M}Q=\begin{pmatrix} A+B & 0 & (c+d)/\sqrt{2}\\ 0 & A-B & (c-d)/\sqrt{2}\\ (e+f)^t/\sqrt{2} & (e-f)^t/\sqrt{2} & g\end{pmatrix}.$$

Write $\operatorname{adj}(A)$ for the classical adjugate of a square matrix A. The bordered determinant formula [109, §3.7.1, Theorem 3.9, p. 47] gives the determinant above as

$$\begin{aligned} &g\det(A+B)\det(A-B)\\ &\quad-\tfrac{1}{2}\det(A-B)(e+f)^t\operatorname{adj}(A+B)(c+d)\\ &\quad-\tfrac{1}{2}\det(A+B)(e-f)^t\operatorname{adj}(A-B)(c-d).\end{aligned}$$

Expanding the right side of (3.6) by the same formula gives twice this expression; the terms containing h cancel. All expressions are polynomials in the matrix entries. The identity therefore includes singular matrices and remains valid over $\mathbb{C}[a_0,\dots,a_D]$. □

Let $u,v\in\mathbb{R}^n$ and $a,b\in\mathbb{R}$. Put $x=(u,-u,a)$, $y=(v,-v,b)$, and

$$E_n=E[u,v],\quad O_n=O[u,v],\qquad E_{n+1}=E[(u,a),(v,b)],\quad O_{n+1}=O[(u,a),(v,b)].$$

In Lemma 3.1, take

$$\begin{aligned} A_{ij}&=\theta(u_i-v_j), & B_{ij}&=\theta(u_i+v_j),\\ c_i&=\theta(u_i-b), & d_i&=\theta(u_i+b),\\ e_j&=\theta(a-v_j), & f_j&=\theta(a+v_j),\\ g&=\theta(a-b), & h&=\theta(a+b).\end{aligned}$$

Since θ is even, (3.6) becomes

$$\det[\theta(x_i-y_j)]=\tfrac{1}{2}\big(\det O_n\det E_{n+1}+\det E_n\det O_{n+1}\big). \tag{3.7}$$

When $a=0$, the last row of O_{n+1} vanishes, so

$$\det[\theta(x_i-y_j)]=\tfrac{1}{2}\det O[u,v]\det E[(u,0),(v,b)],\qquad x=(u,-u,0),\quad y=(v,-v,b). \tag{3.8}$$

The denominators satisfy

$$\delta_{A,N}(u,-u,a)=(-1)^{n(n+1)/2}\delta_{C,n}(u)\delta_{D,n+1}(u,a). \tag{3.9}$$

Indeed, for $\widetilde{x} = (u_1, -u_1, \dots, u_n, -u_n, a)$,

$$\begin{aligned}\delta_{A,N}(\widetilde{x}) &= (2i)^n \prod_i \sin u_i \prod_{i<j} 4(\cos u_i - \cos u_j)^2 \prod_i 2(\cos a - \cos u_i) \\ &= (-1)^n \delta_{C,n}(u)\delta_{D,n+1}(u,a).\end{aligned}$$

The permutation from $\widetilde{x}$ to $(u, -u, a)$ has sign $(-1)^{n(n-1)/2}$, giving (3.9).

An–Yu–Yu proved equality of dimension data for embeddings of G_n and H_n into $\mathrm{SU}(4n+2)$ [6, Theorem 1.5(1)]. Their determinant identity [6, Lemma 5.2, equation (5.1)] is the odd symmetric Toeplitz factorization of [109, §4.5.2, equation (4.5.9), p. 88]. The paired highest-weight factorizations of [116, Proposition 2.5] and [117, arXiv version 2, Proposition 2.5] concern the Weyl-polynomial expressions for twisted dimension data [117, arXiv version 2, §§2.1–2.2]. Heat determinants on compact classical groups are treated in [31, §7.2].

3.2. **The bilateral correction.** Let $\alpha, \beta : P \times P \to \mathrm{U}(1)$ be the characters $\alpha(A, z; B, w) = z$ and $\beta(A, z; B, w) = w$. Identify each with its one-dimensional representation, and put

$$\omega = \alpha\beta^{-1} + \alpha^{-1}\beta - \alpha\beta - \alpha^{-1}\beta^{-1} \quad \text{in } R(P \times P). \tag{3.10}$$

At $z = e^{ia}$, $w = e^{ib}$, its character is $4 \sin a \sin b$.

Theorem 3.2. *For every multiset M of N nonnegative integers,*

$$[\mathcal{V}_G(M)] - [\mathcal{V}_H(M)] = \omega[\mathcal{V}_{J_n'}(M)] \quad \text{in } R(P \times P). \tag{3.11}$$

Proof. Let $u, v \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, and

$$p = (\mathrm{diag}(e^{iu_1}, \dots, e^{iu_n}), e^{ia}), \qquad q = (\mathrm{diag}(e^{iv_1}, \dots, e^{iv_n}), e^{ib}).$$

Retain the matrices $E_n, O_n, E_{n+1}, O_{n+1}$ above, and put

$$\mathcal{D} = \delta_{C,n}(u)\overline{\delta_{C,n}(v)}\delta_{D,n+1}(u,a)\delta_{D,n+1}(v,b).$$

Equations (3.5), (3.7), and (3.9) give, wherever their denominators are nonzero,

$$F_G(i(p), i(q)) - F_H(j(p), j(q)) = \frac{(\det E_n - \det O_n)\det O_{n+1}}{2\mathcal{D}}. \tag{3.12}$$

Put $u^\sigma = (-u_1, u_2, \dots, u_n)$. Replacing u by u^σ leaves E_n and $\delta_{D,n}(u)$ unchanged and negates the first row of O_n. Hence

$$F_{J_n'}(k_-(p), k_+(q)) = \frac{(\det E_n - \det O_n)\det O_{n+1}}{2\delta_{D,n}(u)\delta_{D,n}(v)\delta_{C,n+1}(u,a)\overline{\delta_{C,n+1}(v,b)}}.$$

For $n = 1$, the reflected circle factor is $F_{\mathrm{SO}(2)}(-u, v) = \theta(u + v)$, so the same equality holds. By (3.4),

$$\frac{\delta_{C,n+1}(u,a)\delta_{D,n}(u)}{\delta_{C,n}(u)\delta_{D,n+1}(u,a)} = 2i \sin a. \tag{3.13}$$

The ratio of the auxiliary denominator to $2\mathcal{D}$ is $(2i \sin a)\overline{(2i \sin b)} = 4 \sin a \sin b$. Consequently

$$F_G(i(p), i(q)) - F_H(j(p), j(q)) = \omega(p, q) F_{J_n'}(k_-(p), k_+(q)).$$

The excluded angles satisfy finitely many nontrivial integral angle relations modulo 2π; their complement is dense. Each coefficient is a continuous character sum, so the equality holds for all angles. Every element of P is conjugate to a torus element by an element of $\mathrm{U}(n) \times \{1\}$. The markings respect conjugation, so independent diagonalization of p and q extends the identity to all $p, q \in P$. Simultaneous inversion of p, q gives the characters for the action T, since $\omega(p^{-1}, q^{-1}) = \omega(p, q)$. For a given M, choose $D \geq \max M$ and extract the coefficient of $\prod_{m \in M} a_m$, counting repetitions. Character orthogonality [41, Corollary 35.12] then gives (3.11). □

Put $E^+ = \alpha\beta^{-1} \oplus \alpha^{-1}\beta$ and $E^- = \alpha\beta \oplus \alpha^{-1}\beta^{-1}$. Complete reducibility and (3.11) give

$$\mathcal{V}_G(M) \oplus E^- \otimes \mathcal{V}_{J_n'}(M) \simeq \mathcal{V}_H(M) \oplus E^+ \otimes \mathcal{V}_{J_n'}(M) \tag{3.14}$$

as unitary $P \times P$ modules.

3.3. **The intrinsic identity.** Let $L_n = \mathrm{U}(n)$ act on the left through $A \mapsto i(A, 1)$ or $A \mapsto j(A, 1)$.

Theorem 3.3 (The intrinsic identity). *For every multiset M of N nonnegative integers, the spaces $\mathcal{V}_G(M)$ and $\mathcal{V}_H(M)$ are equivalent unitary $L_n \times P$ modules. In particular, corresponding scalar eigenspaces are equivalent as right-P modules. There is a unitary $S : L^2(G) \to L^2(H)$ sending the constant function 1 on G to the constant function 1 on H and intertwining these actions and the Laplacians, including their self-adjoint domains. A common positive scaling of the metrics preserves these intertwinings.*

Proof. Restrict (3.11) along

$$L_n \times P \longrightarrow P \times P, \qquad (A, q) \longmapsto ((A, 1), q).$$

On this subgroup $\alpha = 1$ and $\omega = 0$. Equality of the two representation classes gives the finite-dimensional unitary equivalences. The squared norms of the half-sums of positive roots satisfy

$$|\rho_A|^2 = |\rho_{C,n}|^2 + |\rho_{D,n+1}|^2 = 2 \sum_{j=1}^{n} j^2. \tag{3.15}$$

Thus each M gives the same Laplace eigenvalue on both sides. Choose a unitary on each multiset component and apply Lemma 2.4. The zero eigenspaces are the constants, whose normalized generators can be matched. Scaling by $s > 0$ replaces both Laplacians by $s^{-1}\Delta$. □

3.4. **The common central algebra.** For $x = (x_1, \dots, x_N) \in \mathbb{C}^N$ and an integer $1 \le r \le N$, set

$$p_r(x) = \sum_{a=1}^{N} x_a^{2r}, \qquad \kappa_r = 2 \sum_{a=1}^{n} a^{2r}, \qquad d_r = (n+1)^{2r} - n^{2r}.$$

For $R = G, H, J_n'$, the shifted Harish–Chandra isomorphism of its complexified Lie algebra, in the Cartan coordinates of (2.4), associates to $p_r - p_r(\rho_R)$ a central differential operator $Z_{R,r}$ on $C^\infty(R) \subset L^2(R)$, of order at most $2r$. Its highest-weight eigenvalue is

$$z_{R,r}(\lambda) = p_r(\lambda + \rho_R) - p_r(\rho_R), \qquad R = G, H, J_n'. \tag{3.16}$$

This convention includes reductive algebras [42, Theorem 14.1(ii)–(v) and Remark 14.2]. On $C^\infty(R)$, $Z_{R,1} = \Delta_R$; moreover, $p_r(\rho_G) = p_r(\rho_H) = \kappa_r$, and $p_r(\rho_{J_n'}) = \kappa_r + d_r$. Put $d = (d_1, \dots, d_N)$. For $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{R}^N$, define

$$E_R(\eta) = \{f \in C^\infty(R) : Z_{R,r} f = \eta_r f \text{ for } 1 \le r \le N\}.$$

It is zero when η is not a joint eigenvalue, and is finite dimensional since $E_R(\eta) \subset E_{\eta_1}(R)$.

Theorem 3.4. *The operators $Z_{R,r}$ on $C^\infty(R) \subset L^2(R)$ are essentially self-adjoint, and their closures strongly commute. With the same notation for these closures, the joint eigenspaces of $(Z_{G,r})_{r=1}^N$ and $(Z_{H,r})_{r=1}^N$ are equivalent $L_n \times P$ modules. The unitary S may be chosen to intertwine all their joint Borel functional calculi, with their domains, and to map $C^\infty(G)$ onto $C^\infty(H)$. For the full bilateral actions,*

$$E_G(\eta) \oplus E^- \otimes E_{J_n'}(\eta - d) \simeq E_H(\eta) \oplus E^+ \otimes E_{J_n'}(\eta - d), \qquad \eta \in \mathbb{R}^N. \tag{3.17}$$

Proof. For every dominant weight λ,

$$|z_{R,r}(\lambda)| \le \left(c_R(\lambda) + |\rho_R|^2\right)^r + p_r(\rho_R) \le C_{R,r}(1 + c_R(\lambda))^r$$

for a constant $C_{R,r} > 0$. Write f_λ for the orthogonal projection of $f \in L^2(R)$ onto $C_\lambda(R)$. The maximal diagonal operator with these eigenvalues is

$$A_{R,r} f = \sum_\lambda z_{R,r}(\lambda) f_\lambda,$$

$$\mathcal{D}(A_{R,r}) = \left\{ f \in L^2(R) : \sum_\lambda |z_{R,r}(\lambda)|^2 \|f_\lambda\|^2 < \infty \right\}.$$

It is self-adjoint, with the finite spectral sums as a core, by the diagonal spectral domain formula of Lemma 2.4. For $f \in C^\infty(R)$, its Laplace spectral truncations converge to f in $H^{2r}(R)$. Since $Z_{R,r}$ has order at most $2r$, their images converge to $Z_{R,r}f$ in $L^2(R)$. The displayed estimate gives convergence also in the graph norm of $A_{R,r}$. Hence $Z_{R,r}f = A_{R,r}f$ on $C^\infty(R)$. The finite spectral sums belong to $C^\infty(R)$, so the closure of $Z_{R,r}$ is $A_{R,r}$. All these diagonal operators have the same orthogonal coefficient-space projections; their spectral projections therefore commute.

Let e_r be the rth elementary symmetric polynomial in $x_1^2, \dots, x_N^2$, with $e_0 = 1$. Newton's identities

$$re_r = \sum_{h=1}^{r} (-1)^{h-1} e_{r-h} p_h, \qquad 1 \le r \le N,$$

determine these polynomials from $p_1, \dots, p_N$. Applied to the real shifted weights, the first N even power sums therefore determine the absolute-coordinate multiset. Theorem 3.3 and (3.14) give the stated joint-space isomorphisms, with eigenvalue vector $\eta - d$ on J_n'. The unitary obtained by summing the multiset equivalences intertwines every Borel function of the joint eigenvalue vector by Lemma 2.4. Since $Z_{R,1} = \Delta_R$, the same lemma gives $S(C^\infty(G)) = C^\infty(H)$. □

Corollary 3.5. *Let $\ell, n_1, \dots, n_\ell$ be positive integers, put $P = \prod_{i=1}^{\ell} P_{n_i}$, and let $K \le P$ be closed. For $I \subseteq [\ell]$, let $J_I = \prod_{i=1}^{\ell} J_{I,i}$ have its reference product metric. Let $\iota_I : P \to J_I$ be the product of i_{n_i} for $i \notin I$ and j_{n_i} for $i \in I$. Let $\tau : K \to \mathrm{U}(V_\tau)$ be a finite-dimensional unitary representation. On $L^2(J_I)$ let $k \in K$ act by $f(g) \mapsto f(g\iota_I(k))$, and on $L^2(J_I) \otimes V_\tau$ take the diagonal action. For $1 \le i \le \ell$ and $1 \le r \le 2n_i + 1$, let $Z_{J_I,i,r}$ act as $Z_{J_{I,i},r}$ on the ith factor and as the identity on the other factors. On $(L^2(J_I) \otimes V_\tau)^K$, these operator families have the same joint spectral multiplicities for every $I \subseteq [\ell]$. There are unitary intertwiners preserving every joint functional-calculus domain. For fixed positive scales $s_1, \dots, s_\ell$, the joint multiplicities and functional-calculus domains are likewise preserved with factor metrics $s_i Q_{J_{I,i}}$ and operators $s_i^{-r} Z_{J_I,i,r}$.*

Proof. Tensor the joint unitaries of Theorem 3.4 with 1_{V_τ} and restrict to the K invariants by Lemma 2.4. In orthonormal dual Cartan coordinates, the change $Q_{J_{I,i}} \mapsto s_i Q_{J_{I,i}}$ multiplies each coordinate by $s_i^{-1/2}$. Both p_r and its value at $\rho_{J_{I,i}}$ are multiplied by s_i^{-r}, giving the same eigenvalue vectors for every I. □

Write $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$, and identify the Cartan coordinate spaces of G and H with $\mathbb{C}^N$ as above. Let S_N permute all coordinates. Let $W(C_n)$ act by arbitrary signed permutations on the first n coordinates, and let $W(D_{n+1})$ act on the last $n+1$ coordinates by signed permutations with an even number of sign changes. Put $W_{BC,N} = \{\pm 1\}^N \rtimes S_N$. The intersection of the two Harish–Chandra invariant algebras is

$$\mathbb{C}[x]^{S_N} \cap \mathbb{C}[x]^{W(C_n)\times W(D_{n+1})} = \mathbb{C}[p_1, \dots, p_N]. \tag{3.18}$$

Indeed, $W(C_n)$ contains a single-coordinate sign change; conjugation by S_N gives every such sign change. The two groups therefore generate $W_{BC,N}$. Its invariants are the symmetric polynomials in the squares, which Newton's identities express in $p_1, \dots, p_N$.

Corollary 3.6. *The inclusions of $\mathbb{C}[p_1, \dots, p_N]$ into the two invariant algebras in* (3.18) *are finite, of generic degrees*

$$2^N \quad \text{and} \quad 2\binom{N}{n},$$

respectively. These are the numbers of points in $\mathbb{C}^N/S_N$ and $\mathbb{C}^N/(W(C_n) \times W(D_{n+1}))$ above the $W_{BC,N}$ orbit of $x \in \mathbb{C}^N$ when $x_a \ne 0$ and $x_a^2 \ne x_b^2$ for $a \ne b$. Under the Harish–Chandra isomorphism, these points are full central characters above the common even-power-sum character.

Proof. The signed-permutation actions are faithful and finite. The polynomial ring $\mathbb{C}[x]$ is finite over the Noetherian ring $\mathbb{C}[x]^{W_{BC,N}}$, and its intermediate invariant subrings are finite over that ring [11, Corollary 1.2.2 and Theorem 1.3.1]. Their fraction-field degrees are the subgroup indices [11, Proposition 1.1.1]. The group orders are $2^N N!$, $N!$, and $(2^n n!)(2^n(n+1)!)$, giving the stated degrees. The stipulated orbit has trivial stabilizer, so its subgroup orbits are counted by the same indices. □

3.5. **Heat traces and maximal subgroups.** Let $t > 0$, $p = (A, z) \in P$, and $q = (B, w) \in P$. Substitute $a_k = e^{-tk^2}$ in the finite identities and let $D \to \infty$. The Gaussian estimate preceding (2.6) gives

$$B_G(t; i(p), i(q)) - B_H(t; j(p), j(q)) = 4 \operatorname{Im} z \operatorname{Im} w \, e^{-Nt} B_{J_n'}(t; k_-(p), k_+(q)). \tag{3.19}$$

Here $d_1 = N$. All $Z_{R,r}$ are nonnegative. Among $2n + 1$ distinct integers, $\{-n, \dots, n\}$ minimizes every even power sum. Among m distinct positive integers the minimizing set is $\{1, \dots, m\}$, and among m distinct nonnegative integers it is $\{0, \dots, m - 1\}$. These are the shifted tuples at zero highest weight for the unitary, symplectic, and orthogonal factors. For $\mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}^N$ with $t_1 > 0$ and $t_r \geq 0$ when $r > 1$,

$$0 \leq \exp\Big(- \sum_r t_r z_{R,r}(\lambda) \Big) \leq e^{-t_1 c_R(\lambda)}.$$

For $R = G, H, J_n'$ and $a, b \in R$, the joint heat trace

$$B_R(\mathbf{t}; a, b) = \operatorname{Tr}_{L^2(R)} \Big(F_{a,b}^* \exp\big(- \sum_r t_r Z_{R,r} \big) \Big)$$

therefore converges absolutely. Evaluating (3.17) at (p^{-1}, q^{-1}) and summing gives

$$\begin{aligned} &B_G(\mathbf{t}; i(p), i(q)) - B_H(\mathbf{t}; j(p), j(q)) \\ &\qquad = \omega(p, q) e^{-\sum_r t_r d_r} B_{J_n'}(\mathbf{t}; k_-(p), k_+(q)). \end{aligned}$$

Lemma 3.7. *Let X be a nonempty closed connected Riemannian manifold. For every isometry $F : X \to X$ and $t > 0$, $\operatorname{Tr}(F^* e^{-t\Delta_X}) > 0$.*

Proof. In dimension zero, X is a point and the trace is 1. Otherwise the scalar heat kernel, relative to Riemannian volume, satisfies $K_t(x, y) > 0$ for $t > 0$ and $x, y \in X$ [51, Corollary 8.12, p. 229]. Hence

$$\operatorname{Tr}(F^* e^{-t\Delta_X}) = \int_X K_t(Fx, x) \, d \operatorname{vol}_X(x) > 0. \qquad \square$$

Proposition 3.8. *The first positive eigenvalue of G and H is N, with multiplicity $2N^2 + 2$. For the action T, its character difference is*

$$\chi_{E_N(G)} - \chi_{E_N(H)} = \omega. \tag{3.20}$$

The defining representations are the unique irreducible representations of $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n + 2)$ with Casimir N; their dimensions are $2n$ and $2n + 2$.

Proof. For G, the shifted tuple is a set S of N distinct integers. If k entries of $S_0 = \{-n, \dots, n\}$ are replaced, then

$$\sum_{s \in S} s^2 - \sum_{s \in S_0} s^2 \geq k((n + 1)^2 - n^2) = kN.$$

Equality at N permits one removal from $\{-n, n\}$ and one insertion from $\{-n - 1, n + 1\}$. The four representations have highest weights

$$(1, 0, \dots, 0), \quad (0, \dots, 0, -1), \quad (1, \dots, 1), \quad (-1, \dots, -1).$$

They are the defining representation, its dual, determinant, and inverse determinant; their dimensions are $N, N, 1, 1$. For $\mathrm{Sp}(n)$, the shifted coordinates are distinct positive integers. The same inequality compares them with $\{1, \dots, n\}$ and permits only $n \mapsto n + 1$ at energy N. For $\mathrm{SO}(2n + 2)$, their absolute values are distinct nonnegative integers; comparison with $\{0, \dots, n\}$ again permits only that replacement. The zero last coordinate has a single sign. In both cases subtraction of ρ gives highest weight $(1, 0, \dots, 0)$. The first product eigenspace is the direct sum of the two defining coefficient spaces. Its dimension is $(2n)^2 + (2n + 2)^2 = 2N^2 + 2$. Sum (3.11) over the multisets of scalar energy N. By (3.16), the auxiliary energy is $N - d_1 = 0$. Its eigenspace consists of the constants, so its character is 1. This gives (3.20). $\square$

Theorem 3.9. *For a closed subgroup $K \le P \times P$, the following conditions are equivalent:*

(1) For every M, the modules $\mathcal{V}_G(M)$ and $\mathcal{V}_H(M)$ are equivalent after restriction to K.

(2) All corresponding scalar eigenspaces are equivalent as K modules.

(3) The eigenspaces at N are equivalent as K modules.

(4) For some $t > 0$, $B_G(t; i(p), i(q)) = B_H(t; j(p), j(q))$ for every $(p, q) \in K$.

(5) $K \subset \ker \alpha^2$ or $K \subset \ker \beta^2$.

The two kernels in (5) are the maximal closed subgroups with these properties. The maximal connected subgroups with these properties are $\ker \alpha$ *and* $\ker \beta$*, each of dimension $2n^2 + 1$. Their intersections with the maximal torus $(\mathbb{T}^n \times \mathbb{T})^2$ have dimension $2n + 1$.*

Proof. Condition (1) implies (2), and (2) implies both (3) and (4). By (3.20), condition (3) makes ω vanish on K. By Lemma 3.7 and (3.19), condition (4) has the same consequence. Thus $K \subset \ker \alpha^2 \cup \ker \beta^2$. A group is not the union of two proper subgroups, so $K \subset \ker \alpha^2$ or $K \subset \ker \beta^2$. Conversely, (5) makes the right side of (3.11) zero on K, giving (1). The two kernels are incomparable, and every subgroup satisfying (1)–(5) lies in one of them. They are therefore maximal. A connected subgroup has connected image in $\{1, -1\}$ and hence trivial image. The connected kernels are the products obtained by omitting one of the two circle factors, giving the stated dimensions. □

Corollary 3.10. *For every closed $K \le P \times P$ and every $\lambda \in \mathbb{R}$,*

$$[E_\lambda(G)|_K] - [E_\lambda(H)|_K] = \omega|_K [E_{\lambda - N}(J_n')|_K]. \tag{3.21}$$

As before, an absent eigenspace is zero. In $R(K)$, the ideal generated by these differences is $(\omega|_K)$, already generated by the difference at N.

Proof. Sum (3.11) over the finitely many multisets of scalar energy λ. The auxiliary energy is $\lambda - N$. Every difference is a multiple of $\omega|_K$, and the difference at N equals $\omega|_K$ by (3.20). □

3.6. **Finite quotients and their flat bundles.** Let $q \in \mathbb{Z}_{\ge 1}$ and $u_i, v_i, A, B \in \mathbb{Z}$ for $1 \le i \le n$. Let K be generated by the element of $(\mathbb{T}^n \times \mathbb{T})^2$ with angles

$$\frac{2\pi}{q}(u_1, \dots, u_n, A; v_1, \dots, v_n, B) \pmod{2\pi}.$$

Condition (5) of Theorem 3.9 is then

$$2A \equiv 0 \pmod q \quad \text{or} \quad 2B \equiv 0 \pmod q. \tag{3.22}$$

For odd q, this is $A \equiv 0 \pmod q$ or $B \equiv 0 \pmod q$.

Set $\iota_G = i$ and $\iota_H = j$. Let $K \le P \times P$ be finite and act freely on both G and H by

$$k \cdot g = \iota_R(p_-) g \iota_R(p_+)^{-1}, \qquad R = G, H, \quad g \in R, \quad k = (p_-, p_+) \in K.$$

Give $K\backslash R$ the quotient Riemannian metric. For a finite-dimensional unitary representation $\tau : K \to \mathrm{U}(V_\tau)$, define the Hermitian bundle $\mathcal{E}_{R,\tau} = (R \times V_\tau)/K$ over $K\backslash R$ by

$$k(g, v) = (k \cdot g, \tau(k)v).$$

The trivial connection on $R \times V_\tau$ descends to a flat Hermitian connection ∇ on $\mathcal{E}_{R,\tau}$. Its smooth sections are the functions $f \in C^\infty(R, V_\tau)$ satisfying $f(k \cdot g) = \tau(k) f(g)$. Write $E_\lambda(\mathcal{E}_{R,\tau})$ for the λ-eigenspace of the nonnegative connection Laplacian $\nabla^*\nabla$. The lifted operator is $\Delta_R \otimes 1_{V_\tau}$, so

$$E_\lambda(\mathcal{E}_{R,\tau}) \simeq (E_\lambda(R) \otimes V_\tau)^K,$$

where the action on the right is $T \otimes \tau$. For irreducible τ,

$$\dim E_\lambda(\mathcal{E}_{R,\tau}) = \dim \mathrm{Hom}_K(V_\tau^*, E_\lambda(R)).$$

Since τ^* ranges over all irreducible representations, Theorem 3.9 gives the equivalence of: equality of the spectra of $\mathcal{E}_{G,\tau}$ and $\mathcal{E}_{H,\tau}$ for every irreducible τ; equality of their multiplicities at N for every irreducible τ; and condition (5) of that theorem. For the cyclic groups above, the last condition is (3.22).

4. Descent, canonical tensors, and cubic Clifford operators

Let J be a compact connected Lie group, let $K \leq J$ be closed, and write $\mathfrak{j} = \mathrm{Lie}(J)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. Fix a positive $\mathrm{Ad}(J)$-invariant inner product Q on $\mathfrak{j}$, and put $\mathfrak{m} = \mathfrak{k}^{\perp_Q}$. Write $\mathrm{pr}_{\mathfrak{k}}$ and $\mathrm{pr}_{\mathfrak{m}}$ for the orthogonal projections, and $[X, Y]_{\mathfrak{a}} = \mathrm{pr}_{\mathfrak{a}}[X, Y]$ for $\mathfrak{a} = \mathfrak{k}, \mathfrak{m}$. The quotient map $p : J \to J/K$ is a Riemannian submersion for the bi-invariant metric on J and the normal metric on J/K. Let $L_g(h) = gh$, and write $\theta_g = (dL_{g^{-1}})_g$ for the left Maurer–Cartan form. The canonical principal connection is $\mathrm{pr}_{\mathfrak{k}}\,\theta$; its associated connections are denoted by ∇^0. For $X \in \mathfrak{j}$ and $f \in C^\infty(J)$, set

$$R_X f(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp(tX)).$$

The same notation denotes componentwise differentiation of vector-valued functions. For a closed Riemannian manifold X, put $E_\lambda(X) = \ker(\Delta_X - \lambda)$, with $E_\lambda(X) = 0$ when λ is absent from the spectrum. Pointwise Hermitian inner products are linear in the first argument. Write $L^2(J)^K$ for the scalar functions invariant under right translation by K.

Lemma 4.1. *For Riemannian probability measures, pullback identifies $L^2(J/K)$ with $L^2(J)^K$, and identifies their closed Dirichlet forms and Laplacians, including their domains. In particular,*

$$\Delta_J(f \circ p) = (\Delta_{J/K} f) \circ p, \qquad f \in C^\infty(J/K). \tag{4.1}$$

Proof. The quotient map has smooth local sections, without a connectedness assumption on K [41, Theorem 3.1(i)]. Every component of a fibre is totally geodesic: for left-invariant fields, $\nabla_X Y = [X, Y]/2$, and $\mathfrak{k}$ is a Lie subalgebra. The submersion identity for minimal fibres [47, Lemma 1.3] gives (4.1).

Fibre integration makes p^* an isometry for probability measures. Local sections and (4.1) give

$$p^* E_\lambda(J/K) = E_\lambda(J)^K.$$

Right translations preserve every eigenspace of Δ_J. The orthogonal sum of the spaces on the right is therefore dense in $L^2(J)^K$. Lemma 2.4, applied to λ and $\sqrt{\lambda}$, gives the operator and form domains. □

Proposition 4.2. *Let a compact Lie group D act freely and isometrically on a closed Riemannian manifold X. Let $B = X/D$ have the quotient metric, let $\pi : X \to B$ be the quotient map, and denote Riemannian volume on X and B by dv_X and dv_B, respectively. Set $v(b) = \mathrm{vol}(\pi^{-1}(b))$ for $b \in B$. The restriction of Δ_X to D-invariant functions is unitarily equivalent to the weighted Laplacian on $L^2(B, v\,dv_B)$,*

$$L_v f = \Delta_B f - df(\mathrm{grad} \log v), \qquad f \in C^\infty(B). \tag{4.2}$$

Here L_v is the self-adjoint operator associated with the closure of $f \mapsto \int_B |df|^2 v\,dv_B$ on $C^\infty(B)$. Consequently, if a closed subgroup $D \leq L_n \times P_n$ acts freely on both G_n and H_n by restriction of the marked left–right actions of Theorem 3.3, the weighted quotient operators have equal spectra. If both orbit-volume functions are constant, the ordinary quotient Laplacians are isospectral.

Proof. The free compact action gives a smooth principal bundle. In local bundle coordinates the vertical metric varies smoothly, so v is smooth and positive. Fibre integration gives, for $f, g \in C^\infty(B)$,

$$\int_X (f \circ \pi)\overline{g \circ \pi}\, dv_X = \int_B f \bar{g}\, v\, dv_B,$$
$$\int_X \langle d(f \circ \pi), d(g \circ \pi)\rangle\, dv_X = \int_B \langle df, dg\rangle v\, dv_B.$$

Integration by parts identifies the latter form with $-v^{-1}\,\mathrm{div}(v\,\mathrm{grad}\, f)$, which is (4.2). Finite sums of invariant Laplace eigenfunctions form a core for the invariant Dirichlet form, by Lemma 2.4. They descend to smooth functions on B. The two displayed identities therefore identify the closed forms and their self-adjoint operators, including their domains.

Let $S : L^2(G_n) \to L^2(H_n)$ be the equivariant unitary of Theorem 3.3, with probability measures. For Riemannian volume measures, $\big(\mathrm{vol}(G_n)/\mathrm{vol}(H_n)\big)^{1/2} S$ is unitary. Its restriction to D-invariant functions, followed by the pullback identifications, intertwines the weighted operators. For constant v, the first-order term vanishes. □

For the product quotients, fix positive integers $r, n_1, \dots, n_r$ and positive real numbers $s_1, \dots, s_r$. Put $P = \prod_{j=1}^r P_{n_j}$ and for $I \subseteq \{1, \dots, r\}$ let $J_I = \prod_{j=1}^r J_{I,j}$, with the factors and metrics of Section 2. Write $\mathfrak{p} = \mathrm{Lie}(P)$ and $\mathfrak{j}_I = \mathrm{Lie}(J_I)$. Let $\iota_I : P \to J_I$ be the product of the prescribed embeddings. Let $K \leq P$ be closed and put $\mathfrak{k} = \mathrm{Lie}(K)$. In J_I and its Lie algebra, identify K and $\mathfrak{k}$ with their images under ι_I and $d\iota_I$. For a finite-dimensional complex Hermitian space W and a unitary representation $\tau : K \to \mathrm{U}(W)$, write $J_I \times_K W$ for the quotient by $(g, w) \cdot k = (g\iota_I(k), \tau(k)^{-1} w)$. For Riemannian probability measures, its L^2 sections identify with $(L^2(J_I) \otimes W)^K$, where k acts by $R_{\iota_I(k)} \otimes \tau(k)$ and $R_h f(g) = f(gh)$. Scalar isospectrality under equivariant descent through minimal-fibre quotients is a special case of [101, arXiv version 2, Theorem 2.7].

Theorem 4.3. *For every closed $K \leq \prod_j P_{n_j}$, the manifolds $M_I = J_I/K$ are scalar-isospectral. For every fixed finite-dimensional unitary K module W, their canonical rough connection Laplacians on $J_I \times_K W$ are isospectral. The unitaries preserve operator domains and commute with fibre intertwiners and finite direct sums. If K^0 acts trivially on W, these connections are flat and their holonomy is trivial on loops that lift to loops in $J_I/K^0 \to J_I/K$.*

Proof. For $I, L \subseteq \{1, \dots, r\}$, tensoring the intertwiners of Theorem 3.3 gives a unitary $S_{IL} : L^2(J_I) \to L^2(J_L)$ commuting with the whole right action of P and intertwining the scalar Laplacians with their domains. Each product eigenspace is the finite orthogonal sum of the tensor products with the prescribed sum of eigenvalues. Haar averaging over K commutes with S_{IL}. Lemma 4.1 gives scalar isospectrality.

For a normal homogeneous quotient J/K and a unitary representation $\tau : K \to \mathrm{U}(W)$, denote the induced canonical connection by ∇^0_τ. A smooth section of $J \times_K W$ is represented by a smooth function $f : J \to W$ satisfying $f(gk) = \tau(k)^{-1} f(g)$. For a Q-orthonormal basis (u_a) of $\mathfrak{k}$, $R_{u_a} f = -d\tau(u_a) f$. Splitting the energy into its horizontal and vertical parts gives, on equivariant functions,

$$\Delta_J = \nabla^{0*}_\tau \nabla^0_\tau + C_\tau, \qquad C_\tau = -\sum_a d\tau(u_a)^2. \tag{4.3}$$

Here Δ_J acts componentwise. The horizontal energy is the canonical connection energy; fibre integration introduces only the constant fibre volume. The invariant form on $\mathfrak{k}$ is preserved by every component of K, so conjugating the orthonormal basis sum shows that C_τ commutes with $\tau(K)$. It is a bounded self-adjoint fibre endomorphism. The algebraic sum of the spaces $(E_\lambda(J) \otimes W)^K$ is a core for the restricted Dirichlet form, by Lemma 2.4. Thus the energy identity proves (4.3) for the associated self-adjoint operators; subtracting C_τ leaves the operator domain unchanged.

The induced forms on $\mathfrak{k}$ agree for all I by Lemma 2.1. Hence $S_{IL} \otimes 1_W$ preserves equivariance and commutes with the same endomorphism C_τ. It gives the required rough intertwiners. Tensoring with the fibre identity commutes with fibre intertwiners and finite direct sums. If $\tau|_{K^0} = 1$, the associated bundle is obtained from the discrete principal K/K^0 bundle $J_I/K^0 \to J_I/K$. Its induced connection pulls back to the trivial connection. □

Theorem 4.4. *Let J_1, J_2 be compact connected metric Lie groups with bi-invariant metrics and embeddings of the same compact group K. Suppose the induced forms on $\mathfrak{k}$ agree. The following are equivalent:*

1. *for every $\lambda \geq 0$, $E_\lambda(J_1)$ and $E_\lambda(J_2)$ are equivalent right-K modules;*
2. *their canonical rough spectra agree for every irreducible unitary representation of the full group K;*
3. *their canonical rough spectra agree for every finite-dimensional unitary representation of K.*

Proof. Let $\tau : K \to \mathrm{U}(V_\tau)$ be irreducible. Full K invariance of C_τ and Schur's lemma give $C_\tau = c_\tau 1_{V_\tau}$ for some $c_\tau \geq 0$, even if $\tau|_{K^0}$ is reducible. The number c_τ is the same for J_1 and J_2. By (4.3), the rough multiplicity at $\lambda - c_\tau$ is

$$\dim(E_\lambda(J_i) \otimes V_\tau)^K = \dim \mathrm{Hom}_K(V_\tau^*, E_\lambda(J_i)).$$

For $\lambda < c_\tau$ this invariant space is zero, since the rough Laplacian is nonnegative. For $\lambda \geq c_\tau$, its dimension is recovered from the rough spectrum; an absent eigenvalue has multiplicity zero. As τ^* runs through all irreducible K modules, (2) determines every irreducible multiplicity in every ambient eigenspace, and proves (1).

Under (1), choose equivariant unitaries on the scalar eigenspaces and take their Hilbert direct sum. For any finite-dimensional unitary representation $\tau : K \to \mathrm{U}(W)$, tensoring with 1_W preserves the invariant spaces and commutes with C_τ. Lemma 2.4 and (4.3) give (3), including operator domains. Condition (3) includes (2). □

4.1. **The tangent representation.** Write $\mathfrak{g}_n$, $\mathfrak{h}_n$ and $\mathfrak{p}_n$ for the Lie algebras of G_n, H_n and P_n, respectively. For a real vector space U, set $U_{\mathbb{C}} = U \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 4.5. *There is a real P_n-equivariant orthogonal isomorphism $F_n : \mathfrak{g}_n \to \mathfrak{h}_n$ satisfying $F_n \circ di_n = dj_n$ on $\mathfrak{p}_n$.*

Proof. Let $V = \mathbb{C}^n$ be the defining $\mathrm{U}(n)$ module, with trivial action of the final circle in $P_n = \mathrm{U}(n)\times\mathrm{U}(1)$. Denote by z the one-dimensional representation $(A, \zeta) \mapsto \zeta$, and abbreviate $V \otimes z^{\pm 1}$ to $Vz^{\pm 1}$. Write Sym^2 and Λ^2 for the symmetric and exterior squares. Since $\bar{V} \simeq V^*$ as unitary modules,

$$\begin{aligned} \mathfrak{g}_{n,\mathbb{C}} \simeq (\mathrm{End}\, V)^{\oplus 2} \oplus V^{\otimes 2} \oplus (V^*)^{\otimes 2} \oplus \mathbb{C} \\ \oplus Vz \oplus Vz^{-1} \oplus V^*z \oplus V^*z^{-1}. \end{aligned} \tag{4.4}$$

The symplectic and orthogonal adjoint representations are the symmetric and exterior squares of the defining representations [41, §18.3, Examples 18.12–18.13]. Hence

$$\mathfrak{sp}(n)_{\mathbb{C}} \simeq \mathrm{Sym}^2 V \oplus \mathrm{End}\, V \oplus \mathrm{Sym}^2 V^*,$$

$$\begin{aligned} \mathfrak{so}(2n+2)_{\mathbb{C}} \simeq \Lambda^2 V \oplus \mathrm{End}\, V \oplus \Lambda^2 V^* \oplus \mathbb{C} \\ \oplus Vz \oplus Vz^{-1} \oplus V^*z \oplus V^*z^{-1}. \end{aligned}$$

The decomposition $\mathrm{Sym}^2 V \oplus \Lambda^2 V \simeq V \otimes V$ equates the ambient complex modules, including $n = 1$, where $\Lambda^2 V = 0$. The embedded real $\mathfrak{p}_n$ has the same invariant metric on both sides. Cancelling its character equates the complexified orthogonal complements. These real complements are therefore isomorphic [110, §3.6, Theorem 5, p. 39]. For a real equivariant isomorphism C between them, C^*C commutes with P_n, so $C(C^*C)^{-1/2}$ is an orthogonal intertwiner. Combine it with the prescribed isometry $di_n u \mapsto dj_n u$. □

Corollary 4.6. *For every closed $K \leq P$, the tangent isotropy modules $\mathfrak{m}_I = \mathfrak{k}^\perp \subset \mathfrak{j}_I$ are orthogonally equivalent as representations of the full group K. All corresponding canonical rough tensor spectra agree, including exterior, symmetric and arbitrary tensor powers, invariant tensor summands, and common finite-dimensional unitary twists.*

Proof. The direct sum of the factorwise maps 1, F_{n_j} or $F_{n_j}^{-1}$ is an isometry between any two products for the prescribed factor scales, is P-equivariant, and identifies the embeddings of $\mathfrak{p}$. It therefore maps $\mathfrak{k}$ and its orthogonal complement onto their counterparts, equivariantly under every component of K. Apply each tensor construction to this identification, then apply Theorem 4.3 to the resulting common fibre module. □

At $o = eK$, the canonical tangent connection has torsion $T^0(X, Y) = -[X, Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$. For the Levi-Civita connection [2, §2],

$$(\nabla^{\mathrm{LC}} - \nabla^0)_X Y = \tfrac{1}{2}[X, Y]_{\mathfrak{m}}, \qquad X, Y \in T_o(J/K) = \mathfrak{m}.$$

The tensor spectra above refer to ∇^0.

4.2. **Equivariant fibres and connection metrics.** For Riemannian submersions with totally geodesic fibres, the horizontal and vertical scalar Laplacians commute [13, Theorem 1.5, p. 183]. The scalar assertion below is the fixed-connection case of [8, Theorem 3.4, p. 9]. For related associated-bundle constructions, see Sutton [100, Proposition 4.1, pp. 712–713] and Bordoni–Marino [19]. Principal connection metrics also occur in Schueth [95, §1, Theorem 2].

Proposition 4.7. *Let a compact Lie group P act isometrically on the right on two closed Riemannian manifolds (F_i, h_i), $i = 1, 2$. All L^2 spaces in this proposition have Riemannian volume measures. Suppose there is a unitary*

$$S : L^2(F_1) \longrightarrow L^2(F_2), \qquad S\Delta_{F_1} = \Delta_{F_2} S, \qquad S R_p^{(1)} = R_p^{(2)} S,$$

where $R_p f(x) = f(xp)$ and the Laplacians are intertwined as self-adjoint operators. Let $\mathcal{Q} \to B$ be a smooth principal right P bundle over a closed Riemannian base (B, g_B), and choose a smooth principal connection.

On

$$E_i = (\mathcal{Q} \times F_i)/P, \qquad (q, x) \cdot p = (qp, xp),$$

let $\pi_i : E_i \to B$ be the projection, and give E_i the connection metric g_i: the chosen connection induces the horizontal distribution, the horizontal metric is the pullback of g_B, and every vertical fibre carries h_i, orthogonal to the horizontal directions. Then (E_1, g_1) and (E_2, g_2) are scalar isospectral. For every finite-rank Hermitian bundle $W \to B$ with a unitary connection, the degree-zero connection Laplacians on $\pi_i^* W \to E_i$ are also unitarily equivalent.

Let $\mathcal{A}$ be an index set. Suppose in addition that self-adjoint operators $Z_{i,a}$ on $L^2(F_i)$, $a \in \mathcal{A}$, commute strongly with one another and with Δ_{F_i}, their spectral projections commute with the P action, and S intertwines $Z_{1,a}$ and $Z_{2,a}$ as self-adjoint operators. Then their fibrewise realizations on E_i are intertwined, including their self-adjoint domains. Their spectral projections commute with those of the total connection Laplacians.

Proof. For scalar functions, apply Ballmann's theorem with the left actions $\mu_i(p)x = xp^{-1}$, the given principal connection on both bundles, and the identity automorphism of $\mathcal{Q}$ for every irreducible representation of P. If B is disconnected, apply the theorem to each component.

Let $W \to B$ be as in the statement, and denote its connection by ∇^W and the pullback connection by $\nabla^{\pi_i^* W}$. Write $\Delta_{E_i, \pi_i^* W} = (\nabla^{\pi_i^* W})^* \nabla^{\pi_i^* W}$ for the connection Laplacian. Let Λ be the common scalar spectrum of F_1 and F_2. For $\lambda \in \Lambda$, put $V_{i,\lambda} = E_\lambda(F_i)$ and $\mathcal{V}_{i,\lambda} = \mathcal{Q} \times_P V_{i,\lambda}$, with the representation $R^{(i)}$ on the fibre. These are finite-rank Hermitian bundles. The diagonal quotient convention gives $u(qp) = R^{(i)}_{p^{-1}} u(q)$. Fibre integration and the orthogonal fibre spectral decomposition give

$$L^2(E_i, \pi_i^* W) \simeq \widehat{\bigoplus_\lambda} L^2(B, \mathcal{V}_{i,\lambda} \otimes W).$$

Each $V_{i,\lambda}$ is P-invariant, so its associated bundle is preserved by the induced connection. Let $\nabla_{i,\lambda}$ be the tensor product of this connection with ∇^W. The fibres of π_i are totally geodesic [13, §2.1, pp. 185–186]. Put $\Delta_{B,W} = (\nabla^W)^* \nabla^W$. Let $U \subset B$ admit a local orthonormal frame (X_a), with horizontal lifts X_a^h. Let $f \in C^\infty(\pi_i^{-1}(U))$ correspond to a smooth section of $\mathcal{V}_{i,\lambda}|_U$, and let $s \in C^\infty(U, W)$. The pullback formula [47, Lemma 1.3, p. 236] and the connection product rule give

$$\begin{aligned} \Delta_{E_i, \pi_i^* W}(f \pi_i^* s) = {} & (\Delta_{E_i} f) \pi_i^* s + f \pi_i^* (\Delta_{B,W} s) \\ & - 2 \sum_a X_a^h(f) \pi_i^* (\nabla^W_{X_a} s). \end{aligned}$$

The vertical Laplacian of f is λf. On the irreducible P summands of $V_{i,\lambda}$, the horizontal Laplacian is the associated connection Laplacian [8, Lemma 3.2, pp. 7–8], and horizontal differentiation is covariant differentiation. Every smooth section of $\mathcal{V}_{i,\lambda} \otimes W$ is locally a finite sum of such products. The tensor connection product rule therefore gives

$$\Delta_{E_i, \pi_i^* W}\big|_{C^\infty(B, \mathcal{V}_{i,\lambda} \otimes W)} = \nabla^*_{i,\lambda} \nabla_{i,\lambda} + \lambda.$$

The operator on the right has a smooth orthonormal eigenbasis on the closed base [38, Theorem 17.1 and §17.1.1, pp. 230–231]. Its sections lift to smooth total eigensections, and the union of these bases over λ is complete in the displayed Hilbert sum. Applying Lemma 2.4 to the finite-dimensional eigenspaces of the operators on the right, then grouping by λ, gives

$$\Delta_{E_i,\pi_i^* W} \simeq \widehat{\bigoplus_{\lambda}} A_{i,\lambda}, \qquad A_{i,\lambda} = \nabla_{i,\lambda}^* \nabla_{i,\lambda} + \lambda. \tag{4.5}$$

The operator domain consists of the tuples $u = (u_\lambda)$ in the displayed Hilbert sum such that

$$u_\lambda \in \mathcal{D}(A_{i,\lambda}), \qquad \sum_{\lambda} \|A_{i,\lambda} u_\lambda\|^2 < \infty.$$

Each restriction $S_\lambda : V_{1,\lambda} \to V_{2,\lambda}$ is a P-equivariant unitary. It induces a parallel unitary on the associated bundles, also after tensoring with 1_W. The Hilbert sum of these maps intertwines the operators in (4.5) and preserves the displayed domains. The decompositions hold on each component of B, without a connectedness assumption on P or F_i.

For the operators $Z_{i,a}$, strong commutation with Δ_{F_i} makes every $V_{i,\lambda}$ a reducing subspace. The restriction of each $Z_{i,a}$ is self-adjoint on this finite-dimensional space, hence is defined everywhere. The commuting restrictions therefore have common orthogonal eigenspaces $V_{i,\lambda,\zeta}$, where $\zeta = (\zeta_a)_{a\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. Only finitely many such joint eigenvalues occur for each λ. Their projections commute with P, so these spaces define parallel associated subbundles, and the decomposition (4.5) refines accordingly. On the corresponding Hilbert sum, the fibrewise realization of $Z_{i,a}$ is multiplication by ζ_a, with domain

$$\left\{ u = (u_{\lambda,\zeta}) \in L^2(E_i, \pi_i^* W) \,:\, \sum_{\lambda,\zeta} |\zeta_a|^2 \|u_{\lambda,\zeta}\|^2 < \infty \right\}.$$

The maps induced by S preserve the joint eigenvalue vectors and these domains. Each joint subbundle is parallel, so its projection reduces the total connection Laplacian. The spectral projections of each fibrewise $Z_{i,a}$ are sums of these reducing projections and therefore commute with the spectral projections of that Laplacian. □

4.3. **Equivariant Clifford modules.** For a smooth Hermitian bundle E over a closed manifold and $s \in \mathbb{R}$, write $H^s(E)$ for the Sobolev space of its sections.

Extend Q complex bilinearly to $\mathfrak{j}_{\mathbb{C}}$. Let $\mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q)$ be the complex Clifford algebra with relations $XY + YX = -2Q(X,Y)1$ for $X, Y \in \mathfrak{m}$. Let W be a finite-dimensional Hermitian space, let $c : \mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q) \to \mathrm{End}\, W$ be a unital representation, and let $\tau : K \to \mathrm{U}(W)$ be a unitary representation of the full group K. Suppose that $c(X)^* = -c(X)$ for $X \in \mathfrak{m}$ and that

$$\tau(k) c(X) \tau(k)^{-1} = c(\mathrm{Ad}(k)X), \qquad k \in K, \quad X \in \mathfrak{m}. \tag{4.6}$$

For $u \in \mathfrak{k}$ and Q-orthonormal bases (e_i) of $\mathfrak{m}$ and (u_a) of $\mathfrak{k}$, put

$$\sigma_c(u) = \tfrac{1}{4} \sum_{i,j} Q([u, e_i], e_j) c(e_i) c(e_j), \qquad \kappa(u) = d\tau(u) - \sigma_c(u), \qquad C_\kappa = -\sum_a \kappa(u_a)^2. \tag{4.7}$$

Lemma 4.8. *The map κ is a skew-Hermitian Lie algebra representation commuting with the Clifford action. Its Casimir C_κ is nonnegative and commutes with $\tau(K)$.*

Proof. For $u, v \in \mathfrak{k}$ and $X \in \mathfrak{m}$, the infinitesimal spin representation satisfies

$$[\sigma_c(u), c(X)] = c([u, X]), \qquad [\sigma_c(u), \sigma_c(v)] = \sigma_c([u, v])$$

[65, (1.10)–(1.12)]. Differentiating (4.6) gives the first commutator also for $d\tau(u)$. Hence $\kappa(u)$ commutes with every Clifford product, in particular every $\sigma_c(v)$. Since $d\tau = \sigma_c + \kappa$, subtracting the Lie bracket identities gives $[\kappa(u), \kappa(v)] = \kappa([u, v])$. Both $d\tau(u)$ and $\sigma_c(u)$ are skew-Hermitian. Conjugation by $\tau(k)$ carries $\kappa(u)$ to $\kappa(\mathrm{Ad}(k)u)$ for every $k \in K$. The invariant orthonormal sum is therefore K-invariant, and $\langle C_\kappa w, w\rangle = \sum_a \|\kappa(u_a) w\|^2$. □

On smooth functions $f : J \to W$ with $f(gk) = \tau(k)^{-1} f(g)$, define

$$D_{J,W} = \sum_i c(e_i) R_{e_i} + v_J, \qquad v_J = \tfrac{1}{2} \sum_{i<j<k} Q([e_i, e_j]_{\mathfrak{m}}, e_k) c(e_i) c(e_j) c(e_k). \tag{4.8}$$

Write $\sigma(u)$ and $\mathbf{v}_J$ for the elements of $\mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q)$ obtained from (4.7) and (4.8) by replacing each $c(e_i)$ by e_i. Thus $c(\sigma(u)) = \sigma_c(u)$ and $c(\mathbf{v}_J) = v_J$. The two contractions in (4.8) are invariant under K, hence descend to $J \times_K W$. On $C^\infty(J, W)$, each R_X is formally skew-adjoint and commutes with every constant fibre endomorphism. Since $c(e_i)$ is skew-Hermitian and a product of three distinct Clifford generators is Hermitian, the descended operator is formally self-adjoint. Its principal symbol is Clifford multiplication, so it is of Dirac type. Since J/K is closed, [9, Theorems 4.4 and 4.11, with empty boundary] give a self-adjoint closure with domain $H^1(J \times_K W)$. Denote this closure again by $D_{J,W}$. For $\mathfrak{a} = \mathfrak{j}, \mathfrak{k}$, let $\rho_{\mathfrak{a}}$ be a half-sum of positive roots, regarded as real weights on a maximal torus as in Section 2. Its norm is taken in the dual metric induced by Q; the roots vanish on the center. Set $b_J = \|\rho_{\mathfrak{j}}\|^2 - \|\rho_{\mathfrak{k}}\|^2$.

Theorem 4.9. *For the above operator,*

$$D_{J,W}^2 = \Delta_J - C_\kappa + b_J 1_W, \qquad \mathcal{D}(D_{J,W}^2) = H^2(J \times_K W). \tag{4.9}$$

Here Δ_J acts componentwise on the equivariant functions. For the quotients J_I/K, identify the tangent isotropy modules by Corollary 4.6, and fix one Hermitian Clifford module (W, c, τ) for this common metric K module. Then the constants b_{J_I} and the spectra of the cubic squares are independent of I. If W has a K-invariant grading for which $c(X)$ is odd, the signed spectra of the cubic operators are independent of I as well.

Proof. Apply Kostant's identities to $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{j}_{\mathbb{C}}$ with bilinear form $-Q$. Both algebras are reductive, the restricted form is nondegenerate, and the orthogonal complement is $\mathfrak{m}_{\mathbb{C}}$. Kostant's spin map is σ, and his cubic element is $-\mathbf{v}_J$ [65, (1.20)–(1.23)]. Since the dual basis to (u_a) is $(-u_a)$, his Theorem 1.91 gives

$$\mathbf{v}_J^2 - \sum_a \sigma(u_a)^2 = b_J 1 \qquad \text{in } \mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q). \tag{4.10}$$

Under $X \mapsto R_X$ and the Clifford action c, Kostant's cubic Dirac element acts as $-D_{J,W}$: its linear contraction is $-\sum_i R_{e_i} c(e_i)$. The ambient Casimir acts as Δ_J, and the diagonal isotropy Casimir acts as $-\sum_a (R_{u_a} + \sigma_c(u_a))^2$. Thus [65, Theorem 2.16] gives, on smooth sections,

$$D_{J,W}^2 = \Delta_J + \sum_a (R_{u_a} + \sigma_c(u_a))^2 + b_J.$$

On equivariant functions $R_u = -d\tau(u)$, so the sum is $-C_\kappa$. The identities hold for arbitrary ranks. The endomorphism C_κ is bounded and self-adjoint. By [9, Theorem 4.4, with empty boundary],

$$\mathcal{D}(D_{J,W}^2) = \{f \in H^1 : D_{J,W} f \in H^1\} = H^2,$$

where the Sobolev spaces are those of $J \times_K W$. By (4.3), the restriction of Δ_J is the canonical rough Laplacian plus the bounded endomorphism C_τ. Its domain is H^2 by elliptic regularity [9, §2.3]; adding $-C_\kappa + b_J$ preserves this domain. Thus (4.9) holds as an identity of self-adjoint operators on H^2.

By (3.15), the ambient root norms agree in each reference block. Products add their squared norms, and scaling a block by s_j divides its dual squared norm by s_j. The common embedded $\mathfrak{k}$ has its common metric, so its root norm agrees as well. The tangent identification makes κ common. Consequently $S_{IL} \otimes 1_W$ commutes with C_κ and intertwines the squares, including their domains.

If ϵ is the grading involution, then $\epsilon D_{J,W} = -D_{J,W} \epsilon$, since both terms in (4.8) have odd Clifford degree. Each positive square eigenvalue λ therefore gives $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ with equal multiplicities. The square also determines the zero multiplicity. This proves equality of the signed spectra. □

4.4. **Exterior modules and spinors.** For $X \in \mathfrak{m}$, let $X^\flat = Q(X, \cdot)$ and let ι_X denote contraction of a differential form with X.

Corollary 4.10. *On the full exterior algebra $W = \Lambda^* \mathfrak{m}^*_{\mathbb{C}}$, with $c(X) = X^\flat \wedge - \iota_X$ and the natural K action, the cubic operators have equal signed spectra on all M_I, also after any common finite-dimensional unitary K twist.*

Proof. The exterior action satisfies (4.6) for every orthogonal transformation. Exterior parity is invariant and Clifford multiplication is odd. Placing a twist in even degree preserves these properties. Apply Theorem 4.9. □

The exterior operator is defined by (4.8). Its cubic term is left Clifford multiplication by the specified three-form. The Hodge operator $d + d^*$ is defined by the Levi-Civita connection. The zero multiplicities of the Hodge Laplacians on two-forms differ in Theorem 12.1.

Suppose that the isotropy representation $\lambda : K \to \mathrm{SO}(\mathfrak{m})$ has a lift $\widetilde{\lambda} : K \to \mathrm{Spin}(\mathfrak{m})$. Let $\Sigma_{\mathfrak{m}}$ be a complex irreducible $\mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q)$ module, with a Hermitian metric for which $c(X)^* = -c(X)$ for $X \in \mathfrak{m}$, and let s be its restricted spin representation. In even dimension this is the full spin module; in odd dimension choose either of the two irreducible Clifford modules. For a finite-dimensional unitary representation $\rho : K \to \mathrm{U}(V_\rho)$, give $\Sigma_{\mathfrak{m}} \otimes V_\rho$ the action $\tau(k) = s(\widetilde{\lambda}(k)) \otimes \rho(k)$ and Clifford multiplication on the first factor. Put $D^{1/3}_{J,\rho} = D_{J,\Sigma_{\mathfrak{m}} \otimes V_\rho}$. The residual action is $\kappa = 1 \otimes d\rho$. With $C_\rho = -\sum_a d\rho(u_a)^2$, Theorem 4.9 gives

$$(D^{1/3}_{J,\rho})^2 = \Delta_J - 1_{\Sigma_{\mathfrak{m}}} \otimes C_\rho + b_J. \tag{4.11}$$

For $t \in \mathbb{R}$, let ∇^t be the J-invariant metric connection whose difference from ∇^0 at $o = eK$ is

$$(\nabla^t - \nabla^0)_X Y = t[X, Y]_{\mathfrak{m}}, \qquad X, Y \in \mathfrak{m}.$$

The twist has its canonical connection. The correction to the canonical spin Dirac operator is

$$\frac{3t}{2} \sum_{i<j<k} Q([e_i, e_j], e_k) c(e_i) c(e_j) c(e_k)$$

[2, §3.1, equation (5)]. Thus $t = 1/3$ gives (4.8). The Levi-Civita value is $t = 1/2$ [2, §2].

Proposition 4.11. *If K^0 is central in K, every M_I admits a homogeneous spin structure. The structures can be chosen from one common isotropy lift. For these structures and every common finite-dimensional unitary K twist, the cubic spin squares have equal spectra. In even dimension their full spinor signed spectra also agree.*

Proof. For the connected reductive complexifications of $\mathrm{U}(2n+1)$, $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+2)$, and their products, the half-sums of positive roots belong to the torus character lattices in (2.4). The adjoint-lifting criterion [62, arXiv version 3, Corollary 4, p. 11] therefore gives algebraic spin lifts. The inverse image of $\mathrm{SO}(\mathfrak{j}_I, Q)$ under the complex spin covering is $\mathrm{Spin}(\mathfrak{j}_I, Q)$. Restriction gives homomorphisms $J_I \to \mathrm{Spin}(\mathfrak{j}_I)$ lifting the ambient adjoint actions. The criterion includes $n = 1$, hence the factor $\mathrm{SO}(4)$.

Centrality of K^0 makes $\mathrm{Ad}(K)$ fix $\mathfrak{k}$ pointwise. The ambient adjoint action is oriented, so its action on $\mathfrak{m}$ is oriented. The inverse image of $1_{\mathfrak{k}} \oplus \mathrm{SO}(\mathfrak{m})$ in $\mathrm{Spin}(\mathfrak{j})$ is the embedded $\mathrm{Spin}(\mathfrak{m})$. Restricting an ambient lift therefore gives an isotropy lift on the full group K. Transport one such lift and the chosen Clifford module by the tangent isometries, giving each tangent space the transported orientation. Equation (4.11) gives the common square spectrum. Even-dimensional full spin modules have the invariant chirality grading, so their cubic operators have equal signed spectra. □

Two lifts differ by a character $K \to \{1, -1\}$; the preceding comparison fixes one common lift. In odd dimension it gives equality of the cubic spin-square spectra.

Proposition 4.12. *For the quotients J_I/K, the elements $\mathbf{v}^2_{J_I}$ correspond under the Clifford isomorphisms induced by the tangent identifications. For one block with $K = P_n$, no P_n-equivariant orthogonal isomorphism $F : \mathfrak{g}_n \to \mathfrak{h}_n$ satisfying $F \circ di_n = dj_n$ carries $\mathbf{v}_{G_n}$ to $\mathbf{v}_{H_n}$ under the induced Clifford isomorphism.*

Proof. Put $C_\sigma = -\sum_a \sigma(u_a)^2$. Equation (4.10) gives $\mathbf{v}_J^2 = b_J - C_\sigma$ in the universal Clifford algebra; both terms on the right are common. For $K = P_n$, let F satisfy the stated equivariance and metric conditions. It preserves the bracket on $\mathfrak{p}_n$ and its action on $\mathfrak{m}$. For $X, Y \in \mathfrak{m}$ and $u \in \mathfrak{p}_n$, invariance of Q gives

$$Q([X, Y]_{\mathfrak{p}_n}, u) = Q([u, X], Y).$$

Thus F preserves the $\mathfrak{p}_n$ component of $[X, Y]$. Using Q to identify three-vectors with three-forms and the vector-space identification $\Lambda^*\mathfrak{m}_{\mathbb{C}} \simeq \mathrm{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q)$, equality of the cubic elements is equality of the alternating forms $\frac{1}{2}Q([X, Y]_{\mathfrak{m}}, Z)$ for $X, Y, Z \in \mathfrak{m}$ [65, (1.16) and (1.23)]. It therefore also preserves the $\mathfrak{m}$ component. Hence F would be a Lie algebra isomorphism. This is impossible: $\mathfrak{u}(2n + 1)$ has a one-dimensional center, whereas $\mathfrak{sp}(n) \oplus \mathfrak{so}(2n + 2)$ has zero center. □

4.5. **Residual flows in rank one.** For $\lambda \geq 0$ and $z \in \mathbb{C}$, take $(1 + \lambda)^z = \exp(z \log(1 + \lambda))$.

Corollary 4.13. *Let $K \leq P_n \times P_n$ be closed and satisfy the equivalent conditions of Theorem 3.9. Every K-equivariant unitary $S : L^2(G_n) \to L^2(H_n)$ intertwining the scalar Laplacians intertwines the generators of each one-parameter subgroup of K, with their domains, and the operators $(1 + \Delta_{G_n})^z$ and $(1 + \Delta_{H_n})^z$ for every $z \in \mathbb{C}$, with their maximal spectral domains.*

Proof. Apply Lemma 2.4 to the K-equivariant scalar eigenspaces in Theorem 3.9. □

For $i \in \{A, B\}$, write $G_A = G_1$, $G_B = H_1$, $\iota_A = i_1$, $\iota_B = j_1$ and $\iota_i(u, v) = \iota_i(e^{iu}, e^{iv})$ for angular arguments. At $n = 1$ the bilateral identity gives, for $t > 0$ and $u, s, v \in \mathbb{T}$,

$$B_{G_A}(t; \iota_A(u, 0), \iota_A(s, v)) = B_{G_B}(t; \iota_B(u, 0), \iota_B(s, v)). \tag{4.12}$$

Let $h = 2\pi/7$ and put

$$H_i = \{\iota_i(0, v) : v \in \mathbb{T}\}, \qquad A = G_A/H_A, \quad B = G_B/H_B.$$

Give A and B their normal quotient metrics and Riemannian probability measures. The common subgroup

$$S_1 = \{(u, 0; 3u, b) : u, b \in \mathbb{T}\} \subset \ker \alpha \tag{4.13}$$

contains the right-circle subgroup and the commuting flow subgroup. Since $\iota_i(3hs, 0)$ commutes with H_i, the latter descends to

$$F_i(s)(gH_i) = \iota_i(hs, 0)g\iota_i(3hs, 0)^{-1}H_i, \qquad s \in \mathbb{R}.$$

On functions pulled back to G_i, its pullback is $F_i(s)^* = T_{(-hs,0;-3hs,0)}$ in the convention of Theorem 3.9.

Corollary 4.14. *There is a unitary $S : L^2(A) \to L^2(B)$ fixing normalized constants, intertwining the Laplacians and the seven-periodic flows $F_i(s)$, and intertwining their generators with domains. In particular, for $s \in \mathbb{R}$ and $t > 0$,*

$$\mathrm{Tr}(F_A(s)^* e^{-t\Delta_A}) = \mathrm{Tr}(F_B(s)^* e^{-t\Delta_B}). \tag{4.14}$$

Proof. Let $U : L^2(G_A) \to L^2(G_B)$ be an S_1-equivariant spectral unitary fixing constants, and let P_i be right-circle averaging. Then $UP_A = P_B U$, and each P_i commutes with the flow and the Laplacian. Restrict U to $\mathrm{ran}\, P_A$ and apply Lemma 4.1. Corollary 4.13 gives the intertwining of the generators with their domains. Since $7h = 2\pi$, one has $F_i(s + 7) = F_i(s)$. The products in (4.14) are trace class and are conjugate by S. □

4.6. **Curvature of normal homogeneous quotients.**

Proposition 4.15. *Let $M = J/K$, where J is a compact connected Lie group and K is a closed subgroup. Give M the metric induced by a positive $\mathrm{Ad}(J)$-invariant form Q, and write $\mathfrak{m} = \mathfrak{k}^{\perp}$ and $o = eK$. Put $b_1(M) = \dim H^1(M;\mathbb{R})$, and let $\mathfrak{z}(\mathfrak{j})$ be the center of $\mathfrak{j}$. Identify T_oM with $\mathfrak{m}$, and regard* Ric *as a self-adjoint endomorphism by means of the Riemannian metric. Then*

$$\ker \mathrm{Ric}_o = \mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}, \qquad \dim \ker \mathrm{Ric}_o = b_1(M).$$

The universal Riemannian cover is

$$\widetilde{M} \cong \mathbb{R}^{b_1(M)} \times M_+,$$

where M_+ is compact and simply connected, and has positive Ricci curvature if it has positive dimension. Moreover,

$$\mathrm{Ric}_M > 0 \quad \Longleftrightarrow \quad \pi_1(M) \text{ is finite.}$$

In dimension zero, M is a point and the Ricci inequality is interpreted as vacuous. For an integer $N \geq 1$, if $J = \mathrm{U}(N)$, $Q(X,Y) = -\operatorname{Re}\operatorname{Tr}(XY)$, and $0 \neq T \in \mathfrak{k}$, then

$$\mathrm{Ric}_{J/K} \geq \frac{|\operatorname{Tr} T|^2}{2Q(T,T)}\, g_{J/K}. \tag{4.15}$$

Proof. All norms and orthogonal projections are taken with respect to Q. For an orthonormal pair $X, Y \in \mathfrak{m}$, the normal homogeneous curvature formula [83, equation (3.6), p. 250] gives

$$\sec(X,Y) = \frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2 + \|[X,Y]_{\mathfrak{k}}\|^2. \tag{4.16}$$

For $X \in \mathfrak{m}$ and an orthonormal basis (e_a) of $\mathfrak{m}$, put

$$A_X = \sum_a \|[X,e_a]_{\mathfrak{m}}\|^2, \qquad C_X = \sum_a \|[X,e_a]_{\mathfrak{k}}\|^2.$$

Then $\mathrm{Ric}(X,X) = A_X/4 + C_X$ for every $X \in \mathfrak{m}$. Let $B_{\mathfrak{j}}(X,Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ be the Killing form, where $\operatorname{ad} X(Y) = [X,Y]$. If (u_α) is an orthonormal basis of $\mathfrak{k}$, invariance of Q gives

$$C_X = \sum_\alpha \|[u_\alpha, X]\|^2, \qquad -B_{\mathfrak{j}}(X,X) = A_X + 2C_X.$$

Thus

$$\mathrm{Ric}(X,X) = -\frac{1}{4}B_{\mathfrak{j}}(X,X) + \frac{1}{2}\sum_\alpha \|[u_\alpha, X]\|^2. \tag{4.17}$$

The Hilbert–Schmidt norm on $\operatorname{End}\mathfrak{j}$ is induced by Q. Since $-B_{\mathfrak{j}}(X,X) = \|\operatorname{ad} X\|_{\mathrm{HS}}^2$, the Ricci tensor is nonnegative and its kernel is $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$.

For $J = \mathrm{U}(N)$, let I_N be the identity matrix, put $Z = iI_N/\sqrt{N}$, and denote the orthogonal projection of X onto $\mathfrak{su}(N)$ by $X_{\mathfrak{su}(N)}$. Since $X \in \mathfrak{m}$ is orthogonal to T, projection onto $T^{\perp}$ gives

$$|Q(X,Z)|^2 \leq \left(1 - \frac{|\operatorname{Tr} T|^2}{NQ(T,T)}\right)\|X\|^2, \qquad \|X_{\mathfrak{su}(N)}\|^2 \geq \frac{|\operatorname{Tr} T|^2}{NQ(T,T)}\|X\|^2.$$

Now $-B_{\mathfrak{u}(N)}(X,X) = 2N\|X_{\mathfrak{su}(N)}\|^2$, so (4.17) implies (4.15). If $N = 1$, the hypothesis $T \neq 0$ forces $\mathfrak{m} = 0$.

Since J is connected, every central $Z \in \mathfrak{m}$ is fixed by $\mathrm{Ad}(J)$, and hence by all of $\mathrm{Ad}(K)$. It therefore defines a global J-invariant field on M, represented by the constant equivariant function $g \mapsto Z$. This field is ∇^0-parallel. Since $[X,Z] = 0$ for $X \in \mathfrak{m}$, the formula for $\nabla^{\mathrm{LC}} - \nabla^0$ makes it Levi-Civita parallel as well; its dual is a real harmonic one-form. Conversely, since M is closed and $\mathrm{Ric}_M \geq 0$, Bochner's theorem [18, Lemma 1, pp. 778–779, and Theorem 1, p. 780] implies that every real harmonic one-form on M is parallel and takes values in the Ricci kernel; orientability is not required. Evaluation of their

metric-dual fields at o therefore identifies the real harmonic one-forms with $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$. Hodge theory gives $\dim \ker \mathrm{Ric}_o = b_1(M)$.

Put $r = b_1(M)$ and $\mathcal{D} = \ker \mathrm{Ric}_M$. A basis of $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$ gives a global parallel frame of $\mathcal{D}$, so $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel. The universal Riemannian cover is complete and simply connected. The de Rham theorem [35, §7, Theorem II, pp. 339–340] gives $\widetilde{M} = \mathbb{R}^r \times M_+$, where M_+ is complete and simply connected and its tangent spaces are the lifts of $\mathcal{D}^\perp$; zero-dimensional factors are interpreted as points. If $\dim M_+ > 0$, let $c > 0$ be the least positive eigenvalue of the Ricci endomorphism at o. Homogeneity gives the same lower bound on $\mathcal{D}^\perp$ at every point. The product Ricci formula therefore gives $\mathrm{Ric}_{M_+} \geq c\, g_{M_+}$. Dimension one is impossible, since its Ricci tensor would vanish. For $\dim M_+ \geq 2$, Myers's theorem [82, Theorem I, p. 404] makes M_+ compact; for $\dim M_+ = 0$, take M_+ to be a point.

In positive dimension, the equivalence of $\mathrm{Ric}_M > 0$ and finiteness of $\pi_1(M)$ is Nash's theorem [83, Proposition 3.4, pp. 249–250], applied to the compact group J and its closed subgroup K. □

5. Finite-quotient heat traces and minimum lengths

For a Riemannian manifold Z, write d_Z, dv_Z, and Δ_Z for its distance, Riemannian volume measure, and nonnegative scalar Laplacian. The L^2 inner products are taken with respect to dv_Z. Put $B_Z(x, r) = \{y \in Z : d_Z(x, y) < r\}$ for $r > 0$, and write $L(\alpha)$ for the length of a piecewise smooth curve α.

For a group G and a finite-dimensional unitary representation $\rho : G \to \mathrm{U}(V_\rho)$ on a Hermitian space V_ρ, put $\chi_\rho(g) = \mathrm{Tr}\, \rho(g)$, and let ρ^* denote the contragredient representation on V_ρ^*. If G is finite, write $\mathrm{Irr}(G)$ for a set of representatives of its irreducible unitary representations. For a homomorphism $u : G' \to G$, put $u^*\rho = \rho \circ u$.

Let X be a closed connected smooth Riemannian manifold of positive dimension, and fix a marking $\Pi = \pi_1(X)$. Write $\ell_X(c)$ for the infimum of lengths of piecewise smooth loops in the free homotopy class of $c \in \Pi$, with $\ell_X(1) = 0$.

Let $\widetilde{X} \to X$ be the universal Riemannian cover, with the marked left deck action of Π. For $\rho : \Pi \to \mathrm{U}(V_\rho)$, set $E_\rho = (\widetilde{X} \times V_\rho)/\Pi$, where $\gamma(x, v) = (\gamma x, \rho(\gamma)v)$. Let ∇_ρ be the descended flat connection and put $\Delta_{X,\rho} = \nabla_\rho^* \nabla_\rho$. Smooth sections of E_ρ correspond to smooth maps $f : \widetilde{X} \to V_\rho$ satisfying

$$f(\gamma x) = \rho(\gamma) f(x). \tag{5.1}$$

A loop whose lift starts at x and ends at γx has parallel transport $\rho(\gamma)^{-1}$ in the trivialization determined by x. Put $\Theta_{X,\rho}(s) = \mathrm{Tr}(e^{-s\Delta_{X,\rho}})$ for $s > 0$. For the trivial one-dimensional representation, $\Delta_{X,1} = \Delta_X$ and $\Theta_X := \Theta_{X,1}$. Representations of a deck group denote their pullbacks to the marked fundamental group when they label flat bundles or flat Laplacians.

5.1. **The finite character transform.** For a finite quotient $q : \Pi \twoheadrightarrow Q$, put $Y_q = (\ker q)\backslash \widetilde{X}$, with the lifted metric and the induced marked left Q-action. For $g \in Q$ and $s > 0$, put

$$R_g f(x) = f(g^{-1}x), \qquad H_{q,g}(s) = \mathrm{Tr}(R_g e^{-s\Delta_{Y_q}}), \qquad \ell_q(g) = \min_{y \in Y_q} d_{Y_q}(y, gy).$$

Invariance of distance under deck transformations gives $\ell_q(hgh^{-1}) = \ell_q(g) = \ell_q(g^{-1})$ for $g, h \in Q$. More generally, let a finite group Γ act freely by isometries on a closed connected Riemannian manifold Z, with quotient X. The flat bundles on X associated to representations of Γ are defined by the induced marked quotient $\Pi \twoheadrightarrow \Gamma$. Put $R_\gamma f(x) = f(\gamma^{-1}x)$. Let λ range over the distinct eigenvalues of Δ_Z, and put $E_\lambda = \ker(\Delta_Z - \lambda)$ over $\mathbb{C}$. Write Hom_Γ for spaces of Γ-equivariant linear maps. For $\gamma \in \Gamma$ and $\rho \in \mathrm{Irr}(\Gamma)$, put

$$a_\lambda(\gamma) = \mathrm{Tr}(R_\gamma|_{E_\lambda}), \qquad m_\rho(\lambda) = \dim \mathrm{Hom}_\Gamma(V_\rho, E_\lambda).$$

The equivariance condition (5.1) gives

$$\ker(\Delta_{X,\rho^*} - \lambda) \simeq (E_\lambda \otimes V_\rho^*)^\Gamma \simeq \mathrm{Hom}_\Gamma(V_\rho, E_\lambda).$$

Scalar complex conjugation is an antilinear automorphism of E_λ commuting with each R_γ. Since the conjugate representation of ρ is isomorphic to ρ^*, the multiplicities of ρ and ρ^* in E_λ are equal. Consequently,

$$m_\rho(\lambda) = \dim\ker(\Delta_{X,\rho^*} - \lambda) = \dim\ker(\Delta_{X,\rho} - \lambda).$$

Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions. Define $D_\gamma, W_\rho \in \mathcal{S}'(\mathbb{R})$ by

$$D_\gamma = \sum_\lambda a_\lambda(\gamma) e^{it\sqrt{\lambda}}, \qquad W_\rho = \sum_\lambda m_\rho(\lambda) e^{it\sqrt{\lambda}}.$$

For $N > \dim Z$, Weyl's law [38, Theorem 17.7] gives

$$\sum_\lambda \dim E_\lambda (1 + \sqrt{\lambda})^{-N} < \infty.$$

Since $|a_\lambda(\gamma)| \le \dim E_\lambda$ and $m_\rho(\lambda) \le \dim E_\lambda$, both series converge in $\mathcal{S}'(\mathbb{R})$. The real form of E_λ gives $a_\lambda(\gamma) \in \mathbb{R}$ and $a_\lambda(\gamma^{-1}) = a_\lambda(\gamma)$. Each a_λ is a class function. Write WF and WF_A for the smooth and analytic wavefront sets of distributions on $\mathbb{R}$ [60, Definitions 8.1.2 and 8.4.3].

Lemma 5.1. *For $s > 0$, $\rho \in \mathrm{Irr}(Q)$, and $g \in Q$,*

$$\Theta_{X,\rho}(s) = \frac{1}{|Q|} \sum_{g \in Q} \overline{\chi_\rho(g)} H_{q,g}(s), \qquad H_{q,g}(s) = \sum_{\rho \in \mathrm{Irr}(Q)} \chi_\rho(g) \Theta_{X,\rho}(s). \tag{5.2}$$

For $\rho \in \mathrm{Irr}(\Gamma)$ and $\gamma \in \Gamma$,

$$W_\rho = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_\rho(\gamma)} D_\gamma, \qquad D_\gamma = \sum_{\rho \in \mathrm{Irr}(\Gamma)} \chi_\rho(\gamma) W_\rho \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{5.3}$$

On every open interval, all W_ρ are smooth if and only if all D_γ are smooth. The same equivalence holds for real analyticity, and for smooth or real analytic germs at any $t \in \mathbb{R}$. Moreover,

$$\bigcup_{\rho \in \mathrm{Irr}(\Gamma)} \mathrm{WF}(W_\rho) = \bigcup_{\gamma \in \Gamma} \mathrm{WF}(D_\gamma), \qquad \bigcup_{\rho \in \mathrm{Irr}(\Gamma)} \mathrm{WF}_A(W_\rho) = \bigcup_{\gamma \in \Gamma} \mathrm{WF}_A(D_\gamma).$$

Proof. Character decomposition and orthogonality [30, §3.3, pp. 20–21] give

$$a_\lambda(\gamma) = \sum_{\rho \in \mathrm{Irr}(\Gamma)} m_\rho(\lambda) \chi_\rho(\gamma), \qquad m_\rho(\lambda) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_\rho(\gamma)} a_\lambda(\gamma).$$

For $(\Gamma, Z) = (Q, Y_q)$, absolute summation against $e^{-s\lambda}$ gives (5.2). Summation against $e^{it\sqrt{\lambda}}$ in $\mathcal{S}'(\mathbb{R})$ gives (5.3).

Each family is a finite constant linear combination of the other. Finite linear combinations preserve smoothness and real analyticity. For either $\mathrm{WF}_* = \mathrm{WF}$ or $\mathrm{WF}_* = \mathrm{WF}_A$,

$$\mathrm{WF}_*(W_\rho) \subseteq \bigcup_{\gamma \in \Gamma} \mathrm{WF}_*(D_\gamma), \qquad \mathrm{WF}_*(D_\gamma) \subseteq \bigcup_{\rho \in \mathrm{Irr}(\Gamma)} \mathrm{WF}_*(W_\rho).$$

Taking unions gives both equalities. □

The scalar half-wave trace on the ρ-isotypic subspace of $L^2(Z)$ is $(\dim V_\rho) W_\rho$. In particular,

$$W_1 = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} D_\gamma, \qquad D_1 = \sum_{\rho \in \mathrm{Irr}(\Gamma)} (\dim V_\rho) W_\rho.$$

5.2. **Heat exponents and attained minima.**

Lemma 5.2. *Let F be an isometry of a closed connected Riemannian n-manifold Z, $n > 0$. Put $R_F f = f \circ F^{-1}$ and $a = \min_{x \in Z} d_Z(x, Fx)$, and let K_s be the heat kernel of Δ_Z. For $s > 0$,*

$$H_F(s) := \operatorname{Tr}(R_F e^{-s\Delta_Z}) = \int_Z K_s(x, Fx)\, dv_Z(x) > 0.$$

There are $c, C, s_0 > 0$ such that, for $0 < s < s_0$,

$$c s^{n/2} e^{-a^2/(4s)} \le H_F(s) \le C s^{-n-1} e^{-a^2/(4s)}. \tag{5.4}$$

Hence $\lim_{s \downarrow 0} -4s \log H_F(s) = a^2$. In particular,

$$\lim_{s \downarrow 0} -4s \log H_{q,g}(s) = \ell_q(g)^2. \tag{5.5}$$

The $H_{q,g}$ are real, positive, and invariant under conjugacy and inversion. For $g \neq 1$, $\ell_q(g) > 0$. There are $a_q, N_q > 0$ such that

$$H_{q,g}(s) = O(s^{-N_q} e^{-a_q/s}) \qquad (s \downarrow 0)$$

uniformly for $g \in Q \setminus \{1\}$.

Proof. Lemma 3.7, applied to F^{-1}, and invariance of Riemannian volume give the trace formula and its positivity. For the scalar Laplacian, the global estimates [74, Theorems 3.5 and 3.8, equation (4.1)] give

$$c_0 s^{-n/2} e^{-d_Z(x,y)^2/(4s)} \le K_s(x, y) \le C_0 s^{-n-1} e^{-d_Z(x,y)^2/(4s)}$$

for all $x, y \in Z$ and $0 < s < s_0$. Integration gives the upper bound in (5.4) with $C = C_0 \operatorname{vol}(Z)$.

Choose x_0 with $d_Z(x_0, Fx_0) = a$. For $x \in B_Z(x_0, s)$,

$$d_Z(x, Fx) \le d_Z(x, x_0) + a + d_Z(Fx_0, Fx) \le a + 2s.$$

After decreasing s_0, one has $\operatorname{vol} B_Z(x_0, s) \ge b\, s^n$ with $b > 0$. Thus

$$H_F(s) \ge c_0 b\ s^{n/2} e^{-(a+2s)^2/(4s)} = c_0 b\ s^{n/2} e^{-a^2/(4s)} e^{-a-s},$$

which gives the lower bound with $c = c_0 b e^{-a-s_0}$. The logarithmic limit follows from (5.4) and $s \log s \to 0$, also when $a = 0$.

Set $Z = Y_q$. For $g, h \in Q$, the identities $K_s(x, y) = K_s(y, x) = K_s(hx, hy)$ give $H_{q,g^{-1}} = H_{q,g}$ and $H_{q,hgh^{-1}} = H_{q,g}$. If $g \neq 1$, the continuous function $y \mapsto d_{Y_q}(y, gy)$ is positive on the compact manifold Y_q, since the deck action is free. For $Q \neq \{1\}$, put $\delta_q = \min_{g \neq 1} \ell_q(g) > 0$. The upper bound gives a constant $A_q > 0$ such that, uniformly for $g \neq 1$,

$$H_{q,g}(s) \le A_q s^{-\dim X - 1} e^{-\delta_q^2/(4s)}.$$

Thus $a_q = \delta_q^2/4$ and $N_q = \dim X + 1$ suffice. If $Q = \{1\}$, any $a_q, N_q > 0$ satisfy the stated estimate. □

Lemma 5.3. *Every nontrivial free homotopy class in X has a positive minimum represented by a smoothly closed geodesic. Only finitely many classes have minimum at most a given $R < \infty$. Moreover, for $g \in Q$,*

$$\ell_q(g) = \min\{\ell_X(c) \,:\, c \in \Pi,\ q(c) \text{ is conjugate to } g\}. \tag{5.6}$$

The minimum is attained, including when $g = 1$.

Proof. Every nontrivial free homotopy class contains a nonconstant closed geodesic of least length [69, Proposition 6.28, p. 174]. Closed geodesics of bounded length on a compact Riemannian manifold represent only finitely many free homotopy classes [34, §2, p. 3647].

Let α be a piecewise smooth loop in the free homotopy class of $c \in \Pi$, where $q(c)$ is conjugate to g. A lift to Y_q joins some y to hy, with h conjugate to g. The covering is a local isometry, so

$$L(\alpha) \ge d_{Y_q}(y, hy) \ge \ell_q(h) = \ell_q(g).$$

Taking the infimum over α gives $\ell_X(c) \geq \ell_q(g)$.

Choose $y \in Y_q$ with $d_{Y_q}(y, gy) = \ell_q(g)$. A minimizing geodesic segment from y to gy projects to a loop in a class c with $q(c)$ conjugate to g. Its length is $d_{Y_q}(y, gy)$; hence

$$\ell_q(g) \leq \ell_X(c) \leq d_{Y_q}(y, gy) = \ell_q(g).$$

This gives (5.6), with the minimum attained. For $g = 1$, the constant loop gives $\ell_q(1) = \ell_X(1) = 0$. □

5.3. **The minimum formula for a finite target.**

Theorem 5.4. *Let M, N be closed connected Riemannian manifolds with marked fundamental groups E, Γ, where Γ is finite. Let $\widetilde{M}$ and $\widetilde{N}$ be their universal Riemannian covers, and let $p : E \twoheadrightarrow \Gamma$ have kernel C. Suppose*

$$\operatorname{Spec} \Delta_{M,p^*\rho} = \operatorname{Spec} \Delta_{N,\rho} \qquad (\rho \in \operatorname{Irr}(\Gamma)).$$

Then $Y_p = C\backslash\widetilde{M}$ and $\widetilde{N}$ have equal Γ-equivariant heat and half-wave traces, and

$$\ell_N(\gamma) = \min\{\ell_M(e) : e \in E,\ p(e) = \gamma\} \qquad (\gamma \in \Gamma). \tag{5.7}$$

If E is finite, let $H_{M,e}$ and $D_{M,e}$, for $e \in E$, denote the equivariant heat and half-wave traces on $\widetilde{M}$, and define $H_{N,\gamma}$ and $D_{N,\gamma}$ on $\widetilde{N}$ in the same way. Then, for $s > 0$ and $\gamma \in \Gamma$,

$$H_{N,\gamma}(s) = \frac{1}{|C|} \sum_{p(e)=\gamma} H_{M,e}(s), \qquad D_{N,\gamma} = \frac{1}{|C|} \sum_{p(e)=\gamma} D_{M,e}. \tag{5.8}$$

Equality of the twisted spectra for irreducibles is equivalent to equality for all finite-dimensional unitary representations of Γ.

Proof. The trivial representation gives equal scalar spectra. If either manifold is a point, both are points: a positive-dimensional closed manifold has unbounded Laplace spectrum. In this case $E = \Gamma = 1$ and the formulas hold. Suppose henceforth that both dimensions are positive.

The covers Y_p and $\widetilde{N}$ are compact, with deck group Γ. Lemma 5.1 gives equality of their equivariant heat and half-wave traces. Equations (5.5) and (5.6) give

$$\ell_N(\gamma) = \min\{\ell_M(e) : e \in E,\ p(e) \text{ is conjugate to } \gamma\},$$

with attainment. If a minimizing e satisfies $p(e) = h\gamma h^{-1}$, choose $a \in E$ with $p(a) = h$. Then

$$p(a^{-1}ea) = \gamma, \qquad \ell_M(a^{-1}ea) = \ell_M(e),$$

which gives (5.7).

Suppose E is finite. Pullback under $\widetilde{M} \to Y_p$, multiplied by $|C|^{-1/2}$, identifies $L^2(Y_p)$ unitarily with $L^2(\widetilde{M})^C$. The orthogonal projector onto this subspace is

$$P_C = \frac{1}{|C|} \sum_{c \in C} R_c.$$

It commutes with the Laplacian and, since C is normal in E, with every R_e. For $p(e) = \gamma$, the restriction of R_e to the C-invariant subspace corresponds to R_γ. Since $R_e P_C = |C|^{-1} \sum_{p(h)=\gamma} R_h$, restriction to each Laplace eigenspace gives

$$\operatorname{Tr}\big(R_\gamma \mid \ker(\Delta_{Y_p} - \lambda)\big) = \frac{1}{|C|} \sum_{p(h)=\gamma} \operatorname{Tr}\big(R_h \mid \ker(\Delta_{\widetilde{M}} - \lambda)\big).$$

Summation with weights $e^{-s\lambda}$ gives the heat identity in (5.8). Summation with weights $e^{it\sqrt{\lambda}}$ in $\mathcal{S}'(\mathbb{R})$ gives its half-wave identity.

Every finite-dimensional unitary representation of Γ is a finite direct sum of irreducibles. Spectral multiplicities add under these sums [30, (3.5)], so equality for irreducibles is equivalent to equality for all finite-dimensional unitary representations. □

Suppose E is finite and C is central. For $\tau \in \operatorname{Irr}(E)$, Schur's lemma gives a character $\zeta_\tau : C \to \mathrm{U}(1)$ such that $\tau(c) = \zeta_\tau(c)\,\mathrm{id}_{V_\tau}$ for $c \in C$. Then τ factors through p if and only if $\zeta_\tau = 1$.

Corollary 5.5. *Under the hypotheses of Theorem 5.4, let $H \le E$ have finite index. Put $\bar{H} = p(H)$, regard $p|_H$ as a map $H \to \bar{H}$, and equip the connected covers*

$$M_H = H\backslash\widetilde{M}, \qquad N_{\bar{H}} = \bar{H}\backslash\widetilde{N}$$

with the lifted Riemannian metrics. If $C \le H$, these covers have equal flat spectra for every finite-dimensional unitary representation of $\bar{H}$, pulled back to H along $p|_H$. If $C \not\subseteq H$, their covering degrees are different and their scalar spectra are unequal.

Proof. Suppose $C \le H$. The restriction of p identifies H/C with $\bar{H}$. For a finite-dimensional unitary representation σ of $\bar{H}$, let $\tau = \operatorname{Ind}_{\bar{H}}^{\Gamma} \sigma$ be the induced representation. The towers

$$Y_p \longrightarrow M_H \longrightarrow M, \qquad \widetilde{N} \longrightarrow N_{\bar{H}} \longrightarrow N$$

have regular composite coverings with group Γ and first coverings with group $\bar{H}$. Apply Sunada's induction identity [99, Lemma 1, pp. 100–101] to the strictly positive operators $1 + \Delta_M$ and $1 + \Delta_N$, and subtract the identity; see also [30, Lemma 3.8.1, pp. 24–25]. The coset operator is unitary for Riemannian densities and commutes with the componentwise scalar Laplacian; no orientation is required. Theorem 5.4 therefore gives

$$\begin{aligned} \operatorname{Spec} \Delta_{M_H,(p|_H)^*\sigma} &= \operatorname{Spec} \Delta_{M,p^*\tau} \\ &= \operatorname{Spec} \Delta_{N,\tau} = \operatorname{Spec} \Delta_{N_{\bar{H}},\sigma}. \end{aligned}$$

For any finite-index $H \le E$, normality of C gives

$$[E : H] = [E : HC][HC : H] = [\Gamma : \bar{H}]\,[C : C \cap H].$$

If $C \not\subseteq H$, then $C \neq 1$, so M has positive dimension. The scalar spectra of M and N are equal; hence N also has positive dimension, and Weyl's law [38, Theorem 17.7] gives equal dimensions and volumes. Consequently

$$\operatorname{vol}(M_H) = [C : C \cap H]\operatorname{vol}(N_{\bar{H}}) > \operatorname{vol}(N_{\bar{H}}).$$

Their leading Weyl coefficients are different, so their scalar spectra are unequal. □

Part II. Closed geodesics and wave traces

The rank-one $L_1 \times P_1$-equivalence intertwines the cyclic actions and residual flows of Corollary 4.14. The geodesic closing equations also involve the exponential periods of G_1 and H_1. The cyclic quotients below have different full period sets: in dimension eight all unitary flat spectra agree under one C_7-marking, while in dimension nine the common half-wave trace is analytic near every period in the difference (Theorems 7.7 and 7.1). Equivariance under the residual flows also gives scalar-isospectral connection metrics on fixed manifolds with primitive periods absent from the other period set, including infinitely many on a simply connected eighteen-manifold (Theorem 8.8).

6. Closing equations and connection metrics

Let $R(s)$ denote rotation through angle s in an oriented real two-plane. Put

$$G_A = \mathrm{U}(3), \qquad G_B = \mathrm{Sp}(1) \times \mathrm{SO}(4).$$

On $\mathfrak{u}(3)$, $\mathfrak{sp}(1)$ and $\mathfrak{so}(4)$, respectively, put

$$Q_A(X,Y) = -\operatorname{Re}\operatorname{Tr}(XY), \qquad Q_C(X,Y) = -\tfrac{1}{2}\operatorname{Re}\operatorname{Tr}(XY), \qquad Q_D(X,Y) = -\tfrac{1}{2}\operatorname{Tr}(XY). \tag{6.1}$$

The first two traces are taken in the defining complex representations; the third is taken in the defining real representation. Write g_{G_A} and g_{G_B} for the corresponding metrics, with g_{G_B} induced by $Q_C \oplus Q_D$.

Identify Sp(1) with the unit quaternions, with standard imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$. For $s, v \in \mathbb{R}$, define the homomorphisms

$$\iota_A(s, v) = \operatorname{diag}(e^{is}, e^{-is}, e^{iv}), \tag{6.2}$$

$$\iota_B(s, v) = (e^{\mathbf{i}s}, \operatorname{diag}(R(s), R(v))). \tag{6.3}$$

The first component on the second line is a unit quaternion. Both homomorphisms are 2π-periodic in each parameter. Throughout,

$$h = \frac{2\pi}{7}, \qquad \zeta = e^{ih}.$$

For $i = A, B$ set

$$\ell_i = \iota_i(h, 0), \qquad r_i = \iota_i(3h, h), \qquad \Phi_i(g) = \ell_i g r_i^{-1},$$

and define orbit quotients

$$M_9 = G_A/\langle \Phi_A \rangle, \qquad N_9 = G_B/\langle \Phi_B \rangle. \tag{6.4}$$

6.1. **Quaternionic coordinates and metric scales.** Write $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathbf{i}$ and $\mathbb{H} = \mathbb{C}\mathbf{j} \oplus \mathbb{C}$, where the ordered bases of the two real planes are $(\mathbf{j}, \mathbf{k})$ and $(1, \mathbf{i})$. For imaginary quaternions u, v, let L_u and R_v denote left multiplication by u and right multiplication by v. The homomorphism

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(4), \qquad (a, b) \,:\, z \longmapsto azb^{-1}, \tag{6.5}$$

is the double covering with kernel $\{(1, 1), (-1, -1)\}$ [103, §4.1, equations (4.1.40)–(4.1.42)]. Its differential is $(u, v) \mapsto L_u - R_v$.

For $a = e^{\mathbf{i}\alpha}$ and $b = e^{\mathbf{i}\beta}$, the two plane angles are

$$(s, v) = (\alpha + \beta, \alpha - \beta). \tag{6.6}$$

Indeed, for $w \in \mathbb{C}$, the identity $\mathbf{j}e^{-\mathbf{i}\beta} = e^{\mathbf{i}\beta}\mathbf{j}$ gives $e^{\mathbf{i}\alpha}(w\mathbf{j})e^{-\mathbf{i}\beta} = e^{\mathbf{i}(\alpha+\beta)}w\mathbf{j}$, whereas $e^{\mathbf{i}\alpha}we^{-\mathbf{i}\beta} = e^{\mathbf{i}(\alpha-\beta)}w$. One has

$$-\tfrac{1}{2}\operatorname{Tr}_{\mathbb{R}}(L_u^2) = 2|u|^2, \qquad -\tfrac{1}{2}\operatorname{Tr}_{\mathbb{R}}(R_v^2) = 2|v|^2, \qquad \operatorname{Tr}_{\mathbb{R}}(L_u R_v) = 0.$$

The first two identities follow from $u^2 = -|u|^2$, $v^2 = -|v|^2$. For the third identity, bilinearity reduces to $u, v \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. If $u = v$, the diagonal entries of $L_u R_u$ in the quaternionic basis consist of $-1, -1, 1, 1$. If $u \neq v$, quaternion multiplication sends every basis vector to a signed different basis vector, so all four diagonal entries are zero. Since $L_u R_v = R_v L_u$, expansion of $-\frac{1}{2}\operatorname{Tr}_{\mathbb{R}}(L_u - R_v)^2$ gives the pullback of Q_D:

$$2(|u|^2 + |v|^2). \tag{6.7}$$

The defining complex matrix of an imaginary quaternion u has eigenvalues $i|u|$ and $-i|u|$. Hence $Q_C(u, u) = -\frac{1}{2}\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(u^2) = |u|^2$. Let $S^3(R)$ denote the round three-sphere of radius $R > 0$. Consequently G_B has the Riemannian double cover

$$S^3(1) \times S^3(\sqrt{2}) \times S^3(\sqrt{2}).$$

6.2. **The finite actions and their torus coordinates.** Both ℓ_i and r_i have order seven, so $\Phi_i^7 = I$. For $1 \leq j \leq 6$, a fixed point g would satisfy $g^{-1}\ell_i^j g = r_i^j$, so the two factors would be conjugate in G_i.

On the unitary side,

$$\det \ell_A^j = 1, \qquad \det r_A^j = e^{ijh} \neq 1.$$

Thus they are not conjugate. On the other side, their first components in Sp(1) have angles jh and $3jh$. Conjugacy of unit quaternions identifies the angles up to sign modulo 2π. It would require $j \equiv \pm 3j \pmod 7$, impossible for $j \neq 0$. Thus both cyclic actions are free and have exactly seven elements.

With their bi-invariant metrics, G_A, G_B are globally symmetric and have nonnegative sectional curvature; their geodesics are $g\exp(tV)$ [103, Lemma E.4.1, Proposition E.4.2 and equation (E.4.13)]. The finite free quotients are therefore closed real-analytic locally symmetric nine-manifolds of nonnegative sectional curvature.

The torus coordinates are $i\,\mathrm{diag}(x_1,x_2,x_3)$ on $\mathfrak{u}(3)$, $\mathbf{i}x_0$ on $\mathfrak{sp}(1)$, and $\mathrm{diag}(x_1R'(0),x_2R'(0))$ on $\mathfrak{so}(4)$. Equations (6.1) give Euclidean squared norm in these coordinates. The full torus period lattices are

$$2\pi\mathbb{Z}^3 \quad \text{on both groups.} \tag{6.8}$$

For SO(4), each real two-plane has independent period 2π. The spin covering group instead has plane-angle period lattice $\{2\pi(a,b) : a,b\in\mathbb{Z},\ a\equiv b \pmod 2\}$, since both $(s+v)/2$ and $(s-v)/2$ must be in $2\pi\mathbb{Z}$ there.

6.3. **Scalar spectra.** Let $B_G(t;L,R)$ denote the bilateral heat trace in (2.6).

Corollary 6.1. *The manifolds M_9, N_9 have equal scalar Laplace spectra, including multiplicities.*

Proof. The invariant projection is $\frac{1}{7}\sum_{j=0}^{6}\Phi_i^{j*}$. Finite-quotient descent in Section 3.6 gives

$$\Theta_{M_9}(t)=\frac{1}{7}\sum_{j=0}^{6}B_{G_A}(t;\ell_A^j,r_A^j),\qquad \Theta_{N_9}(t)=\frac{1}{7}\sum_{j=0}^{6}B_{G_B}(t;\ell_B^j,r_B^j).$$

Equation (4.12), at $(u,s,v)=(jh,3jh,jh)$, equates each summand. Lemma 2.3 then equates the spectral multiplicities; its absolute-convergence hypothesis follows by bounding their differences by the two scalar heat traces. □

6.4. **Circle quotients and marked connection spectra.** The induced metric on either parameter torus is $2\,ds^2+dv^2$. Let

$$H_i=\{\iota_i(0,v) : v\in\mathbb{R}/(2\pi\mathbb{Z})\},\qquad X_i=G_i/H_i\quad(i=A,B),\qquad X_A=A,\quad X_B=B,$$

with the normal homogeneous submersion metrics g_A, g_B. Write $\mathfrak{h}_i=\mathrm{Lie}(H_i)$. Set

$$r_{i,0}=\iota_i(3h,0),\quad \phi_i(gH_i)=\ell_i g r_{i,0}^{-1}H_i. \tag{6.9}$$

The elements $r_{i,0}$ centralize H_i, so these are well-defined isometries. Put

$$M_8=A/\langle\phi_A\rangle,\qquad N_8=B/\langle\phi_B\rangle. \tag{6.10}$$

The period-seven flows extending these deck transformations are

$$F_i(s)(gH_i)=\iota_i(hs,0)g\iota_i(3hs,0)^{-1}H_i,\qquad \phi_i=F_i(1). \tag{6.11}$$

Let $K_i(x)=\frac{d}{ds}\big|_{s=0}F_i(s)x$. For a vector field K, write also $Kf=df(K)$; thus $\frac{d}{ds}\big|_{s=0}F_i(s)^*f=K_if$.

Lemma 6.2. *The spaces A,B are simply connected. The actions in* (6.10) *are free of order seven.*

Proof. The map

$$\mathrm{U}(3)/H_A\longrightarrow\mathrm{SU}(3),\qquad gH_A\longmapsto g\ \mathrm{diag}(1,1,(\det g)^{-1})$$

is well defined: right multiplication by $\mathrm{diag}(1,1,z)$ multiplies $\det g$ by z and cancels in the displayed expression. Its inverse is $g\mapsto gH_A$. Both maps are smooth. Thus $A\simeq\mathrm{SU}(3)$. The fibration $\mathrm{SU}(2)\to\mathrm{SU}(3)\to S^5$, given by the first column, and its homotopy sequence give $\pi_1(A)=\pi_2(A)=0$ [53, Corollary 4.9 and Theorem 4.41].

Write $H_B=\{1\}\times H_D$, where $H_D=\{\mathrm{diag}(I_2,R(v)) : v\in\mathbb{R}\}\le\mathrm{SO}(4)$. The group H_D is the full stabilizer of the ordered frame $(\mathbf{j},\mathbf{k})$: an element fixing these two vectors restricts to a rotation on their oriented orthogonal complement. Transitivity on ordered orthonormal two-frames therefore gives $\mathrm{SO}(4)/H_D\simeq V_2(\mathbb{R}^4)$, where $V_2(\mathbb{R}^4)=\{(x,y)\in S^3\times S^3 : \langle x,y\rangle=0\}$ is the Stiefel manifold of orthonormal two-frames. Let $S^2=\{z\in\mathrm{Im}\,\mathbb{H} : |z|=1\}$. The maps

$$(x,y)\longmapsto(x,x^{-1}y),\qquad (x,z)\longmapsto(x,xz)$$

identify the Stiefel manifold with $S^3\times S^2$: $\mathrm{Re}(x^{-1}y)=\langle x,y\rangle=0$ and $|x^{-1}y|=1$. Consequently $B\simeq S^3\times S^3\times S^2$, with $\pi_1(B)=0$ and $\pi_2(B)=\mathbb{Z}$ [53, Corollaries 4.9 and 4.25]. A full rotation in H_B lifts in the spin group from $(1,1)$ to $(-1,-1)$, so $\pi_1(H_B)\to\pi_1(G_B)=C_2$ is reduction modulo two.

If ϕ_A^j fixed a coset for $1 \le j \le 6$, then ℓ_A^j would be conjugate to $r_{A,0}^j b$ for some $b = \mathrm{diag}(1, 1, e^{iv})$. Determinants force $e^{iv} = 1$. The remaining eigenvalue sets would require

$$\{j, -j, 0\} = \{3j, -3j, 0\} \quad \text{in } \mathbb{Z}/7\mathbb{Z},$$

which is impossible. On B, a fixed coset would require the first quaternionic components $e^{\mathbf{i}jh}$ and $e^{3\mathbf{i}jh}$ to be conjugate. This requires $j \equiv \pm 3j \pmod 7$, also impossible. Both actions have seventh power the identity, hence have order seven and are free. The markings of the quotient fundamental groups send $1 \in C_7$ to the displayed deck generator. □

The simply connected product presentation is

$$B = \frac{S^3(1) \times S^3(\sqrt{2}) \times S^3(\sqrt{2})}{\widetilde{H}}, \qquad \widetilde{H} = \{(1, e^{\mathbf{i}v}, e^{-\mathbf{i}v}) : v \in \mathbb{R}/(2\pi\mathbb{Z})\}. \tag{6.12}$$

Under the product of the identity on $\mathrm{Sp}(1)$ and (6.5), an element $(1, e^{\mathbf{i}v}, e^{-\mathbf{i}v})$ maps to $(1, \mathrm{diag}(I_2, R(2v)))$. Thus $\widetilde{H}$ maps onto H_B and contains the covering kernel $(1, -1, -1)$ at $v = \pi$. It is the full preimage of H_B and is connected. Lifts of ℓ_B and $r_{B,0}$, respectively, are

$$\widetilde{\ell} = (e^{\mathbf{i}h}, e^{\mathbf{i}h/2}, e^{\mathbf{i}h/2}), \qquad \widetilde{r} = (e^{3\mathbf{i}h}, e^{3\mathbf{i}h/2}, e^{3\mathbf{i}h/2}). \tag{6.13}$$

Their seventh powers equal the central element $(1, -1, -1)$. Their two-sided action therefore has seventh power the identity; its projection to B has order seven by Lemma 6.2, so it also has order seven.

6.4.1. *The quotient metric and its curvature.* Lemma 4.1 identifies the quotient Laplacians with the restrictions of the group Laplacians to right-H_i-invariant functions, including their form and operator domains. The curvature identities of Proposition 4.15 apply to both quotients. The Frobenius inequality gives $\|[X, Y]\| \le 2\|X\|\|Y\|$ on $\mathfrak{u}(3)$. For imaginary quaternions, $|uv - vu| \le 2|u||v|$. Thus the corresponding constants are 2 for Q_C and $\sqrt{2}$ for each spin factor with metric $2|\cdot|^2$. Squaring and summing over the product factors gives $\|[X, Y]\| \le 2\|X\|\|Y\|$ on $\mathrm{Lie}(G_B)$. Equation (4.16) therefore places every sectional curvature of A and B in $[0, 4]$.

On a round three-sphere of radius R, the Ricci endomorphism is $2R^{-2}I$. The three covering spheres in Section 6 therefore give $\mathrm{Ric}_{G_B} \ge g_{G_B}$. Equation (4.17) implies $\mathrm{Ric}_B \ge g_B$. For $A = \mathrm{U}(3)/H_A$, the circle generator $T = i\,\mathrm{diag}(0, 0, 1)$ satisfies $\mathrm{Tr}\, T = i$ and $Q_A(T, T) = 1$. Equation (4.15) therefore gives

$$\mathrm{Ric}_A \ge \tfrac{1}{2} g_A, \qquad \mathrm{Ric}_B \ge g_B.$$

The group G_B has volume

$$\frac{1}{2}\,\mathrm{Vol}(S^3(1))\,\mathrm{Vol}(S^3(\sqrt{2}))^2 = 32\pi^6.$$

Equation (4.12) at the identity gives the same group heat trace on G_A. The leading term $\mathrm{Vol}(G)(4\pi t)^{-\dim G/2}$ of the scalar heat trace [30, Section 3.10] consequently gives $\mathrm{Vol}(G_A) = 32\pi^6$. The circles H_i have length 2π, so

$$\mathrm{Vol}(A) = \mathrm{Vol}(B) = 16\pi^5. \tag{6.14}$$

The finite quotients have volume $16\pi^5/7$. Put $Q_B = Q_C \oplus Q_D$, $\mathfrak{m}_i = \mathfrak{h}_i^{\perp_{Q_i}}$, and let $\mathrm{pr}_{\mathfrak{m}_i}$ denote the Q_i-orthogonal projection. Write $g_{X_A} = g_A$ and $g_{X_B} = g_B$. The exponential slice gives an analytic local section $\sigma : U \to G_i$ of $G_i \to X_i$, where $U \subset X_i$ is open. For $x \in U$ and $V, W \in T_x X_i$,

$$g_{X_i}(V, W) = Q_i\big(\mathrm{pr}_{\mathfrak{m}_i}(\sigma(x)^{-1} d\sigma_x V), \mathrm{pr}_{\mathfrak{m}_i}(\sigma(x)^{-1} d\sigma_x W)\big).$$

The metric coefficients are analytic. The finite quotient maps are local analytic isometries, so M_8, N_8 inherit real-analytic metrics.

6.4.2. *The seven flat line bundles.* Since $\phi_i = F_i(1)$, Corollary 4.14 gives

$$\operatorname{Tr}(\phi_A^{j*} e^{-t\Delta_A}) = \operatorname{Tr}(\phi_B^{j*} e^{-t\Delta_B}) \quad (j \in C_7,\ t > 0). \tag{6.15}$$

For $r \in \{0, \dots, 6\}$, put $\chi_r(j) = e^{2\pi i r j/7}$ for $j \in C_7$. By Lemma 6.2, the maps $A \to M_8$ and $B \to N_8$ are universal covers with the displayed C_7 markings. The deck operators in Lemma 5.1 are $R_{i,j} = (\phi_i^{-j})^*$. Equation (6.15), with $-j$ in place of j, equates their equivariant heat traces. Formula (5.2) therefore gives

$$\Theta_{M_8,\chi_r}(t) = \Theta_{N_8,\chi_r}(t) \qquad (0 \le r \le 6,\ t > 0).$$

Lemma 2.3 gives equality of the corresponding spectral multiplicities. Every finite-dimensional unitary representation of C_7 is an orthogonal sum of these characters. Hence

$$\operatorname{Spec}(\Delta_{M_8,\rho}) = \operatorname{Spec}(\Delta_{N_8,\rho}) \qquad \big(\rho : C_7 \to \mathrm{U}(V_\rho)\big),$$

with the same marking and with multiplicities.

6.5. **Closing equations for normal homogeneous quotients.** Let G be a Lie group with a bi-invariant metric, let $H \le G$ be closed, and put $\mathfrak{g} = \operatorname{Lie}(G)$ and $\mathfrak{h} = \operatorname{Lie}(H)$. Give G/H its normal homogeneous metric. For $g \in G$ and $V \in \mathfrak{h}^\perp$, the curve $g\exp(tV)$ is a horizontal geodesic of constant speed $|V|$ [103, Proposition E.4.2]. Its projection $\gamma(t) = g\exp(tV)H$ is a geodesic [69, Problem 5-6(b), equation (5.27)]. Every initial tangent vector on G/H has a horizontal representative, so these are all its geodesics. The segment $0 \le t \le 1$ therefore has length $|V|$.

Lemma 6.3. *Let $\ell, r \in G$, suppose that r centralizes H, and set $\Phi(gH) = \ell g r^{-1} H$. For $j \in \mathbb{Z}$, the equalities $\gamma(1) = \Phi^j(\gamma(0))$ and $\dot\gamma(1) = d\Phi^j_{\gamma(0)}\dot\gamma(0)$ hold if and only if, for some $b \in H$,*

$$V \perp \mathfrak{h}, \qquad e^V = g^{-1}\ell^j g\, r^{-j} b, \qquad \operatorname{Ad}_{r^j b^{-1}} V = V. \tag{6.16}$$

For $H = \{1\}$ these equations reduce to

$$e^V = g^{-1}\ell^j g\, r^{-j}, \qquad \operatorname{Ad}_{r^j} V = V. \tag{6.17}$$

For $V = 0$ they mean that gH is fixed by Φ^j. If $V \neq 0$ and Φ generates a finite free action, they characterize smoothly closed quotient geodesics of length $|V|$.

Proof. Put $a = \ell^j g r^{-j}$. Endpoint equality is $ge^V = ab$ for some $b \in H$. At the representative a, the terminal velocity is represented by $\operatorname{Ad}_b V$, since $ab\exp(tV)H = a\exp(t\operatorname{Ad}_b V)H$. The transformed initial velocity is represented by $\operatorname{Ad}_{r^j} V$. Both vectors lie in $\mathfrak{h}^\perp$, on which the quotient differential is injective. Since r centralizes H, the elements r^j and b commute. Thus $\operatorname{Ad}_b V = \operatorname{Ad}_{r^j} V$ is equivalent to the last equation in (6.16). Conversely the equations give both endpoint and velocity equality. Uniqueness of the geodesic equation then extends the segment smoothly across every translated copy. Every quotient geodesic lifts horizontally, so all initial data are covered. Setting $H = \{1\}$ gives (6.17). □

Coordinates for the closing equations. For G_A and G_B, the torus exponential has kernel $2\pi\mathbb{Z}^3$ in the coordinates of (6.8). On the covering group $\widetilde{G}_B = \mathrm{Sp}(1)^3$, quaternionic angular coordinates (α, β, γ) project to

$$(x_0, x_1, x_2) = (\alpha, \beta + \gamma, \beta - \gamma), \qquad |V|^2 = \alpha^2 + 2\beta^2 + 2\gamma^2.$$

The regular cyclic quotients. For $1 \le j \le 6$, the three unitary eigenvalues of r_A^j have distinct residues $3j, -3j, j$ modulo seven. If $R = \operatorname{diag}(z_1, z_2, z_3)$, then

$$[V, R]_{ab} = (z_b - z_a)V_{ab}.$$

Thus $[V, r_A^j] = 0$ forces every off-diagonal entry of V to vanish. For the first factor of G_B, write $r = e^{\mathbf{i}\alpha}$ with $\sin\alpha \neq 0$. An imaginary quaternion u commutes with r precisely when $[u, \mathbf{i}] = 0$,

or equivalently $u \in \mathbb{R}\mathbf{i}$. In the orthogonal factor, the four complex eigenvalues of r_B^j have residues $3j, -3j, j, -j$, all distinct. A commuting real operator preserves each complex eigenline and therefore each of the two real rotation planes. Its skew-symmetric restriction to either plane is a real multiple of that plane's rotation generator. Hence the Lie-algebra centralizer is the Lie algebra of the specified two-plane torus. Equation (6.17) therefore makes V toral on both groups for every nonidentity deck power.

For a subgroup $T \le G$, write $N_G(T) = \{g \in G : gTg^{-1} = T\}$ for its normalizer. For $i = A, B$, let T_i be the maximal torus specified in Section 6.2. Put $\mathfrak{t}_i = \mathrm{Lie}(T_i)$ and $\mathcal{W}_i = N_{G_i}(T_i)/T_i$. Identify $\mathfrak{t}_i$ with $\mathbb{R}^3$ by those Euclidean coordinates. The group $\mathcal{W}_i$ acts on $\mathfrak{t}_i$ by conjugation. Put

$$a_A = (1, -1, 0), \quad b_A = (3, -3, 1), \qquad a_B = (1, 1, 0), \quad b_B = (3, 3, 1).$$

Thus $\ell_i = \exp(ha_i)$ and $r_i = \exp(hb_i)$. The orbit of a_A under the unitary Weyl group is its six coordinate permutations. The orbit of a_B under the product Weyl group is

$$\mathcal{W}_B a_B = \{(\epsilon, \eta, 0), (\epsilon, 0, \eta) : \epsilon, \eta \in \{1, -1\}\}. \tag{6.18}$$

Indeed the first coordinate may change sign, and even signed permutations in the last two coordinates take $(1, 0)$ to $(\pm 1, 0)$ or $(0, \pm 1)$.

Proposition 6.4. *Put $P_A = M_9$ and $P_B = N_9$. For $i = A, B$,*

$$\mathrm{Lsp}(P_i) = \left\{ h|m| \; : \; \begin{array}{l} m \in \mathbb{Z}^3 \setminus \{0\}, \text{ and either } m \in 7\mathbb{Z}^3, \\ \text{or } m \equiv j(wa_i - b_i) \pmod 7 \text{ for some } 1 \le j \le 6, \; w \in \mathcal{W}_i \end{array} \right\}. \tag{6.19}$$

In particular both length sets are locally discrete in $(0, \infty)$.

Proof. For $1 \le j \le 6$, the centralizer calculations give $V \in \mathfrak{t}_i$. Equation (6.17) makes $\exp V\, r_i^j \in T_i$ conjugate to ℓ_i^j. The groups G_i are compact and connected, so conjugacy in T_i is equivalence under $\mathcal{W}_i$ [29, Proposition 0.1]. Thus $\exp V\, r_i^j = w(\ell_i^j)$ for some $w \in \mathcal{W}_i$. By (6.8),

$$V = hm, \qquad m \equiv j(wa_i - b_i) \pmod 7.$$

Conversely, choose $g \in N_{G_i}(T_i)$ with $g^{-1}\ell_i g = w(\ell_i)$ and any integer lift m of the congruence. The vector $V = hm$ commutes with r_i^j, and both equations in (6.17) hold.

For $j = 0$, one has $\exp V = 1$. The closure of $\exp(\mathbb{R}V)$ is a torus, and [103, Proposition 6.A.3 and equations (6.A.12), (6.A.14), (6.A.16)] places it in a conjugate of T_i. Hence V is conjugate to a vector of $\mathfrak{t}_i$ with coordinates $2\pi k$, $k \in \mathbb{Z}^3$. Conjugation preserves the norm, so these lengths are exactly $h|m|$ with $m \in 7\mathbb{Z}^3 \setminus \{0\}$. Conversely, every such torus vector has exponential one. For $T > 0$, a length at most T has squared length $h^2 n$ with $1 \le n \le T^2/h^2$; only finitely many lengths occur. □

Lemma 6.5. *Let G be a compact Lie group with a bi-invariant metric, let $T \le G$ be a maximal torus, put $\mathfrak{t} = \mathrm{Lie}(T)$, and let $n \in N_G(T)$. Suppose that a finite free cyclic action is generated by $x \mapsto \ell x r^{-1}$ with $\ell, r \in T$. The coset Tn descends to an embedded totally geodesic flat torus in the quotient. Every primitive closed geodesic contained in this torus is embedded.*

Proof. The map $t \mapsto tn$ identifies the restricted deck action with translation by $\delta = \ell n r^{-1} n^{-1} \in T$, since $\ell(tn)r^{-1} = t\delta n$. Therefore the image is the quotient torus $T/\langle\delta\rangle$. Its map into the ambient quotient is injective: two representatives in Tn with the same image differ by one of these restricted deck translations. It is an immersion because both quotient maps are local isometries, and it is an embedding because its domain is compact and the ambient quotient is Hausdorff. For a bi-invariant metric, $\nabla_X Y = \frac{1}{2}[X, Y]$ on left-invariant fields [103, equation (E.4.11)]. Its restriction to the abelian Lie algebra $\mathfrak{t}$ vanishes; hence Tn is flat and totally geodesic, and these local properties descend. Write $T/\langle\delta\rangle = \mathfrak{t}/\Lambda$, where Λ is its period lattice. A nonconstant closed geodesic has the form $t \mapsto x + tV$ (mod Λ), with $x, V \in \mathfrak{t}$ and $V \neq 0$. Its periods form the subgroup $\{t \in \mathbb{R} : tV \in \Lambda\} = p\mathbb{Z}$, where $p > 0$ is its least period. It therefore induces an injective immersion of $\mathbb{R}/p\mathbb{Z}$, which is an embedding by compactness. □

6.6. **Twisted geodesics at rational flow times.** Let F be an isometry of a Riemannian manifold X. A geodesic $\gamma : [0,1] \to X$ is F-twisted if

$$\gamma(1) = F(\gamma(0)), \qquad \dot\gamma(1) = dF_{\gamma(0)}\dot\gamma(0).$$

A constant geodesic is F-twisted precisely when its value is a fixed point of F.

Lemma 6.6. *Let $q \in \mathbb{Z}_{>0}$, $\gcd(q,6) = 1$, $N = 7q$, and $h_q = 2\pi/N = h/q$. For every integer j, every $F_A(j/q)$- or $F_B(j/q)$-twisted geodesic has squared length in $h_q^2\mathbb{Z}_{\ge 0}$. On either A or B, every nonconstant ordinary closed geodesic has length at least 2π and squared length in $4\pi^2\mathbb{Z}_{>0}$.*

Proof. Since $F_i(s+7) = F_i(s)$, the twisting map depends only on j modulo N. Assume $j \in \{0,\dots,N-1\}$. By Lemma 6.3, a twisted segment has a horizontal velocity V and, for some $g \in G_i$ and $b \in H_i$,

$$R_i = \iota_i(3jh_q, 0)b^{-1}, \qquad e^V R_i = g^{-1}\iota_i(jh_q, 0)g, \qquad [V, R_i] = 0.$$

Since $\gcd(N,6) = 1$, the relation $6j \equiv 0 \pmod N$ forces $j = 0$. For $j \neq 0$, the angle $3jh_q$ is therefore neither zero nor π modulo 2π, and $e^{3ijh_q} \neq e^{-3ijh_q}$.

On A, write $V = iC$ with C Hermitian. Horizontality is $C_{33} = 0$. For some $v \in \mathbb{R}/(2\pi\mathbb{Z})$, the effective right element in (6.16) is

$$R = R_A = \operatorname{diag}(e^{3ijh_q}, e^{-3ijh_q}, e^{iv}).$$

The first two entries are distinct. If all three entries are distinct, $[C,R] = 0$ makes C diagonal and horizontality makes its third entry zero. For each of its first two eigenvalues θ, the corresponding eigenvalue of $e^{iC}R$ is $e^{i(\theta+3\epsilon jh_q)}$ for an $\epsilon \in \{1,-1\}$. If the last entry of R equals one of the first two, its eigenspaces have dimensions two and one, and C preserves them. On each of these two blocks, $R = e^{3i\epsilon jh_q}I$ for one of the same two signs. In either case each eigenvalue on a block with right phase $e^{3i\epsilon jh_q}$ satisfies

$$e^{i(\theta+3\epsilon jh_q)} \in \{e^{ijh_q}, e^{-ijh_q}, 1\}, \qquad \theta = h_q(t - 3\epsilon j + Nn)$$

for some $t \in \{j,-j,0\}$ and $n \in \mathbb{Z}$. In the case of three distinct right eigenvalues, the remaining eigenvalue of C is zero. Thus all three eigenvalues of C belong to $h_q\mathbb{Z}$, and $|V|^2 = \operatorname{Tr}(C^2) \in h_q^2\mathbb{Z}_{\ge 0}$.

On B, write $V = (u, V_D) \in \mathfrak{sp}(1) \oplus \mathfrak{so}(4)$. The symplectic component u centralizes the nonreal quaternion $e^{3\mathbf{i}jh_q}$, so it is $\theta_C\mathbf{i}$. Its exponential equation gives $\theta_C + 3jh_q \equiv \pm jh_q \pmod{2\pi}$ and hence $\theta_C \in h_q\mathbb{Z}$. In SO(4), for some $v \in \mathbb{R}/(2\pi\mathbb{Z})$, the effective right element is

$$R_D = \operatorname{diag}(R(\alpha), R(v)), \qquad \alpha = 3jh_q.$$

If $v \not\equiv \pm\alpha \pmod{2\pi}$, the two real planes have disjoint complex spectra. A commuting real matrix preserves both planes; its skew-symmetric restriction to each plane is a multiple of the plane-rotation generator. This remains true when $v = 0$ or π, since the second eigenspace is the second real plane and $\sin\alpha \neq 0$. Write the two angular speeds as θ, η. Horizontality is $\eta = 0$. The complex eigenvalues of $e^{V_D}R_D$ are those of a rotation through jh_q and a fixed plane. Hence $\theta + \alpha$ is congruent to 0, jh_q, or $-jh_q$ modulo 2π. Since $\alpha = 3jh_q$ and $2\pi = Nh_q$, one has $\theta \in h_q\mathbb{Z}$, and $|V_D|^2 = \theta^2$.

If $v \equiv \pm\alpha \pmod{2\pi}$, put $J = (R_D - \cos\alpha\, I)/\sin\alpha$. Then $J^2 = -I$ and $J^TJ = I$. The equations $[V_D, R_D] = 0$ and $[V_D, J] = 0$ are equivalent. A real skew-symmetric operator commuting with J is skew-Hermitian on the complex two-dimensional space $(\mathbb{R}^4, J)$. The unitary spectral theorem therefore gives its complex eigenvalues as $i\theta_1, i\theta_2$, with $\theta_1, \theta_2 \in \mathbb{R}$. On this complex space $R_D = e^{i\alpha}I$, so the complex eigenvalues of $e^{V_D}R_D$ are $e^{i(\theta_1+\alpha)}$ and $e^{i(\theta_2+\alpha)}$. Together with their complex conjugates, these give the spectrum of the complexification of the real operator, which by the closing equation is $\{e^{ijh_q}, e^{-ijh_q}, 1, 1\}$. Thus each $\theta_a + \alpha$ is congruent to 0, jh_q, or $-jh_q$ modulo 2π, and both θ_a are multiples of h_q. Each eigenvalue $-\theta_a^2$ of V_D^2 occurs twice over $\mathbb{R}$, whence

$$|V_D|^2 = -\tfrac{1}{2}\operatorname{Tr}_{\mathbb{R}}(V_D^2) = \theta_1^2 + \theta_2^2.$$

For $j = 0$ on A, $R = \operatorname{diag}(1, 1, e^{iv})$. If $e^{iv} \neq 1$, commutation splits off the third coordinate and $C_{33} = 0$ leaves the third eigenvalue of $e^{iC}R$ unequal to one, a contradiction. Thus $R = I$ and the eigenvalues of C are multiples of 2π.

For $j = 0$ on B, the symplectic exponential must be one. The orthogonal effective element is $\operatorname{diag}(I_2, R(v))$. When $v \neq 0$ modulo 2π, the spectra of its two plane restrictions are disjoint, even at $v = \pi$, so its Lie-algebra centralizer is block diagonal. The last horizontal angular speed is zero, so its exponential cannot cancel $R(v)$. Therefore $R_D = I$ and $e^{V_D} = I$. Its two rotation speeds and the magnitude of the symplectic vector are multiples of 2π. On either space the resulting squared norm is

$$|V|^2 = 4\pi^2(n_1^2 + n_2^2 + n_3^2), \qquad n_1, n_2, n_3 \in \mathbb{Z}.$$

For $V \neq 0$ this is at least $4\pi^2$, and for $V = 0$ it is zero. □

6.7. **Connection metrics and flat connection spectra.**

Lemma 6.7. *Let (W_i, g_i), $i = 0, 1$, be closed Riemannian manifolds with Killing fields K_i. Suppose that a unitary map $T : L^2(W_0, dv_{g_0}) \to L^2(W_1, dv_{g_1})$ intertwines their scalar Laplacians as self-adjoint operators and the one-parameter groups generated by K_0, K_1. Let (Z, b) be a closed Riemannian manifold, let α be a smooth real one-form on Z, and let $E \to Z$ be a finite-rank Hermitian bundle with a metric connection ∇^E. For $v, w \in T_xW_i$ and $X, Y \in T_zZ$, set*

$$G_i(v + X, w + Y) = g_i\big(v + \alpha(X)K_i, w + \alpha(Y)K_i\big) + b(X, Y). \tag{6.20}$$

Then $T \otimes I$, where I is the identity on $L^2(Z, E)$, intertwines the connection Laplacians on the pullbacks of E to $(W_i \times Z, G_i)$, including their self-adjoint domains. In particular, the scalar Laplacians are unitarily equivalent.

Proof. Let Ψ_i be the flow of K_i, and let P be the closure of $\{(\Psi_0(t), \Psi_1(t)) : t \in \mathbb{R}\}$ in $\operatorname{Isom}(W_0, g_0) \times \operatorname{Isom}(W_1, g_1)$. The isometry groups are compact Lie groups [43, Theorem 2.1]; their connected abelian subgroup P is therefore a compact torus, possibly trivial. Put $\mathfrak{p} = \operatorname{Lie}(P)$ and write $\exp(t\xi) = (\Psi_0(t), \Psi_1(t))$, with $\xi \in \mathfrak{p}$, and regard the two actions as right actions. Strong continuity extends the equivariance of T to P.

Let $\theta_P = p^{-1}dp$ be the left Maurer–Cartan form on P. Pull θ_P and α back to the trivial principal bundle $Z \times P$, and take the connection $\omega = \theta_P - \alpha\xi$. Under the identification $[((z, p), x)] \mapsto (xp^{-1}, z)$, its horizontal vectors are $(-\alpha(X)K_i, X)$. Its connection metric is therefore (6.20). Proposition 4.7, with coefficient bundle E, gives the unitary $T \otimes I$ and the stated equivalence, including the operator domains.

The shear $A_i(v, X) = (v + \alpha(X)K_i, X)$ has determinant one and $G_i = A_i^*(g_i \oplus b)$. Hence $dv_{G_i} = dv_{g_i}\,dv_b$ and, for a smooth section f of the pullback of E, write $d_{W_i}f$ for its derivative in the fibre variable and $\nabla^E_Z f$ for its covariant derivative in the base variable; put $K_i f = d_{W_i}f(K_i)$. If ∇ denotes the pullback connection, its Dirichlet form is $\mathcal{E}_i(f) = \int_{W_i \times Z} |\nabla f|^2_{G_i}\, dv_{G_i}$, and

$$\mathcal{E}_i(f) = \int_{W_i \times Z} \big(|d_{W_i}f|^2_{g_i} + |\nabla^E_Z f - \alpha K_i f|^2_b\big)\, dv_{g_i}\, dv_b. \tag{6.21}$$

□

Proposition 6.8. *Let (W, g_W) be closed and connected, and let $(\widetilde{W}, \widetilde{g}_W)$ be its universal Riemannian cover. Let Ψ be an isometric flow of period seven with generator K, and let $L > 0$. Write $dv_W = dv_{g_W}$, and let s be the coordinate on $\mathbb{R}$. For $d \in \mathbb{Z}_{>0}$, the invariant field ∂_s and one-form ds descend to $\mathbb{R}/d\mathbb{Z}$. Write $[s]_d$ for the class of s modulo $d\mathbb{Z}$. On $P = W \times (\mathbb{R}/\mathbb{Z})$, for $\tau \in \mathbb{R}$, $v, w \in T_xW$ and $a, b \in \mathbb{R}$, define*

$$g_\tau(v + a\partial_s, w + b\partial_s) = g_W(v + \tau aK, w + \tau bK) + L^2 ab.$$

This metric is isometric to the quotient of $(W \times \mathbb{R}, g_W + L^2 ds^2)$ by $T_\tau(x, s) = (\Psi(\tau)x, s + 1)$, and its Riemannian volume density is $L\, dv_W\, ds$. Its Riemannian universal cover is $(\widetilde{W} \times \mathbb{R}, \widetilde{g}_W + L^2 ds^2)$.

For $d \in \mathbb{Z}_{>0}$, let $P_d = W \times (\mathbb{R}/d\mathbb{Z})$ and $\pi_d : P_d \to P$, $\pi_d(x,[s]_d) = (x,[s]_1)$. Give P_d the metric $\pi_d^* g_\tau$. For $\vartheta \in \mathbb{R}$, define the character $\rho_\vartheta : d\mathbb{Z} \to \mathrm{U}(1)$ by $\rho_\vartheta(kd) = e^{2\pi i k\vartheta}$, $k \in \mathbb{Z}$. Let $E_\vartheta \to P_d$ be the pullback from $\mathbb{R}/d\mathbb{Z}$ of the flat line bundle labelled by ρ_ϑ in (5.1). *Let $(\lambda_a, ihr_a)_{a\in\mathcal{I}}$, $h = 2\pi/7$, be the joint eigenvalues of (Δ_W, K), indexed by a complete orthonormal joint eigenbasis. Then $r_a \in \mathbb{Z}$ and the complete connection spectrum is*

$$E_{a,m} = \lambda_a + \frac{4\pi^2}{L^2}\left(\frac{m+\vartheta}{d} - \frac{\tau r_a}{7}\right)^2, \qquad (a,m) \in \mathcal{I} \times \mathbb{Z}. \tag{6.22}$$

Thus equality of joint eigenspace multiplicities for two flows on (W, g_W) gives equality of all these spectra, with every fixed finite-dimensional unitary representation of $d\mathbb{Z}$ and every $d \geq 1$.

For $\tau = 1/q$, $q \in \mathbb{Z}_{>0}$, the product $(W \times \mathbb{R}/(7q\mathbb{Z}), g_W + L^2 ds^2)$ is a Riemannian cover of degree $7q$. Write $\mathrm{inj}(X,g)$ *for the injectivity radius of a Riemannian manifold (X,g). For every τ,*

$$\mathrm{inj}(P, g_\tau) \geq \min\{\mathrm{inj}(W, g_W), L/2\} > 0. \tag{6.23}$$

All covariant derivatives of curvature, with norms taken in g_τ, are bounded independently of τ. Moreover, $g_\tau \to g_W + L^2 ds^2$ in C^∞ as $\tau \to 0$. If W, g_W, K are real analytic, the metrics are jointly analytic in space and τ.

Proof. Translation in s makes $\langle T_\tau \rangle$ free and properly discontinuous. Write $[y,s]$ for the class of (y,s) in $(W \times \mathbb{R})/\langle T_\tau \rangle$. The maps

$$(x,[s]_1) \longmapsto [\Psi(\tau s)x, s], \qquad [y,s] \longmapsto (\Psi(-\tau s)y, [s]_1) \tag{6.24}$$

are inverse diffeomorphisms. For $\Xi_\tau(x,s) = (\Psi(\tau s)x, s)$,

$$d\Xi_\tau(v + a\partial_s) = \big(d\Psi(\tau s)(v + \tau a K), a\partial_s\big).$$

For the projection $\pi_\infty : W \times \mathbb{R} \to P$, $\pi_\infty(x,s) = (x,[s]_1)$, the isometry of $\Psi(\tau s)$ gives $\Xi_\tau^*(g_W + L^2 ds^2) = \pi_\infty^* g_\tau$. The determinant-one shear gives $dv_{g_\tau} = L\, dv_W\, ds$. The map $q_\tau : W \times \mathbb{R} \to P$, $q_\tau(x,s) = (\Psi(-\tau s)x, [s]_1)$, is a Riemannian covering from the product metric, with deck group $\langle T_\tau \rangle$. Composing the universal covering map of $W \times \mathbb{R}$ with q_τ gives the stated simply connected Riemannian cover.

For the line bundle on P_d, take $Z = \mathbb{R}/d\mathbb{Z}$, $b = L^2 ds^2$, and $\alpha = \tau\, ds$ in (6.21). In the parallel frame on $\mathbb{R}$, sections satisfy $f(x, s+d) = e^{2\pi i\vartheta} f(x,s)$, and the closed form is the closure of

$$\mathcal{E}_\tau(f,f) = \int_0^d \int_W \left(|d_W f|^2 + L^{-2}|\partial_s f - \tau K f|^2\right) L\, dv_W\, ds. \tag{6.25}$$

The initial domain consists of smooth equivariant functions on $W \times \mathbb{R}$, so every derivative has the same boundary multiplier. Unitarity of that multiplier cancels the boundary terms in integration by parts, so ∂_s is formally skew-adjoint on this smooth domain. The flow preserves volume, so K is also formally skew-adjoint. Moreover, K commutes with ∂_s and Δ_W. The associated differential operator is

$$\Delta_\tau = \Delta_W - L^{-2}(\partial_s - \tau K)^2. \tag{6.26}$$

Diagonalize the circle action on each finite-dimensional Laplace eigenspace. Its period seven gives a complete orthonormal joint basis v_a with $Kv_a = ihr_a v_a$, $r_a \in \mathbb{Z}$. The sections

$$e_{a,m}(x,s) = (Ld)^{-1/2} v_a(x) e^{2\pi i(m+\vartheta)s/d}$$

form a complete orthonormal basis of $L^2(P_d, E_\vartheta)$, consisting of smooth sections, and (6.26) gives their eigenvalues (6.22). Lemma 2.4, applied to this basis with diagonal eigenvalues $E_{a,m}$ and $\sqrt{E_{a,m}}$, gives

$$\mathcal{D}(\mathcal{E}_\tau) = \left\{ \sum c_{a,m} e_{a,m} \,:\, \sum (1 + E_{a,m})|c_{a,m}|^2 < \infty \right\},$$

$$\mathcal{D}(\Delta_\tau) = \left\{ \sum c_{a,m} e_{a,m} \,:\, \sum (1 + E_{a,m}^2)|c_{a,m}|^2 < \infty \right\}.$$

For $R \geq 0$, if $E_{a,m} \leq R$, then $\lambda_a \leq R$ and $|m + \vartheta - d\tau r_a/7| \leq Ld\sqrt{R}/(2\pi)$; hence only finitely many pairs occur. A finite-dimensional unitary representation ρ of $d\mathbb{Z}$ splits into the eigenspaces of $\rho(d)$. Its connection spectrum is the multiset union of the corresponding character spectra, repeated with those eigenspace multiplicities.

At $\tau = 1/q$, $T_\tau^{7q}(x, s) = (x, s + 7q)$. The covering map in the fixed coordinates of P is

$$(x, [s]_{7q}) \longmapsto (\Psi(-s/q)x, [s]_1).$$

It pulls back $g_{1/q}$ to the product metric and has exactly $7q$ preimages over each point. For $n \neq 0$, the deck transformation T_τ^n of q_τ has displacement at least L in the product metric. If $0 < a < \min\{\mathrm{inj}(W, g_W), L/2\}$, the product exponential is injective and nonsingular on each tangent ball of radius a. Two endpoints in such a ball cannot differ by a nontrivial deck transformation, since their distance is less than $2a < L$. The quotient exponential is therefore injective and nonsingular there, proving (6.23).

Local isometry to the fixed product gives all curvature bounds. With $\eta = g_W(K, \cdot)$, the exact tensor difference is

$$g_\tau - (g_W + L^2 ds^2) = \tau(ds \otimes \eta + \eta \otimes ds) + \tau^2 |K|^2 ds \otimes ds. \tag{6.27}$$

For each integer $m \geq 0$, its C^m norm with respect to the fixed product metric $g_W + L^2 ds^2$ is bounded by $C_m(|\tau| + \tau^2)$, where C_m is independent of τ. Under the analytic hypotheses, its coefficients are jointly real analytic in space and τ. □

Equip $W = A \times B$ with $g_W = g_A \oplus g_B$, and put $\Psi_0(s)(x, y) = (F_A(s)x, y)$ and $\Psi_1(s)(x, y) = (x, F_B(s)y)$. Their generators are $K_0 = (K_A, 0)$ and $K_1 = (0, K_B)$. Let S be the unitary of Corollary 4.14. By (6.14), the volumes of A and B agree, so S is also unitary for their Riemannian volume measures. For $f \in L^2(A)$ and $g \in L^2(B)$, put

$$U(f \otimes g) = S^{-1}g \otimes Sf.$$

This extends to a unitary U on $L^2(W)$, with $U^2 = I$ and

$$U\Delta_W = \Delta_W U, \qquad UK_0 = K_1 U. \tag{6.28}$$

The identities hold on the joint eigenbasis and hence on the respective spectral domains by Lemma 2.4. Since S fixes normalized constants, $U1 = 1$. Proposition 6.8 gives equal flat spectra for the suspension metrics of Ψ_0, Ψ_1, for every $d\mathbb{Z}$ and every finite-dimensional unitary representation of that group. Since W is simply connected, these are all connected finite covers and all finite-dimensional unitary representations of their full fundamental groups. For $d, e \in \mathbb{Z}_{>0}$, the covering $P_{de} \to P_d$, $(x, [s]_{de}) \mapsto (x, [s]_d)$, induces $de\mathbb{Z} \hookrightarrow d\mathbb{Z}$ on fundamental groups. Pullback along this covering sends the flat bundle of a representation ρ of $d\mathbb{Z}$ to that of $\rho|_{de\mathbb{Z}}$.

7. Length sets in dimensions eight and nine

Let $\mathcal{S}_3 = \{a^2 + b^2 + c^2 \,:\, a, b, c \in \mathbb{Z}\}$.

Theorem 7.1 (Different period sets in dimension nine). *The two actions in* (6.4) *are free of order seven. The quotients are smooth closed connected real-analytic nine-manifolds, locally symmetric and of nonnegative sectional curvature. They satisfy*

$$\mathrm{Spec}(\Delta_{M_9}) = \mathrm{Spec}(\Delta_{N_9}), \qquad h\sqrt{14} \in \mathrm{Lsp}(N_9) \setminus \mathrm{Lsp}(M_9). \tag{7.1}$$

The length $h\sqrt{14}$ on N_9 is realized by an embedded primitive geodesic. Moreover, $\mathrm{Lsp}(M_9) \subsetneq \mathrm{Lsp}(N_9)$ *and*

$$\mathrm{Lsp}(N_9) \setminus \mathrm{Lsp}(M_9) = \{h\sqrt{n} \,:\, n \in \mathcal{S}_3,\ 7 \mid n,\ 49 \nmid n\}. \tag{7.2}$$

As $T \to +\infty$,

$$\#\big((\mathrm{Lsp}(N_9) \setminus \mathrm{Lsp}(M_9)) \cap (0, T]\big) = \frac{5}{4\pi^2} T^2 + O(\log(2 + T)). \tag{7.3}$$

The common scalar half-wave trace is real analytic near every length in the difference. The difference also contains the embedded primitive lengths

$$h\sqrt{(49k-2)^2+10}, \qquad k \in \mathbb{Z}_{\geq 0}. \tag{7.4}$$

7.1. **A primitive length absent from the unitary quotient.**

Lemma 7.2. *Let $q \in \mathbb{Z}_{>0}$. If $h\sqrt{q} \in \mathrm{Lsp}(M_9)$ and $7 \mid q$, then $49 \mid q$.*

Proof. For a nonidentity deck element, Proposition 6.4 gives $j \in \{1,\dots,6\}$, $w \in \mathcal{W}_A$, and $m \in \mathbb{Z}^3$ such that

$$q = |m|^2 \equiv j^2|wa_A - b_A|^2 = j^2(21 - 2\langle wa_A, b_A\rangle) \pmod 7.$$

The scalar product is a difference of two distinct entries of $(3,-3,1)$, so it belongs to $\{\pm 2, \pm 4, \pm 6\}$. Since 2 and j are invertible in $\mathbb{Z}/7\mathbb{Z}$, the residue $-2j^2\langle wa_A, b_A\rangle$ is nonzero. Thus $7 \mid q$ is incompatible with any nonidentity deck element. The remaining case has $m \in 7\mathbb{Z}^3$, and hence $49 \mid q$. □

Corollary 7.3. *The lengths in* (7.4) *are pairwise distinct and belong to* $\mathrm{Lsp}(N_9) \setminus \mathrm{Lsp}(M_9)$. *Each has an embedded primitive representative on N_9.*

Proof. Put $J = \mathrm{diag}(1,-1,1,-1) \in \mathrm{SO}(4)$ and $g = (1, J) \in N_{G_B}(T_B)$. In the torus coordinates of Section 6.2, put

$$V_k = h(49k-2, 3, -1) \in \mathfrak{t}_B \qquad (k \in \mathbb{Z}_{\geq 0}). \tag{7.5}$$

Let $\gamma_k(t)$ be the image of $g\exp(tV_k)$ in N_9. Conjugation by J reverses the two rotation angles, so $g^{-1}\ell_B g$ has torus angles $(h,-h,0)$. Since

$$(49k-2, 3, -1) \equiv (-2, -4, -1) \pmod 7,$$

one has

$$\exp V_k = g^{-1}\ell_B g\, r_B^{-1}, \qquad \mathrm{Ad}_{r_B} V_k = V_k.$$

By Lemma 6.3, γ_k is a geodesic of period one. A smaller period $0 < t < 1$ would give $\exp(tV_k) = g^{-1}\ell_B^j g\, r_B^{-j}$ for some $j \in \mathbb{Z}$. The last coordinate gives $e^{-ith} = e^{-ijh}$, hence $t - j \in 7\mathbb{Z}$, a contradiction. Thus the least period is one. Lemma 6.5 implies that γ_k induces an embedding $\mathbb{R}/\mathbb{Z} \to N_9$.

Its squared length is

$$|V_k|^2 = h^2\big((49k-2)^2 + 10\big), \qquad (49k-2)^2 + 10 \equiv 14 \pmod{49}.$$

Lemma 7.2 excludes this length from M_9. Finally,

$$(49(k+1)-2)^2 - (49k-2)^2 = 49(98k+45) > 0,$$

so the lengths are distinct. □

7.2. **The length-set difference in dimension nine.** Let $\mathfrak{S}_3$ be the symmetric group on three letters, and let $\mathfrak{W} = \{\pm 1\}^3 \rtimes \mathfrak{S}_3$ act on $\mathbb{Z}^3$ by signed coordinate permutations and on $(\mathbb{Z}/7\mathbb{Z})^3$ by reduction modulo seven. For $m \in \mathbb{Z}^3$, define $\nu(m) = (a,b,c)$ by taking the least absolute residue of each coordinate and arranging the values so that $0 \leq a \leq b \leq c \leq 3$. This is the unique ordered representative of the $\mathfrak{W}$-orbit of m mod 7. If $\nu(m) = \nu(m')$, some $w \in \mathfrak{W}$ satisfies $wm \equiv m' \pmod 7$. Since w preserves the Euclidean norm, for every $C \subset (\mathbb{Z}/7\mathbb{Z})^3$ the sets

$$\{|m|^2 : m \in \mathbb{Z}^3,\ m \bmod 7 \in C\}, \qquad \{|m|^2 : m \in \mathbb{Z}^3,\ m \bmod 7 \in \mathfrak{W}C\}$$

are equal: if $m \bmod 7 = wc$ with $c \in C$, then $w^{-1}m$ has residue c and squared norm $|m|^2$. Thus the squared norms in (6.19) are determined by the representatives $\nu(j(wa_i - b_i))$, with $i \in \{A,B\}$, $1 \leq j \leq 6$, and $w \in \mathcal{W}_i$, together with the identity residue $(0,0,0)$.

Write abc for the ordered triple (a,b,c). Multiplication by 2 modulo seven induces the permutation

$$\kappa(0) = 0, \qquad \kappa(1) = 2, \qquad \kappa(2) = 3, \qquad \kappa(3) = 1$$

on least absolute residues. Let $C(a,b,c)$ be the increasing rearrangement of $(\kappa(a),\kappa(b),\kappa(c))$. Thus

$$C\nu(m) = \nu(2m).$$

Since $(\mathbb{Z}/7\mathbb{Z})^\times = \{\pm1,\pm2,\pm4\}$ and $\nu(-m) = \nu(m)$, multiplication of a fixed deck residue by $j \in \{1,\dots,6\}$ gives exactly its C-orbit. The following four C-orbits occur for both groups:

$$\begin{aligned} \mathcal{O}_1 &= \{033, 011, 022\}, & \mathcal{O}_2 &= \{133, 112, 223\}, \\ \mathcal{O}_3 &= \{023, 013, 012\}, & \mathcal{O}_4 &= \{233, 113, 122\}. \end{aligned}$$

Indeed, applying $1 \mapsto 2 \mapsto 3 \mapsto 1$ to each displayed triple, then arranging its entries in increasing order, gives its successor. In the following table, the two entries in the last column and last row are distinct Weyl images; every other entry occurs once.

ν	$\lvert m\rvert^2 \pmod 7$	$wa_A - b_A$	$wa_B - b_B$
033	4	$(-4,3,0)$	$(-4,-3,0)$
133	5	$(-4,4,-1)$	$(-4,-4,-1)$
023	6	$(-3,2,0)$	$(-2,-3,0)$
233	1	$(-3,4,-2)$	$(-4,-3,-2)$
122	2	$(-2,2,-1)$	$(-2,-2,-1)$
223	3	$(-2,3,-2)$	$(-2,-3,-2)$
123	0	none	$(-4,-2,-1),\ (-2,-4,-1)$

(7.6)

Subtracting b_A from the six permutations of a_A, and b_B from the eight vectors in (6.18), gives all entries. The first six types generate $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$, whereas $C(123) = 123$. The identity deck transformation contributes 000 on both sides. For $i = A, B$, let $\mathcal{R}_i$ be the set of ordered representatives of the residues occurring in (6.19). Then

$$\begin{aligned} \mathcal{R}_A &= \{000\} \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4, \\ \mathcal{R}_B &= \mathcal{R}_A \cup \{123\}, \end{aligned} \tag{7.7}$$

Proposition 6.4 and norm preservation give

$$\begin{aligned} \mathrm{Lsp}(M_9) &= \{h|m| \,:\, m \in \mathbb{Z}^3 \setminus \{0\},\ \nu(m) \in \mathcal{R}_A\}, \\ \mathrm{Lsp}(N_9) &= \{h|m| \,:\, m \in \mathbb{Z}^3 \setminus \{0\},\ \nu(m) \in \mathcal{R}_B\}. \end{aligned}$$

Thus $\mathrm{Lsp}(M_9) \subset \mathrm{Lsp}(N_9)$.

The only square residues modulo seven are $0, 1, 2, 4$. The set $\{1, 2, 4\}$ is disjoint from its negative; hence a zero coordinate in a triple of square residues summing to zero forces all three to be zero. Two equal nonzero entries a would require the nonsquare $-2a$ as the third. Thus

$$7 \mid |m|^2 \quad \Longleftrightarrow \quad \nu(m) = 000 \text{ or } \nu(m) = 123. \tag{7.8}$$

If $7 \mid q$ and $49 \nmid q$, every three-square representation therefore has representative 123: the alternative $m \in 7\mathbb{Z}^3$ would force $49 \mid q$. For $q \in \mathcal{S}_3$ such a representation exists, and (7.7) gives a length on N_9 but not on M_9. Conversely, a squared norm occurring on N_9 and not on M_9 must arise from the additional representative 123, so it is divisible by seven. The Gauss–Legendre criterion [98, Introduction, equation (1.2), arXiv v14] gives, for $n \in \mathbb{Z}_{\geq 0}$,

$$n \notin \mathcal{S}_3 \quad \Longleftrightarrow \quad n = 4^a(8b+7) \quad \text{for some } a, b \in \mathbb{Z}_{\geq 0}.$$

If $q = 49u > 0$, the maximal power 4^a dividing u also is the maximal power dividing $49u$, and $49 \equiv 1 \pmod 8$. Hence $49u \in \mathcal{S}_3$ if and only if $u \in \mathcal{S}_3$. A representation $u = x^2 + y^2 + z^2$ gives $q = (7x)^2 + (7y)^2 + (7z)^2$, an identity-deck representation on M_9. Consequently a squared norm occurring only on N_9 is divisible by seven and not by 49, and (7.2) follows.

For a positive integer n, put $v_7(n) = \max\{a \in \mathbb{Z}_{\geq 0} \,:\, 7^a \mid n\}$. For $R \geq 0$, put

$$A(R) = \#\{n \in \mathcal{S}_3 \,:\, 0 < n \leq R,\ v_7(n) = 1\}.$$

The number of positive integers at most R with seven-adic valuation one is

$$\lfloor R/7\rfloor - \lfloor R/49\rfloor = \frac{6R}{49} + O(1).$$

The three-square exceptions with that valuation are precisely $4^a(8b+7)$, where $a, b \in \mathbb{Z}_{\geq 0}$ and $v_7(8b+7) = 1$. Their exponents a are unique, because $8b+7$ is odd. For $X \geq 7$, define

$$E_7(X) = \#\{b \in \mathbb{Z}_{\geq 0} : 8b+7 \leq X,\ v_7(8b+7) = 1\}.$$

The integers counted by $E_7(X)$ satisfy $0 \leq b \leq (X-7)/8$ and lie in the six residue classes $0, 7, 14, 21, 28, 35$ modulo 49; the class 42 gives divisibility by 49. Hence $E_7(X) = 6X/392 + O(1)$, with an absolute error bound. For $R \geq 7$, the unique exceptional representations give

$$A(R) = \lfloor R/7\rfloor - \lfloor R/49\rfloor - \sum_{a=0}^{\lfloor \log_4(R/7)\rfloor} E_7(R/4^a).$$

There are $O(\log(2+R))$ terms, and

$$\sum_{a\geq 0} \frac{6}{392\,4^a} = \frac{1}{49}.$$

In the omitted tail, the first value of $R/4^a$ is less than seven; the tail is therefore $O(1)$. It follows that

$$A(R) = \frac{5R}{49} + O(\log(2+R)).$$

Since $h\sqrt{n} \leq T$ is equivalent to $n \leq 49T^2/(4\pi^2)$, this gives (7.3).

7.3. **Regularity of the common wave trace.** For a closed Riemannian manifold P, put

$$W_P(t) = \operatorname{Tr}(e^{it\sqrt{\Delta_P}}), \qquad W_P^-(t) = W_P(-t) = \overline{W_P(t)}, \qquad W_P,\, W_P^- \in \mathcal{S}'(\mathbb{R}).$$

The polynomial spectral bound in Section 5, applied to the trivial deck group on each connected component, gives convergence in $\mathcal{S}'(\mathbb{R})$. Equal spectral multiplicities give equal wave distributions.

The Chazarain–Duistermaat–Guillemin Poisson relation [23, 37], in the half-wave form of [102, §1.2, equation (1.1), arXiv v1], gives

$$\operatorname{sing\,supp} W_P \subset \{0\} \cup \overline{\operatorname{Lsp}(P) \cup (-\operatorname{Lsp}(P))}. \tag{7.9}$$

If the metric on P is real analytic, the scalar inclusion for analytic singular support follows from [37, Corollary 1.2 and the following paragraph, p. 45], applied to the analytic operator Δ_P of order two and its square root. The zero eigenspace contributes a constant. Proposition A.3 gives the same inclusion for finite-rank unitary flat bundles.

The metrics on M_9, N_9 are real analytic and scalar-isospectral. Corollary A.5 makes their common half-wave trace real analytic near every length in $\operatorname{Lsp}(N_9) \setminus \operatorname{Lsp}(M_9)$. Proposition 6.4 gives $\operatorname{Lsp}(M_9) \subset h\sqrt{\mathbb{Z}_{>0}}$. Proposition A.3 therefore gives

$$W_{M_9} = W_{N_9} \in C^\omega\big((h\sqrt{13}, h\sqrt{15})\big), \tag{7.10}$$

since the only integer strictly between 13 and 15 is 14, excluded by Lemma 7.2.

Proof of Theorem 7.1. Section 6.2 gives freeness and the geometric properties of the quotients. Corollary 6.1 gives their scalar isospectrality. Corollary 7.3 gives the primitive family, including $h\sqrt{14}$ at $k = 0$. Section 7.2 proves (7.2) and (7.3). The real analyticity follows from Corollary A.5. □

Corollary 7.4. *The metrics on M_9 and N_9 are clean. At every $\ell \in \operatorname{Lsp}(N_9) \setminus \operatorname{Lsp}(M_9)$, every coefficient in the clean wave-trace expansion vanishes after summing the component contributions with the same asymptotic degree.*

Proof. The groups G_A, G_B are globally symmetric by Section 6.2, hence clean by [102, Theorem 1.8, arXiv v1]. Their finite actions are free, so Lemma A.1 makes both quotients clean. Fix $\ell \in \mathrm{Lsp}(N_9) \setminus \mathrm{Lsp}(M_9)$ and choose $\chi \in C_c^\infty(\mathbb{R})$ equal to one near ℓ, with support in a neighborhood on which $W_{N_9}^-$ is analytic and containing no other period. Such a neighborhood exists by Corollary A.5 and Proposition 6.4. Let Φ_t denote the geodesic flow on SN_9. For each component $\Theta \subset \mathrm{Fix}(\Phi_\ell)$, put $d_\Theta = \dim \Theta$. Denote by $c_{\Theta,r}$ its contribution at degree $(d_\Theta - 1)/2 - r$ in the expansion of $\langle W_{N_9}^-(t), \chi(t) e^{is(t-\ell)} \rangle$ as $s \to +\infty$, where $r \in \mathbb{Z}_{\geq 0}$ [102, §1.2, equations (1.5)–(1.6), arXiv v1]. Since $\chi W_{N_9}^- \in C_c^\infty(\mathbb{R})$, this transform is rapidly decreasing. There are finitely many components, and the degrees form a discrete set bounded above. Uniqueness of asymptotic expansions gives, for every $p \in \frac{1}{2}\mathbb{Z}$,

$$\sum_{\substack{\Theta,\ r \in \mathbb{Z}_{\geq 0} \\ (d_\Theta - 1)/2 - r = p}} c_{\Theta,r} = 0.$$

Complex conjugation gives the same vanishing for W_{N_9}. □

7.3.1. *The fixed components at $h\sqrt{14}$.* Put $G = G_B$, with the bi-invariant metric (6.1), and $T = T_B$. Put $\mathfrak{g} = \mathrm{Lie}(G)$ and identify $\mathfrak{t} = \mathrm{Lie}(T)$ with $\mathbb{R}^3$ in the Euclidean torus coordinates of Section 6.2; the period lattice is $2\pi\mathbb{Z}^3$. Let $\mathcal{W} = N_G(T)/T$ be the Weyl group and $\exp_T : \mathfrak{t} \to T$ the exponential map. For $x \in G$, write $C_G(x) = \{g \in G : gx = xg\}$. Set

$$h = \frac{2\pi}{7}, \qquad a = (1, 1, 0), \qquad b = (3, 3, 1).$$

$$l = \exp_T(ha), \quad r = \exp_T(hb), \quad \Psi(g) = lgr^{-1} = \Phi_B(g).$$

The cyclic group $\Gamma = \langle \Psi \rangle$ has order seven and acts freely. Let $N = N_9 = G/\Gamma$, and let $p : SG \longrightarrow SN$ denote the induced unit-tangent covering. Let $\overline{\Phi}_t$ be the geodesic flow on SN, and put $\ell = h\sqrt{14}$.

Proposition 7.5. *The fixed set of the time-ℓ geodesic flow on SN consists of exactly twelve connected components. Each is a three-dimensional torus and consists of initial velocities of embedded primitive closed geodesics. There are exactly two components associated with each nonidentity deck transformation Ψ^j, $1 \leq j \leq 6$. Their free-loop Morse indices are*

$$(2, 0), \quad (2, 4), \quad (4, 2), \quad (4, 2), \quad (2, 4), \quad (2, 0),$$

in the order $j = 1, \dots, 6$. The two components associated with each j therefore have opposite Duistermaat–Guillemin phases.

Proof. The fixed set. For $1 \leq j \leq 6$, both l^j and r^j are regular torus elements. For the symplectic factor, their adjoint root angles are respectively $2jh$ and $6jh$. For the orthogonal factor, their two positive-root angles are respectively (jh, jh) and $(2jh, 4jh)$. None is a multiple of 2π. Their group centralizers are the connected torus T. For the orthogonal factor of l^j, one real plane rotates through $jh \not\equiv 0, \pi$, while the second plane is fixed. A commuting orthogonal map preserves these planes. Its restriction to the rotating plane is a rotation and has determinant one; its restriction to the fixed plane must consequently have determinant one as well. The centralizer in $\mathrm{SO}(4)$ is therefore $\mathrm{SO}(2) \times \mathrm{SO}(2)$, which is connected. For r^j its four distinct complex eigenlines give the same centralizer, and each nonreal quaternionic component has centralizer $\{e^{\mathbf{i}t} : t \in \mathbb{R}\}$.

By Lemma 6.3, a geodesic $\gamma(t) = g \exp(tV)$, $0 \leq t \leq 1$, closes through Ψ^j if and only if

$$\exp V = g^{-1} l^j g\, r^{-j}, \qquad \mathrm{Ad}_{r^j} V = V.$$

Since $C_G(r^j) = T$, the second equality gives $V \in \mathfrak{t}$. The first then places $g^{-1} l^j g$ in T. Torus conjugacy and the period lattice, as in Proposition 6.4, give

$$V = hm, \qquad m \in \mathbb{Z}^3, \qquad m \equiv j(a' - b) \pmod 7, \qquad g^{-1} l^j g = \exp_T(hja')$$

for some $a' \in \mathcal{W}a$, where

$$\mathcal{W}a = \{(\epsilon, \eta, 0), (\epsilon, 0, \eta) \,:\, \epsilon, \eta \in \{1, -1\}\}.$$

If $|m|^2 = 14$, the absolute coordinates of m are $1, 2, 3$. Indeed each coordinate has absolute value at most three, and $14 = 9 + 4 + 1$ is the only decomposition into three such squares. Consequently each coordinate is its unique representative modulo seven in $\{-3, \dots, 3\}$.

Table (7.6) shows that exactly two Weyl images have residue type 123:

$$a^{(1)} = (-1, 1, 0), \qquad a^{(2)} = (1, -1, 0).$$

Multiplication by $j \neq 0$ permutes the three nonzero absolute residues, so these are the only two images contributing to $|m|^2 = 14$ for every j. For $1 \leq j \leq 6$ and $\nu \in \{1, 2\}$, define $m_{j,\nu} \in \mathbb{Z}^3$ by reducing $j(a^{(\nu)} - b)$ coordinatewise to $\{-3, \dots, 3\}$.

Choose a Weyl representative $g_\nu \in N_G(T)$ with $g_\nu^{-1} l g_\nu = \exp_T(h a^{(\nu)})$. For every $j \in \{1, \dots, 6\}$, the set of allowed base points for (j, ν) is exactly the torus coset $T g_\nu$: the equality $g^{-1} l^j g = g_\nu^{-1} l^j g_\nu$ is equivalent to $g g_\nu^{-1} \in C_G(l^j) = T$. In left trivialization $SG = G \times \{v \in \mathfrak{g} \,:\, |v| = 1\}$, the corresponding lifted component is

$$\widetilde{\Theta}_{j,\nu} = \left\{ \left(g, \frac{h m_{j,\nu}}{\ell} \right) \,:\, g \in T g_\nu \right\} \subset SG.$$

Put $c_\nu = l g_\nu r^{-1} g_\nu^{-1} \in T$. For $z \in T$,

$$\Psi(z g_\nu) = z c_\nu g_\nu, \qquad \mathrm{Ad}_r\, m_{j,\nu} = m_{j,\nu}.$$

Thus the deck action preserves each lifted component and acts there by translation by c_ν. Freeness and order seven give $\Theta_{j,\nu} := p(\widetilde{\Theta}_{j,\nu}) \simeq T/\langle c_\nu \rangle$, a three-torus. Different labels cannot be identified: the deck group preserves them, and freeness makes the closing deck element unique at a given lifted initial velocity. No component is associated with the identity deck transformation, because the identity case of Proposition 6.4 gives minimum nonzero length $2\pi > \ell$. Consequently

$$\mathrm{Fix}(\overline{\Phi}_\ell) = \coprod_{j=1}^{6} \coprod_{\nu=1}^{2} \Theta_{j,\nu}, \qquad \Theta_{j,\nu} \simeq (\mathbb{R}/\mathbb{Z})^3.$$

The index form. Consider the energy

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot\gamma(t)|^2 \, dt$$

on the Sobolev path space $\{\gamma \in H^1([0,1], G) \,:\, \gamma(1) = \Psi^j \gamma(0)\}$. A sufficiently small neighborhood of a loop in N has unique nearby lifts after choosing its initial point in one evenly covered neighborhood. The closing deck element is locally constant, since it takes values in the finite discrete group. Projection consequently identifies this local path space with a neighborhood in the free loop space of N and preserves energy. Its Hessian index is therefore the ordinary free-loop index of the projected closed geodesic. Along $\gamma(t) = g \exp(tV)$ with $V = hm$, left-trivialize a variation field as $Y(t) = dL_{\gamma(t)} y(t)$ and put $A = \mathrm{ad}_V$. Write $B_j = \mathrm{Ad}_{r^j}$. The tangent space to this path space consists of H^1 fields with boundary condition

$$y(1) = \mathrm{Ad}_{r^j}\, y(0).$$

The bi-invariant connection and curvature [103, equations (E.4.11)–(E.4.13)] give

$$\nabla_t Y \longleftrightarrow y' + \tfrac{1}{2} A y, \qquad R(Y, \dot\gamma)\dot\gamma \longleftrightarrow -\tfrac{1}{4} A^2 y.$$

Let I_j be the Hessian of E at γ on the twisted path space. The free-loop energy formula [90, §5, author version, p. 14] applies to the projected loop and pulls back under the local isometry $G \to N$. Equivalently, its endpoint terms cancel because Ψ^j is an isometry and $\dot\gamma(1) = d\Psi^j \dot\gamma(0)$. Thus

$$\begin{aligned} I_j(y,y) &= \int_0^1 \left(|y' + \tfrac{1}{2}Ay|^2 + \tfrac{1}{4}\langle A^2 y, y\rangle \right) dt \\ &= \int_0^1 \left(|y'|^2 + \langle y', Ay\rangle \right) dt, \end{aligned}$$

because $A^* = -A$. The Jacobi equation [69, Theorem 10.1] and its twisted endpoint conditions become

$$y'' + Ay' = 0, \qquad y(1) = B_j y(0), \qquad y'(1) = B_j y'(0).$$

Indeed, $AB_j = B_j A$, so matching the covariant derivatives $y' + \frac{1}{2}Ay$ is equivalent to the last condition.

The toral directions give the periodic form $\int_0^1 |y'|^2 dt$. Its index is zero and its kernel is the three-dimensional space of constant toral fields. With e_1, e_2, e_3 the standard dual basis of the displayed torus coordinates, the three positive roots are

$$\alpha_1 = 2e_1, \qquad \alpha_2 = e_2 - e_3, \qquad \alpha_3 = e_2 + e_3.$$

For a positive root α, let $\mathfrak{g}_\alpha$ denote its real root plane. The orthogonal decomposition is

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3}, \qquad \dim_{\mathbb{R}} \mathfrak{g}_{\alpha_k} = 2.$$

Identify $\mathfrak{g}_\alpha$ isometrically with $\mathbb{C}$ so that A acts by $ih\alpha(m)$ and B_j by $e^{ihj\alpha(b)}$. Put $\omega_n = h(7n + j\alpha(b))$. Every twisted H^1 field on this plane has the expansion

$$z(t) = \sum_{n\in\mathbb{Z}} c_n e^{i\omega_n t}, \qquad \sum_n (1+\omega_n^2)|c_n|^2 < \infty.$$

Multiplication by $e^{-ihj\alpha(b)t}$ identifies this expansion with the periodic Fourier expansion. Parseval's identity gives

$$I_j(z,z) = \sum_{n\in\mathbb{Z}} \omega_n \big(\omega_n + h\alpha(m)\big)|c_n|^2.$$

The series converges absolutely under the displayed H^1 condition. The coefficient of $|c_n|^2$ is

$$\lambda_{j,m,\alpha,n} = h^2\big(7n + j\alpha(b)\big)\big(7n + j\alpha(b) + \alpha(m)\big). \tag{7.11}$$

Each mode has real dimension two. The index is therefore twice the number of negative coefficients. Replacing α by $-\alpha$ and n by $-n$ leaves this list unchanged.

Here $\big(\alpha_1(b), \alpha_2(b), \alpha_3(b)\big) = (6, 2, 4)$. Neither factor in (7.11) can vanish: the first is nonzero modulo seven, and the second is congruent to $j\alpha(a^{(\nu)})$ modulo seven. All the roots of $a^{(1)}$ and $a^{(2)}$ are nonzero modulo seven. Thus every kernel field has $y(t) = y_0 \in \mathfrak{t}$. It is the variation field of $\gamma_\epsilon(t) = g\exp(\epsilon y_0)\exp(tV)$. These variations move the initial point in Tg_ν, leave the toral velocity and its norm fixed, and exhaust its three tangent directions. Conversely, a vector in $\ker(D_v\overline{\Phi}_\ell - I)$, with $v \in \Theta_{j,\nu}$, lifts to a Jacobi field satisfying the two twisted boundary conditions after rescaling $[0,\ell]$ to $[0,1]$. Uniqueness for the Jacobi initial-value problem [69, Proposition 10.2] and the constant toral variations give

$$\ker(D_v\overline{\Phi}_\ell - I) = T_v\Theta_{j,\nu}.$$

Let

$$q_\alpha(j,m) = \#\{n \in \mathbb{Z} \,:\, (7n + j\alpha(b))(7n + j\alpha(b) + \alpha(m)) < 0\}.$$

Equivalently, $q_\alpha(j,m)$ counts the integers satisfying

$$\min\{0, -\alpha(m)\} < 7n + j\alpha(b) < \max\{0, -\alpha(m)\}.$$

The Morse index is $\mu(j,m) = 2\sum_{\alpha\in\{\alpha_1,\alpha_2,\alpha_3\}} q_\alpha(j,m)$. Writing $x = j\alpha(b)$, $u = \min(0,-\alpha(m))$, and $v = \max(0,-\alpha(m))$, the number of integers in the strict interval $(u-x)/7 < n < (v-x)/7$ is

$$q_\alpha(j,m) = \max\left\{0, \left\lceil\frac{v-x}{7}\right\rceil - \left\lfloor\frac{u-x}{7}\right\rfloor - 1\right\}.$$

Put $q(j,m) = (q_{\alpha_1}(j,m), q_{\alpha_2}(j,m), q_{\alpha_3}(j,m))$ and $\mu_{j,\nu} = \mu(j,m_{j,\nu})$. The resulting integers are

j	$m_{j,1}$	$q(j,m_{j,1})$	$\mu_{j,1}$	$m_{j,2}$	$q(j,m_{j,2})$	$\mu_{j,2}$
1	$3,-2,-1$	$(1,0,0)$	2	$-2,\ 3,-1$	$(0,0,0)$	0
2	$-1,\ 3,-2$	$(0,1,0)$	2	$3,-1,-2$	$(1,0,1)$	4
3	$2,\ 1,-3$	$(1,1,0)$	4	$1,\ 2,-3$	$(0,1,0)$	2
4	$-2,-1,\ 3$	$(1,1,0)$	4	$-1,-2,\ 3$	$(0,1,0)$	2
5	$1,-3,\ 2$	$(0,1,0)$	2	$-3,\ 1,\ 2$	$(1,0,1)$	4
6	$-3,\ 2,\ 1$	$(1,0,0)$	2	$2,-3,\ 1$	$(0,0,0)$	0

Each triple in a q column is ordered by $(\alpha_1,\alpha_2,\alpha_3)$. In particular $i^{-\mu_{j,1}} = -i^{-\mu_{j,2}}$ for every j.

Primitivity and embeddedness. Every positive closed-geodesic length on N is $h\sqrt{n}$ for an integer $n \geq 1$, by the closing equations. If a geodesic of length $h\sqrt{14}$ were a k-fold iterate with $k \geq 2$, its primitive length would give $14 = k^2 n$ for a positive integer n. The squarefreeness of 14 excludes this. Moreover, the image of every displayed geodesic lies in Tg_ν, because g_ν normalizes T. Lemma 6.5 applies to this invariant coset. The primitive projected straight lines in its quotient torus are embedded, so all geodesics in the twelve components are embedded. □

7.3.2. *Cancellation for each deck transformation.* Write $\Psi_A = \Phi_A$, $\Psi_B = \Phi_B$ for the cyclic isometries of G_A, G_B with their metrics (6.1). For the direct pullback $F^*f = f\circ F$, put

$$W_{i,j}(t) = \operatorname{Tr}\left((\Psi_i^j)^* e^{-it\sqrt{\Delta_{G_i}}}\right), \qquad i\in\{A,B\},\quad j\in\{0,\dots,6\}.$$

Time reversal gives

$$W_{i,j}(-t) = \operatorname{Tr}\left((\Psi_i^j)^* e^{it\sqrt{\Delta_{G_i}}}\right).$$

Let $K_i^-(t,x,y)$ be the Schwartz kernel of $e^{-it\sqrt{\Delta_{G_i}}}$. The diagonal kernel in this convention is $K_i^-(t,\Psi_i^j x,x)$. Its positive-time stationary points are precisely geodesics satisfying the endpoint and tangent conditions through Ψ_i^j. The index j is the deck index in the preceding table.

Proposition 7.6 (Complete cancellation in each deck class). *For every $j\in\{1,\dots,6\}$, the complete singular expansions of the two clean components $\Theta_{j,1}$ and $\Theta_{j,2}$ cancel coefficient by coefficient at $\ell = h\sqrt{14}$. Each component has a strictly positive Duistermaat–Guillemin mass, and the two masses associated with the same j are equal.*

Proof. The maximal-equivariance identity, specialized as in Equation (4.12) to $(u,s,v) = (jh, 3jh, jh)$, gives

$$\operatorname{Tr}\left((\Psi_A^j)^* e^{-t_0\Delta_{G_A}}\right) = \operatorname{Tr}\left((\Psi_B^j)^* e^{-t_0\Delta_{G_B}}\right) \qquad (t_0>0)$$

for each individual j. Put $E_{i,\lambda} = \ker(\Delta_{G_i}-\lambda)$, with $E_{i,\lambda} = \{0\}$ if λ is absent. For every $\lambda \geq 0$,

$$\left|\operatorname{Tr}((\Psi_i^j)^*|_{E_{i,\lambda}})\right| \leq \dim E_{i,\lambda}.$$

The finite heat traces give absolute convergence. Lemma 2.3 therefore gives

$$\operatorname{Tr}((\Psi_A^j)^*|_{E_{A,\lambda}}) = \operatorname{Tr}((\Psi_B^j)^*|_{E_{B,\lambda}}) \qquad (\lambda\geq 0).$$

The distributional convergence in Section 5 then gives $W_{A,j} = W_{B,j}$ in $\mathcal{S}'(\mathbb{R})$.

Put $I = (h\sqrt{13}, h\sqrt{15})$. By Section 7.3, $I\cap \operatorname{Lsp}(M_9) = \emptyset$. Proposition A.3 makes every flat half-wave trace over M_9 real analytic on $I\cup(-I)$. For $r\in\{0,\dots,6\}$ and $j\in C_7$, put $\chi_r(j) = e^{2\pi i r j/7}$. Let W_{M_9,χ_r}

denote the positive-sign half-wave trace of the flat line bundle obtained by composing χ_r with the epimorphism $\pi_1(M_9) \twoheadrightarrow C_7$ defined by $G_A \to M_9$. The convention $R_\gamma f = f \circ \gamma^{-1}$ in Lemma 5.1 gives

$$W_{A,j}(t) = D_{\Psi_A^{-j}}(-t) = \sum_{r=0}^{6} \chi_r(-j)\, W_{M_9,\chi_r}(-t).$$

Thus $W_{A,j} = W_{B,j} \in C^\omega(I)$.

The finite-cover trace identity is

$$W_N^-(t) = \frac{1}{7} \sum_{j=0}^{6} W_{B,j}(t).$$

Microlocally near ℓ, the term $\frac{1}{7} W_{B,j}$ receives exactly the two contributions from $\Theta_{j,1}$ and $\Theta_{j,2}$. Identify SN with the unit cotangent bundle S^*N by the metric. Choose zeroth-order pseudodifferential operators $A_{j,\nu}$ on N such that $A_{j,\nu} - \mathrm{Id}$ is microlocally smoothing near the positive conic extension of $\Theta_{j,\nu}$, and $A_{j,\nu}$ is microlocally smoothing near the other fixed components. Pullback of the kernels near the diagonal in evenly covered charts, followed by deck averaging, gives deck-invariant operators $\widetilde{A}_{j,\nu}$ whose descent agrees with $A_{j,\nu}$ modulo smoothing operators. For $W_{j,\nu}^{\mathrm{loc}} = \mathrm{Tr}(A_{j,\nu}\, e^{-it\sqrt{\Delta_N}})$, finite-cover averaging gives

$$W_{j,\nu}^{\mathrm{loc}} = \frac{1}{7} \sum_{k=0}^{6} \mathrm{Tr}(\widetilde{A}_{j,\nu} (\Psi^k)^*\, e^{-it\sqrt{\Delta_G}}) \pmod{C^\infty}.$$

Smoothing errors have rapidly decreasing spectral matrix coefficients and hence smooth traces. At $\Theta_{j,\nu}$, only $k = j$ has stationary points. The sum $A_{j,1} + A_{j,2}$ equals the identity modulo microlocally smoothing operators near both components associated with Ψ^j. The localized clean trace formula [102, §1.2, equations (1.5)–(1.6), arXiv v1] gives

$$W_{j,1}^{\mathrm{loc}} + W_{j,2}^{\mathrm{loc}} = \frac{1}{7} W_{B,j} \pmod{C^\infty \text{ near } \ell}. \tag{7.12}$$

The common factor $1/7$ is positive. Changing the cutoffs away from these components changes only a smooth remainder.

By Proposition 6.4, $I \cap \mathrm{Lsp}(N) = \{\ell\}$. Choose $\chi \in C_c^\infty(I)$ equal to one near ℓ and supported where (7.12) holds. Define the shifted Fourier transform by

$$\alpha_{j,\nu}(s) = \left\langle W_{j,\nu}^{\mathrm{loc}}(t), \chi(t) e^{is(t-\ell)} \right\rangle.$$

All components have dimension three. Their shifted, localized Fourier amplitudes therefore have expansions

$$\alpha_{j,\nu}(s) \sim \sum_{r=0}^{\infty} C_{j,\nu,r}\, s^{1-r} \qquad (s \longrightarrow +\infty).$$

Let $d\mu_\ell$ be the positive Duistermaat–Guillemin measure on each component, and put

$$D_{j,\nu} = \int_{\Theta_{j,\nu}} d\mu_\ell > 0.$$

The leading-coefficient formula [102, §1.2, equation (1.6), arXiv v1] gives a constant $\kappa_3 \in \mathbb{C} \setminus \{0\}$, independent of j, ν, such that

$$C_{j,\nu,0} = \kappa_3\, i^{-\mu_{j,\nu}} D_{j,\nu}.$$

Here $\mu_{j,\nu}$ is the free-loop Morse index computed in Proposition 7.5; κ_3 depends only on the Fourier and density conventions and the component dimension.

By (7.12) and smoothness of $W_{B,j}$, $\alpha_{j,1}(s) + \alpha_{j,2}(s) = O(s^{-M})$ for every $M > 0$. Uniqueness of the asymptotic expansion gives

$$C_{j,1,r} + C_{j,2,r} = 0 \qquad (r \geq 0).$$

For $r = 0$, the identity $i^{-\mu_{j,1}} = -i^{-\mu_{j,2}}$ implies $D_{j,1} = D_{j,2} > 0$. □

7.4. **The eight-dimensional quotients.**

Theorem 7.7 (Equal flat spectra and different period sets). *The spaces A, B are simply connected, and the actions in* (6.10) *are free of order seven. Mark each quotient fundamental group by sending $1 \in C_7$ to its displayed deck generator. Then*

$$\operatorname{Spec}(\Delta_{M_8,\rho}) = \operatorname{Spec}(\Delta_{N_8,\rho}) \quad \text{for every finite-dimensional unitary representation } \rho \text{ of } C_7, \tag{7.13}$$

whereas

$$3h \in \operatorname{Lsp}(N_8) \setminus \operatorname{Lsp}(M_8). \tag{7.14}$$

The geodesic of length $3h$ constructed on N_8 is embedded and primitive. For $P = M_8, N_8$, the quotient metric g_P is real analytic and satisfies

$$0 \le \sec_{g_P} \le 4, \qquad \operatorname{Ric}_{g_P} \ge \tfrac{1}{2} g_P, \qquad \operatorname{Vol}(M_8) = \operatorname{Vol}(N_8) = \frac{16\pi^5}{7}.$$

With $b_2(X) = \dim_{\mathbb{Q}} H^2(X;\mathbb{Q})$, one has $\pi_2(M_8) = 0$, $\pi_2(N_8) = \mathbb{Z}$, $b_2(M_8) = 0$, and $b_2(N_8) = 1$.

Let U be a simply connected nilpotent Lie group, $\Lambda \le U$ a cocompact discrete subgroup, and g_U a left-invariant metric. If an automorphism Φ of U satisfies

$$\Phi(\gamma) \in \{u\gamma u^{-1} \,:\, u \in U\} \qquad (\gamma \in \Lambda),$$

then g_U and $\Phi^* g_U$ on $\Lambda\backslash U$ have equal spectra on every finite-rank unitary flat bundle [48, Theorem 2.2 and Remark 2.3].

7.4.1. *Exclusion of the length $3h$ on M_8.* Let $\overline{\gamma}$ be a nonconstant closed geodesic on M_8, parametrized on $[0,1]$ with constant speed. Its lift to A has the form $g\exp(tV)H_A$, with $g \in G_A$ and $V \perp \mathfrak{h}_A$, and closes through ϕ_A^j for some $j \in \{0,\dots,6\}$. Write $V = iC$, with C Hermitian. The condition $V \perp \mathfrak{h}_A$ is equivalent to

$$C_{33} = 0, \tag{7.15}$$

and the squared length is $|V|^2 = \operatorname{Tr}(C^2)$, the sum of the squares of the eigenvalues of C. Set $\zeta = e^{2\pi i/7}$. Lemma 6.3 gives $b \in H_A$ and $v \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$R = r_{A,0}^j b^{-1} = \operatorname{diag}(\zeta^{3j}, \zeta^{-3j}, e^{iv}), \qquad [C,R] = 0, \qquad e^{iC}R = g^{-1}\ell_A^j g.$$

For $j = 0$, Lemma 6.6 gives length at least $2\pi > 3h$ for every nonconstant closed geodesic.

Let $1 \le j \le 6$. The first two diagonal entries of R are distinct. There are exactly the following possibilities.

Three distinct eigenvalues. If the third eigenvalue differs from both, C is diagonal. Equation (7.15) makes its third entry zero, and the first two entries have the form hm, hn with $m, n \in \mathbb{Z}$. Neither can be zero: otherwise ζ^{3j} or ζ^{-3j} would belong to $\{\zeta^j, \zeta^{-j}, 1\}$, contrary to $3 \not\equiv 0, \pm 1 \pmod 7$. If $m, n \neq 0$ and $m^2 + n^2 = 9$, each square is at most eight, so $m^2, n^2 \in \{1, 4\}$. Their sum is then $2, 5$, or 8, a contradiction. Thus this case does not give length $3h$.

A repeated eigenvalue. Suppose the third eigenvalue equals $\zeta^{3\sigma j}$, with $\sigma \in \{1, -1\}$. Its eigenspace is spanned by the third coordinate and the first coordinate when $\sigma = 1$, or the second when $\sigma = -1$. Let the eigenvalues of C on this two-dimensional space be hm, hn, and its eigenvalue on the remaining line be hr. The scalars m, n, r are integers because R is scalar on each block and $e^{iC}R$ has eigenvalues $\zeta^j, \zeta^{-j}, 1$.

None of m, n, r is zero: the repeated and the remaining eigenvalues of R are both excluded from that set. The equation $|V| = 3h$ would give

$$m^2 + n^2 + r^2 = 9. \tag{7.16}$$

Every nonzero square is at least one, so each of the three squares is at most seven and belongs to $\{1, 4\}$. If k of them equal four, their sum is $3 + 3k = 9$, whence $k = 2$. Thus their absolute values are

$1, 2, 2$. In the coordinate order placing the third coordinate last, the repeated Hermitian block is $\left(\begin{smallmatrix} a & z \\ \overline{z} & 0 \end{smallmatrix}\right)$, with $a \in \mathbb{R}$ and $z \in \mathbb{C}$. Hence

$$h^2 mn = \det \begin{pmatrix} a & z \\ \overline{z} & 0 \end{pmatrix} = -|z|^2 < 0, \tag{7.17}$$

since $m, n \neq 0$.

Since $e^{iC}R$ is conjugate to ℓ_A^j, its exponent residues satisfy the multiset identity

$$\{m + 3\sigma j, n + 3\sigma j, r - 3\sigma j\} = \{j, -j, 0\} \quad \text{in } \mathbb{Z}/7\mathbb{Z}.$$

Put $S = m + n + r$. Taking sums and sums of squares gives

$$S + 3\sigma j \equiv 0 \pmod 7,$$
$$(m + 3\sigma j)^2 + (n + 3\sigma j)^2 + (r - 3\sigma j)^2 \equiv 2j^2 \pmod 7.$$

Equation (7.16) and the first congruence give

$$0 \equiv 9 + 6\sigma j(S - 2r) + 25j^2 \equiv 9 + 7j^2 - 12\sigma jr \equiv 2 + 2\sigma jr \pmod 7.$$

Thus $\sigma jr \equiv -1 \pmod 7$, and hence

$$Sr \equiv 3 \pmod 7. \tag{7.18}$$

If $|r| = 1$, the opposite signs of m, n, both of magnitude two, give $Sr = 1$. If $|r| = 2$, then $|m|, |n|$ are one and two with opposite signs, giving $Sr = 4 \pm 2 \in \{2, 6\}$. Neither agrees with (7.18). Thus neither repeated-eigenvalue case gives length $3h$.

Thus $3h \notin \mathrm{Lsp}(M_8)$. For an arbitrary closed length, Lemma 6.6 at $q = 1$ gives

$$\mathrm{Lsp}(M_8) \subset \{h\sqrt{q} \,:\, q \in \mathbb{Z}_{>0}\}. \tag{7.19}$$

7.4.2. *An embedded primitive geodesic on N_8.* In the product presentation (6.12), the metric norm on imaginary quaternion triples is

$$|(v_0, v_1, v_2)|^2 = |v_0|^2 + 2|v_1|^2 + 2|v_2|^2.$$

Put

$$g = \left(1, \mathbf{j}, \frac{1 - \mathbf{k}}{\sqrt{2}}\right), \qquad V = h(\mathbf{i}, 0, -2\mathbf{j}), \qquad b = (1, e^{-\mathbf{i}h}, e^{\mathbf{i}h}) \in \widetilde{H}. \tag{7.20}$$

The Lie algebra of $\widetilde{H}$ is spanned by $(0, \mathbf{i}, -\mathbf{i})$, which is orthogonal to V. Thus V is horizontal. Quaternion multiplication gives

$$\mathbf{j}^{-1}\mathbf{i}\mathbf{j} = -\mathbf{i}, \qquad \left(\frac{1 - \mathbf{k}}{\sqrt{2}}\right)^{-1} \mathbf{i} \left(\frac{1 - \mathbf{k}}{\sqrt{2}}\right) = \mathbf{j}.$$

Consequently

$$g^{-1}\widetilde{\ell}^3 g = (e^{3\mathbf{i}h}, e^{-3\mathbf{i}h/2}, e^{3\mathbf{j}h/2}),$$

while

$$R := \widetilde{r}^3 b^{-1} = (e^{9\mathbf{i}h}, e^{11\mathbf{i}h/2}, e^{7\mathbf{i}h/2}) = (e^{9\mathbf{i}h}, e^{11\mathbf{i}h/2}, -1).$$

The first component of $e^V R$ is $e^{10\mathbf{i}h} = e^{3\mathbf{i}h}$; the second is $e^{11\mathbf{i}h/2} = e^{-3\mathbf{i}h/2}$. The third is

$$e^{-2\mathbf{j}h}(-1) = e^{\mathbf{j}(\pi - 2h)} = e^{3\mathbf{j}h/2}.$$

The first right component commutes with its velocity, the second velocity is zero, and the third right component is central. Consequently

$$e^V R = g^{-1}\widetilde{\ell}^3 g, \qquad \mathrm{Ad}_R\, V = V.$$

Put $\gamma(t) = g\exp(tV)\widetilde{H}$ for $0 \le t \le 1$. By Lemma 6.3, its projection to N_8 is a smoothly closed geodesic with closing deck transformation ϕ_B^3. Its length is

$$|V| = h\sqrt{1 + 2\cdot 4} = 3h.$$

Suppose the points at $0 \le t_1 < t_2 < 1$ have the same image in N_8. The first quaternionic component, on which $\widetilde{H}$ acts trivially, would satisfy

$$e^{\mathbf{i}ht_2} = e^{\mathbf{i}jh}e^{\mathbf{i}ht_1}e^{-3\mathbf{i}jh}$$

for some $j \in \{0,\dots,6\}$. Thus $t_2 - t_1 + 2j \in 7\mathbb{Z}$, forcing $t_2 - t_1$ to be an integer, contrary to $0 < t_2 - t_1 < 1$. The induced map from $\mathbb{R}/\mathbb{Z}$ is an injective immersion; compactness of $\mathbb{R}/\mathbb{Z}$ makes it an embedding. A multiple traversal would identify two times in $[0,1)$, so the geodesic is primitive. Its lift to the simply connected space B ends at ϕ_B^3 of its initial point, so its marked homotopy class is $3 \in C_7$.

7.4.3. *Topology and marked wave traces.* Lemma 6.2 and the covering isomorphism [53, Proposition 4.1] give

$$\pi_2(M_8) = \pi_2(A) = 0, \qquad \pi_2(N_8) = \pi_2(B) = \mathbb{Z}. \tag{7.21}$$

Each connected component of a common finite cover would have second homotopy group both zero and $\mathbb{Z}$, which is impossible. The isotopies $F_i(t)$, $0 \le t \le 1$, give $\phi_i^* = I$ on $H^*(X_i;\mathbb{Q})$. Since $X_i \to X_i/C_7$ is a regular covering of degree seven, [53, Proposition 3G.1] gives

$$H^*(X_i/C_7;\mathbb{Q}) \simeq H^*(X_i;\mathbb{Q})^{C_7} = H^*(X_i;\mathbb{Q}).$$

The spaces A, B are simply connected, so [53, Theorem 4.32] gives $H_2(A;\mathbb{Z}) = 0$ and $H_2(B;\mathbb{Z}) = \mathbb{Z}$. The universal coefficient theorem [53, Theorem 3.2] then gives

$$b_2(M_8) = 0, \qquad b_2(N_8) = 1. \tag{7.22}$$

Proof of Theorem 7.7. Lemma 6.2 gives simple connectedness, freeness and the markings. Section 6.4.2 gives (7.13). The curvature bounds, volumes and real analyticity follow from Section 6.4. Sections 7.4.1 and 7.4.2 give (7.14) and the embedded primitive geodesic. Equations (7.21) and (7.22) give the homotopy groups and Betti numbers. □

The simply connected manifolds A, B are orientable, and the deck isometries, being isotopic to the identity, preserve orientation. Thus M_8, N_8 are closed oriented manifolds. By the Hodge theorem, the numbers in (7.22) are the zero-eigenvalue multiplicities of their Hodge Laplacians on two-forms [30, Example 3.1.2].

Both metrics are real analytic, and the bundles in (5.1) have constant transition matrices in parallel frames. Proposition A.3 therefore gives real analyticity of each trace on every open interval disjoint from zero and the signed closed-geodesic lengths.

By (7.19) and the exclusion of $3h$, no length of M_8 lies in

$$(h\sqrt{8}, h\sqrt{10}). \tag{7.23}$$

Consequently, for every integer $r \ge 1$ and representation $\rho : C_7 \to \mathrm{U}(r)$,

$$\mathrm{Tr}(e^{it\sqrt{\Delta_{M_8,\rho}}}) = \mathrm{Tr}(e^{it\sqrt{\Delta_{N_8,\rho}}}) \in C^\omega((h\sqrt{8}, h\sqrt{10})).$$

Since $\phi_i^{j*} = R_{\phi_i^{-j}}$, Lemma 5.1 gives

$$\mathrm{Tr}(\phi_i^{j*}e^{it\sqrt{\Delta_{X_i}}}) \in C^\omega((h\sqrt{8}, h\sqrt{10})) \qquad (j \in C_7).$$

Lemma 7.8. *No $F_A(j)$-twisted geodesic has length $3h$ for any integer j. The $F_B(3)$-twisted geodesic in (7.20) has length $3h$ and projects to an embedded curve in N_8. Each of A and B has an $F_i(3)$-twisted geodesic of squared length $2h^2$.*

Proof. Every $F_A(j)$-twisted geodesic projects to a closed geodesic on M_8, so Section 7.4.1 excludes length $3h$. The $F_B(3)$-twisted geodesic of length $3h$ in (7.20) has an embedded projection by Section 7.4.2. For $i = A, B$, put

$$V_i = \left.\frac{d}{dt}\right|_{t=0} \iota_i(ht, 0).$$

The metric $2\,ds^2 + dv^2$ on the parameter torus gives $V_i \perp \mathfrak{h}_i$ and $|V_i|^2 = 2h^2$. In G_i,

$$e^{V_i} = \iota_i(h, 0) = \iota_i(-6h, 0) = \ell_i^3 r_{i,0}^{-3}, \qquad \mathrm{Ad}_{r_{i,0}^3}\, V_i = V_i.$$

Lemma 6.3, with $g = b = 1$, gives an $F_i(3)$-twisted geodesic of length $h\sqrt{2}$. □

Corollary 7.9. *For the marked class* $3 \in C_7$*,*

$$\ell_{M_8}(3) = \ell_{N_8}(3) = h\sqrt{2}, \qquad 3h \in \mathrm{Lsp}(N_8) \setminus \mathrm{Lsp}(M_8).$$

The geodesic of length $3h$ *in* (7.20) *is embedded and primitive.*

Proof. By Lemma 5.3, the nontrivial class $3 \in C_7$ has a shortest representative that is a closed geodesic. Its closing exponent is $j = 3$. In the three-distinct-eigenvalue case of Section 7.4.1, its two nonzero horizontal eigenvalues belong to $h\mathbb{Z}$, so its squared length is at least $2h^2$. In a repeated-eigenvalue case, all three eigenvalues are nonzero elements of $h\mathbb{Z}$, so the squared length is at least $3h^2$. Lemma 7.8 supplies a class-three geodesic of length $h\sqrt{2}$ on M_8. Theorem 5.4, with the identity epimorphism of C_7 and the equal flat spectra of Theorem 7.7, gives the same minimum on N_8. Sections 7.4.1 and 7.4.2 exclude $3h$ on M_8 and give the primitive embedded geodesic of length $3h$ on N_8. □

8. Finite covers and simply connected manifolds

8.1. **Products and marked finite covers.**

Theorem 8.1. *Let* F *be any smooth closed connected Riemannian manifold, and put*

$$P = M_8 \times F, \qquad Q = N_8 \times F, \qquad \Gamma = C_7 \times \pi_1(F).$$

Fix the left deck action of $\pi_1(F)$ *on its universal cover* $\widetilde{F}$*, and put* $\widetilde{P} = A \times \widetilde{F}$ *and* $\widetilde{Q} = B \times \widetilde{F}$*. For each finite-index subgroup* $\Lambda \le \Gamma$*, put*

$$P_\Lambda = \Lambda\backslash(A \times \widetilde{F}), \qquad Q_\Lambda = \Lambda\backslash(B \times \widetilde{F}),$$

where (j, γ) *acts as* (ϕ_A^j, γ) *or* (ϕ_B^j, γ)*, respectively. Fix a point in each universal cover, and take its projections as basepoints at every level. These points determine the identifications* $\pi_1(P_\Lambda) = \Lambda = \pi_1(Q_\Lambda)$*. For every* $d \in \mathbb{Z}_{>0}$ *and every unitary representation* $\rho : \Lambda \to \mathrm{U}(d)$*,*

$$\mathrm{Spec}(\Delta_{P_\Lambda,\rho}) = \mathrm{Spec}(\Delta_{Q_\Lambda,\rho}). \tag{8.1}$$

For finite-index $\Lambda' \le \Lambda \le \Gamma$*, the natural covering maps on both sides induce the inclusion* $\Lambda' \hookrightarrow \Lambda$*; the pullback of the bundle labelled by* ρ *is labelled by* $\rho|_{\Lambda'}$*. If every nonconstant smoothly closed geodesic on* F *has length greater than* $3h$*, then* $3h \in \mathrm{Lsp}(Q) \setminus \mathrm{Lsp}(P)$*.*

Proof. Write $\Pi = \pi_1(F)$, and let $\eta : \Gamma \to \mathrm{U}(V)$ be a unitary representation on a finite-dimensional Hermitian space V. For $a \in \{0, \dots, 6\}$, put $\chi_a(j) = e^{2\pi i a j/7}$, $j \in C_7$, and let V_a be the χ_a-eigenspace of the C_7 action. Each V_a is invariant under Π; denote its representation by $\sigma_a : \Pi \to \mathrm{U}(V_a)$. Then

$$V = \bigoplus_{a=0}^{6} V_a, \qquad \eta|_{V_a} = \chi_a \otimes \sigma_a.$$

For each nonzero summand, the connection Laplacian on the corresponding exterior-product bundle is

$$\Delta_{M_8,\chi_a} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{F,\sigma_a}.$$

The factor eigenbases and Lemma 2.4 give its complete spectrum, with multiplicities, and its self-adjoint domain. For $t > 0$, Tonelli's theorem gives

$$\Theta_{M_8\times F,\eta}(t) = \sum_{\substack{0\le a\le 6\\ \bar V_a\neq 0}} \Theta_{M_8,\chi_a}(t)\Theta_{F,\sigma_a}(t). \tag{8.2}$$

Theorem 7.7 gives $\Theta_{M_8,\chi_a}(t) = \Theta_{N_8,\chi_a}(t)$ for every a and $t > 0$. Thus $\Theta_{P,\eta}(t) = \Theta_{Q,\eta}(t)$ for every finite-dimensional unitary representation η of Γ.

Let $\rho : \Lambda \to \mathrm{U}(d)$ be the given subgroup representation and put $E = \mathbb{C}^d$. The induced representation $\tau = \mathrm{Ind}_\Lambda^\Gamma \rho$ acts on

$$\mathcal{V} = \{v : \Gamma \to E : v(\lambda\gamma) = \rho(\lambda)v(\gamma) \text{ for } \lambda \in \Lambda,\ \gamma \in \Gamma\}$$

by $(\tau(\delta)v)(\gamma) = v(\gamma\delta)$, $\delta \in \Gamma$. Choose representatives of $\Lambda\backslash\Gamma$. Its Hermitian norm is

$$\|v\|^2 = \sum_{\Lambda\gamma\in\Lambda\backslash\Gamma} |v(\gamma)|^2,$$

and its dimension is $[\Gamma : \Lambda]d$. For a smooth ρ-equivariant function $f : \widetilde{P} \to E$, the finite-index section isomorphism in [91, proof of Theorem 2.6, pp. 158–160], written with the convention (5.1), is

$$\widehat{f}(x)(\gamma) = f(\gamma x), \qquad f(x) = \widehat{f}(x)(1).$$

The induced section satisfies $\widehat{f}(\delta x) = \tau(\delta)\widehat{f}(x)$, $\delta \in \Gamma$. Let D be a measurable fundamental domain for Γ on $\widetilde{P}$. Integrating with the Riemannian volume densities gives

$$\int_D \|\widehat{f}(x)\|^2\, dx = \sum_{\Lambda\gamma\in\Lambda\backslash\Gamma} \int_{\gamma D} |f|^2 = \int_{\Lambda\backslash\widetilde{P}} |f|^2.$$

The same identity holds for the Dirichlet forms, since the deck maps are local isometries and the lifted flat connections are trivial. The section map is complex-linear and preserves these Hermitian norms. It therefore identifies the closed forms and their self-adjoint degree-zero Laplacians:

$$\mathrm{Spec}(\Delta_{P_\Lambda,\rho}) = \mathrm{Spec}(\Delta_{P,\mathrm{Ind}_\Lambda^\Gamma \rho}). \tag{8.3}$$

The same construction applies to Q. Equation (8.2) for $\eta = \tau$ gives equal heat traces on P_Λ and Q_Λ; Lemma 2.3 gives (8.1).

For $\Lambda' \le \Lambda$, a based loop has the same lift and endpoint in both quotients of the universal cover, so the induced map on fundamental groups is inclusion. The map

$$[(x,v)]_{\Lambda'} \longmapsto \big([x]_{\Lambda'}, [(x,v)]_\Lambda\big)$$

identifies the bundle for $\rho|_{\Lambda'}$ with the pullback bundle. Replacing (x,v) by $(\lambda' x, \rho(\lambda')v)$, $\lambda' \in \Lambda'$, leaves both target classes unchanged. In lifted trivializations the map is the identity on E; it is invertible and preserves the flat connection.

By [69, Problem 5-7(a), p. 146], a product curve is a geodesic if and only if both components are geodesics. For a smoothly closed product geodesic, both components are smoothly closed and have constant speeds; its squared length is the sum of their squared lengths. Under the stated hypothesis on F, a closed product geodesic of length $3h$ has constant F component. It would therefore give a geodesic of length $3h$ on M_8, contrary to Theorem 7.7. On N_8, the geodesic of length $3h$ paired with a constant point of F gives one on Q. □

For $F = (\mathbb{R}/(2\pi\mathbb{Z}), dx^2)$ and $\alpha \in \mathbb{R}$, the boundary condition $f(x + 2\pi) = e^{2\pi i\alpha} f(x)$ has the complete eigenbasis $e^{i(m+\alpha)x}$, $m \in \mathbb{Z}$, with eigenvalues $(m + \alpha)^2$. For $k \in \mathbb{Z}_{>0}$ and $\Lambda = C_7 \times k\mathbb{Z}$, the marked covers are

$$P_\Lambda = M_8 \times \mathbb{R}/(2\pi k\mathbb{Z}), \qquad Q_\Lambda = N_8 \times \mathbb{R}/(2\pi k\mathbb{Z}).$$

The character sending the generator $k \in k\mathbb{Z}$ to $e^{2\pi i\alpha}$ has circle eigenvalues $(m+\alpha)^2/k^2$, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$; the spectrum depends only on α mod $\mathbb{Z}$. A nonconstant closed geodesic in this circle has length at least $2\pi k$. Since $3h < 2\pi k$, every closed product geodesic of length $3h$ has constant circle component. Consequently $3h \in \mathrm{Lsp}(Q_\Lambda) \setminus \mathrm{Lsp}(P_\Lambda)$ for each $k \geq 1$. The circle direction has zero Ricci curvature.

The smooth manifolds underlying A and B are $A \simeq \mathrm{SU}(3)$ and $B \simeq S^3 \times S^3 \times S^2$. Put

$$W = A \times B, \quad g_W = g_A + g_B, \quad L = h\,2^{1/4}, \quad P = W \times (\mathbb{R}/\mathbb{Z}).$$

Set $K_0 = (K_A, 0)$ and $K_1 = (0, K_B)$, and write $s \in \mathbb{R}/\mathbb{Z}$ for the last coordinate. For $\sigma \in \{0,1\}$, $\tau \in \mathbb{R}$, $x \in W$, $v, w \in T_xW$, and $a, b \in \mathbb{R}$, define

$$g_{\sigma,\tau}(v + a\partial_s, w + b\partial_s) = g_W(v + \tau a K_\sigma, w + \tau b K_\sigma) + L^2 ab. \tag{8.4}$$

These are Riemannian metrics on the fixed smooth manifold

$$P \simeq \mathrm{SU}(3) \times S^3 \times S^3 \times S^2 \times S^1, \qquad \pi_1(P) = \mathbb{Z}, \tag{8.5}$$

marked by the last coordinate.

Theorem 8.2. *For every $\tau \in \mathbb{R}$, $d, r \in \mathbb{Z}_{>0}$, and every unitary representation $\rho : d\mathbb{Z} \to \mathrm{U}(r)$, let $\Delta_{g_{\sigma,\tau},d,\rho}$ be the flat connection Laplacian on $P_d = W \times (\mathbb{R}/d\mathbb{Z})$ with the pullback metric $g_{\sigma,\tau}$ and marking* (8.5)*. Then*

$$\mathrm{Spec}(\Delta_{g_{0,\tau},d,\rho}) = \mathrm{Spec}(\Delta_{g_{1,\tau},d,\rho}). \tag{8.6}$$

These equalities hold with the marking (8.5) *and the subgroup inclusions $de\mathbb{Z} \subset d\mathbb{Z}$ for integers $d, e \geq 1$. Every metric has Riemannian universal cover $(W \times \mathbb{R}, g_W + L^2 ds^2)$.*

If $q \geq 1$ is an integer and $\gcd(q,6) = 1$, the pair at $\tau = 1/q$ has a common connected finite Riemannian cover of degree $7q$ over each, but

$$\ell_q = 3h\sqrt{1+q^2\sqrt{2}} \in \mathrm{Lsp}(P, g_{1,1/q}) \setminus \mathrm{Lsp}(P, g_{0,1/q}). \tag{8.7}$$

The length ℓ_q is represented by an embedded primitive geodesic of marked winding $3q$. The families (8.4) *are real analytic in space and parameter, agree at $\tau = 0$, and have volume $L(16\pi^5)^2$ for every parameter. On $|\tau| \leq 1$ they have uniform positive injectivity radius and bounds on every covariant derivative of curvature. Their sectional curvatures belong to $[0,4]$; their Ricci tensors have a one-dimensional kernel and positive eigenvalues at least $1/2$.*

For each τ, the minimum length of a loop in every nonzero marked free homotopy class agrees for the two metrics.

Equation (6.28) and Proposition 6.8 give (8.6), with the stated markings and subgroup inclusions. Since W is simply connected, the same proposition gives the common universal cover $(W \times \mathbb{R}, g_W + L^2 ds^2)$. As $q \to \infty$, both metrics at $\tau = 1/q$ converge in C^∞ to $g_W + L^2 ds^2$, while $7q$ and ℓ_q tend to infinity.

8.2. **Common finite covers and unequal length sets.** By Proposition 6.8, for $q \in \mathbb{Z}_{>0}$ and $\tau = 1/q$ the product

$$C_q = (W \times \mathbb{R}/(7q\mathbb{Z}), g_W + L^2 ds^2) \tag{8.8}$$

is a common Riemannian cover of degree $7q$ of the two metrics. Write $T_{\sigma,\tau}(x,s) = (\Psi_\sigma(\tau)x, s+1)$. Let $\gamma_B(t) = g\exp(tV)\widetilde{H} \in B$, $0 \leq t \leq 1$, where g, V are given in (7.20). By Section 7.4.2, this geodesic has length $3h$ and is $F_B(3)$-twisted.

Lemma 8.3. *Let q be a positive integer coprime to six, let $C > 0$, and let u be the coordinate on $\mathbb{R}$. The product curve $t \mapsto (\gamma_B(t), 3qt)$, $0 \leq t \leq 1$, induces an embedded primitive geodesic in*

$$\mathcal{B}(q,C) = (B \times \mathbb{R}, g_B + C^2 du^2)/\langle (x,u) \mapsto (F_B(1/q)x, u+1)\rangle.$$

Its length is $3\sqrt{h^2 + q^2C^2}$ and its winding is $3q$. For $C = L$, denote the manifold by $\mathcal{B}_q$ and the induced curve by $\Gamma_q : \mathbb{R}/\mathbb{Z} \to \mathcal{B}_q$; its length is $\ell_q = 3h\sqrt{1+q^2\sqrt{2}}$.

Proof. The endpoints and tangents of the product curve match through the $3q$th deck transformation, and its squared speed is $9h^2 + 9q^2C^2$. If two points at parameter difference $0 < \delta < 1$ were identified, their real-coordinate difference would give $k = 3q\delta \in \mathbb{Z}$. The first quaternionic coordinate in (7.20), unaffected by H_B, then gives

$$\delta + 2k/q \in 7\mathbb{Z}.$$

Substitution gives $\delta \in \mathbb{Z}$, a contradiction. The curve has nonzero speed and is injective on the parameter circle. Compactness makes it an embedding, and injectivity excludes a smaller period. □

Fix henceforth $q \in \mathbb{Z}_{>0}$ with $\gcd(q, 6) = 1$.

Every closed geodesic on the suspension lifts to a product geodesic on $A \times B \times \mathbb{R}$. If its winding is $k \in \mathbb{Z}$, its real-coordinate increment is k and its squared length is the sum of its two fibre squared lengths and k^2L^2. For $g_{0,1/q}$ the A curve is twisted by $F_A(k/q)$ and the B curve is ordinarily closed. Lemma 6.6 implies that its squared length has the form

$$h^2\left(\frac{R}{q^2} + k^2\sqrt{2}\right), \qquad R \in \mathbb{Z}_{\geq 0}. \tag{8.9}$$

The squared lengths for $g_{1,1/q}$ have the same form (8.9).

The metric $g_{1,1/q}$ has the geodesic given by Γ_q and a constant A-component, of squared length ℓ_q^2. Suppose the same length occurred for $g_{0,1/q}$. Equality with (8.9) gives

$$\left(\frac{R}{q^2} - 9\right) + (k^2 - 9q^2)\sqrt{2} = 0.$$

Both coefficients are rational. Since $\sqrt{2} \notin \mathbb{Q}$, they vanish:

$$k^2 = 9q^2, \qquad R = 9q^2.$$

The total fibre squared length is then $9h^2 < 4\pi^2$. By Lemma 6.6, its ordinarily closed B component must be constant. Its A component would be an $F_A(3)$- or $F_A(-3)$-twisted geodesic of length $3h$, contradicting Lemma 7.8. Thus $\ell_q \notin \mathrm{Lsp}(P, g_{0,1/q})$, and (8.7) follows for every $q \geq 1$ with $\gcd(q, 6) = 1$.

Lemma 8.3 gives its embeddedness and primitivity. Its marked homotopy class is $3q \in \mathbb{Z}$.

The set in (8.9) is locally finite: a length at most T satisfies $|k| \leq T/L$ and $0 \leq R \leq q^2T^2/h^2$, permitting only finitely many integer pairs. There is therefore an open interval $I_q \subset (0, \infty)$ containing ℓ_q and disjoint from $\mathrm{Lsp}(P, g_{0,1/q})$. Every closed geodesic on a finite Riemannian cover projects to a closed geodesic of the same length, so I_q is disjoint from the length set of every such cover of $(P, g_{0,1/q})$. The metrics are real analytic. Corollary A.5, applied to the isospectral flat Laplacians in (8.6), gives

$$\mathrm{Tr}(e^{it\sqrt{\Delta_{g_{0,1/q},d,\rho}}}) = \mathrm{Tr}(e^{it\sqrt{\Delta_{g_{1,1/q},d,\rho}}}) \in C^\omega(I_q)$$

for every $d, r \in \mathbb{Z}_{>0}$ and every unitary representation $\rho : d\mathbb{Z} \to \mathrm{U}(r)$. The interval I_q is independent of d and ρ.

8.2.1. *A common quotient.* Write $N = 7q$ and $C_N = \mathbb{Z}/N\mathbb{Z}$, and let T_0, T_1 denote $T_{0,1/q}$, $T_{1,1/q}$. They commute, and for $a, b \in \mathbb{Z}$,

$$T_0^a T_1^b(x_A, x_B, s) = \big(F_A(a/q)x_A, F_B(b/q)x_B, s + a + b\big).$$

Put $h_q = h/q = 2\pi/N$. For $j \in \{1, \dots, N-1\}$, if $F_A(j/q)$ fixes gH_A, then

$$g^{-1}\iota_A(jh_q, 0)g = \iota_A(3jh_q, 0)b \qquad \text{for some } b \in H_A.$$

Taking determinants gives $\det b = 1$, hence $b = \mathrm{Id}$. Equality of eigenvalue multisets gives

$$\{j, -j, 0\} = \{3j, -3j, 0\} \quad \text{in } \mathbb{Z}/N\mathbb{Z}.$$

For $0 < j < N$, both multisets have three distinct entries, since $\gcd(N, 6) = 1$. Their zero entries coincide, so $j \equiv 3j$ or $j \equiv -3j \pmod N$. Thus $2j \equiv 0$ or $4j \equiv 0 \pmod N$, contrary to the oddness of N. A fixed point of $F_B(j/q)$ would give $\cos(jh_q) = \cos(3jh_q)$ in the first quaternionic factor, and hence the same two congruences. Thus both actions are fixed-point-free for $0 < j < N$, and $F_i(a/q) = \mathrm{Id}$ exactly when $N \mid a$.

The maps T_0, T_1 descend to $C_q = W \times \mathbb{R}/(N\mathbb{Z})$; retain the same notation for their descents. The homomorphism $(a, b) \mapsto T_0^a T_1^b$ has kernel $N\mathbb{Z} \oplus N\mathbb{Z}$. Its image is therefore $C_N \times C_N$. If an element of this image has a fixed point, its translation coordinate gives $a + b \equiv 0 \pmod N$. If $a \not\equiv 0$, its A component is fixed-point-free; if $a \equiv 0$, then also $b \equiv 0$. Hence the finite action is free. The quotient $C_q/(C_N \times C_N)$ is a smooth common base. Each intermediate quotient by $\langle T_0 \rangle$ or $\langle T_1 \rangle$ has degree N over that base.

Put $D = C_N \times C_N$. For a prime ℓ, write $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$. Since W is simply connected, $H_1(C_q; \mathbb{F}_\ell) \cong \mathbb{F}_\ell$. Each element $T_0^a T_1^b$ is isotopic to the identity through

$$(x_A, x_B, [s]_N) \longmapsto (F_A(ta/q)x_A, F_B(tb/q)x_B, [s + t(a + b)]_N), \qquad 0 \le t \le 1.$$

The D-action on this homology group is therefore trivial. In particular, it contains neither the regular module $\mathbb{F}_\ell[D]$, of dimension N^2, nor either permutation module $\mathbb{F}_\ell[D/\langle T_\sigma \rangle]$, of dimension N. Neither permutation module is a quotient of this homology group either: every linear map from a one-dimensional space has rank at most one, whereas $N \ge 7$. Thus this covering diagram fails the regular-module condition for homological wideness and the weaker permutation-module condition, in both its quotient and submodule forms [30, Theorem 6.4.1, §7.1, and Definition 8.1.1].

8.3. **Minimum lengths in free homotopy classes and curvature bounds.** Put $X_A = A$, $X_B = B$, and $\Delta_i = \Delta_{X_i}$ for $i \in \{A, B\}$. Let d_{X_i} denote Riemannian distance on X_i. For $s \in \mathbb{R}$, define

$$d_i(s) = \min_{x \in X_i} d_{X_i}(x, F_i(s)x).$$

The minimum exists by compactness. Since $F_i(s)$ is an isometry, $d_i(-s) = d_i(s)$. For $t > 0$, Lemma 5.2 applied to $F_i(-s)$ gives the positive traces

$$\Theta_i(t, s) = \mathrm{Tr}(F_i(s)^* e^{-t\Delta_i}) > 0$$

and

$$-4t \log \Theta_i(t, s) \longrightarrow d_i(s)^2 \qquad (t \downarrow 0). \tag{8.10}$$

The flow identity (4.14) therefore gives $d_A(s) = d_B(s)$ for every real s.

For $k \in \mathbb{Z} \setminus \{0\}$, define $\ell_{\min,\sigma,\tau}(k)$ as the infimum of lengths of piecewise smooth loops in the marked free homotopy class k. The universal cover $W \times \mathbb{R}$ is simply connected and its deck group is the marked abelian group $\mathbb{Z}$, so this is exactly the class of paths with endpoint $T_{\sigma,\tau}^k$ times their initial point. Let d_{prod} be the distance of $g_W + L^2 ds^2$ on $W \times \mathbb{R}$. For $z = (x_A, x_B, s)$,

$$d_{\mathrm{prod}}(z, T_{\sigma,\tau}^k z)^2 = k^2 L^2 + \begin{cases} d_{X_A}(x_A, F_A(\tau k)x_A)^2, & \sigma = 0, \\ d_{X_B}(x_B, F_B(\tau k)x_B)^2, & \sigma = 1, \end{cases}$$

Consequently minimizing displacement gives

$$\ell_{\min,\sigma,\tau}(k)^2 = k^2 L^2 + d_A(\tau k)^2 = k^2 L^2 + d_B(\tau k)^2 \quad (k \neq 0). \tag{8.11}$$

Indeed, every loop of winding k lifts to a path joining a point to its kth deck translate, so its length is at least the minimum displacement in (8.11). Compactness of the fiber attains that displacement, and a minimizing product segment projects to a loop of winding k with that length. Hence it is the class minimum. Lemma 5.3 supplies a smoothly closed representative; its length is positive because it is at least $|k|L$. This proves equality of the two marked minimum-length functions for every τ.

By Proposition 6.8, both metrics are locally isometric to the fixed product $W \times \mathbb{R}$. Thus their curvature and all its covariant derivatives have the product bounds. In the coordinates (6.24), the real direction is $\partial_s - \tau K_\sigma$, so

$$\ker \mathrm{Ric}_{g_{\sigma,\tau}} = \mathbb{R}(\partial_s - \tau K_\sigma), \qquad 0 \le \sec(g_{\sigma,\tau}) \le 4.$$

The other eigenvalues of the Ricci endomorphism relative to $g_{\sigma,\tau}$ are those of (W, g_W), hence at least $1/2$. The same proposition gives the positive uniform injectivity bound and C^m convergence estimate, and (6.14) gives volume $L(16\pi^5)^2$. Together with (8.6) and (8.7), these identities prove Theorem 8.2.

8.4. **Infinite length-set differences on a simply connected manifold.** Schueth constructed continuous isospectral families on $S^n \times S^3 \times S^3$, $n \ge 4$, arbitrarily close to the standard product metric [93, Main Theorem and following paragraph, p. 288]. There are also positive-Ricci isospectral families on $\mathrm{SO}(n)$, $n \ge 8$, and $\mathrm{SU}(n)$, $n \ge 6$ [94, Remarks 3.12(i)–(ii), author version].

Retain the eight-dimensional normal homogeneous manifolds (A, g_A) and (B, g_B), the seven-periodic flows F_A, F_B, and their infinitesimal generators K_A, K_B from Section 6.4. The volumes of A and B agree by (6.14). Corollary 4.14 therefore gives a unitary operator for their Riemannian volume measures,

$$U : L^2(A, dv_A) \longrightarrow L^2(B, dv_B), \qquad U\Delta_A = \Delta_B U, \qquad UK_A = K_B U.$$

The domains satisfy

$$U\mathcal{D}(\Delta_A) = \mathcal{D}(\Delta_B), \qquad U\mathcal{D}(K_A) = \mathcal{D}(K_B).$$

By Theorem 7.7, the fixed manifold

$$M = A \times B \times S^2$$

is simply connected and has dimension eighteen.

8.4.1. *A prescribed length in an analytic family.* For a fixed real-analytic metric on a closed manifold, the length set is locally finite [92, Proposition 1.2].

Lemma 8.4. *Let P be a closed real-analytic manifold, let $I \subset \mathbb{R}$ be an open interval, and let $(g_s)_{s\in I}$ be a jointly real-analytic family of Riemannian metrics. Fix a compact interval $J \subset I$ and $\ell > 0$. Suppose that every closed g_s-geodesic $\gamma : \mathbb{R}/\mathbb{Z} \to P$ of constant speed ℓ, for $s \in J$, satisfies*

$$\int_0^1 (\partial_s g_s)_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt > 0. \tag{8.12}$$

Then $\{s \in J : \ell \in \mathrm{Lsp}(P, g_s)\}$ is finite.

Proof. If no such geodesic occurs on J, the exceptional set is empty. Otherwise, write $J = [s_-, s_+]$ and choose an auxiliary Riemannian metric g_{ref} on P. Compactness of $J \times P$ gives a number $c > 0$ such that $g_s \ge c\, g_{\mathrm{ref}}$ for $s \in J$. Let X_s be the geodesic spray of g_s and Φ_s^t its flow. The vector field $(s, v) \mapsto (0, X_s(v))$ on $I \times TP$ is real analytic. Its flow is analytic on its domain [21, Theorem 1]. Since P is closed, every g_s is complete; thus $(s, v) \mapsto \Phi_s^1(v)$ is defined and real analytic on $I \times TP$. The set

$$Z_\ell = \{(s, v) \in J \times TP : \Phi_s^1(v) = v,\ g_s(v, v) = \ell^2\}$$

is closed and semianalytic in $I \times TP$. The fixed-point condition is the inverse image of the analytic diagonal in $TP \times TP$ under $(s, v) \mapsto (\Phi_s^1(v), v)$, the speed condition is one analytic equation, and $s_- \le s \le s_+$ gives two analytic inequalities. Its vectors satisfy $|v|_{g_{\mathrm{ref}}} \le \ell/\sqrt{c}$, so Z_ℓ is compact. The uniformization theorem [15, Theorem 0.1] gives a real-analytic manifold N and a proper analytic map $\psi : N \to I \times TP$ with image Z_ℓ. Since $N = \psi^{-1}(Z_\ell)$, properness makes N compact; hence N has finitely many connected components.

Let $p : N \to I$ be the first component of ψ, and let $\pi_{TP} : TP \to P$ be the tangent-bundle projection. For a smooth curve $a \mapsto n(a)$ in N, write $\psi(n(a)) = (s(a), v(a))$ and set

$$\gamma_a(t) = \pi_{TP}(\Phi_{s(a)}^t(v(a))).$$

For a Riemannian metric g on P and a smooth loop $\gamma : \mathbb{R}/\mathbb{Z} \to P$, put

$$E_g(\gamma) = \frac{1}{2}\int_0^1 g(\dot\gamma, \dot\gamma)\, dt.$$

Each γ_a is a smoothly closed geodesic with $E_{g_{s(a)}}(\gamma_a) = \ell^2/2$. It is a critical point of $E_{g_{s(a)}}$ on the free-loop space [90, §5, author-version pp. 13–14]. The chain rule gives

$$0 = \frac{d}{da}E_{g_{s(a)}}(\gamma_a) = \frac{s'(a)}{2}\int_0^1 (\partial_s g_{s(a)})(\dot\gamma_a, \dot\gamma_a)\, dt.$$

Hypothesis (8.12) gives $s'(a) = 0$. Every tangent vector to N is the velocity of such a curve, so $dp = 0$. Thus p is constant on each connected component of N, and $p(N) = \{s \in J : \ell \in \mathrm{Lsp}(P, g_s)\}$ is finite. □

8.4.2. *Norms and periods of twisted geodesics.* Lemma 6.6 gives

$$\mathrm{Lsp}(A, g_A) \cup \mathrm{Lsp}(B, g_B) \subseteq \{2\pi\sqrt{N} : N \in \mathbb{Z}_{>0}\}. \tag{8.13}$$

For $\beta > 0$, a closed geodesic in $(A\times B, g_A \oplus \beta g_B)$ has squared length $4\pi^2(N+\beta D)$ for some $N, D \in \mathbb{Z}_{\geq 0}$.

Put $\zeta = e^{ih}$, where $h = 2\pi/7$. For $j \in \mathbb{Z}$, write

$$L_j = \mathrm{diag}(\zeta^j, \zeta^{-j}, 1), \qquad R_j = \mathrm{diag}(\zeta^{3j}, \zeta^{-3j}, 1).$$

Thus $F_A(j)(gH_A) = L_j g R_j^{-1} H_A$. By Lemma 7.8, no $F_A(j)$-twisted geodesic has length $3h$ for an integer j.

Lemma 8.5. *For $r \in \mathbb{Z}_{\geq 0}$ and $k = 2^r$, there is no $F_A(3k)$- or $F_A(-3k)$-twisted geodesic of length $3kh$.*

Proof. Suppose such a geodesic exists, and put $j = 3k$ or $j = -3k$. Write it as $t \mapsto g\exp(itC)H_A$, $0 \leq t \leq 1$, with $g \in G_A$ and a Hermitian matrix $C \in M_3(\mathbb{C})$ satisfying

$$C_{33} = 0, \qquad \mathrm{Tr}(C^2) = 9k^2h^2.$$

By the horizontal closing criterion (6.16), for some $b \in H_A$,

$$S = R_j b^{-1}, \qquad [C, S] = 0, \qquad e^{iC}S = g^{-1}L_j g. \tag{8.14}$$

The proof of Lemma 6.6, with $q = 1$, gives all three eigenvalues of C in $h\mathbb{Z}$: $j = \pm 3\cdot 2^r$ is nonzero modulo seven, so the first two diagonal entries of S are distinct.

Write these eigenvalues as hm_1, hm_2, hm_3. They satisfy

$$m_1^2 + m_2^2 + m_3^2 = 9\cdot 4^r.$$

An integral square is zero or one modulo four. Since at most three such residues occur, divisibility of their sum by four forces every residue to be zero. Induction on r gives $k \mid m_a$ for each $a \in \{1,2,3\}$. Choose $a_0 \in \{1,\dots,6\}$ with $a_0 k \equiv 1 \pmod 7$. On an eigenvector with eigenvalue hkz, where $z \in \mathbb{Z}$, one has $e^{ihz} = e^{ia_0 hkz}$; hence

$$e^{iC/k} = (e^{iC})^{a_0}.$$

Since e^{iC} and S commute, taking the a_0th power of $e^{iC}S = g^{-1}L_j g$ gives

$$e^{iC/k}S^{a_0} = g^{-1}L_{j/k}g, \qquad S^{a_0} = R_{j/k}b^{-a_0}, \qquad [C/k, S^{a_0}] = 0.$$

Here $j/k = 3$ or -3, and the identities follow from $a_0 j \equiv j/k \pmod 7$. Together with $(C/k)_{33} = 0$, these are the closing and horizontality equations for an $F_A(j/k)$-twisted geodesic of length $3h$, contrary to Lemma 7.8. □

Let

$$\mathcal{N} = \{n \in \mathbb{Z}_{>0} : n \equiv 1,2,3,5,6 \pmod 8\}. \tag{8.15}$$

Lemma 8.6. *For each $n \in \mathcal{N}$ there are an odd positive integer d and a closed geodesic $c_n : \mathbb{R} \to A$ of speed $2\pi\sqrt{n}$ such that*

$$c_n(t+\delta) = c_n(t) \quad \Longleftrightarrow \quad d\delta \in \mathbb{Z} \qquad (t, \delta \in \mathbb{R}). \tag{8.16}$$

Proof. The residue classes defining $\mathcal{N}$ contain no integer of the form $4^a(8b+7)$, with $a, b \in \mathbb{Z}_{\geq 0}$. The three-square theorem [98, Introduction, equation (1.2), arXiv v14] therefore gives $n = m_1^2 + m_2^2 + m_3^2$ with $m_a \in \mathbb{Z}$. Then $d = \gcd(|m_1|, |m_2|, |m_3|)$ is odd, since $4 \nmid n$. By changing signs, arrange $\min_a m_a \leq 0 \leq \max_a m_a$. Choose nonnegative t_1, t_2, t_3 with $t_1 + t_2 + t_3 = 1$ and $\sum_a t_a m_a = 0$. For an explicit choice, if some $m_a = 0$, put all weight at that index. Otherwise, for $m_p < 0 < m_q$, set $t_p = m_q/(m_q - m_p)$, $t_q = -m_p/(m_q - m_p)$, and the remaining weight to zero. Take a unitary matrix V whose third column is $(\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3})^{\mathsf{T}}$. Then

$$C_n = 2\pi V^* \operatorname{diag}(m_1, m_2, m_3) V, \qquad (C_n)_{33} = 0.$$

The group geodesic $t \mapsto \exp(itC_n)$ is horizontal. Its projection $c_n(t) = \exp(itC_n)H_A$ is therefore a geodesic of speed $2\pi\sqrt{n}$. Since $\exp(iC_n) = \mathrm{Id}$, it is closed with period one.

Equality of its values at t and $t+\delta$ is equivalent to $\exp(i\delta C_n) \in H_A$. Write this exponential as $\operatorname{diag}(1, 1, e^{iv})$, with $v \in \mathbb{R}$. If $e^{iv} \neq 1$, commutation with C_n forces C_n to preserve the third coordinate line, on which its eigenvalue is $(C_n)_{33} = 0$. Its exponential would then act as one on that line, a contradiction. Thus $\exp(i\delta C_n) = \mathrm{Id}$. Equivalently, $\delta m_a \in \mathbb{Z}$ for each a, which is equivalent to $d\delta \in \mathbb{Z}$ by Bézout's identity. □

Let $\mathcal{B}_q$ and Γ_q be the suspension and embedded primitive geodesic of Lemma 8.3.

8.4.3. *The metrics and their spectra.* On $W = A \times B$, put

$$g_{W,0,\beta} = g_A \oplus \beta g_B, \qquad g_{W,1,\beta} = \beta g_A \oplus g_B, \qquad K_0 = (K_A, 0), \quad K_1 = (0, K_B), \qquad \beta > 0.$$

Let x, y, z be the restrictions of the Euclidean coordinate functions to the unit sphere $S^2 \subset \mathbb{R}^3$, and let b_* be its round metric of curvature one. Define

$$R = \frac{L}{2\pi}, \qquad \alpha_\tau = \frac{\tau}{2\pi}(x\,dy - y\,dx), \qquad b_s = R^2(1 + sz^2)b_*, \qquad \tau \in \mathbb{R}, \quad s > -1. \tag{8.17}$$

For $\sigma \in \{0, 1\}$, $p \in W$, $b \in S^2$, $v, w \in T_pW$, and $X, Y \in T_bS^2$, set

$$G_{\sigma,\beta,\tau,s}(v+X, w+Y) = g_{W,\sigma,\beta}\big(v + \alpha_\tau(X)K_\sigma, w + \alpha_\tau(Y)K_\sigma\big) + b_s(X, Y). \tag{8.18}$$

These are jointly real-analytic metrics in space and the parameters. Their positive definiteness follows directly from the invertible shear in Lemma 6.7 and $1 + sz^2 \geq \min\{1, 1+s\} > 0$ on S^2.

Lemma 8.7. *For every $\beta > 0$, $\tau \in \mathbb{R}$, and $s > -1$, the metrics $G_{0,\beta,\tau,s}$ and $G_{1,\beta,\tau,s}$ are scalar-isospectral.*

Proof. Let T be the unitary operator on $L^2(A \times B, dv_A\,dv_B)$ determined, for $f \in L^2(A, dv_A)$ and $g \in L^2(B, dv_B)$, by

$$T(f \otimes g) = U^{-1}g \otimes Uf.$$

On tensor products of joint eigenfunctions,

$$T(\Delta_A + \beta^{-1}\Delta_B) = (\beta^{-1}\Delta_A + \Delta_B)T.$$

Lemma 2.4 gives this identity on the self-adjoint domains. The flow intertwining of Corollary 4.14 also gives

$$T(F_A(t)^* \otimes \mathrm{Id}) = (\mathrm{Id} \otimes F_B(t)^*)T \qquad (t \in \mathbb{R}).$$

Both vertical volume forms are $\beta^4 dv_A\,dv_B$, since $\dim A = \dim B = 8$. Thus T is unitary for the two vertical metrics. Apply Lemma 6.7 with base (S^2, b_s), one-form α_τ, and the trivial complex line bundle with its trivial connection. □

The reflection $(x, y, z) \mapsto (x, y, -z)$ preserves b_s and α_τ. Its product with the identity of W is an isometry of (8.18). Its connected fixed set $\Sigma = W \times \{z = 0\}$ is totally geodesic by [69, Problem 8-32, p. 262]. In the equatorial coordinate $(x, y, z) = (\cos 2\pi u, \sin 2\pi u, 0)$, $u \in \mathbb{R}/\mathbb{Z}$,

$$b_s = L^2 du^2, \qquad \alpha_\tau = \tau\, du.$$

For $p \in W$, $v, w \in T_p W$, and $a, b \in \mathbb{R}$, the induced metric is

$$\widehat{G}_{\sigma,\beta,\tau}(v + a\partial_u, w + b\partial_u) = g_{W,\sigma,\beta}(v + \tau a K_\sigma, w + \tau b K_\sigma) + L^2 ab. \tag{8.19}$$

It is independent of s. For $\sigma = 1$ and $\tau = 1/q$, Proposition 6.8, applied to (B, g_B) and F_B with scale L, identifies this equatorial metric with $(A, \beta g_A) \times \mathcal{B}_q$. The isometry is

$$(x_A, x_B, [u]) \longmapsto (x_A, [F_B(u/q)x_B, u]),$$

where $u \in \mathbb{R}$ is any representative of $[u] \in \mathbb{R}/\mathbb{Z}$.

8.4.4. *Infinitely many embedded primitive geodesics.* Choose $\beta > 0$ with $1, \beta, \sqrt{2}$ linearly independent over $\mathbb{Q}$. For $q \in \mathbb{Z}_{>0}$ coprime to six, $r \in \mathbb{Z}_{\geq 0}$, and $n \in \mathcal{N}$, let $\Lambda_{r,n,q,\beta} > 0$ be defined by

$$k_r = 2^r, \qquad \Lambda^2_{r,n,q,\beta} = h^2\big(9k_r^2 + 49\beta n + 9q^2 k_r^2 \sqrt{2}\big). \tag{8.20}$$

Theorem 8.8 (Infinitely many primitive lengths). *For each such (q, β), there is a countable set $E_{q,\beta} \subset (-1, \infty)$ such that, for $s \in (-1, \infty) \setminus E_{q,\beta}$, the two real-analytic scalar-isospectral metrics on the fixed simply connected eighteen-manifold M satisfy*

$$\Lambda_{r,n,q,\beta} \in \mathrm{Lsp}(M, G_{1,\beta,1/q,s}) \setminus \mathrm{Lsp}(M, G_{0,\beta,1/q,s}) \quad (r \in \mathbb{Z}_{\geq 0},\ n \in \mathcal{N}). \tag{8.21}$$

Each length $\Lambda_{r,n,q,\beta}$ is represented by an embedded primitive geodesic on $(M, G_{1,\beta,1/q,s})$, and the common half-wave trace is real analytic on an open interval about that length. The lengths are pairwise distinct. As $T \to +\infty$,

$$\#\{(r, n) \in \mathbb{Z}_{\geq 0} \times \mathcal{N} \,:\, \Lambda_{r,n,q,\beta} \leq T\} = \frac{5}{32\beta\pi^2 \log 2}\, T^2 \log T + O_{q,\beta}(T^2). \tag{8.22}$$

Moreover, there is a product metric g_* on M with positive Ricci curvature such that every C^∞ neighborhood of g_* contains a pair $(G_{0,\beta,1/q,s}, G_{1,\beta,1/q,s})$ with positive Ricci curvature for some q, β as above and some $s \in (-1, \infty) \setminus E_{q,\beta}$.

Proof. Abbreviate $G_{\sigma,s} = G_{\sigma,\beta,1/q,s}$. Identify $(\Sigma, G_{1,s}|_\Sigma)$ with $(A, \beta g_A) \times \mathcal{B}_q$ by the preceding isometry, and put

$$\gamma_{r,n}(t) = (c_n(t), \Gamma_q(k_r t)), \qquad t \in \mathbb{R}/\mathbb{Z}.$$

Both components are affinely parametrized geodesics. Thus $\gamma_{r,n}$ is a geodesic of squared length (8.20). By Lemma 8.6, equality at parameter difference δ in its first component is equivalent to $d\delta \in \mathbb{Z}$, for some odd positive integer d. Embeddedness of Γ_q gives equality in the second component precisely when $k_r\delta \in \mathbb{Z}$. Since $\gcd(d, k_r) = 1$, Bézout's identity implies $\delta \in \mathbb{Z}$. The curve $\gamma_{r,n}$ has nonzero constant speed, so it is an injective immersion of the compact parameter circle and hence an embedding. Injectivity also excludes every smaller positive period. Since Σ is totally geodesic, $\gamma_{r,n}$ is an embedded primitive geodesic of $G_{1,\beta,1/q,s}$ for every $s > -1$.

Suppose that the same length belongs to $\mathrm{Lsp}(\Sigma, G_{0,s}|_\Sigma)$. Parametrize such a geodesic at constant speed on $\mathbb{R}/\mathbb{Z}$. Its lift has real-coordinate displacement $m \in \mathbb{Z}$; its A-component is $F_A(m/q)$-twisted, and its B-component is closed or constant. Indeed, the untwisting isometry identifies the universal cover of this equatorial metric with the product $(A \times B \times \mathbb{R}, g_A \oplus \beta g_B \oplus L^2 du^2)$; the mth deck map is $(F_A(m/q), \mathrm{Id}, u \mapsto u + m)$. Lifting a smooth closed geodesic gives a product geodesic whose endpoints and tangents match through precisely this deck map. For every $m \in \mathbb{Z}$, including zero, Lemma 6.6 therefore gives the squared length

$$h^2\big(q^{-2}N + 49\beta D + m^2\sqrt{2}\big), \qquad N, D \in \mathbb{Z}_{\geq 0}. \tag{8.23}$$

Rational independence of $1, \beta, \sqrt{2}$, together with (8.20), gives

$$N = 9q^2k_r^2, \qquad D = n, \qquad m = \pm 3qk_r.$$

The A-component would be $F_A(\pm 3k_r)$-twisted with length $3k_rh$, contrary to Lemma 8.5.

For fixed (r, n), put

$$E_{r,n} = \{s > -1 \,:\, \Lambda_{r,n,q,\beta} \in \mathrm{Lsp}(M, G_{0,s})\}.$$

Let $J \subset (-1, \infty)$ be a compact interval. Pulling base tensors back to M, one has $\partial_s G_{0,s} = R^2 z^2 b_* \geq 0$. For $s \in J$, let $\gamma = (w, b) : \mathbb{R}/\mathbb{Z} \to M$ be a closed $G_{0,s}$-geodesic of constant speed $\Lambda_{r,n,q,\beta}$. If the integral in (8.12) were zero, continuity would give

$$z(b(t))^2 |\dot{b}(t)|^2_{b_*} = 0 \qquad (t \in \mathbb{R}/\mathbb{Z}).$$

The base curve b then lies in the equator or is constant. Indeed, on each connected component of the inverse image of either open hemisphere, $\dot{b} = 0$. Its value is constant there, and any such nonempty component is also closed in the parameter circle, since its constant value is strictly inside that hemisphere. It must therefore be the entire circle.

In the equatorial case the curve, already an ambient geodesic, is also an intrinsic geodesic of Σ, since its tangential covariant acceleration is zero. This contradicts the preceding exclusion from $\mathrm{Lsp}(\Sigma, G_{0,s}|_\Sigma)$. If b is constant, variations of w with the same base point have energy $\frac{1}{2}\int_0^1 (g_A \oplus \beta g_B)(\dot{w}, \dot{w})\, dt$. Since γ is a critical point of the ambient energy, w is an intrinsic closed geodesic of $(W, g_A \oplus \beta g_B)$. By (8.13), its squared length belongs to

$$49h^2\{N' + \beta D' \,:\, N', D' \in \mathbb{Z}_{\geq 0}\}.$$

This cannot equal (8.20), whose coefficient of $\sqrt{2}$ is nonzero. Thus (8.12) holds, and Lemma 8.4 gives $\#(E_{r,n} \cap J) < \infty$. The compact intervals $[-1 + 1/a, a]$, $a \in \mathbb{Z}_{>0}$, exhaust $(-1, \infty)$, so each $E_{r,n}$ is countable. Put

$$E_{q,\beta} = \bigcup_{\substack{r \in \mathbb{Z}_{\geq 0} \\ n \in \mathcal{N}}} E_{r,n}.$$

This set is countable, and every $s \in (-1, \infty) \setminus E_{q,\beta}$ satisfies (8.21) for all (r, n).

If two quantities in (8.20) are equal, the coefficients of $\sqrt{2}$ give $4^r = 4^{r'}$, and the coefficients of β then give $n = n'$. Hence the lengths $\Lambda_{r,n,q,\beta}$ are distinct. Corollary A.5 makes the common half-wave trace of the scalar-isospectral analytic metrics $G_{0,\beta,1/q,s}$ and $G_{1,\beta,1/q,s}$ real analytic near each excluded positive length.

For $T \geq 0$, put

$$H(T) = \#\{(r, n) \in \mathbb{Z}_{\geq 0} \times \mathcal{N} \,:\, \Lambda_{r,n,q,\beta} \leq T\},$$

and set

$$X = (T/h)^2, \qquad A_q = 9(1 + q^2\sqrt{2}), \qquad B_\beta = 49\beta.$$

The five residue classes defining $\mathcal{N}$ give

$$\#(\mathcal{N} \cap [1, Y]) = \tfrac{5}{8}Y + O(1) \qquad (Y \geq 0),$$

with absolute error at most five. Consequently, for $X \geq A_q$ and $K = \lfloor \log_4(X/A_q) \rfloor$,

$$\begin{aligned} H(T) &= \sum_{r=0}^{K} \left(\frac{5}{8B_\beta}(X - A_q 4^r) + O(1) \right) \\ &= \frac{5}{8B_\beta}\left((K+1)X - \frac{A_q(4^{K+1} - 1)}{3} \right) + O(K+1) \\ &= \frac{5}{8B_\beta \log 4} X \log X + O_{q,\beta}(X). \end{aligned}$$

Here $K+1=\log X/\log 4+O_q(1)$, $A_q 4^{K+1}\le 4X$, and the sum of the residue-class errors has absolute value at most $5(K+1)$. Substituting $h^2=4\pi^2/49$ gives (8.22).

For $j\in\mathbb{Z}_{>0}$, take $q_j=6j+1$ and put $\beta_j=1+\sqrt{3}/q_j$. The numbers $1,\beta_j,\sqrt{2}$ are linearly independent over $\mathbb{Q}$, since $1,\sqrt{3},\sqrt{2}$ are. Choose

$$s_j\in(0,1/q_j)\setminus E_{q_j,\beta_j}.$$

Such a choice is possible because the exceptional set is countable. Put

$$g_*=g_A\oplus g_B\oplus R^2 b_*.$$

All factor tensors below are pulled back to M. Put

$$\alpha_*=\frac{x\,dy-y\,dx}{2\pi},\qquad \eta_0=g_A(K_A,\cdot),\quad \eta_1=g_B(K_B,\cdot),\qquad h_0=g_B,\quad h_1=g_A.$$

Expanding (8.18) gives

$$\begin{aligned}G_{\sigma,\beta,\tau,s}-g_*&=(\beta-1)h_\sigma+\tau(\eta_\sigma\otimes\alpha_*+\alpha_*\otimes\eta_\sigma)\\&\quad+\tau^2 g_W(K_\sigma,K_\sigma)\alpha_*\otimes\alpha_*+sR^2z^2b_*.\end{aligned}$$

All tensors on the right are fixed and smooth on the compact manifold. Taking their $C^m(g_*)$ norms gives, for every $m\ge 0$,

$$\|G_{\sigma,\beta,\tau,s}-g_*\|_{C^m(g_*)}\le C_m\big(|\beta-1|+|\tau|+\tau^2+|s|\big)$$

when $1/2\le\beta\le 2$, $|\tau|\le 1$, and $-1<s\le 1$. Thus $G_{\sigma,\beta_j,1/q_j,s_j}\to g_*$ in C^∞ for both values of σ. Put $\rho=\min\{1/2,R^{-2}\}>0$. By Theorem 7.7,

$$\mathrm{Ric}_{g_*}\ge\rho g_*.$$

By continuity of the Ricci tensor in the C^2 topology,

$$\mathrm{Ric}_{G_{\sigma,\beta_j,1/q_j,s_j}}\ge(\rho/2)g_*>0$$

for both values of σ and all sufficiently large j. □

Remark 8.9. The counting function in (8.22) is that of the subset $\{\Lambda_{r,n,q,\beta}:r\ge 0,\ n\in\mathcal{N}\}$ of the ordinary length-set difference. The spectral equality is

$$\mathrm{Spec}(\Delta_{G_{0,\beta,\tau,s}})=\mathrm{Spec}(\Delta_{G_{1,\beta,\tau,s}})\qquad(\beta>0,\ \tau\in\mathbb{R},\ s>-1),$$

with the same parameters on both sides.

8.5. **Short geodesics in products.**

Lemma 8.10. *Let $(L_0,g_0),(L_1,g_1)$ and $(X,h_X),(Y,h_Y)$ be two pairs of scalar-isospectral closed connected Riemannian manifolds of positive dimension. Fix $T>0$ and choose $c>0$ such that*

$$2c\min\{\mathrm{inj}(X,h_X),\mathrm{inj}(Y,h_Y)\}>T.\tag{8.24}$$

Set $P_0=(L_0\times X,g_0+c^2h_X)$ and $P_1=(L_1\times Y,g_1+c^2h_Y)$. Then P_0,P_1 are scalar-isospectral and

$$\mathrm{Lsp}(P_i)\cap(0,T]=\mathrm{Lsp}(L_i,g_i)\cap(0,T]\qquad(i=0,1).$$

A closed geodesic of L_i, regarded as a curve in P_i with constant second component, remains embedded and primitive whenever it has these properties in L_i.

Proof. The scale c exists by positivity of the injectivity radii [69, Lemma 6.16, p. 166]. Common scaling sends each second-factor eigenvalue μ to $c^{-2}\mu$. The tensor-product decomposition in the proof of Theorem 8.1 gives equal spectra on P_0, P_1.

By [69, Problem 5-7(a), p. 146], the nonconstant closed geodesics of P_i are precisely the products of factor geodesics parametrized affinely on $\mathbb{R}/\mathbb{Z}$, with at most one constant component. Their squared lengths are the sums of the factor squared lengths. Thus, for $(Z_0, h_0) = (X, h_X)$ and $(Z_1, h_1) = (Y, h_Y)$,

$$\operatorname{Lsp}(P_i) = \left\{ \sqrt{a^2 + c^2 b^2} \;:\; \begin{array}{l} a \in \operatorname{Lsp}(L_i, g_i) \cup \{0\}, \quad b \in \operatorname{Lsp}(Z_i, h_i) \cup \{0\}, \\ (a, b) \neq (0, 0) \end{array} \right\}. \tag{8.25}$$

Every nonconstant closed geodesic on (Z_i, h_i) has length at least $2\operatorname{inj}(Z_i, h_i)$ [69, Proposition 6.11, p. 161]. Thus (8.24) forces the second component of every closed product geodesic of length at most T to be constant. Its first component is a closed geodesic of the same positive length. Conversely, every closed geodesic of the first factor gives a product geodesic of the same length. This proves the set identities. Equality of two points, or occurrence of a smaller period, for a curve with constant second component is equivalent to the same property for its first component; its velocity is $(\dot\gamma_1, 0)$. □

9. Exact length differences and their asymptotics

9.1. **Locally symmetric suspensions.** Let g_{G_A} and g_{G_B} be the bi-invariant Riemannian metrics induced by (6.1) on $G_A = \mathrm{U}(3)$ and $G_B = \mathrm{Sp}(1) \times \mathrm{SO}(4)$. Set

$$W_G = G_A \times G_B, \qquad g_{W_G} = g_{G_A} \oplus g_{G_B}, \qquad P_G = W_G \times (\mathbb{R}/\mathbb{Z}), \qquad h = \frac{2\pi}{7}, \quad L = h2^{1/4}.$$

The metrics g_A, g_B on the quotients A, B remain as in Section 6.4. For $i \in \{A, B\}$, $u \in \mathbb{R}$ and $x \in G_i$, define the seven-periodic isometric flow

$$\mathcal{F}_i(u)(x) = \iota_i(hu, 0)x\iota_i(3hu, hu)^{-1}.$$

On W_G, put

$$\Psi_0^G(u) = \mathcal{F}_A(u) \times \mathrm{id}_{G_B}, \qquad \Psi_1^G(u) = \mathrm{id}_{G_A} \times \mathcal{F}_B(u),$$

and let Z_σ be the generator of Ψ_σ^G, for $\sigma \in \{0, 1\}$. Let s denote the coordinate on $\mathbb{R}$; the field ∂_s and the one-form ds descend to $\mathbb{R}/\mathbb{Z}$. For $x \in W_G$, $v, w \in T_x W_G$, $a, b \in \mathbb{R}$ and $\tau \in \mathbb{R}$, set

$$g_{\sigma,\tau}^G(v + a\partial_s, w + b\partial_s) = g_{W_G}(v + \tau a Z_\sigma(x), w + \tau b Z_\sigma(x)) + L^2 ab. \tag{9.1}$$

Proposition 9.1. *For every $\tau \in \mathbb{R}$, the metrics $g_{0,\tau}^G$ and $g_{1,\tau}^G$ are scalar-isospectral, real analytic, locally symmetric, and have nonnegative sectional curvature and the same Riemannian universal cover. For $d \in \mathbb{Z}_{>0}$, put $P_{G,d} = W_G \times (\mathbb{R}/d\mathbb{Z})$ and let $\pi_d : P_{G,d} \to P_G$ be given by $\pi_d(x, [s]_d) = (x, [s]_1)$, where $[s]_d$ denotes the class modulo $d\mathbb{Z}$. For every $r \in \mathbb{Z}_{>0}$ and unitary representation $\rho : d\mathbb{Z} \to \mathrm{U}(r)$, the two connection Laplacians on the pullback to $P_{G,d}$ of the flat bundle over $\mathbb{R}/d\mathbb{Z}$ associated with ρ are isospectral for the metrics $\pi_d^* g_{0,\tau}^G$ and $\pi_d^* g_{1,\tau}^G$. For $\tau = 1/q$, $q \in \mathbb{Z}_{>0}$, they have a common connected finite Riemannian cover of degree $7q$ and their geodesic flows are clean.*

Proof. The closed flow subgroup

$$\mathcal{K} = \{(ht, 0; 3ht, ht) \;:\; t \in \mathbb{R}\} \subset S_1$$

is contained in $\ker \alpha$ by (4.13). Theorem 3.9 gives a $\mathcal{K}$-equivariant unitary $S : L^2(G_A) \to L^2(G_B)$ intertwining the scalar Laplacians. The measures may be taken to be the Riemannian volume measures: both group volumes are $32\pi^6$ by Section 6.4. For $f \in L^2(G_A)$ and $g \in L^2(G_B)$, the map

$$T(f \otimes g) = S^{-1}g \otimes Sf$$

extends to a unitary on $L^2(W_G)$ and satisfies

$$T\Delta_{W_G} = \Delta_{W_G} T, \qquad T\Psi_0^G(u)^* = \Psi_1^G(u)^* T \quad (u \in \mathbb{R}).$$

The Laplacian identity holds on its self-adjoint domain by Lemma 2.4. The full-flow identity gives $TZ_0 = Z_1 T$ with the generator domains. Proposition 6.8, applied to Ψ_0^G and Ψ_1^G, gives the scalar and connection spectra and local isometry to $(W_G \times \mathbb{R}, g_{W_G} + L^2 ds^2)$. If $(\widetilde{W}_G, \widetilde{g}_{W_G})$ is the Riemannian universal cover of (W_G, g_{W_G}), both universal covers are $(\widetilde{W}_G \times \mathbb{R}, \widetilde{g}_{W_G} + L^2 ds^2)$. For $\tau = 1/q$, the same proposition gives the common finite cover

$$(W_G \times \mathbb{R}/(7q\mathbb{Z}), g_{W_G} + L^2 ds^2)$$

of degree $7q$. Its deck generator is induced by $(x, s) \mapsto (\Psi_\sigma^G(1/q)x, s + 1)$; this action is free of order $7q$ because of its circle component. The product metric is real analytic, locally symmetric, and has nonnegative sectional curvature. The flows Ψ_σ^G are analytic, so the local isometries in Proposition 6.8 are analytic. The metrics $g_{\sigma,\tau}^G$ therefore have these three properties. The compact product cover is a Lie group with a bi-invariant metric, hence is globally symmetric and clean [102, arXiv v1, §1.3 and Theorem 1.8, p. 7]. Lemma A.1 applies to its free finite isometric quotients. □

For $\sigma \in \{0, 1\}$, write $g_\sigma = g_{\sigma,1}^G$ and, for $T \geq 0$, set

$$N_\sigma(T) = \#\big(\mathrm{Lsp}(P_G, g_\sigma) \cap (0, T]\big).$$

Thus N_σ counts distinct positive lengths.

Theorem 9.2 (Exact period differences). *The length sets satisfy* $\mathrm{Lsp}(P_G, g_0) \subsetneq \mathrm{Lsp}(P_G, g_1)$*, with*

$$\mathrm{Lsp}(P_G, g_1) \setminus \mathrm{Lsp}(P_G, g_0) = \left\{ h\sqrt{7m + \sqrt{2}\,k^2} \,:\, m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\} \right\}. \tag{9.2}$$

Every length in this difference is a clean period of (P_G, g_1) *at which the common scalar half-wave trace is real analytic. As* $T \to \infty$*,*

$$N_0(T) = \frac{253}{12\,2^{1/4}\pi^3} T^3 + \frac{1}{8\pi^2} T^2 + O(T), \tag{9.3}$$

$$N_1(T) = \frac{295}{12\,2^{1/4}\pi^3} T^3 + \frac{1}{8\pi^2} T^2 + O(T). \tag{9.4}$$

Consequently,

$$\#\big((\mathrm{Lsp}(P_G, g_1) \setminus \mathrm{Lsp}(P_G, g_0)) \cap (0, T]\big) = \frac{7}{2\,2^{1/4}\pi^3} T^3 + O(T), \tag{9.5}$$

and

$$\lim_{T\to\infty} \frac{\#((\mathrm{Lsp}(P_G, g_1) \setminus \mathrm{Lsp}(P_G, g_0)) \cap (0, T])}{N_1(T)} = \frac{42}{295}.$$

9.1.1. *The norm sets for a fixed deck element.* Recall that $\mathcal{S}_3 = \{x^2 + y^2 + z^2 \,:\, x, y, z \in \mathbb{Z}\}$, and let ν be the residue type defined in Section 7.2. For $i \in \{A, B\}$ and $j \in \mathbb{Z}/7\mathbb{Z}$, define

$$Q_{i,j} = \left\{ h^{-2}\,\mathrm{length}(\gamma)^2 \,:\, \begin{array}{l} \gamma : [0,1] \to G_i \text{ is an affinely parametrized geodesic,} \\ \gamma(1) = \mathcal{F}_i(j)\gamma(0), \\ \dot\gamma(1) = d\mathcal{F}_i(j)_{\gamma(0)}\dot\gamma(0) \end{array} \right\}.$$

Seven-periodicity makes this definition independent of the integer representative of j. Constant geodesics are included when they satisfy these equations.

The norm-preserving action of signed coordinate permutations in Section 7.2 leaves each squared-norm set unchanged. The fixed-j calculation in the proof of Proposition 6.4 gives $Q_{i,j} \subset \mathbb{Z}_{\geq 0}$ and

$$Q_{A,0} = Q_{B,0} = 49\mathcal{S}_3.$$

For $j = 1$, Table (7.6) gives the six A types, whose squared norms have the six nonzero residues modulo seven. The B types are exactly these six together with 123; the multiplicity two of the latter does not affect a norm set. For general $j \neq 0$, multiply each residue vector by j before taking its type.

Signed coordinate permutations commute with scalar multiplication. Moreover, multiplication by any $j \neq 0$ permutes the three nonzero least absolute residues $\{1, 2, 3\}$. Therefore, with

$$\mathcal{H} = \{|v|^2 \,:\, v \in \mathbb{Z}^3,\ \nu(v) = 123\},$$

one has

$$Q_{B,j} = Q_{A,j} \,\dot\cup\, \mathcal{H} \qquad (j \neq 0). \tag{9.6}$$

The union is disjoint: every parameter in $Q_{A,j}$ is nonzero modulo seven, while every parameter in $\mathcal{H}$ is divisible by seven. The six A types, after multiplication by j, have all six nonzero squared norm residues exactly once.

For subsets $E, F \subset \mathbb{R}$, write $E + F = \{e + f \,:\, e \in E,\ f \in F\}$. For $j \in \mathbb{Z}/7\mathbb{Z}$, put

$$F_{A,j} = Q_{A,j} + 49\mathcal{S}_3, \qquad F_{B,j} = Q_{B,j} + 49\mathcal{S}_3.$$

The squared lengths of the $\Psi_0^G(j)$-twisted geodesics on W_G form $h^2 F_{A,j}$; those of the $\Psi_1^G(j)$-twisted geodesics form $h^2 F_{B,j}$. Indeed, the squared lengths in the factor with trivial monodromy form $h^2\, 49\mathcal{S}_3$ by (6.8). Lagrange's four-square theorem [111, Theorem 11.4.3, p. 171] implies $\mathcal{S}_3 + \mathcal{S}_3 = \mathbb{Z}_{\geq 0}$. Hence

$$F_{A,0} = F_{B,0} = 49\mathbb{Z}_{\geq 0}. \tag{9.7}$$

9.1.2. *Representations by vectors outside* $7\mathbb{Z}^3$.

Lemma 9.3. *For every $n \in \mathcal{S}_3 \setminus \{0\}$, there are $x, y, z \in \mathbb{Z}$, not all divisible by seven, such that $n = x^2 + y^2 + z^2$. Consequently*

$$\mathcal{H} = \{n \in \mathcal{S}_3 \,:\, n > 0,\ 7 \mid n\}. \tag{9.8}$$

Proof. Let $n \in \mathcal{S}_3 \setminus \{0\}$. Write $n = 4^a u$, with $a \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{Z}_{>0}$ and $4 \nmid u$. The three-square criterion [111, Theorem 14.3.8, p. 226] gives $u \not\equiv 0, 4, 7 \pmod 8$. Gauss's primitive representation theorem [111, Theorem 30.1.3, p. 542] then gives $u = x^2 + y^2 + z^2$ with $\gcd(x, y, z) = 1$. The vector $2^a(x, y, z)$ has squared norm n and coordinate gcd 2^a, so it lies outside $7\mathbb{Z}^3$.

By (7.8), a vector of squared norm divisible by seven has residue type 000 or 123. Thus a vector outside $7\mathbb{Z}^3$ whose squared norm is a multiple of seven has least absolute residue type 123. This proves (9.8). □

Lemma 9.4. *One has the exact sum-set identity*

$$\mathcal{H} + 49\mathcal{S}_3 = 7\mathbb{Z}_{>0} \setminus \{7, 28\}. \tag{9.9}$$

Proof. Every member of the left-hand side is a positive multiple of seven. Neither 7 nor 28 belongs to it: both are smaller than 49 and neither is a sum of three squares.

Let $n \in 7\mathbb{Z}_{>0} \setminus \{7, 28\}$. If $n \in \mathcal{S}_3$, then $n = n + 49 \cdot 0 \in \mathcal{H} + 49\mathcal{S}_3$ by (9.8). Otherwise the three-square criterion gives

$$n = 4^a(8b + 7) = 7 \cdot 4^a(8d + 1), \qquad a, b, d \in \mathbb{Z}_{\geq 0},$$

since $7 \mid n$ implies $7 \mid b$ and hence $b = 7d$. If $a = 0$, the exclusion $n \neq 7$ gives $n \geq 63$, and $n - 49 \equiv 6 \pmod 8$. If $a = 1$, the exclusion $n \neq 28$ gives $n \geq 252$, and $n - 49 \equiv 3 \pmod 8$. In both cases $n - 49$ is a positive three-square integer divisible by seven, hence belongs to $\mathcal{H}$. If $a \geq 2$, then $n \geq 112$ and $n - 98 \equiv 6 \pmod 8$. Thus $n - 98$ is again a positive member of $\mathcal{H}$. Since $1, 2 \in \mathcal{S}_3$, these three cases prove the reverse inclusion. □

Since $F_{A,j} \cap 7\mathbb{Z} = \emptyset$ for $j \neq 0$, (9.6) and (9.9) give

$$F_{B,j} = F_{A,j} \,\dot\cup\, (7\mathbb{Z}_{>0} \setminus \{7, 28\}) \qquad (j \neq 0). \tag{9.10}$$

9.1.3. *Eventual completeness of the nonzero residue classes.*

Lemma 9.5. *For every $j \neq 0$ and every integer $n \geq 7203$,*

$$n \in F_{A,j} \quad \Longleftrightarrow \quad 7 \nmid n.$$

In particular $F_{B,j}$ contains every integer at least 7203. Uniformly in $j \neq 0$ and $X \geq 0$,

$$\#(F_{A,j} \cap [0, X]) = \frac{6}{7}X + O(1), \qquad \#(F_{B,j} \cap [0, X]) = X + O(1). \tag{9.11}$$

Proof. Fix $j \in (\mathbb{Z}/7\mathbb{Z}) \setminus \{0\}$ and $n \in \mathbb{Z}$ with $n \geq 7203$ and $7 \nmid n$. Choose $r \in \mathbb{Z}^3$ whose residue is allowed for $Q_{A,j}$ and satisfies $|r|^2 \equiv n \pmod 7$. Such a residue exists because the six allowed types have the six nonzero squared-norm residues. Choose $i \in \{1, 2, 3\}$ with $7 \nmid r_i$, and let e_i be the ith standard coordinate vector. For $t \in \mathbb{Z}$,

$$|r + 7te_i|^2 \equiv |r|^2 + 14r_i t \pmod{49}.$$

Since $2r_i$ is invertible modulo seven, there is a $t \in \mathbb{Z}$ such that $r' = r + 7te_i$ satisfies $|r'|^2 \equiv n \pmod{49}$. Let $b \in \{0, 1, 2, 3\}$ be congruent to $n - 1$ modulo four, and choose $\varepsilon \in \{0, 1\}^3$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = b$. By the Chinese remainder theorem, there is an $x \in \mathbb{Z}^3$ such that

$$x \equiv r' \pmod{49}, \qquad x \equiv \varepsilon \pmod 2, \qquad |x_i| \leq 49 \quad (i = 1, 2, 3).$$

The last inequalities follow by choosing each coordinate modulo 98 in $[-49, 48]$. Thus $|x|^2 \leq 3 \cdot 49^2 = 7203$, the residue of x is allowed for $Q_{A,j}$, and

$$u := \frac{n - |x|^2}{49} \in \mathbb{Z}_{\geq 0}, \qquad u \equiv n - b \equiv 1 \pmod 4.$$

Since $u \equiv 1 \pmod 4$, it is not of the form $4^\alpha(8\delta + 7)$ with $\alpha, \delta \in \mathbb{Z}_{\geq 0}$. The three-square criterion gives $u \in \mathcal{S}_3$. Hence $n = |x|^2 + 49u \in F_{A,j}$. Conversely, $F_{A,j} \cap 7\mathbb{Z} = \emptyset$. Equation (9.10) gives $[7203, \infty) \cap \mathbb{Z} \subset F_{B,j}$. Both sets differ from the indicated unions of residue classes by a subset of $[0, 7203) \cap \mathbb{Z}$. Counting complete blocks of seven and the remaining block therefore gives (9.11), with an absolute error bound independent of the six nonzero values of j. □

9.1.4. *Suspension lengths and their counting functions.*

Proof of Theorem 9.2. For $j \in \mathbb{Z}/7\mathbb{Z}$, put $F_{0,j} = F_{A,j}$ and $F_{1,j} = F_{B,j}$. By Proposition 6.8, an affinely parametrized closed geodesic $\gamma : \mathbb{R}/\mathbb{Z} \to (P_G, g_\sigma)$ of winding $k \in \mathbb{Z}$ lifts to

$$t \longmapsto (\gamma_A(t), \gamma_B(t), s_0 + kt), \qquad 0 \leq t \leq 1,$$

where $s_0 \in \mathbb{R}$ and the two group components are geodesics whose endpoint positions and tangents match through $\Psi_\sigma^G(k)$. Conversely, these conditions give a smooth closed geodesic after projection. The product metric gives squared length $h^2 n + L^2 k^2$ with $n \in F_{\sigma, k \bmod 7}$. In the residue description, $v \mapsto -v$ changes j to $-j$ and preserves $|v|^2$, so $F_{\sigma,-j} = F_{\sigma,j}$. Put $c = \sqrt{2} = L^2/h^2$. The squared lengths are therefore precisely

$$h^2(n + ck^2), \qquad k \in \mathbb{Z}_{\geq 0}, \quad n \in F_{\sigma, k \bmod 7}, \quad (n, k) \neq (0, 0). \tag{9.12}$$

The parametrization by (n, k) is injective: rational independence of 1 and $\sqrt{2}$ gives $n = n'$ and $k^2 = (k')^2$, and nonnegative k, k' are then equal. Equations (9.7) and (9.10) prove both length-set inclusion and (9.2). The difference is nonempty, for instance for $m = 2$ and $k = 1$.

For any fixed $a \in \mathbb{Z}_{>0}$ and $c > 0$, define, for $R \geq 0$,

$$\begin{aligned} S_{a,c}(R) &:= \sum_{\substack{l \in \mathbb{Z}_{>0} \\ l \leq \sqrt{R}/(a\sqrt{c})}} (R - ca^2 l^2) \\ &= \frac{2}{3a\sqrt{c}} R^{3/2} - \frac{R}{2} + O_{a,c}(\sqrt{R}) \qquad (R \to \infty). \end{aligned} \tag{9.13}$$

Indeed, with $x = \sqrt{R}/(a\sqrt{c})$, $m = \lfloor x \rfloor$ and $\theta = x - m \in [0,1)$, the finite-sum formula gives

$$\begin{aligned} S_{a,c}(R) &= Rm - \frac{ca^2 m(m+1)(2m+1)}{6} \\ &= \frac{2}{3a\sqrt{c}} R^{3/2} - \frac{R}{2} + a\sqrt{c}\left(-\theta^2 + \theta - \frac{1}{6}\right)\sqrt{R} + O_{a,c}(1). \end{aligned}$$

Now take $c = \sqrt{2}$, $R = (T/h)^2$ and $K = \sqrt{R/c}$. It follows that

$$\sum_{\substack{1 \le k \le K \\ 7 \nmid k}} (R - ck^2) = S_{1,c}(R) - S_{7,c}(R) = \frac{4}{7\sqrt{c}} R^{3/2} + O(\sqrt{R}),$$

$$\sum_{\substack{1 \le k \le K \\ 7 \mid k}} (R - ck^2) = S_{7,c}(R) = \frac{2}{21\sqrt{c}} R^{3/2} - \frac{R}{2} + O(\sqrt{R}).$$

Write $d_0 = 6/7$ and $d_1 = 1$. For $1 \le k \le K$ with $7 \nmid k$, equation (9.11) gives $d_\sigma(R - ck^2) + O(1)$ lengths. For $7 \mid k$, equation (9.7) gives exactly $\lfloor (R - ck^2)/49 \rfloor + 1$ lengths. The 1 includes the zero fibre parameter, since the winding is positive. For $k = 0$ there are $\lfloor R/49 \rfloor$ lengths, since zero total length is excluded. Summing over at most $\lfloor K \rfloor$ positive windings gives

$$N_\sigma(T) = d_\sigma\big(S_{1,c}(R) - S_{7,c}(R)\big) + \frac{1}{49} S_{7,c}(R) + \frac{R}{49} + O(\sqrt{R}).$$

Consequently,

$$N_\sigma(T) = \frac{2}{3\sqrt{c}} \left(\frac{6}{7} d_\sigma + \frac{1}{343}\right) R^{3/2} + \frac{R}{98} + O(\sqrt{R}).$$

The parentheses equal $253/343$ for $\sigma = 0$ and $295/343$ for $\sigma = 1$. Substitution of $h = 2\pi/7$ proves (9.3)–(9.5).

For a condition P, let $\mathbf{1}_{\{P\}}$ be 1 when P holds and 0 otherwise. The difference has the exact count

$$\sum_{\substack{1 \le k \le \sqrt{R/c} \\ 7 \nmid k}} \left(\left\lfloor \frac{R - ck^2}{7} \right\rfloor - \mathbf{1}_{\{R - ck^2 \ge 7\}} - \mathbf{1}_{\{R - ck^2 \ge 28\}} \right).$$

The two removed fibre parameters are precisely those excluded in (9.2).

Proposition 9.1 gives closed real-analytic scalar-isospectral manifolds with clean geodesic flows. Corollary A.5, applied to their trivial flat line bundles and the difference (9.2), gives real analyticity of the common half-wave trace near each length in that difference. □

9.1.5. *Base covers and nearby metrics.* For $\sigma \in \{0,1\}$ and $p \in \mathbb{Z}_{>0}$, let $g_{\sigma,p}^{\mathrm{cov}}$ be the pullback of $g_\sigma = g_{\sigma,1}^G$ to $P_{G,p} = W_G \times (\mathbb{R}/p\mathbb{Z})$.

Corollary 9.6. *If $7 \nmid p$, the two lifted length sets are nested and their difference is*

$$\left\{ h\sqrt{7m + \sqrt{2}\, p^2 k^2} \;:\; m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1,4\} \right\}.$$

As $T \to \infty$, the number of its members in $(0,T]$ is

$$\frac{7}{2p\, 2^{1/4} \pi^3} T^3 + O_p(T),$$

and its proportion among all positive distinct lengths of $g_{1,p}^{\mathrm{cov}}$ tends to $42/295$. If $7 \mid p$, the two lifted metrics are isometric to $(W_G \times \mathbb{R}/p\mathbb{Z},\, g_{W_G} + L^2 ds^2)$.

Proof. A geodesic of base winding $k \in \mathbb{Z}$ in this cover has real displacement pk and monodromy $\Psi_\sigma^G(pk)$. If $7 \nmid p$, multiplication by p permutes the nonzero residues modulo seven. The difference for a fixed deck transformation (9.10) is independent of the nonzero residue, and the fibre densities in (9.11) are uniform in it. The proof of Theorem 9.2 therefore applies with $c = \sqrt{2}\,p^2$. The leading coefficients of both length counts, and of their difference, are divided by p, proving the stated count and ratio. If $7 \mid p$, then $\Psi_\sigma^G(p) = \mathrm{id}_{W_G}$. The untwisting map (6.24) descends modulo $p\mathbb{Z}$ and identifies each lifted metric with the stated product metric. □

With basepoints at the group identities and the zero circle coordinate, the product identifications give

$$\pi_1(W_G) = \mathbb{Z} \times C_2, \qquad C_2 = \mathbb{Z}/2\mathbb{Z},$$

$$\pi_1(P_G) = (\mathbb{Z} \times C_2) \times \mathbb{Z}, \qquad \pi_1(P_{G,p}) = (\mathbb{Z} \times C_2) \times p\mathbb{Z},$$

where $\pi_1(P_{G,p})$ is identified with its image under the covering homomorphism. The representations in Proposition 9.1 are precisely the finite-dimensional unitary representations trivial on $\pi_1(W_G)$.

Lemma 9.7. *Let G be a compact Lie group with a bi-invariant metric and Lie algebra $\mathfrak{g}$. Let $\ell, r, g \in G$, $N \in \mathbb{Z}_{>0}$ and $V \in \mathfrak{g}$. Suppose $\ell^N = r^N = 1$ and the geodesic $g\exp(tV)$, $0 \le t \le 1$, satisfies the position and tangent conditions*

$$\exp V = g^{-1}\ell g r^{-1}, \qquad \mathrm{Ad}_r\, V = V.$$

Then $\exp(NV) = 1$. In particular, let $i \in \{A, B\}$, $j \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$. Every $\mathcal{F}_i(j/q)$-twisted geodesic on G_i, with the metric (6.1)*, has squared length in $(h/q)^2\mathbb{Z}_{\ge 0}$.*

Proof. The tangent condition implies that r commutes with $\exp V$. Hence

$$\exp(NV) = (\exp V\, r)^N r^{-N} = g^{-1}\ell^N g = 1.$$

For G_i, where $i \in \{A, B\}$, and the twist $\mathcal{F}_i(j/q)$, the translating elements are $\ell = \iota_i(hj/q, 0)$ and $r = \iota_i(3hj/q, hj/q)$; both have $(7q)$th power one. Thus $\exp(7qV) = 1$. Conjugacy into a maximal torus, as in the identity-deck case of Proposition 6.4, places $7qV$ in the period lattice $2\pi\mathbb{Z}^3$ of (6.8). Its squared norm is $4\pi^2 N'$ for some $N' \in \mathbb{Z}_{\ge 0}$, so $|V|^2 = (h/q)^2 N'$. □

Corollary 9.8. *For each $q \in \mathbb{Z}_{>0}$, the metrics $g^G_{0,1/q}$ and $g^G_{1,1/q}$ on the fixed manifold P_G have a length-set difference containing*

$$\begin{aligned} \mathcal{D}_q &= \left\{ h\sqrt{7m + \sqrt{2}\,q^2k^2} \;:\; m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\} \right\} \\ &\subseteq \mathrm{Lsp}(P_G, g^G_{1,1/q}) \setminus \mathrm{Lsp}(P_G, g^G_{0,1/q}). \end{aligned} \tag{9.14}$$

As $T \to \infty$,

$$\#(\mathcal{D}_q \cap (0, T]) = \frac{7}{2q\,2^{1/4}\pi^3}\,T^3 + O_q(T). \tag{9.15}$$

Each member is a clean period of the second metric at which the common half-wave trace is real analytic. Both metric sequences converge in C^∞ to $g_{W_G} \oplus L^2 ds^2$ as $q \to \infty$.

Proof. For $k \in \mathbb{Z}$, winding qk gives monodromy $\Psi_\sigma^G(k)$ and squared circle-component length $L^2q^2k^2 = h^2\sqrt{2}\,q^2k^2$. Equations (9.7) and (9.10), together with the irrationality of $\sqrt{2}$, give $\mathcal{D}_q$ as the difference between the two sets of lengths whose winding is divisible by q.

For an arbitrary winding $l \in \mathbb{Z}$, Lemma 9.7 and (6.8) put the full squared length in the form

$$h^2\left(q^{-2}N + \sqrt{2}\,l^2\right), \qquad N \in \mathbb{Z}_{\ge 0}.$$

Suppose this equals the square of a member of $\mathcal{D}_q$, so that

$$q^{-2}N + \sqrt{2}\,l^2 = 7m + \sqrt{2}\,q^2k^2, \qquad m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\}.$$

Irrationality gives $l^2 = q^2k^2$ and $q^{-2}N = 7m$. Thus $l = \pm qk$, and (9.10) excludes this length from $\mathrm{Lsp}(P_G, g^G_{0,1/q})$. Equations (9.13) for $a = 1, 7$ and $c = \sqrt{2}\,q^2$ give (9.15) for every integer $q \geq 1$. Proposition 9.1 gives cleanliness and scalar isospectrality of the two closed real-analytic manifolds. Corollary A.5, applied to their trivial flat line bundles and (9.14), gives real analyticity of the common trace near every member of $\mathcal{D}_q$. The C^∞ convergence is the case $\tau = 1/q \to 0$ of Proposition 6.8. □

At $q = 1$, the set $\mathcal{D}_1$ is the entire length-set difference by Theorem 9.2.

The metrics below are real analytic, with the parameter ranges of the cited theorems. Theorem 8.8 also gives eighteen-dimensional pairs of positive Ricci curvature arbitrarily C^∞-close to a product metric.

Theorem	Dimension and topology	Equal spectra	Length-set difference
7.7	Dimension 8; $\pi_1 = C_7$	Every marked unitary flat spectrum	$3h$; equal class-3 minima by Corollary 7.9
7.1	Dimension 9; locally symmetric	Scalar spectrum	Exact difference (7.2)
8.2	Dimension 17; one manifold, $\pi_1 = \mathbb{Z}$	Every marked unitary flat spectrum on every finite cover	Embedded primitive length (8.7), at $\tau = 1/q$ with $\gcd(q, 6) = 1$
8.8	Dimension 18; one simply connected manifold	Scalar spectrum	Infinitely many embedded primitive lengths (8.21)
9.2	Dimension 19; one locally symmetric manifold	Flat spectra pulled back from the base circle	At $\tau = 1$: difference (9.2); counts (9.3)–(9.4)

For the nearby nineteen-dimensional pairs, the length-set difference contains $\mathcal{D}_q$ of Corollary 9.8.

Part III. Central quotients and topology

For the products J_I and a closed $K \leq \prod_j P_{n_j}$, the groups $\pi_1(J_I/K)$ are central extensions of the finite group $\pi_0(K)$ by finitely generated abelian groups (Lemma 10.2). Conversely, every nonempty finite list of such extensions over one finite group has a common-isotropy realization whose boundary homomorphisms are the prescribed quotient maps. The realizations can be chosen to have equal canonical rough spectra for every common finite-dimensional unitary isotropy representation and pairwise different higher rational homotopy groups (Theorem 11.1).

10. Prescribed central quotients

10.1. Central kernels and covering groups.

Lemma 10.1. *For $m \geq 1$, the determinant identifies $\pi_1(U(m))$ with $\mathbb{Z}$, whereas $SU(m)$ and $Sp(m)$ are simply connected. These groups have zero π_2. For $m \geq 3$, $\pi_1(SO(m)) = C_2$ and $\pi_2(SO(m)) = 0$; a full rotation in one coordinate plane generates $\pi_1(SO(m))$. For every compact connected Lie group L and maximal torus $T \leq L$, the homomorphism $\pi_1(T) \to \pi_1(L)$ is onto.*

Proof. For a compact connected Lie group, $\pi_2 = 0$, and maximal-torus inclusion identifies π_1 with the integral lattice modulo the coroot lattice [20, Chapter V, Theorem 7.1, p. 223]. The classical lattices in [20, Chapter V, §6] give the stated groups: the two lattices coincide for $SU(m)$ and $Sp(m)$; their quotient for $U(m)$ is $\mathbb{Z}$, with quotient map the coordinate sum; for $SO(m)$ the coroot lattice consists of the integer vectors with even coordinate sum. The coordinate sum for $U(m)$ is induced by the determinant, and a full coordinate-plane rotation in $SO(m)$ represents the nonzero class. The low-dimensional cases follow from $U(1) = S^1$, $SU(1) = \{1\}$, $Sp(1) = SU(2) = S^3$ and $SO(3) = \mathbb{RP}^3$; see [20, Chapter I, (6.18)–(6.19)]. □

Lemma 10.2. *Let J be a compact connected Lie group and $K \le J$ closed. Let $\widetilde{J} \to J$ be the universal covering homomorphism and $\widetilde{K}$ the inverse image of K. Then*

$$\pi_1(J/K) \cong \pi_0(\widetilde{K}), \qquad 1 \longrightarrow C \longrightarrow \pi_1(J/K) \xrightarrow{\partial} \pi_0(K) \longrightarrow 1,$$

where

$$C = \operatorname{coker}\bigl(\pi_1(K^0) \longrightarrow \pi_1(J)\bigr)$$

is finitely generated and central. Fundamental groups are based at the identity or its coset. The product of two based loops traverses the first loop and then the second. The boundary homomorphism sends a loop to the component of the inverse endpoint of its lift starting at the identity.

Proof. The homotopy sequence of the homogeneous fibration $\widetilde{K} \to \widetilde{J} \to J/K$ [41, Theorem 3.1(i) and Remark 3.4] gives the first bijection. If two loops have lifts a, b starting at the identity and ending at $u, v \in \widetilde{K}$, their concatenation lifts as a followed by $b\,u$, with endpoint vu. Since $(vu)^{-1} = u^{-1}v^{-1}$, inverse endpoints give a group isomorphism. The homotopy sequence of $K \to J \to J/K$ gives the exact sequence and the formula for C. The kernel of $\widetilde{J} \to J$ is central [41, Proposition 2.11(ii)]; its image in $\pi_0(\widetilde{K})$ is C. Finally, $\pi_1(J)$ is finitely generated abelian, and K has finitely many components. Thus C is finitely generated and has finite index. □

Lemma 10.3. *Let $1 \to A \to E \to \Gamma \to 1$ be central, with A finitely generated and Γ finite. For every real vector space V, regarded as an additive group, restriction $\operatorname{Hom}(E, V) \to \operatorname{Hom}(A, V)$ is an isomorphism. In particular, $\dim_{\mathbb{R}} \operatorname{Hom}(E, \mathbb{R}) = \operatorname{rank} A$.*

Proof. Take trivial coefficients in V. For the regular covering $K(A,1) \to K(E,1)$, restriction is pullback on H^1 under $H^1(K(E,1); V) = \operatorname{Hom}(E, V)$ and $H^1(K(A,1); V) = \operatorname{Hom}(A, V)$. Centrality of A makes the deck action on $H^1(K(A,1); V)$ trivial. The transfer identities [53, §3.G, pp. 321–322] therefore make restriction invertible, with inverse $|\Gamma|^{-1}$ times transfer. The last assertion follows from $\operatorname{Hom}(A, \mathbb{R}) \simeq \mathbb{R}^{\operatorname{rank} A}$. □

10.2. **The central extension.** Fix the groups, embeddings, and metrics of Section 2, with all factor scales equal to one.

Theorem 10.4 (Prescribed central quotients). *Let $1 \to A \to E \xrightarrow{p} \Gamma \to 1$ be a central extension, where Γ is finite and A is finitely generated. Fix a generating tuple $c_1, \dots, c_r$ of A, with $r \ge 1$; identity entries and repetitions are allowed. For $I \subseteq [r]$, put $A_I = \langle c_j : j \in I \rangle$. There are positive odd integers $n_1 < \cdots < n_r$ and a closed subgroup $K \le \prod_{j=1}^{r} P_{n_j}$, with K^0 central in K, having the following properties. Put*

$$J_I = \prod_{j=1}^{r} J_{I,j}, \qquad J_{I,j} = \begin{cases} H_{n_j}, & j \in I, \\ G_{n_j}, & j \notin I, \end{cases} \qquad M_I = J_I/K.$$

(1) There are isomorphisms

$$\pi_1(M_I) \simeq E/A_I, \qquad \pi_0(K) \simeq \Gamma,$$

under which the boundary homomorphism is the quotient $E/A_I \to \Gamma$. For $I \subseteq I'$, the quotient homomorphism $E/A_I \to E/A_{I'}$ is induced by a homomorphism of lifted stabilizers. Distinct subsets give different higher rational homotopy groups, even when the fundamental groups are isomorphic.

(2) The quotients are closed connected real analytic normal homogeneous manifolds with nonnegative sectional curvature and

$$\dim \ker \operatorname{Ric}_{M_I} = \operatorname{rank}(A/A_I).$$

(3) Their scalar spectra agree, as do their canonical rough connection spectra for every common finite-dimensional unitary K-representation. Their metric tangent isotropy modules are isomorphic, giving the same spectra for all canonical natural tensor connections. They admit compatible

homogeneous spin structures for which the cubic spin squares are isospectral for every common finite-dimensional unitary K-twist. The graded cubic exterior operators have equal signed spectra for every such twist, with the conventions of Section 4.

(4) *For every $L \le \Gamma$, the corresponding connected covers have degree $[\Gamma : L]$, fundamental groups $p^{-1}(L)/A_I$, and equal flat spectra for every finite-dimensional unitary L-representation pulled back along $p^{-1}(L)/A_I \to L$.*

Proof. The compact group.

Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{Z}^r$, and set

$$\phi : \mathbb{Z}^r \twoheadrightarrow A, \quad \phi(e_j) = c_j, \qquad \Lambda = \ker \phi, \quad W = \operatorname{span}_{\mathbb{R}} \Lambda, \quad V = \mathbb{R}^r/W, \quad T = \mathbb{R}^r/\Lambda.$$

Put $a = \operatorname{rank} A$ and $h = r - a$. Then Λ is a lattice of rank h in W, and W/Λ is a compact torus. Define

$$\iota : A \longrightarrow T, \qquad \iota(\phi(n)) = [n]_\Lambda \quad (n \in \mathbb{Z}^r).$$

This is an injective homomorphism. Its image is closed, since its inverse image in $\mathbb{R}^r$ is $\mathbb{Z}^r$, and discrete, since the image of an open ball centered at zero with radius less than $1/2$ meets it only at zero.

Give E the discrete topology and form the Lie group

$$\overline{K} = (E \times T)/\{(c, -\iota(c)) : c \in A\}.$$

The denominator is closed, discrete and central, so the quotient map is a covering homomorphism. The maps $e \mapsto [e, 0]$ and $t \mapsto [1, t]$ embed E and T, respectively. The embedded E is closed and discrete, since its inverse image is $E \times \iota(A)$. The two subgroups intersect in A, with $[c, 0] = [1, \iota(c)]$. For $x \in \mathbb{R}^r$, write $[1, x] = [1, [x]_\Lambda]$. The homomorphism

$$\bar{p} : \overline{K} \longrightarrow \Gamma, \qquad \bar{p}([e, t]) = p(e)$$

has kernel T. Hence $\overline{K}^0 = T$, this identity component is central, and $\pi_0(\overline{K}) = \Gamma$.

Let $q_V : \mathbb{R}^r \to V$ be the quotient map, let $\bar{q}_V : T \to V$ be its descent, and put $\ell = \bar{q}_V \circ \iota : A \to V$. By Lemma 10.3, ℓ extends uniquely to a homomorphism $B_E : E \to V$. The homomorphism

$$\beta : \overline{K} \longrightarrow V, \qquad \beta([e, t]) = B_E(e) + \bar{q}_V(t)$$

is well defined, since $B_E(c) - \bar{q}_V(\iota(c)) = 0$ for $c \in A$. Choose a linear section $s : V \to \mathbb{R}^r$ of q_V, and put $K_c = \ker \beta$. Multiplication gives an isomorphism of Lie groups

$$V \times K_c \longrightarrow \overline{K}, \qquad (v, k) \longmapsto [1, s(v)]k,$$

whose inverse sends b to $(\beta(b), [1, -s(\beta(b))]b)$. The restriction $p_c = \bar{p}|_{K_c} : K_c \to \Gamma$ is onto: $[1, -s(\beta(b))]b \in K_c$ has the same $\bar{p}$-image as b. Moreover,

$$\ker p_c = K_c \cap T = W/\Lambda = K_c^0.$$

Thus $\pi_0(K_c) = \Gamma$, and K_c is compact, being a finite union of translates of W/Λ.

The embedding.

The homomorphism

$$z : \overline{K} \longrightarrow \mathbb{R}^r/\mathbb{Z}^r \simeq \mathbb{T}^r, \qquad z([e, [x]_\Lambda]) = [x]_{\mathbb{Z}^r}$$

is well defined and onto, with kernel the embedded E. Its coordinate characters restrict to continuous homomorphisms $z_j : K_c \to U(1)$, given by $z_j([e, [x]_\Lambda]) = e^{2\pi i x_j}$.

Choose a basis $\chi_1, \dots, \chi_h$ of the character lattice of K_c^0. Let ρ be the direct sum of the regular representation of Γ, pulled back by p_c, and $\operatorname{Ind}_{K_c^0}^{K_c} \chi_\alpha$ for $1 \le \alpha \le h$. The finite-index induction of the proof of Theorem 8.1 has dimension $|\Gamma|$ for each χ_α; it is continuous here because K_c^0 is open and χ_α is continuous. Centrality gives

$$\operatorname{Ind}_{K_c^0}^{K_c} \chi_\alpha \Big|_{K_c^0} \simeq \chi_\alpha^{\oplus |\Gamma|}.$$

The kernel of the regular summand is K_c^0, and the χ_α separate its points. Therefore

$$\rho : K_c \longrightarrow U(d), \qquad d = |\Gamma|(h+1),$$

is faithful. For $h = 0$, the regular summand alone is faithful.

Put $K = K_c \times \mathbb{T}^r$, and let $w_j : K \to U(1)$ be the j-th auxiliary coordinate character. The component homomorphism is $p_K : K \to \Gamma$, $p_K(k, w) = p_c(k)$, and $K^0 = (W/\Lambda) \times \mathbb{T}^r$ is central in K. Set $n_j = 2d + 2j - 1$, extend ρ and z_j trivially over $\mathbb{T}^r$, and define

$$A_j = \rho \oplus \rho \oplus w_j \oplus 1^{\oplus(2j-2)}, \qquad k \longmapsto ((A_j(k), z_j(k)))_{j=1}^r \in \prod_{j=1}^r P_{n_j}.$$

Here $1^{\oplus(2j-2)}$ is the trivial representation of dimension $2j-2$. An element of the kernel has trivial K_c-coordinate by faithfulness of ρ, and every w_j-coordinate is one. This homomorphism therefore embeds K as a closed subgroup.

Fundamental groups.

The full determinant pullback is

$$\widetilde{K}_c = \{(k, x) \in K_c \times \mathbb{R}^r : z(k) = [x]_{\mathbb{Z}^r}\}.$$

The formula

$$\Psi : \widetilde{K}_c \longrightarrow E, \qquad \Psi(k, x) = k[1, -x]$$

defines a continuous homomorphism, since T is central and $z(k[1,-x]) = 0$. Given $e \in E$, choose $x \in \mathbb{R}^r$ with $q_V(x) = -B_E(e)$ and put $k = e[1, x]$. Then $k \in K_c$ and $\Psi(k, x) = e$, so Ψ is onto. Its kernel is

$$\{([1, x], x) : x \in W\} \cong W.$$

Since this kernel is connected and E is discrete, $\widetilde{K}_c^0 = \ker \Psi$ and $\pi_0(\widetilde{K}_c) = E$.

For $I \subseteq [r]$, put $I^c = [r] \setminus I$, and let $\mathbb{R}^I$ and $\mathbb{Z}^I$ denote the coordinate subgroups supported in I. Write x_{I^c} and z_{I^c} for coordinate projections, and set

$$\widetilde{K}_{c,I} = \{(k, x_{I^c}) \in K_c \times \mathbb{R}^{I^c} : z_{I^c}(k) = [x_{I^c}]_{\mathbb{Z}^{I^c}}\}.$$

Coordinate deletion gives a surjective covering homomorphism $F_I : \widetilde{K}_c \to \widetilde{K}_{c,I}$, with kernel $\{(1, x) : x \in \mathbb{Z}^I\}$. For $x \in \mathbb{Z}^I$,

$$\Psi(1, x) = [1, -x] = \phi(x)^{-1}.$$

Hence $\Psi(\ker F_I) = A_I$, and Ψ induces a continuous surjective homomorphism $\Psi_I : \widetilde{K}_{c,I} \to E/A_I$. Put $W_I = F_I(\widetilde{K}_c^0)$. If $\Psi(k, x) \in A_I$, choose $y \in \mathbb{Z}^I$ with $\Psi(1, y) = \Psi(k, x)$. Then $(k, x)(1, y)^{-1} \in \ker \Psi$, so $\ker \Psi_I = W_I$. This kernel is connected. On the identity component, parametrized by $x \mapsto ([1, x], x)$, the kernel of F_I is $\Lambda \cap \mathbb{Z}^I$. Since F_I is open and $\widetilde{K}_c^0$ is an open subgroup, it induces an isomorphism of Lie groups

$$W_I \simeq W/(\Lambda \cap \mathbb{Z}^I),$$

and the following commutative diagram has exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W & \longrightarrow & \widetilde{K}_c & \xrightarrow{\Psi} & E & \longrightarrow & 1 \\ & & \downarrow & & \downarrow F_I & & \downarrow & & \\ 1 & \longrightarrow & W_I & \longrightarrow & \widetilde{K}_{c,I} & \xrightarrow{\Psi_I} & E/A_I & \longrightarrow & 1. \end{array}$$

Since E/A_I is discrete and W_I is connected,

$$\pi_0(\widetilde{K}_{c,I}) \simeq E/A_I. \tag{10.1}$$

For $I \subseteq I'$, coordinate deletion gives $F_{I,I'} : \widetilde{K}_{c,I} \to \widetilde{K}_{c,I'}$, and $\Psi_{I'} F_{I,I'}$ is Ψ_I followed by $E/A_I \to E/A_{I'}$. These homomorphisms compose under successive inclusions of subsets.

Define the determinant covering group

$$\widehat{J}_I = \{(g,x) \in J_I \times \mathbb{R}^{I^c} \,:\, \det(g_j) = e^{2\pi i x_j} \text{ for } j \notin I\}.$$

It is connected: each covered unitary factor is isomorphic to $SU(2n_j+1)\times\mathbb{R}$, and every remaining factor is connected. By Lemma 10.1, $\pi_1(\widehat{J}_I) \simeq (C_2)^I$. Its inverse image of K is $\widehat{K}_I \simeq \widetilde{K}_{c,I} \times \mathbb{T}^r$, because $\det(i_{n_j}(A_j, z_j)) = z_j$ for $j \notin I$. The projection induces $\widehat{J}_I/\widehat{K}_I \simeq M_I$. For $I \subseteq I'$, the homomorphism between these lifted stabilizers is $F_{I,I'} \times \mathrm{id}_{\mathbb{T}^r}$.

For $j \in I$, take the loop in $\widehat{K}_I^0$ with K_c-coordinate 1, all real coordinates zero, $w_k = 1$ for $k \neq j$, and $w_j(t) = e^{2\pi i t}$, $0 \le t \le 1$. Along this loop all z_k equal one, all A_k with $k \neq j$ are the identity, and $\det A_j(t) = e^{2\pi i t}$. Equation (2.7) therefore identifies its image in $\pi_1(\widehat{J}_I)$ with the j-th coordinate generator. The first arrow in

$$\pi_1(\widehat{K}_I^0) \longrightarrow (C_2)^I \longrightarrow \pi_1(M_I) \longrightarrow \pi_0(\widehat{K}_I) \longrightarrow 1$$

is onto. Exactness and (10.1) give $\pi_1(M_I) \simeq E/A_I$. The projection $\widehat{K}_I \to K$ induces $E/A_I \to \Gamma$, since $p(\Psi(k,x)) = p_c(k)$. With the inverse-endpoint convention of Lemma 10.2, this is the boundary homomorphism $\pi_1(M_I) \to \pi_0(K)$.

Geometry and spectra.

Theorem 4.3, applied to K, gives equal scalar and canonical rough connection spectra. For $L \le \Gamma$, put $K_L = p_K^{-1}(L)$. Then $K_L^0 = K^0$, and $J_I/K_L \to J_I/K$ is a connected cover of degree $[\Gamma : L]$. Let $\widehat{K}_{I,L}$ be the inverse image of K_L in $\widehat{J}_I$. The homomorphism

$$\widehat{K}_{I,L} \longrightarrow p^{-1}(L)/A_I, \qquad (k, x_{I^c}, w) \longmapsto \Psi_I(k, x_{I^c})$$

is onto and has connected kernel $W_I \times \mathbb{T}^r$. The auxiliary circle loops belong to $\widehat{K}_{I,L}^0$ and again map onto $(C_2)^I$. Consequently,

$$\pi_1(J_I/K_L) \simeq p^{-1}(L)/A_I.$$

A finite-dimensional unitary representation of $L = \pi_0(K_L)$, pulled back by $p_K|_{K_L}$, is trivial on K_L^0. Theorem 4.3 applied to K_L and this representation gives the stated flat spectral equalities.

The quotients are closed, connected and real analytic. Their metrics are normal homogeneous, so $\sec \ge 0$. Proposition 4.15 and Lemma 10.3 give

$$\dim\ker \mathrm{Ric}_{M_I} = b_1(M_I) = \dim_{\mathbb{R}} \mathrm{Hom}(E/A_I, \mathbb{R}) = \mathrm{rank}(A/A_I),$$

since $1 \to A/A_I \to E/A_I \to \Gamma \to 1$ is central and Γ is finite.

The dimensions are $\dim K = h + r$ and

$$\dim M_I = \sum_{j=1}^{r} (2n_j+1)^2 - (h+r).$$

Thus the dimension depends only on $|\Gamma|$, r, and $\mathrm{rank}\, A$.

The integers n_j are odd and strictly increasing, and K^0 is toral. By Theorem 12.8,

$$s_j(I) := \dim_{\mathbb{Q}}(\pi_{4n_j+1}(M_I) \otimes \mathbb{Q}) = \#\{k \in \{j,\dots,r\} \,:\, k \notin I\} \quad (1 \le j \le r).$$

Set $s_{r+1}(I) = 0$. Then $s_j(I) - s_{j+1}(I) = 1$ if $j \notin I$, and zero otherwise; hence these homotopy groups determine I. Since K^0 is central in K, Corollaries 4.6 and 4.10 and Proposition 4.11 give the common metric tangent module, the tensor connection spectra, compatible spin structures, equal cubic spin squares, and equal signed cubic exterior spectra, with the common twists stated in the theorem. □

10.3. **Fundamental groups of compact homogeneous spaces.** A group is *central-by-finite* if its center has finite index.

Proposition 10.5. *The fundamental group of a closed connected Riemannian homogeneous manifold is finitely generated and central-by-finite. Conversely, each finitely generated central-by-finite group occurs as the fundamental group of a compact normal homogeneous member of the product construction. For every central extension $1 \to A \to E \to \Gamma \to 1$ with A finitely generated and Γ finite, the members with fundamental groups E and Γ have equal scalar and canonical homogeneous connection spectra, and equal flat spectra for bundles pulled back from the same finite-dimensional unitary representation of Γ.*

Proof. The first assertion is Wilking's theorem [114, Theorem 2.2(a),(d), p. 132]. Conversely, let E be finitely generated and central-by-finite, and let $A \le Z(E)$ have finite index. Then A is finitely generated. Put $\Gamma = E/A$, and apply Theorem 10.4 to a generating tuple of A, taking an identity entry if $A = 1$. The members $M_{\varnothing}$ and $M_{[r]}$ have fundamental groups E and $E/A = \Gamma$, respectively, and the stated spectra. □

Wilking embeds a finite extension of a lattice as a discrete cocompact subgroup of a Lie group by pushing out the lattice into its real span [114, §2.2, pp. 135–136]. For each central extension and generating tuple, Theorem 10.4 realizes all E/A_I with one compact isotropy group and the canonical spectral equalities of Section 4; the homomorphisms $E/A_I \to E/A_{I'}$ arise from coordinate deletions between lifted stabilizers. Colantonio and Lauret [27, Theorems 1.5 and 4.4] construct strongly isospectral spherical space forms with nonisomorphic fundamental groups.

11. Fundamental groups and dimension bounds

11.1. The common finite quotient.

Theorem 11.1 (Fundamental-group lists with common isotropy). *Let $E_1, \dots, E_m$ be a nonempty finite list of groups. The following conditions are equivalent.*

1. *There are compact connected Lie groups J_i and embeddings of one compact Lie group K in all the J_i such that $\pi_1(J_i/K) \simeq E_i$.*
2. *There are a finite group Γ and surjections $p_i : E_i \longrightarrow \Gamma$ whose kernels A_i are finitely generated central subgroups of E_i.*
3. *Condition (1) holds in the finite-product family of Theorem 10.4. The quotients have a common scalar spectrum, and for every $i \neq j$ there is an integer $q \ge 2$ such that*

$$\dim_{\mathbb{Q}}\big(\pi_q(J_i/K) \otimes \mathbb{Q}\big) \neq \dim_{\mathbb{Q}}\big(\pi_q(J_j/K) \otimes \mathbb{Q}\big).$$

For any prescribed data in (2), the family in (3) may be chosen with $\pi_0(K) \simeq \Gamma$ and boundary homomorphisms p_i. Its metrics are real analytic and normal homogeneous, with nonnegative sectional curvature. Their Ricci nullities are $\operatorname{rank} A_i = b_1(J_i/K)$*; their Ricci tensors are positive exactly when E_i is finite.*

The canonical rough Laplacians are isospectral for every common finite-dimensional unitary K-module, including the natural tensor modules of the common tangent isotropy representation. There are compatible homogeneous spin structures for which the cubic spin squares are isospectral after every common finite-dimensional unitary K twist. The graded cubic exterior operators have equal signed spectra under the same twists.

Let $\vartheta : K \to \Gamma$ be the component homomorphism. For every $L \le \Gamma$, put $K_L = \vartheta^{-1}(L)$. The connected covers $J_i/K_L \to J_i/K$ have degree $[\Gamma : L]$ and fundamental groups $p_i^{-1}(L)$. They satisfy the same spectral equalities for every common finite-dimensional unitary K_L-module. In particular, the flat bundles associated to every common finite-dimensional unitary representation of L have isospectral connection Laplacians.

Proof. If (1) holds, Lemma 10.2, with $\Gamma = \pi_0(K)$, gives (2).

Suppose (2) holds. Put

$$E = \{(e_1, \dots, e_m) \in \textstyle\prod_i E_i \,:\, p_1(e_1) = \cdots = p_m(e_m)\}, \qquad p((e_i)) = p_i(e_i).$$

Then $p : E \to \Gamma$ is a central extension with kernel $A = \prod_i A_i$. The projection $q_i : E \to E_i$ is onto and has kernel $\prod_{j \neq i} A_j$, and $p_i q_i = p$. Choose a finite generating tuple for each coordinate subgroup

$A_i \le A$, allowing the empty tuple when $A_i = 1$. Concatenate these tuples and append m identity elements. Denote the resulting tuple by $c_1, \dots, c_r$, the index set of the tuple from A_i by B_i, and the appended indices by $d_1, \dots, d_m$. Thus

$$[r] = B_1 \sqcup \cdots \sqcup B_m \sqcup \{d_1, \dots, d_m\}, \qquad r = \sum_i |B_i| + m \ge 1.$$

For $1 \le i \le m$, set

$$I_i = \Big(\bigcup_{j \ne i} B_j\Big) \cup \{d_i\}, \qquad A_{I_i} = \langle c_j \,:\, j \in I_i \rangle = \prod_{j \ne i} A_j.$$

Theorem 10.4 gives a common subgroup K and quotients J_i/K, with $J_i = J_{I_i}$, for which

$$\pi_1(J_i/K) \simeq E/A_{I_i} \simeq E_i.$$

Their boundary maps are p_i, because $p_i q_i = p$. If $i \ne j$, then $d_i \in I_i \setminus I_j$; the higher rational homotopy groups therefore differ by Theorem 10.4(1). The geometric and spectral conclusions are Theorem 10.4(2)–(3), since $A/A_{I_i} \simeq A_i$.

For $L \le \Gamma$, the group $K_L = \vartheta^{-1}(L)$ is closed, $K_L^0 = K^0$, and $[K : K_L] = [\Gamma : L]$. The covering maps are local isometries, since K_L and K have the same Lie algebra. Theorem 10.4(4) gives

$$\pi_1(J_i/K_L) \simeq p^{-1}(L)/A_{I_i} \simeq p_i^{-1}(L).$$

Apply Theorem 4.3 directly to K_L and any common finite-dimensional unitary K_L-module. The tangent isotropy identifications and spin lifts restrict from K to K_L. Corollary 4.6, Theorem 4.9, and Corollary 4.10 give the tensor and cubic spectral equalities on the covers. A representation of $L = K_L/K^0$ has zero differential, so its canonical connection is flat by Theorem 4.3. Finally, (3) implies (1). □

Corollary 11.2. *Every finite list of finitely generated abelian groups occurs as the fundamental groups of one such isospectral family together with a simply connected member. Conversely, every group in a family as in Theorem 11.1(1) containing a simply connected member is abelian. The first Betti numbers, and hence the dimensions of the Euclidean factors of the universal covers, can be prescribed by any finite list of nonnegative integers.*

Proof. For existence, apply Theorem 11.1 with $\Gamma = 1$ to the given list augmented by 1. Conversely, a simply connected member forces $\pi_0(K) = 1$, so all the other fundamental groups are central boundary kernels. Apply Proposition 4.15 to obtain the Euclidean factors of the universal covers. □

Proposition 11.3. *Let J be a compact connected Lie group, and let $H_0, H_1 \le J$ be closed subgroups. Let $\mathrm{Irr}(J)$ be a set of representatives of the equivalence classes of irreducible finite-dimensional continuous complex representations of J. For such a representation V and a subgroup $H \le J$, let V^H be its subspace of H-fixed vectors. Suppose that H_0 and H_1 have identical dimension data:*

$$\dim V^{H_0} = \dim V^{H_1} \qquad (V \in \mathrm{Irr}(J)).$$

Then J/H_0 is simply connected if and only if J/H_1 is simply connected. Their fundamental groups are simultaneously finite or infinite; in the finite case they have equal orders.

Proof. By [6, Theorem 1.3, p. 60], maximal tori $T_i \le H_i^0$ are conjugate in J, and $|\pi_0(H_0)| = |\pi_0(H_1)|$. Lemma 10.1 gives surjectivity of $\pi_1(T_i) \to \pi_1(H_i^0)$.

Put $I_i = \mathrm{im}(\pi_1(H_i^0) \to \pi_1(J))$. These are the images of $\pi_1(T_i)$. The tori are conjugate, and inner automorphisms of connected J are homotopic to the identity through maps fixing the identity. Hence $I_0 = I_1 = I$. Lemma 10.2 gives

$$1 \longrightarrow \pi_1(J)/I \longrightarrow \pi_1(J/H_i) \longrightarrow \pi_0(H_i) \longrightarrow 1.$$

Hence $\pi_1(J/H_i)$ is finite exactly when $\pi_1(J)/I$ is finite, and then

$$|\pi_1(J/H_i)| = |\pi_1(J)/I|\,|\pi_0(H_i)|.$$

It is trivial exactly when both factors are trivial. □

11.2. **Dimension bounds.** In this subsection, $J = J_I$ and $K \le P = \prod_{j=1}^{r} P_{n_j}$ are as in Section 2, with $r \ge 1$, $n_j \ge 1$, and K closed. The subgroup K is fixed when I varies. For a compact Lie group H, let $\operatorname{rank} H$ be the dimension of a maximal torus of H^0. For a finitely generated abelian group C, let $d(C)$ denote its minimal number of generators; set $d(1) = 0$.

Theorem 11.4. *Put*

$$C = \ker\bigl(\pi_1(J/K) \longrightarrow \pi_0(K)\bigr)$$

for the boundary homomorphism of Lemma 10.2. This group is finitely generated abelian. Put $d = d(C)$. Then

$$r \ge d, \qquad \dim(J/K) \ge \sum_j (3n_j^2 + 4n_j) + d \ge 7r + d \ge 8d. \tag{11.1}$$

If $\pi_1(J/K)$ is nonabelian, then some $n_j \ge 2$, and the last bound improves to $\dim(J/K) \ge 8d + 13$.

Proof. Choose a maximal torus T_K of K^0, contained in a maximal torus T_P of P, and write

$$L_K = \pi_1(T_K), \qquad L_P = \pi_1(T_P).$$

The inclusion $L_K \subset L_P$ is primitive: its quotient is the lattice of the torus T_P/T_K and is free. The map $\pi_1(T_K) \to \pi_1(K^0)$ is onto by Lemma 10.1.

If J has u unitary and h symplectic–orthogonal coordinates, $u + h = r$, then

$$B = \pi_1(J) = \mathbb{Z}^u \oplus (C_2)^h.$$

Let $f : L_P \to B$ be induced by $T_P \hookrightarrow P \hookrightarrow J$. The map $\pi_1(T_P) \to \pi_1(P)$ is onto by Lemma 10.1. The map $\pi_1(P) \to B$ is onto by (2.7): in a G_n-factor it is the z-winding, and in an H_n-factor it is the sum of the $\mathrm{U}(n)$ determinant winding and the z-winding, modulo two. Hence f is onto. Lemma 10.2 therefore gives

$$C = B/f(L_K).$$

Thus C is a quotient both of the r-generated group B and of the free abelian group L_P/L_K. Consequently

$$d \le r, \qquad d \le \operatorname{rank} P - \operatorname{rank} K.$$

Let $\mathfrak{t}_P, \mathfrak{t}_K, \mathfrak{p}, \mathfrak{k}$ be the Lie algebras of T_P, T_K, P, K. By [20, Chapter IV, Theorem 2.3(i), p. 165], the centralizer of T_K in K^0 is T_K. Since $\mathfrak{t}_P$ centralizes T_K,

$$\mathfrak{t}_P \cap \mathfrak{k} = \mathfrak{t}_K, \qquad \mathfrak{t}_P/\mathfrak{t}_K \hookrightarrow \mathfrak{p}/\mathfrak{k}.$$

Consequently

$$\dim P - \dim K \ge \operatorname{rank} P - \operatorname{rank} K \ge d.$$

Since $\dim G_n = \dim H_n = (2n+1)^2$ and $\dim P_n = n^2 + 1$, this proves (11.1).

If all $n_j = 1$, then P is a torus contained in a maximal torus T of J. By Lemma 10.1, the map $\pi_1(T) \to \pi_1(J)$ is onto. Hence the full inverse image $\widetilde{T}$ in the universal covering group is connected. Its Lie algebra is abelian, so $\widetilde{T}$ is abelian. The full inverse image $\widetilde{K}$ lies in $\widetilde{T}$; thus $\pi_1(J/K) \simeq \pi_0(\widetilde{K})$ is abelian by Lemma 10.2. Thus a nonabelian fundamental group requires some $n_j \ge 2$, whence

$$\sum_j (3n_j^2 + 4n_j) \ge 7(r-1) + 20 = 7r + 13.$$

□

Theorem 11.5 (Sharp dimension for prescribed abelian groups). *Let A be a nontrivial finitely generated abelian group. Among pairs $J_I/K, J_{I'}/K$ with $r, n_j \ge 1$ and one closed subgroup $K \le \prod_{j=1}^{r} P_{n_j}$, the least dimension of an isospectral pair with fundamental groups A and 1 is $8d(A)$.*

Proof. The simply connected member forces K to be connected by Lemma 10.2. The other member has central boundary kernel A, so Theorem 11.4 gives $\dim(J_I/K) \geq 8d(A)$.

Write $A = \mathbb{Z}^a \oplus \bigoplus_{j=1}^{t} C_{q_j}$, where $a, t \geq 0$, $q_j \geq 2$, and $q_j \mid q_{j+1}$ for $1 \leq j < t$. Then $d(A) = a + t$. Corollary 11.12, with $n = 1$, gives an eight-dimensional pair with fundamental groups $C_{q_j}, 1$ for each finite factor, and one with groups $\mathbb{Z}, 1$ for each free factor. Take the product of these circles as the common subgroup of $(P_1)^{a+t}$. The product pair has dimension $8(a + t)$. □

Corollary 11.6. *Let E be finite and $1 \neq C \leq Z(E)$. Among the product realizations for which the all-unitary boundary is the specified map $E \to E/C$, the least number of factors is $d(C)$.*

Proof. The all-unitary member has central boundary kernel C, so Theorem 11.4 gives $r \geq d(C)$. A generating tuple of length $d(C)$ in Theorem 10.4 attains this number. □

Example 11.7. Let Q_8 and D_8 denote the quaternion and dihedral groups of order eight. The groups Q_8, D_8, and C_2^2 occur in one isospectral family with common isotropy: their maps to C_2^2 are the two quotients by the centers and the identity map. Their kernels are central, so Theorem 11.1 applies.

11.3. **A primitive torus and all its coordinate quotients.** Coordinate quotients of a free abelian group admit realizations with $n_j = 1$ for every j.

Theorem 11.8. *Let $\Phi : \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}^a$ be a surjective homomorphism, where $r \geq 1$ and $0 \leq a \leq r$. Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{Z}^r$, and put $c_j = \Phi(e_j)$. There is a connected torus $K \leq (P_1)^r$ such that, with $n_j = 1$ for every j, the quotients $M_I = J_I/K$ have dimension $7r + a$ and*

$$\pi_1(M_I) \simeq \mathbb{Z}^a/\langle c_j : j \in I\rangle \qquad (I \subseteq [r]).$$

All 2^r quotients have one scalar spectrum and the same canonical rough connection spectrum for every common finite-dimensional unitary K-representation. Their normal homogeneous metrics are real analytic and have nonnegative sectional curvature. The Ricci nullity at each member is the rank of its displayed fundamental group.

Proof. Put $\Lambda = \ker\Phi$ and $W = \operatorname{span}_{\mathbb{R}}\Lambda$. Since $\mathbb{Z}^r/\Lambda \simeq \mathbb{Z}^a$ is torsion-free, Λ is primitive and $W \cap \mathbb{Z}^r = \Lambda$. Consequently the torus

$$T_\Lambda = W/\Lambda \longrightarrow \mathbb{R}^r/\mathbb{Z}^r \simeq \mathbb{T}^r$$

embeds as a closed connected subtorus of dimension $r - a$. Define $z_j : T_\Lambda \to \mathrm{U}(1)$ by $z_j([x]) = e^{2\pi i x_j}$ for $x = (x_1, \dots, x_r) \in W$. Let $w_j : \mathbb{T}^r_w \to \mathrm{U}(1)$ be the coordinate characters of an auxiliary copy $\mathbb{T}^r_w$ of $\mathbb{T}^r$. Embed $K = T_\Lambda \times \mathbb{T}^r_w$ by

$$(x, y) \longmapsto \big((w_j(y), z_j(x))\big)_{j=1}^{r} \in (P_1)^r.$$

The characters $z_1, \dots, z_r$ jointly separate points of T_Λ, as do $w_1, \dots, w_r$ on $\mathbb{T}^r_w$. Thus this homomorphism is injective and has closed image. Equivalently K is the kernel of the torus homomorphism $(P_1)^r \to \mathbb{T}^a$ defined by applying the integer matrix Φ to the z-coordinates.

Because K is connected, $\pi_1(M_I)$ is the cokernel of the map from $\pi_1(K)$ to the ambient fundamental group. In the coordinates $\pi_1(K) = \Lambda \oplus \mathbb{Z}^r$, equation (2.7) gives the map, with $I^c = [r] \smallsetminus I$,

$$(\lambda, m) \longmapsto \big((\lambda_j)_{j\notin I}, (\lambda_j + m_j \bmod 2)_{j\in I}\big) \in \mathbb{Z}^{I^c} \oplus (\mathbb{Z}/2)^I.$$

The images of the m_j loops contain $\{0\} \oplus (\mathbb{Z}/2)^I$. Let $\mathrm{pr}_{I^c} : \mathbb{Z}^r \to \mathbb{Z}^{I^c}$ be coordinate projection, and put $\mathbb{Z}^I = \bigoplus_{j\in I} \mathbb{Z}e_j$. The remaining cokernel is

$$\mathbb{Z}^{I^c}/\,\mathrm{pr}_{I^c}\Lambda \simeq \mathbb{Z}^r/(\Lambda + \mathbb{Z}^I) \simeq \mathbb{Z}^a/\langle c_j : j \in I\rangle.$$

For $I \subseteq I'$, coordinate projection induces the homomorphism

$$\mathbb{Z}^a/\langle c_j : j \in I\rangle \longrightarrow \mathbb{Z}^a/\langle c_j : j \in I'\rangle, \qquad [x] \longmapsto [x].$$

Each ambient factor has dimension nine, whereas $\dim K = (r-a)+r$. The quotient dimension is therefore $9r-(2r-a) = 7r+a$. Theorem 4.3 gives the scalar and canonical connection spectra. The normal quotient is real analytic, and (4.16) gives nonnegative sectional curvature. Proposition 4.15 identifies its Ricci nullity with the rank of the displayed abelian fundamental group. □

Corollary 11.9. *Let $p_1, \dots, p_r$ be distinct primes, with $r \geq 1$, and put $N = \prod_j p_j$. There are 2^r pairwise nonhomeomorphic scalar-isospectral normal homogeneous manifolds of dimension $7r+1$, with fundamental groups*

$$\pi_1(M_\emptyset) = \mathbb{Z}, \qquad \pi_1(M_I) = C_{N/\prod_{i\in I} p_i} \quad (\emptyset \neq I \subseteq [r]).$$

Here C_1 denotes the trivial group. Every member has nonnegative sectional curvature. If $I \neq \emptyset$, then M_I has positive Ricci curvature, whereas $M_\emptyset$ has Ricci nullity one. Among families J_I/K with one closed subgroup $K \leq \prod_j P_{n_j}$, the least dimension for this prescribed list is $7r+1$.

Proof. Take $a = 1$ and $c_j = N/p_j$. Their greatest common divisor is one: a prime dividing all of them would divide N, but the entry indexed by that prime omits it. This also holds for $r = 1$, when the single entry is one. Thus they define the surjection required in Theorem 11.8. For nonempty I,

$$\gcd(c_i : i \in I) = N/\prod_{i\in I} p_i.$$

A prime p_k divides every selected entry exactly when $k \notin I$, proving the equality. Different nonempty subsets give different finite group orders, and the empty subset gives the infinite group $\mathbb{Z}$. All 2^r groups are therefore different, which proves nonhomeomorphism. The member with fundamental group $\mathbb{Z}$ has Ricci nullity one; the others have positive Ricci curvature.

For optimality, a product with r' factors has at most $2^{r'}$ ambient choices, so $r' \geq r$. The simply connected member makes the common isotropy connected. Thus the member with fundamental group $\mathbb{Z}$ has central boundary kernel $\mathbb{Z}$, and Theorem 11.4 gives the dimension bound $7r'+1 \geq 7r+1$. The quotients above attain this value. □

Example 11.10 (Four members in dimension fifteen). Take $\Phi = (2,3) : \mathbb{Z}^2 \to \mathbb{Z}$. An explicit common subgroup is $K = \mathbb{T}^3$, embedded by

$$(u, w_1, w_2) \longmapsto \big((w_1, u^3), (w_2, u^{-2})\big) \in P_1 \times P_1.$$

The joint powers of u are faithful because $\gcd(3,2) = 1$. For $I = \emptyset, \{1\}, \{2\}, \{1,2\}$, the fundamental groups are $\mathbb{Z}, C_2, C_3, 1$, respectively. The common dimension is $18-3 = 15$. By Corollary 11.9, this dimension is least among the families J_I/K for that four-group list. For three primes, the same corollary gives eight members in dimension twenty-two; primes $2, 3, 5$ give $\mathbb{Z}, C_{15}, C_{10}, C_6, C_5, C_3, C_2, 1$.

11.4. **Primitive circles and their cyclic quotients.**

Theorem 11.11. *Let $n \geq 1$ and let $(a_1, \dots, a_n, b) \in \mathbb{Z}^{n+1}$ be primitive. Equip G_n and H_n with the metrics of (2.2). In P_n set*

$$k(\theta) = \big(\mathrm{diag}(e^{ia_1\theta}, \dots, e^{ia_n\theta}), e^{ib\theta}\big), \qquad K = \{k(\theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

Put $\mathbf{a} = (a_1, \dots, a_n)$, $s = \sum_j a_j + b$, $D = 2\sum_j a_j^2 + b^2$, and

$$X_{n;\mathbf{a},b} = G_n/K, \qquad Y_{n;\mathbf{a},b} = H_n/K.$$

These are real analytic normal homogeneous manifolds of dimension $4n(n+1)$ with nonnegative sectional curvature and the common canonical spectra of Theorem 4.3. Their homotopy groups are

$$\pi_1(X_{n;\mathbf{a},b}) = \mathbb{Z}/b\mathbb{Z}, \qquad \pi_2(X_{n;\mathbf{a},b}) = \begin{cases} 0, & b \neq 0, \\ \mathbb{Z}, & b = 0, \end{cases}$$

$$\pi_1(Y_{n;\mathbf{a},b}) = \begin{cases} \mathbb{Z}/2, & s \text{ even}, \\ 1, & s \text{ odd}, \end{cases} \qquad \pi_2(Y_{n;\mathbf{a},b}) = \mathbb{Z}.$$

Here $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. The Ricci tensor of $Y_{n;\mathbf{a},b}$ is positive. On the other member,

$$\mathrm{Ric}_{X_{n;\mathbf{a},b}} \geq \frac{b^2}{2D}\, g_{X_{n;\mathbf{a},b}},$$

and its Ricci tensor is positive exactly when $b \neq 0$. The circle generators satisfy

$$Q_G(di_n\dot{k}(0), di_n\dot{k}(0)) = Q_H(dj_n\dot{k}(0), dj_n\dot{k}(0)) = D. \tag{11.2}$$

If $b \neq 0$, put $F = i_n(K) \cap \mathrm{SU}(2n+1)$. Then

$$F \cong \mathbb{Z}/|b|\mathbb{Z}, \qquad X_{n;\mathbf{a},b} \cong \mathrm{SU}(2n+1)/F, \qquad b_2(X_{n;\mathbf{a},b}) = 0.$$

If s is odd, then $b_2(Y_{n;\mathbf{a},b}) = 1$. In particular $X_{n;\mathbf{a},1}$ is diffeomorphic to $\mathrm{SU}(2n+1)$.

Proof. Write $X = X_{n;\mathbf{a},b}$ and $Y = Y_{n;\mathbf{a},b}$. Primitivity gives integers r_j, r with $\sum r_j a_j + rb = 1$. If every $a_j\theta, b\theta$ is in $2\pi\mathbb{Z}$, so is θ; thus K is an embedded circle. Its squared generator norm is D by (2.3), and the quotient dimension is $(2n+1)^2 - 1 = 4n(n+1)$. Theorem 4.3 and (4.16) give the common canonical spectra and nonnegative sectional curvature. For $J = G_n$ or $J = H_n$, Lemma 10.1 gives

$$0 \to \pi_2(J/K) \to \mathbb{Z} \xrightarrow{k_*} \pi_1(J) \to \pi_1(J/K) \to 1. \tag{11.3}$$

By (2.7), k_* is multiplication by b into $\mathbb{Z}$ or by s into C_2. Their kernels and cokernels give the table; the latter kernel is either $\mathbb{Z}$ or $2\mathbb{Z}$, both infinite cyclic.

The Lie algebra of H_n is semisimple; hence the Ricci tensor of Y is positive by Proposition 4.15. For the generator $T = di_n\dot{k}(0)$ in $\mathfrak{u}(2n+1)$, $\mathrm{Tr}\, T = ib$ and $Q_G(T,T) = D$. Equation (4.15) gives the Ricci bound. If $b = 0$, the nonzero central vector iI_{2n+1} is horizontal and belongs to the Ricci kernel. When $b \neq 0$, the determinant $e^{ib\theta}$ maps K onto $\mathrm{U}(1)$. Thus $\mathrm{SU}(2n+1)$ acts transitively, with stabilizer F of order $|b|$, and $\mathrm{SU}(2n+1)/F \cong X$ by [41, Corollary 3.13]. Since $\pi_1(\mathrm{SU}(2n+1)) = \pi_2(\mathrm{SU}(2n+1)) = 0$, the Hurewicz theorem gives $H_2(\mathrm{SU}(2n+1);\mathbb{Z}) = 0$ [53, Theorem 4.32]. The universal coefficient theorem gives $H^2(\mathrm{SU}(2n+1);\mathbb{Q}) = 0$ [53, Theorem 3.2]. The finite covering $\mathrm{SU}(2n+1) \to X$ induces an injection on rational cohomology [53, Proposition 3G.1], so $b_2(X) = 0$. If s is odd, then Y is simply connected and $\pi_2(Y) = \mathbb{Z}$; the Hurewicz and universal coefficient theorems give $b_2(Y) = 1$. □

Corollary 11.12. *For all integers $n, q \geq 1$, the choice $\mathbf{a} = (q-1, 0, \dots, 0)$, $b = q$ gives the pair $X = X_{n;\mathbf{a},b}$, $Y = Y_{n;\mathbf{a},b}$ of dimension $4n(n+1)$ with*

$$\pi_1(X) = \mathbb{Z}/q\mathbb{Z}, \quad \pi_1(Y) = 1, \quad \pi_2(X) = 0, \quad \pi_2(Y) = \mathbb{Z}, \quad b_2(X) = 0, \quad b_2(Y) = 1.$$

Both normal metrics have positive Ricci curvature. The choice $\mathbf{a} = (1, 0, \dots, 0)$, $b = 0$ gives fundamental groups $\mathbb{Z}$ and 1*, with positive Ricci curvature on the latter. For fixed n, the scalar spectra of the finite cyclic pairs are distinct as q varies. For $n = 1$, put*

$$K_q = \{(e^{i(q-1)\theta}, e^{iq\theta}) \,:\, \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \leq P_1, \qquad X_q = G_1/K_q, \quad Y_q = H_1/K_q.$$

For $\mu_q = \{\mathrm{diag}(z, z^{-1}, 1) \,:\, z \in \mathrm{U}(1),\ z^q = 1\}$, these eight-dimensional manifolds satisfy

$$X_q \cong \mathrm{SU}(3)/\mu_q, \qquad Y_q \cong S^2 \times S^3 \times S^3.$$

Proof. The two primitive vectors have $s = 2q-1$ and $s = 1$, respectively. Theorem 11.11 gives the displayed homotopy groups and Betti numbers and positive Ricci curvature. Fibre integration, (11.2), and Weyl's law [38, Theorem 17.7] give

$$\mathrm{Vol}\, X = \mathrm{Vol}\, Y = \frac{\mathrm{Vol}(G_n, Q_G)}{2\pi\sqrt{2(q-1)^2 + q^2}}.$$

Put $D_q = 2(q-1)^2 + q^2$. Since $D_{q+1} - D_q = 6q - 1 > 0$, the volumes, and therefore the scalar spectra, are distinct for distinct q. For $n = 2$ the unitary circle is

$$\{\mathrm{diag}(e^{i(q-1)\theta}, 1, e^{-i(q-1)\theta}, 1, e^{iq\theta}) \,:\, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}; \tag{11.4}$$

the corresponding circle in H_2 has rotation weights $(q-1, 0, q)$ in its SO(6) component. Both quotients have dimension 24. For $n = 1$, the finite unitary stabilizer is μ_q because $a_1 = q - 1$.

For the topology of Y_q, lift to

$$\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \longrightarrow \mathrm{Sp}(1) \times \mathrm{SO}(4).$$

In each SU(2) factor, write e^{it} for $\mathrm{diag}(e^{it}, e^{-it})$. Choose torus angles so that (β, γ) maps to two-plane rotations $(\beta + \gamma, \beta - \gamma)$. The kernel is $\{(1,1,1),(1,-1,-1)\}$. The full preimage of K_q is the connected primitive circle

$$\widetilde{K}_q = \{(e^{i(2q-2)t}, e^{i(2q-1)t}, e^{-it}) \,:\, t \in \mathbb{R}/2\pi\mathbb{Z}\}. \tag{11.5}$$

It contains the covering kernel at $t = \pi$ and maps onto K_q twice, with $\theta = 2t$, so the displayed circle is the full inverse image of K_q.

Up to conjugation and permutation of the three factors, the weights of $\widetilde{K}_q$ are

$$\begin{cases} (1,1,0), & q = 1, \\ (2q-1, 2q-2, 1), & q \geq 2. \end{cases}$$

These triples are primitive, with nonnegative entries in nonincreasing order. Thus $Y_q \cong \mathrm{SU}(2)^3/\widetilde{K}_q$ is diffeomorphic to $S^2 \times S^3 \times S^3$ by [4, arXiv version 2, Theorem A(a2)]. □

Proposition 11.13. *Let $n \geq 1$, $\mathbf{a} = (1, 0, \dots, 0)$, $b = 1$, and set $X = X_{n;\mathbf{a},b}$, $Y = Y_{n;\mathbf{a},b}$ as in Theorem 11.11. Then X is diffeomorphic to $\mathrm{SU}(2n+1)$ and $\pi_1(Y) = \mathbb{Z}/2$. Both metrics have positive Ricci curvature. The pullback g of the normal metric under $\mathrm{SU}(2n+1) \to X$, $u \mapsto uK$, is left invariant. At the identity, put $T = di_n(\mathrm{diag}(i, 0, \dots, 0), i)$, so that $Q_G(T,T) = 3$. For $U, V \in \mathfrak{su}(2n+1)$,*

$$g(U,V) = Q_G(U,V) - \frac{1}{3} Q_G(U,T) Q_G(V,T).$$

It is not bi-invariant.

Proof. For $b = 1$ and $s = 2$, Theorem 11.11 gives $X \cong \mathrm{SU}(2n+1)$, $\pi_1(Y) = C_2$, and positive Ricci curvature. The metric formula is the orthogonal projection away from the vertical line $\mathbb{R}T$. Write $T_0 = T - (\mathrm{Tr}\, T)/(2n+1) I_{2n+1} \neq 0$. For $U \in \mathfrak{su}(2n+1)$ one has $Q_G(U,T) = Q_G(U,T_0)$, so $Q_G - g$ has rank one on $\mathfrak{su}(2n+1)$. If g were bi-invariant, the kernel of $Q_G - g$ would be a nonzero proper ideal by [41, Proposition 14.12], contrary to the simplicity of $\mathfrak{su}(2n+1)$. □

For a nontrivial finitely generated abelian group A, the products with $n = 1$ attain dimension $8d(A)$, as in Theorem 11.5. For every $n \geq 1$, the vector $\mathbf{a} = 0, b = 1$ in Theorem 11.11 gives two simply connected members with second Betti numbers zero and one.

Proposition 11.14. *Every quotient J_I/K with at least one factor has dimension at least seven. In dimension seven the only possible parameters are $r = 1, n_1 = 1, K = P_1 = \mathbb{T}^2$, so both quotients are simply connected. In dimension eight, if either choice is simply connected, K is a primitive circle with weights (a,b), and*

$$\pi_1(G_1/K) = \mathbb{Z}/b\mathbb{Z}, \qquad \pi_1(H_1/K) = \begin{cases} 1, & a+b \text{ odd}, \\ C_2, & a+b \text{ even}. \end{cases}$$

Here $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. Exactly one member is simply connected precisely when either $a+b$ is odd and $|b| \neq 1$, or $|b| = 1$ and a is odd. Both quotient Ricci tensors are positive precisely when $b \neq 0$.

Proof. Theorem 11.4 gives $\dim(J_I/K) \geq \sum_j (3n_j^2 + 4n_j) \geq 7r$. Dimension at most eight forces $r = n_1 = 1$. In dimension seven a closed subgroup of $P_1 = \mathbb{T}^2$ of dimension two is all of P_1. By (2.7), both maps from $\pi_1(P_1)$ to the ambient fundamental groups are onto; hence both quotients are simply connected. In dimension eight a simply connected quotient forces K to be connected by Lemma 10.2. Since $\dim K = 1$, it is a primitive circle in P_1. Theorem 11.11, with $n = 1$, gives the fundamental groups and the Ricci criterion. The unitary quotient is simply connected exactly when $|b| = 1$, and the other quotient exactly when $a + b$ is odd. These conditions give the stated alternatives. □

11.5. **Free torus actions.**

Proposition 11.15. *Let $r \ge 1$, $n_j \ge 1$, and $K \le \prod_{j=1}^{r} P_{n_j}$ be closed. Equip J_I with the product metric of Section 2. Every quotient J_I/K admits a free isometric action of a torus of dimension r, with flat totally geodesic orbits. For fixed K and matched block scales, the acting torus and its orbit metric are independent of I. Every such quotient has Euler characteristic zero and bounds unorientedly; if it is orientable, it bounds orientedly.*

Proof. For one factor and $t \in \mathbb{R}/2\pi\mathbb{Z}$, put

$$\ell_G(t) = \operatorname{diag}(e^{it}I_n, e^{it}I_n, 1), \qquad \ell_H(t) = \big(\operatorname{diag}(e^{it}I_n, e^{-it}I_n), r(e^{-it}I_n) \oplus I_2\big).$$

These circles centralize the respective images of P_n. Let $Z_G = \dot\ell_G(0)$ and $Z_H = \dot\ell_H(0)$. For $X \in \mathfrak{u}(n)$ and $\beta \in \mathbb{R}$, write $u = (X, i\beta) \in \mathfrak{p}_n$ and $\operatorname{Tr} X = i\alpha$, where $\alpha \in \mathbb{R}$. The two components of Z_H have inner products α and $-\alpha$ with the corresponding components of $dj_n u$; the z-component contributes zero. Thus

$$Q_G(Z_G, di_n u) = Q_H(Z_H, dj_n u) = 0, \qquad Q_G(Z_G, Z_G) = Q_H(Z_H, Z_H) = 2n.$$

In either ambient group, membership of $\ell_G(t)$ or $\ell_H(t)$ in the image of P_n requires $e^{it} = e^{-it}$ and $z = 1$. The intersection is therefore given by $t = 0, \pi$, corresponding to $(I_n, 1)$ and $(-I_n, 1)$ in P_n.

Let $\ell_I : \mathbb{R}^r \to J_I$ be the product of these homomorphisms, and put $L_I = \ell_I(\mathbb{R}^r)$. Define

$$D = \left\{ \varepsilon \in (\mathbb{Z}/2)^r \,:\, \big((-1)^{\varepsilon_j} I_{n_j}, 1\big)_{j=1}^{r} \in K \right\}, \qquad \Lambda_D = \{\pi m \,:\, m \in \mathbb{Z}^r,\ m \bmod 2 \in D\}.$$

The torus L_I centralizes K. Right multiplication is isometric, and at every gK its stabilizer in $\mathbb{R}^r$ is

$$\{t \,:\, g\ell_I(t)K = gK\} = \{t \,:\, \ell_I(t) \in K\} = \Lambda_D.$$

Hence $T_D = \mathbb{R}^r/\Lambda_D$ acts freely. For block scales $s_j > 0$, its orbit metric on the covering space $\mathbb{R}^r$ is

$$\sum_{j=1}^{r} 2s_j n_j \, dt_j^2.$$

The lattice and inner product are independent of I.

The subgroup KL_I is compact, and $J_I/K \to J_I/(KL_I)$ is a principal T_D-bundle. Its fibres are flat tori with the displayed metric. Their geodesics are $g\exp(tZ)K$, with $Z \in \operatorname{Lie} L_I \subseteq \mathfrak{k}^{\perp}$. These are geodesics of the normal homogeneous quotient by Section 6.5; the fibres are therefore totally geodesic.

Put $M = J_I/K$ and choose a primitive circle $S \le T_D$. The fundamental vector field of its free action has no zero; the Poincaré–Hopf theorem gives $\chi(M) = 0$. Identify S with $\mathrm{U}(1)$ and put $B = \{z \in \mathbb{C} \,:\, |z| \le 1\}$. Let $W = M \times_S B$ be the disk bundle of the associated complex line bundle $M \times_S \mathbb{C} \to M/S$, with $(m\cdot s, z) \sim (m, sz)$ [26, §2.1.2, p. 23, and §9.1, p. 230]. Its boundary diffeomorphism is

$$\partial W = M \times_S S \xrightarrow{\cong} M, \qquad [m, z] \longmapsto m \cdot z.$$

If M is oriented, the connected group S preserves its orientation. Orient M/S so that a positive circle tangent followed by a positive base frame gives the orientation of M. Orient W by the complex fibre followed by the base. The displayed diffeomorphism preserves the boundary orientation. □

11.6. **Prescribed finite abelian groups and finitely many periods.**

Corollary 11.16. *Let D be a nontrivial finite abelian group, and let $d(D)$ be its least number of generators. Fix a pair $(L_0, g_0), (L_1, g_1)$ from Theorem 8.8 with positive Ricci curvature, and a finite nonempty subset F of the lengths in* (8.21) *for this pair. There are closed connected real-analytic scalar-isospectral manifolds P_0, P_1 of dimension $18 + 8d(D)$ such that*

$$\operatorname{Ric}_{P_i} > 0, \qquad \pi_1(P_0) \simeq D, \qquad \pi_1(P_1) = 1, \qquad F \subset \operatorname{Lsp}(P_1) \setminus \operatorname{Lsp}(P_0).$$

Every $\ell \in F$ is the length of an embedded primitive closed geodesic on P_1. The common half-wave trace is real analytic near each $\ell \in F$. For nontrivial cyclic D, the dimension is 26.

Proof. Theorem 11.5 gives real analytic normal homogeneous manifolds $(X, h_X), (Y, h_Y)$ of dimension $8d(D)$ with equal scalar spectra and fundamental groups D and 1. Their Ricci tensors are positive by Proposition 4.15, because their fundamental groups are finite. Choose $c > 0$ with

$$2c \min\{\mathrm{inj}(X, h_X), \mathrm{inj}(Y, h_Y)\} > \max F.$$

Put $P_0 = (L_0 \times X, g_0 + c^2 h_X)$ and $P_1 = (L_1 \times Y, g_1 + c^2 h_Y)$. Lemma 8.10, with $T = \max F$, gives the spectral identity, the length exclusions, and the embedded primitive representatives. Since the first factors are simply connected, the product fundamental groups are D and 1. The Ricci tensor of a product is the sum of the Ricci tensors of its factors; constant scaling does not change the latter as covariant tensors. It is therefore positive on every nonzero product tangent vector. The product metrics are real analytic. Apply Corollary A.5 to the trivial flat unitary line bundles on P_0, P_1 and each $\ell \in F$. □

For $D = 1$, take the original eighteen-dimensional pair (L_0, L_1). For nontrivial D, the scale c depends on the finite set F.

12. Seven-dimensional topology and prescribed fundamental groups

Identify U(1) with the diagonal maximal torus of SU(2) by $z \mapsto \mathrm{diag}(z, z^{-1})$, and define $\chi : \mathrm{U}(1)^3 \to \mathrm{U}(1)$ by $\chi(u, v, w) = uv^{-1}w^{-1}$. Put

$$S^1_{(1,-1,0)} = \{\mathrm{diag}(z, z^{-1}, 1) : z \in \mathrm{U}(1)\}.$$

For $n = 1$ and $K = P_1 = \mathbb{T}^2$, the quotients in Theorem 4.3 are

$$A = A_7 = G_1/i_1(P_1) \cong \mathrm{SU}(3)/S^1_{(1,-1,0)},$$
$$B = B_7 = H_1/j_1(P_1) \cong \mathrm{SU}(2)^3/\ker\chi.$$

Give them the quotient metrics induced by Q_G and Q_H in (2.2). The first manifold is the Aloff–Wallach space $N_{1,-1}$; the second is $Q^{1,1,1}$, after Weyl changes of signs in the last two SU(2) factors.

Both winding maps in (2.7) are onto, so both quotients are simply connected. The determinant kernel SU(3) is transitive on A, with stabilizer $S^1_{(1,-1,0)}$. On the finite cover $\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to H_1$, choose 2π-periodic torus angles α, β, γ so that the last two factors give real rotations $\beta + \gamma$ and $\beta - \gamma$. The inverse image of $j_1(P_1)$ is $\alpha = \beta + \gamma \pmod{2\pi}$, namely $\ker\chi$. The character $(1, -1, -1)$ is primitive, so its kernel is a connected torus; it contains the covering kernel $\{(1, 1, 1), (1, -1, -1)\}$.

For a closed connected manifold, formality over $\mathbb{Q}$ is equivalent to formality over $\mathbb{R}$, since its cohomology is finite-dimensional [97, arXiv v5, Corollary 4.17 and §4.8].

Theorem 12.1 (Seven-dimensional topology). *The scalar-isospectral manifolds A and B have dimension seven, nonnegative sectional curvature, and positive Ricci curvature. The manifold A is rationally formal. The manifold B has a nonzero triple Massey product and is not rationally formal. Their second Betti numbers are respectively* 1 *and* 2*, so the zero eigenvalue of the Hodge Laplacian on two-forms has different multiplicities. Moreover,*

$$H^4(A; \mathbb{Z}) = 0, \qquad H^4(B; \mathbb{Z}) = \mathbb{Z}/2.$$

Dimension seven is the smallest dimension in which there is a pair of simply connected closed manifolds exactly one of which is rationally formal.

Proof. The manifold $A = N_{1,-1}$ is formal by [44, arXiv v3, Theorem 5.3]. Proposition C.24 gives the second Betti numbers and the integral fourth cohomology groups. Let $p : B \to (S^2)^3$ be the circle bundle in that proposition, and let $x_1, x_2, x_3 \in H^2((S^2)^3; \mathbb{Z})$ be the pullbacks of the positive generators of the three sphere factors. Its first Chern class is $x_1 + x_2 + x_3$. This is the bundle in [17, arXiv v3, proof of Theorem 3.2, case $n = 3$].

Let $u, v \in H^2(B; \mathbb{Q})$ be the rational images of p^*x_1, p^*x_2. Since $H^4(B; \mathbb{Q}) = 0$, the product $\langle u, u, v\rangle \subset H^5(B; \mathbb{Q})$ is defined. Poincaré duality gives $H^3(B; \mathbb{Q}) = 0$, so its indeterminacy $uH^3(B; \mathbb{Q}) + H^3(B; \mathbb{Q})v$

is zero. By naturality under extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$, its unique element maps to the nonzero Massey product computed in [17, arXiv v3, proof of Theorem 3.2, case $n = 3$]. It is therefore nonzero over $\mathbb{Q}$, and B is not formal. Hodge theory gives the different zero multiplicities on two-forms.

Scalar isospectrality follows from Theorem 4.3 with the trivial representation. The normal homogeneous metrics have nonnegative sectional curvature. Their Ricci curvatures are positive by Proposition 4.15, since A and B are simply connected. Every simply connected closed manifold of dimension at most six is formal by [86, Proposition 4.6, p. 574]. □

Corollary 12.2. *For every finite-dimensional unitary P_1-module, A and B have equal canonical rough tensor spectra and equal signed spectra of the cubic operators on the full exterior algebra, with that homogeneous twist. Their Hodge two-form spectra differ. They have unique spin structures, and their cubic spin squares have equal spectra for every such twist. The least eigenvalue of the untwisted cubic spin square is 2, with multiplicity 2.*

Proof. Corollaries 4.6 and 4.10 give the tensor and exterior spectral equalities. The Hodge two-form spectra differ by Theorem 12.1. Proposition 4.11 applies to the torus P_1 and gives the spin-square equalities. Simple connectivity makes the spin structures unique.

Let Σ be the common complex spin module, and let $D_{J,\Sigma}$ be the untwisted operator in (4.8), for $J = G_1$ or H_1. In the 2π-periodic angles a, b of $P_1 = \mathrm{U}(1) \times \mathrm{U}(1)$, the tangent module is a fixed line and three rotation planes of weights $2a, a + b, a - b$. The eight spin weights are their half-sums with independent signs. Thus the character of Σ is

$$2 + e^{2ia} + e^{-2ia} + e^{i(a+b)} + e^{-i(a+b)} + e^{i(a-b)} + e^{-i(a-b)},$$

and $\dim \Sigma^{P_1} = 2$. For $\mathfrak{j} = \mathrm{Lie}(J)$, the reference metrics give $\|\rho_{\mathfrak{j}}\|^2 = 2$. Since the isotropy is abelian, (4.11) gives

$$D_{J,\Sigma}^2 = \Delta_J + 2.$$

The kernel of Δ_J on equivariant spinors is the space of constant maps with values in Σ^{P_1}. Hence the least square eigenvalue is 2, with multiplicity 2.

The metric on P_1 is $2\,da^2 + db^2$. The Casimir of its action on Σ therefore has eigenvalues $0, 2, 3/2$, with multiplicities $2, 2, 4$. On this module, (4.10) gives eigenvalues $2, 0, 1/2$ for the square of the cubic endomorphism v_J in (4.8), with the same multiplicities. □

12.1. **Formality of the dimension-datum quotients.**

Proposition 12.3. *Let $n \geq 1$ be an integer and put $N = 2n + 1$. The quotients $\mathrm{SU}(2N)/\mathrm{U}(N)$ and $\mathrm{SU}(2N)/(\mathrm{Sp}(n) \times \mathrm{SO}(2n + 2))$ in An–Yu–Yu's construction* [6, Theorem 1.5] *are both rationally formal, for the defining embeddings in that theorem.*

Proof. Let H be either isotropy group. It is connected and has rank N. For a Lie group L, write BL for its classifying space. Let T be the standard maximal torus of H, and let $x_1, \dots, x_N \in H^2(BT; \mathbb{R})$ be the first Chern classes of the line bundles associated to its coordinate characters. Let $c_k \in H^{2k}(B\,\mathrm{SU}(2N); \mathbb{R})$, $2 \leq k \leq 2N$, be the universal Chern classes. The defining representation of $\mathrm{SU}(2N)$ has weights $\pm x_1, \dots, \pm x_N$ on T. Hence every odd c_k restricts to zero in $H^*(BT; \mathbb{R})$. The restriction $H^*(BH; \mathbb{R}) \to H^*(BT; \mathbb{R})$ is injective [66, arXiv v2, §2], so these odd classes also restrict to zero in $H^*(BH; \mathbb{R})$.

Order the polynomial generators of $H^*(B\,\mathrm{SU}(2N); \mathbb{R})$ as

$$c_2, c_4, \dots, c_{2N}, \qquad c_3, c_5, \dots, c_{2N-1}.$$

The restrictions of the last $N - 1$ generators belong to the ideal generated by those of the first $N = \mathrm{rank}\, H$ generators. Thus the quotient is of Cartan type [66, arXiv v2, §2.3], and is formal over $\mathbb{R}$ by [66, arXiv v2, §3, Theorem 6 and Remark 2]. The rational–real equivalence above gives rational formality. □

12.2. **Products and all unitary flat spectra.**

Theorem 12.4. *Let Z be a closed connected Riemannian manifold. Fix basepoints and put $\Pi = \pi_1(Z)$. Give $Z \times A$ and $Z \times B$ their product metrics. The product projections identify their fundamental groups with Π. Under these identifications:*

(1) For every finite-dimensional unitary representation of Π, the corresponding flat connection Laplacians on sections are isospectral, including multiplicities.
(2) The same equality holds on every pair of corresponding connected finite covers, for every finite-dimensional unitary representation of their common fundamental group.
(3) The minimum length in each free homotopy class equals the minimum length of the corresponding class in Z.
(4) The second Betti numbers of the products differ by one, so the products are not homeomorphic. The product $Z \times B$ is not rationally formal. If Z is rationally formal, then so is $Z \times A$.

Proof. Fix a finite-dimensional Hermitian space V and a unitary representation $\rho : \Pi \to \mathrm{U}(V)$. Since A and B are simply connected, the associated flat bundles are pullbacks from Z. For $F = A$ or $F = B$, their connection Laplacians satisfy

$$\Delta_{Z\times F,\rho} = \Delta_{Z,\rho} \otimes 1 + 1 \otimes \Delta_F.$$

Tensor products of eigenbases give the sums of the eigenvalues; their multiplicities are the sums of the products of the corresponding multiplicities. The scalar isospectrality of A and B gives (1). For a finite-index subgroup $L \leq \Pi$, let $Z_L \to Z$ be the corresponding connected Riemannian covering. The product covers are $Z_L \times A$ and $Z_L \times B$. The same formula for Z_L and any finite-dimensional unitary representation of L gives (2).

For any closed connected simply connected Riemannian manifold F, projection to Z does not increase length, and the lift $\gamma \mapsto (\gamma, f_0)$, for a fixed $f_0 \in F$, preserves length and the fundamental-group marking. Thus

$$\ell_{Z\times F}(c) = \ell_Z(c), \qquad c \in \Pi. \tag{12.1}$$

A constant-fibre lift of a minimizing loop in Z attains this minimum, by Lemma 5.3; the trivial class has a constant representative. Taking $F = A$ and $F = B$ gives (3).

The Künneth formula and $H^1(A;\mathbb{Q}) = H^1(B;\mathbb{Q}) = 0$ give

$$b_2(Z \times A) = b_2(Z) + 1, \qquad b_2(Z \times B) = b_2(Z) + 2.$$

A product of connected manifolds is formal over $\mathbb{R}$ if and only if both factors are formal over $\mathbb{R}$ [45, arXiv v3, Lemmas 2.10–2.11, pp. 7–8]. Theorem 12.1 and the rational–real equivalence give (4). □

Corollary 12.5. *For every finite group E, there is a pair of closed normal homogeneous manifolds with marked fundamental group E, nonnegative sectional curvature and positive Ricci curvature, exactly one of which is rationally formal. Their flat connection Laplacians are isospectral for every finite-dimensional unitary representation of E. This equality holds on every pair of corresponding connected finite covers for all finite-dimensional unitary representations of their common fundamental group. All marked minimum lengths agree. For $E = 1$, dimension seven is possible. If E embeds in $\mathrm{SU}(N)$, dimension $N^2 + 6$ is possible.*

Proof. Choose a faithful unitary representation $\sigma : E \to \mathrm{U}(d)$, where $d \geq 1$. Then $\sigma \oplus \det(\sigma)^{-1} : E \to \mathrm{SU}(d+1)$ is faithful. For any embedding $E \hookrightarrow \mathrm{SU}(N)$ with $N \geq 2$, let $p : \mathrm{SU}(N) \to Z = \mathrm{SU}(N)/E$ be the quotient map and give Z a normal metric. The boundary convention of Lemma 10.2 identifies $\pi_1(Z)$ with E. Each deck transformation is a right translation and is homotopic to the identity through right translations. Therefore [53, Proposition 3G.1, p. 321] gives

$$p^* : H^*(Z;\mathbb{R}) \xrightarrow{\simeq} H^*(\mathrm{SU}(N);\mathbb{R}).$$

Pullback on the real de Rham algebras is consequently a quasi-isomorphism. The connected Lie group $\mathrm{SU}(N)$ is formal [86, p. 573]; hence Z is formal over $\mathbb{R}$, and therefore over $\mathbb{Q}$.

Apply Theorem 12.4. Products of normal homogeneous metrics are normal homogeneous. Their sectional curvatures are nonnegative, and their Ricci curvatures are positive by Proposition 4.15, since their fundamental group is finite. Their dimension is $(N^2-1)+7 = N^2+6$. For $E = 1$, take Z to be a point; this also gives the asserted dimension when $N = 1$. □

Corollary 12.6. *For every closed oriented hyperbolic surface Σ_g of genus $g \geq 2$, the nine-dimensional manifolds $\Sigma_g \times A$ and $\Sigma_g \times B$ have equal marked flat connection spectra on all corresponding connected finite covers, for every finite-dimensional unitary representation of the covering fundamental group. The minimum lengths in corresponding free homotopy classes agree. The product $\Sigma_g \times A$ is rationally formal, whereas $\Sigma_g \times B$ is not.*

Proof. The oriented surface Σ_g admits a Kähler structure and is formal over $\mathbb{R}$ by [33, §6, Main Theorem (i), p. 270]. It is therefore formal over $\mathbb{Q}$ by the rational–real equivalence above. Apply Theorem 12.4. □

12.3. **Higher homotopy groups.**

Lemma 12.7. *Let J be a connected Lie group and $K \leq J$ a closed subgroup whose identity component K^0 is a torus. Based at the identity and its coset, the quotient map induces an isomorphism*

$$\pi_q(J) \longrightarrow \pi_q(J/K), \qquad q \geq 3.$$

Proof. For $q \geq 2$, one has $\pi_q(K) = \pi_q(K^0) = 0$. The homotopy exact sequence of $K \to J \to J/K$ [53, Theorem 4.41, p. 376] gives the result. □

Theorem 12.8. *Let $r \geq 1$ be an integer, let $n_1 < \cdots < n_r$ be odd positive integers, and let $K \leq \prod_{j=1}^{r} P_{n_j}$ be a closed subgroup whose identity component is a torus. For $I \subseteq \{1, \dots, r\}$, let M_I be the quotient in Theorem 4.3. Put*

$$s_j(I) = \dim_{\mathbb{Q}}\big(\pi_{4n_j+1}(M_I) \otimes \mathbb{Q}\big) \quad (1 \leq j \leq r), \qquad s_{r+1}(I) = 0.$$

Then, for $1 \leq j \leq r$,

$$s_j(I) = \#\{k \,:\, j \leq k \leq r,\ k \notin I\}, \qquad s_j(I) - s_{j+1}(I) = \begin{cases} 1, & j \notin I, \\ 0, & j \in I. \end{cases}$$

Consequently all 2^r manifolds have pairwise nonisomorphic graded rational higher homotopy groups, even when their fundamental groups are isomorphic.

Proof. The nonzero rational homotopy groups of $\mathrm{SU}(2n+1)$ occur in degrees $3, 5, \dots, 4n+1$, each with dimension one [105, arXiv v1, Theorem 2, §3.1 and Appendix B, (28)]. The finite cover $\mathrm{U}(1) \times \mathrm{SU}(2n+1) \to G_n = \mathrm{U}(2n+1)$ adds one degree-one generator. For odd $n \geq 3$, the C_n and D_{n+1} formulas in [105, arXiv v1, Appendix B, (29)–(30)] give the following degrees for H_n, counted with their rational homotopy multiplicities: two copies of

$$3, 7, \dots, 4n-1$$

and one further degree $2n+1$. The latter contributes an additional copy when it already occurs in the list. For $n = 1$, the finite cover $\mathrm{SU}(2)^3 \to H_1$ gives three degree-three generators.

For odd n, all these degrees for H_n are congruent to 3 modulo 4. Thus $\pi_{4k+1}(H_n) \otimes \mathbb{Q} = 0$ for $k \geq 1$, whereas $\pi_{4k+1}(G_n) \otimes \mathbb{Q}$ has dimension one if $k \leq n$ and is zero otherwise. Lemma 12.7 and the product formula for higher homotopy groups give the stated finite sums. Their successive differences recover I. □

Theorem 12.9. *Fix a finite group E, an integer $d \geq 1$, and a faithful representation $\sigma : E \to \mathrm{U}(d)$. Let m be the least odd integer with $m \geq d$. For each integer $r \geq 1$, put $n_j = m + 2(j-1)$, $1 \leq j \leq r$. There is*

an embedding $K = E \times \mathbb{T}^r \hookrightarrow \prod_{j=1}^r P_{n_j}$ for which the 2^r quotients $M_I = J_I/K$ are mutually isospectral closed normal homogeneous manifolds of dimension

$$D_r = 4\sum_{j=1}^{r} n_j(n_j+1) = \frac{16}{3}r^3 + (8m-4)r^2 + \left(4m^2 - 4m - \frac{4}{3}\right) r.$$

They have marked fundamental group E, pairwise nonisomorphic graded rational higher homotopy groups, nonnegative sectional curvature and positive Ricci curvature. Their canonical rough connection Laplacians are isospectral for every common finite-dimensional unitary K-module. In particular, their marked flat connection spectra agree for every finite-dimensional unitary representation of E. For every subgroup $L \leq E$, the corresponding connected covers have equal flat connection spectra for every finite-dimensional unitary representation of L. As $r \to \infty$, the number of homotopy types in this family is

$$2^r = 2^{(3/16)^{1/3} D_r^{1/3} + O_m(1)},$$

where $O_m(1)$ is bounded by a constant depending only on m.

Proof. For $e \in E$ and $t = (t_1, \dots, t_r) \in \mathbb{T}^r$, define the jth component of the embedding of K by

$$A_j(e,t) = \begin{cases} \sigma(e), & n_j = d, \\ \sigma(e) \oplus I_{n_j - d}, & n_j > d, \end{cases} \qquad z_j(e,t) = t_j, \qquad 1 \leq j \leq r.$$

Faithfulness of σ makes the map injective, and its compact image is closed. Its identity component is $\{1\} \times \mathbb{T}^r$, which is central in K.

By (2.7), the map on fundamental groups from K^0 to any ambient product is the product of the maps

$$\begin{aligned} &\mathbb{Z} \longrightarrow \mathbb{Z}, \quad l \longmapsto l && (J_{I,j} = G_{n_j}), \\ &\mathbb{Z} \longrightarrow \mathbb{Z}/2, \quad l \longmapsto l \bmod 2 && (J_{I,j} = H_{n_j}). \end{aligned}$$

It is surjective. Lemma 10.2, with its inverse-endpoint boundary convention, therefore gives $\pi_1(M_I) \cong \pi_0(K) = E$ for every I.

Theorem 4.3 gives the canonical rough spectra. A finite-dimensional unitary representation of E, inflated to K, is trivial on K^0, so its canonical connection is flat. The same surjectivity makes J_I/K^0 simply connected. These associated bundles therefore exhaust the flat unitary bundles with the stated E-marking. For $L \leq E$, replace K by $L \times \mathbb{T}^r$. This gives the corresponding connected cover, and Theorem 4.3 applies to every finite-dimensional unitary representation of L. Theorem 12.8 distinguishes the graded rational higher homotopy groups of the 2^r base manifolds. Their normal metrics have nonnegative sectional curvature; their finite fundamental groups give positive Ricci curvature by Proposition 4.15.

Finally, $\dim G_n = \dim H_n = (2n+1)^2$ and $\dim K = r$, so $\dim M_I = 4\sum_{j=1}^r n_j(n_j+1)$. Substituting $n_j = m + 2(j-1)$ gives the displayed polynomial. In particular, $D_r = (16/3)r^3 + O_m(r^2)$, whence $r = (3D_r/16)^{1/3} + O_m(1)$ and the stated count. □

For each fixed r, these 2^r manifolds have one spectrum in dimension D_r. Both the spectrum and the dimension may depend on r.

13. Unitary flat spectra and Riemannian holonomy

Ejiri [39, Theorem 2.2 and the corollary in §4] proved the scalar spectral comparison for compact warped products and constructed nonisometric irreducible examples. All flat Laplacians below act on sections of finite-rank Hermitian bundles with unitary flat connection. Write Hol^0 for the restricted Levi–Civita holonomy group.

Theorem 13.1 (Equality of all unitary flat spectra). *Let (F_0, h_0) and (F_1, h_1) be closed connected simply connected scalar-isospectral Riemannian manifolds of dimension $d \geq 1$. Let (B, g_B) be a closed connected Riemannian manifold of dimension $b \geq 1$, and let $f : B \to (0, \infty)$ be smooth. Put*

$$X_i = B \times F_i, \qquad g_i = g_B + f^2 h_i, \qquad i = 0, 1.$$

Fix basepoints in B, F_0, F_1. The projections identify both fundamental groups with $\pi_1(B)$. For every finite-dimensional Hermitian space V_ρ and unitary representation $\rho : \pi_1(B) \to \mathrm{U}(V_\rho)$, the corresponding flat connection Laplacians on (X_0, g_0) and (X_1, g_1) have identical spectra, including multiplicities. These equalities hold on every pair of corresponding connected finite covers, for every unitary flat twist on those covers.

If $\mathrm{Hess}_{g_B} f$ *is nondegenerate at one point of B, then*

$$\mathrm{Hol}^0(X_i, g_i) = \mathrm{SO}(b + d), \qquad i = 0, 1.$$

Under this Hessian hypothesis, their universal Riemannian covers have no nontrivial de Rham product decomposition. If B is orientable, their full holonomy groups are also $\mathrm{SO}(b + d)$. If the Hessian is nondegenerate on a dense subset of B, no nonempty open subset of either total manifold is a nontrivial Riemannian product.

Proof. Let $\pi_i : X_i \to B$ be the projection, and let (E_ρ, ∇^ρ) be the Hermitian flat bundle of ρ. Put $\Delta_{B,\rho} = (\nabla^\rho)^* \nabla^\rho$, where the adjoint is taken with respect to g_B and dv_B. Since F_i is simply connected, every unitary flat bundle on X_i is isomorphic, with its connection, to a pullback from B under the marking induced by π_i.

Choose a unitary map $S : L^2(F_0, dv_{h_0}) \to L^2(F_1, dv_{h_1})$ which maps each scalar eigenspace onto the eigenspace with the same eigenvalue. The volume density of g_i is $f^d \, dv_B \, dv_{h_i}$. Hence

$$L^2(X_i, \pi_i^* E_\rho; dv_{g_i}) \simeq L^2(B, E_\rho; f^d dv_B) \widehat{\otimes} L^2(F_i, dv_{h_i}),$$

where $\widehat{\otimes}$ is the Hilbert tensor product, and $U_\rho = I \otimes S$ is unitary between these spaces. For the unwarped product metrics $g_B + h_i$, the same map is unitary and intertwines the product connection Laplacians

$$P_i = \Delta_{B,\rho} \otimes I + I \otimes \Delta_{F_i}.$$

The spectral Sobolev norms of $1 + P_i$ therefore give

$$U_\rho H^{2k}(X_0, \pi_0^* E_\rho) = H^{2k}(X_1, \pi_1^* E_\rho), \qquad k \in \mathbb{Z}_{\geq 0},$$

and U_ρ maps smooth sections onto smooth sections. The elliptic estimates and the spectral theorem apply to these Hermitian bundles [38, Theorem 16.7 and §17.1.1, pp. 216 and 230–231].

For the weighted base measure $f^d dv_B$, the horizontal operator is

$$f^{-d} (\nabla^\rho)^* (f^d \nabla^\rho) = \Delta_{B,\rho} - d \nabla^\rho_{\mathrm{grad}_{g_B} \log f}.$$

The vertical inverse metric is $f^{-2} h_i^{-1}$, so on smooth sections

$$\Delta_{X_i,\rho} = \Delta_{B,\rho} - d \nabla^\rho_{\mathrm{grad}_{g_B} \log f} + f^{-2} \Delta_{F_i}. \tag{13.1}$$

The base derivatives and multiplication by base functions commute with U_ρ. Thus U_ρ intertwines these differential expressions. On each compact manifold, the Sobolev spaces are independent of the smooth metric. The self-adjoint domains of the warped elliptic connection Laplacians are H^2 [38, §17.1.1 and §17.3, pp. 231 and 241]. Consequently

$$U_\rho \Delta_{X_0,\rho} U_\rho^{-1} = \Delta_{X_1,\rho},$$

with equality of domains.

Let $L \leq \pi_1(B)$ have finite index, and let $p_L : B_L \to B$ be the corresponding connected cover. The corresponding cover of X_i is $B_L \times F_i$, with metric $p_L^* g_B + (f \circ p_L)^2 h_i$. The preceding argument applies on B_L to every finite-dimensional unitary representation of L.

Fix i, write $g = g_i$, and let ∇ be its Levi–Civita connection. Choose $p \in B$ where the Hessian is nondegenerate and $q \in F_i$. At (p, q), let H and V be the horizontal and vertical tangent subspaces, with their inner products induced by g. Define the invertible self-adjoint map $A : H \to H$ by

$$g_B(AX, Y) = f(p)^{-1} \operatorname{Hess}_{g_B} f(X, Y).$$

For tangent vectors X, U, Z, put $(X \wedge U)Z = g(U, Z)X - g(X, Z)U$, and take the convention

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

The warped curvature formulas [16, Lemma 7.4(2),(4)], translated to this convention, give, for $X, Y \in H$ and $U, W \in V$,

$$R(X, U)Y = g_B(AX, Y)U, \qquad R(X, U)W = -g(U, W)AX.$$

Hence

$$R(X, U) = -(AX) \wedge U. \tag{13.2}$$

By the Ambrose–Singer theorem [5, §2, Theorem 2 and Lemma 4, pp. 438 and 441], these curvature endomorphisms belong to the restricted holonomy Lie algebra. Since A is onto, that algebra contains every $X \wedge U$ with $X \in H$ and $U \in V$. Choose orthonormal bases $e_1, \dots, e_b$ of H and $u_1, \dots, u_d$ of V. The identities

$$[e_a \wedge u_j, e_c \wedge u_j] = -e_a \wedge e_c, \qquad [e_a \wedge u_j, e_a \wedge u_k] = -u_j \wedge u_k$$

give all horizontal and vertical skew maps as well. The Lie algebra is therefore $\mathfrak{so}(b + d)$, and connectedness of restricted holonomy gives $\operatorname{Hol}^0(X_i, g_i) = \mathrm{SO}(b + d)$. The universal Riemannian cover has this full holonomy group. A nontrivial de Rham factor would give a nonzero proper invariant tangent subspace, contrary to irreducibility of the standard $\mathrm{SO}(b + d)$ representation. The simply connected fibers are orientable. If B is orientable, so is X_i; its full holonomy is contained in $\mathrm{SO}(b + d)$ and hence equals it.

If the Hessian is nondegenerate on a dense subset of B, every nonempty open subset of X_i contains a point at which the curvature endomorphisms generate $\mathfrak{so}(b + d)$. The tangent subspaces to the factors of a Riemannian product are parallel and invariant under all curvature endomorphisms. The same irreducibility excludes a nontrivial product on that open subset. □

Lemma 13.2. *Let (B, g_B) be a closed Riemannian manifold of positive dimension. There are positive smooth functions $f : B \to \mathbb{R}$ with a nondegenerate Hessian at some point, arbitrarily close to 1 in C^∞. If B and g_B are real analytic, f may be chosen real analytic with its Hessian nondegenerate on a dense open subset.*

Proof. Suppose first that B is connected. In a normal coordinate ball centered at p, choose a smooth function ψ with compact support which equals $1 - \sum_{a=1}^{\dim B} (x^a)^2$ near p. Extend it by zero to B. Then $\operatorname{Hess}_{g_B} \psi(p) = -2g_B(p)$. For sufficiently small $\varepsilon > 0$, the function $f = 1 + \varepsilon\psi$ is positive and has a nondegenerate Hessian at p. It tends to 1 in C^∞ as $\varepsilon \downarrow 0$.

If B and g_B are real analytic, the real Laplace eigenfunctions are analytic by elliptic analytic regularity [22, Theorem 1.1, p. 2], applied to $\Delta_B - \lambda$ for each eigenvalue λ. The spectral partial sums of ψ converge in every Sobolev space, and hence in C^2 [38, Theorems 12.16, 16.7 and 17.1]. Choose one such sum ϕ for which $\operatorname{Hess}_{g_B} \phi(p)$ is nondegenerate, and put $f = 1 + \varepsilon\phi$. For sufficiently small $\varepsilon > 0$, this function is positive and analytic. The analytic function $\det(g_B^{-1} \operatorname{Hess}_{g_B} f)$ is nonzero at p; its nonzero set is therefore open and dense in the connected manifold B. With ϕ fixed, $f \to 1$ in C^∞ as $\varepsilon \downarrow 0$. If B is disconnected, apply the construction on its finitely many components, with a common sufficiently small ε. □

Corollary 13.3. *Let A_7, B_7 be the simply connected scalar-isospectral seven-manifolds of Theorem 12.1, with the metrics specified there. For every closed connected positive-dimensional Riemannian manifold (B, g_B), there are warped metrics on $B \times A_7$ and $B \times B_7$ with restricted holonomy $\mathrm{SO}(\dim B + 7)$ and equal marked unitary flat spectra, on the manifolds and on every pair of corresponding connected finite covers.*

If $\mathrm{Ric}_{g_B} > 0$*, the common warping function may be chosen so that both total Ricci tensors are positive. Their second Betti numbers are*

$$b_2(B \times A_7) = b_2(B) + 1, \qquad b_2(B \times B_7) = b_2(B) + 2.$$

They are therefore not homotopy equivalent. The product $B \times B_7$ *is not rationally formal; if* B *is rationally formal, then* $B \times A_7$ *is rationally formal. If* B *is aspherical, the universal covers have second Betti numbers* 1 *and* 2*, respectively, so the two manifolds have no common connected Riemannian cover. Their scalar curvatures are pullbacks of the same function on* B*.*

For $B = S^1$*, the dimension is eight and the fundamental group is* $\mathbb{Z}$*. For an oriented closed surface* $B = \Sigma_g$*,* $g \geq 2$*, the dimension is nine, the full holonomy group is* $\mathrm{SO}(9)$*, the fundamental group is* $\pi_1(\Sigma_g)$*, and the second Betti numbers are* 2 *and* 3*. The product* $\Sigma_g \times A_7$ *is rationally formal, whereas* $\Sigma_g \times B_7$ *is not. If the surface metric is hyperbolic and the warping function is chosen real analytic as in Lemma 13.2, neither total metric is a nontrivial Riemannian product on any nonempty open subset.*

Proof. Put $F_0 = A_7$, $F_1 = B_7$, and denote their metrics by h_0, h_1. Choose f as in Lemma 13.2, and put $g_i = g_B + f^2 h_i$. Theorem 13.1 gives the holonomy and spectral equalities. Suppose that $\mathrm{Ric}_{g_B} > 0$. The product metrics $g_i^0 = g_B + h_i$ have positive Ricci curvature by Theorem 12.1. Compactness gives $c > 0$ such that $\mathrm{Ric}_{g_i^0} \geq c g_i^0$ for $i = 0, 1$. By continuity of the Ricci tensor in the C^2 topology, for f sufficiently close to 1 in C^2 one has $\mathrm{Ric}_{g_B + f^2 h_i} \geq (c/2) g_i^0 > 0$ for both i. Such functions, with the required Hessian condition, exist by Lemma 13.2. The Betti numbers and formality statements follow from Theorem 12.4.

If B is aspherical, its universal cover $\widetilde{B}$ is contractible. The universal covers $\widetilde{B} \times A_7$ and $\widetilde{B} \times B_7$ then have second Betti numbers 1 and 2. A common connected cover would have a universal cover homeomorphic to both, which is impossible.

The homogeneous fiber metrics have constant scalar curvature. Their equal scalar spectra give equal volumes and equal integrated scalar curvatures [47, Theorem A.1(a),(c), p. 239], hence the same constant scalar curvature σ. The warped curvature formulas [16, Lemma 7.4], with $\Delta_B = -\operatorname{tr}_{g_B} \mathrm{Hess}_{g_B}$, give

$$\mathrm{Scal}_{g_i} = \mathrm{Scal}_{g_B} + f^{-2}\sigma + 14 f^{-1} \Delta_B f - 42 f^{-2} |df|^2_{g_B}.$$

This expression is a base function independent of i. The circle is formal. Corollary 12.6 gives the formality of Σ_g, and $b_2(\Sigma_g) = 1$. The circle and the surfaces of genus at least two are aspherical, giving the stated specializations. □

Remark 13.4. In Theorem 13.1, a smooth warping function with a nondegenerate Hessian at one point may be constant on a nonempty open set $U \subset B$. The metric on $U \times F_i$ is then a product, although the restricted holonomy on X_i is $\mathrm{SO}(b + d)$. On a circle of length $L > 0$, with arclength coordinate $s \in \mathbb{R}/L\mathbb{Z}$, the function

$$f(s) = 1 + \varepsilon \cos(2\pi s/L), \qquad 0 < \varepsilon < 1,$$

has nondegenerate Hessian except at $s = L/4, 3L/4$ modulo L. It therefore gives the exclusion of product open subsets in Theorem 13.1.

Part IV. Finite holonomy and least lengths

For a closed connected Riemannian manifold X of positive dimension, marked finite-holonomy spectra determine the deck heat traces on every finite regular cover. After division by the deck-group orders, these traces define central heat measures on the profinite completion; the finite-quotient minima have supremum equal to the minimum of ℓ_X on each nonempty profinite conjugacy fiber (Theorems 14.4 and 14.2). If the fundamental group is conjugacy separable, every nontrivial class minimum is recovered and is a singularity of some finite-holonomy flat half-wave trace (Proposition 15.5). For the C_7-pair of Theorem 7.7, these minima agree while the full period sets differ.

14. Profinite heat measures and the conjugacy envelope

Let X be a closed connected Riemannian manifold of positive dimension with a fixed marking, and put $\Pi = \pi_1(X)$. For each finite quotient $q : \Pi \twoheadrightarrow Q$, retain Y_q, $H_{q,g}$ and ℓ_q from Section 5. The finite quotients are indexed by the normal finite-index subgroups of Π, ordered by reverse inclusion. For a finite-dimensional unitary representation $\rho : \Pi \to \mathrm{U}(V_\rho)$ on a Hermitian space V_ρ, put $\chi_\rho = \operatorname{Tr}\rho$ and, for $t > 0$,

$$\Theta_X(t) = \operatorname{Tr}(e^{-t\Delta_X}), \qquad \Theta_{X,\rho}(t) = \operatorname{Tr}(e^{-t\Delta_{X,\rho}}).$$

Write $K = \widehat{\Pi}$ for the profinite completion, $\iota : \Pi \to K$ for its canonical map, and $\widehat{q} : K \to Q$ for the continuous extension of q. The group K is compact and metrizable, since Π is finitely generated.

Let $C(\Pi)$ be the set of conjugacy classes in Π, and write $[\gamma]_\Pi$ for the class of $\gamma \in \Pi$ and $c^{-1} = \{\gamma^{-1} : \gamma \in c\}$ for $c \in C(\Pi)$. The class functions ℓ_X and χ_ρ also denote the corresponding functions on $C(\Pi)$. For $c \in C(\Pi)$ and $\gamma \in c$, put

$$O_c = \mathrm{Conj}_K(\iota(\gamma)).$$

This orbit is independent of the representative γ. Put $c \approx d$ if $O_c = O_d$, and call an equivalence class for $\approx$ a *profinite conjugacy fiber*; write $[c]_\approx$ for the fiber containing c. The map ι need not be injective. For $z \in K$, let δ_z be the Dirac probability at z.

14.1. The exact conjugacy envelope.

Definition 14.1. For $z \in K$, define

$$\widehat{\ell}_X(z) = \sup_q \ell_q(\widehat{q}(z)) \in [0, \infty].$$

Theorem 14.2 (The profinite conjugacy envelope). *The function $\widehat{\ell}_X$ is lower semicontinuous, invariant under conjugacy and inversion, and zero precisely at 1. For every finite $R \geq 0$,*

$$\{z : \widehat{\ell}_X(z) \leq R\} = \bigcup_{\substack{c \in C(\Pi) \\ \ell_X(c) \leq R}} O_c, \tag{14.1}$$

a finite union of compact conjugacy orbits. Consequently, for $z \in K$ and $\gamma \in \Pi$,

$$\widehat{\ell}_X(z) = \min\{\ell_X(c) : c \in C(\Pi),\ z \in O_c\}, \tag{14.2}$$

$$\sup_q \ell_q(q(\gamma)) = \min\{\ell_X(\delta) : \delta \in \Pi,\ q(\delta) \sim_Q q(\gamma) \text{ for every finite quotient } q\}. \tag{14.3}$$

The minimum of an empty set is ∞. For every $\gamma \in \Pi$, a finite quotient attains the supremum in (14.3). *For every finite $R \geq 0$, one quotient attains it for all γ with $\ell_X(\gamma) \leq R$.*

Proof. Every continuous finite quotient of K restricts to a quotient of Π, since $\iota(\Pi)$ is dense. By [115, Lemma 2.1, arXiv version 2], two elements of K are conjugate if and only if their images are conjugate in every finite quotient. Each orbit is compact as the image of K under conjugation.

Every $\ell_q \circ \widehat{q}$ is locally constant and invariant under conjugacy and inversion. Their supremum is lower semicontinuous. A nonidentity z survives in a finite quotient, whose nonidentity deck minimum is positive. Thus $\widehat{\ell}_X$ vanishes precisely at 1. By (5.6), every orbit on the right of (14.1) lies in its left side. Conversely, suppose $\widehat{\ell}_X(z) \leq R$. The set $S_R = \{c \in C(\Pi) : \ell_X(c) \leq R\}$ is finite by Lemma 5.3. If z belongs to none of the orbits O_c, $c \in S_R$, choose a finite quotient separating it from each such orbit and take a common refinement. By (5.6), its deck minimum at $\widehat{q}(z)$ exceeds R, a contradiction. This proves (14.1). Taking $R = \widehat{\ell}_X(z)$ when finite proves (14.2); if it is infinite, z belongs to no O_c. Profinite conjugacy gives (14.3).

For $\gamma \in \Pi$, put $a = \widehat{\ell}_X(\iota(\gamma)) \leq \ell_X(\gamma)$. If $a > 0$, separate γ simultaneously from the finitely many classes of minimum length less than a. The resulting quotient has deck minimum at least a and, by definition of the supremum, at most a. If $a = 0$, every quotient already attains it. Refinement cannot decrease a deck minimum, by (5.6); a common refinement therefore works for every class in the finite set S_R. □

A class $c \in C(\Pi)$ is *conjugacy distinguished* if $[c]_{\approx} = \{c\}$. Equivalently, every $\gamma \in c$ and $\delta \notin c$ have nonconjugate images in some finite quotient. The group Π is *conjugacy separable* if every class has this property.

Corollary 14.3. *For a fixed metric, $c \in C(\Pi)$ and $\gamma \in c$, one has $\widehat{\ell}_X(\iota(\gamma)) = \ell_X(c)$ exactly when $\ell_X(c) = \min_{d \approx c} \ell_X(d)$. Equality holds for every class exactly when ℓ_X is constant on each profinite conjugacy fiber; in particular it holds if Π is conjugacy separable.*

Proof. Apply (14.2) on each fiber. Conjugacy separability makes every fiber a singleton. □

14.2. **Central heat measures and their class masses.** A measure on K is *central* if it is invariant under conjugation.

Theorem 14.4. *For every $t > 0$ there is a unique positive finite central Borel measure $\lambda_{X,t}$ on K with*

$$(\widehat{q})_* \lambda_{X,t}(\{g\}) = \frac{H_{q,g}(t)}{|Q|}. \tag{14.4}$$

Its mass is $\Theta_X(t)$. The probability $\nu_{X,t} = \lambda_{X,t}/\Theta_X(t)$ is invariant under conjugacy and inversion. For every finite-image unitary representation ρ of Π, regarded also as a continuous representation of K,

$$\int_K \overline{\chi_\rho(z)}\, d\nu_{X,t}(z) = \frac{\Theta_{X,\rho}(t)}{\Theta_X(t)}. \tag{14.5}$$

The complete marked finite-image flat spectra are equivalent to $(\lambda_{X,t})_{t>0}$, or to $(\Theta_X, (\nu_{X,t})_{t>0})$.

Proof. Let $q' : \Pi \twoheadrightarrow Q'$ refine q, and let $p : Q' \twoheadrightarrow Q$ satisfy $q = p \circ q'$. Put $C = \ker p$. Pullback under $Y_{q'} \to Y_q$, multiplied by $|C|^{-1/2}$, identifies $L^2(Y_q)$ unitarily with $L^2(Y_{q'})^C$. The finite-cover trace identity in the proof of Theorem 5.4, applied to $Y_{q'} \to Y_q$, gives

$$H_{q,g}(t) = \frac{1}{|C|} \sum_{p(h)=g} H_{q',h}(t). \tag{14.6}$$

The trivial-character case of (5.2) gives $|Q|^{-1} \sum_g H_{q,g}(t) = \Theta_X(t)$. Thus

$$\nu_{q,t}(\{g\}) = \frac{H_{q,g}(t)}{|Q|\Theta_X(t)}$$

defines compatible probabilities on the finite quotients. Apply the projective measure theorem [46, 418M]: the projections $\widehat{q} : K \to Q$ are continuous, onto, compatible, and separate points; the compact set K has full image measure in every quotient. The resulting Radon probability, restricted to the Borel sets, is $\nu_{X,t}$. Every finite Borel measure on the compact metrizable space K is regular, so uniqueness holds among Borel measures. Set $\lambda_{X,t} = \Theta_X(t)\nu_{X,t}$. Its invariance under conjugacy and inversion follows from the corresponding invariances of $H_{q,g}$.

Equation (14.5) is (5.2). The spectrum of the trivial representation gives $\Theta_X(t)$. Finite-group character inversion determines every central pushforward of $\nu_{X,t}$ from these moments, hence determines $\lambda_{X,t} = \Theta_X(t)\nu_{X,t}$. Conversely the measures determine every twisted heat trace. Lemma 2.3 determines its spectrum: the union of two candidate spectra is locally finite in $[0, \infty)$, and convergence of their heat traces at any $t_0 > 0$ gives the required absolute summability of the multiplicity difference. □

Remark 14.5. For an irreducible finite-image unitary representation ρ, Schur's lemma and (14.5) give

$$\int_K \rho(z^{-1})\, d\nu_{X,t}(z) = \frac{\Theta_{X,\rho}(t)}{\dim \rho\, \Theta_X(t)} \mathrm{Id}\,.$$

If $K = \{1\}$, then $\nu_{X,t} = \delta_1$ for every metric. Thus the family $(\nu_{X,t})_{t>0}$ alone need not determine Θ_X.

14.2.1. *Holonomy and orbital decomposition.* The vector-bundle Feynman–Kac formula [88, Section 8, equation (34), p. 204], with zero potential and time parameter $2t$, gives the flat holonomy identity for $e^{-t\Delta}$. For a continuous nonnegative killing rate $k \not\equiv 0$, Le Jan's integrated character identity for flat connections on trivial bundles is [71, §3, proof of Proposition 5 and final display]. There the characters and twisted zeta functions are divided by the representation dimension, and the generator is $-\Delta/2 - k$. Here $\chi_\rho = \operatorname{Tr}\rho$ and $\Theta_{X,\rho}$ is the heat trace on the associated flat bundle of rank $\dim\rho$. For graph covers and finite-quotient approximation of delocalized traces, see [70, Sections 4.1 and 4.3] and [112, Lemmas 2.4 and 2.6].

Let dv_X be Riemannian volume on X, and let $K_X(t,x,y)$ be its scalar heat kernel. For $t>0$ and $x \in X$, let $Q_t^{x,x}$ be the unnormalized Brownian bridge of duration t for generator $-\Delta_X$, reparametrized on $[0,1]$; its mass is $K_X(t,x,x)$. For a continuous loop ω, let $[\omega] \in C(\Pi)$ be its free homotopy class. For $c \in C(\Pi)$, put

$$b_c(t) = \int_X Q_t^{x,x}\{\omega : [\omega] = c\}\, dv_X(x).$$

In the uniform topology on the continuous loop space, each class is open: loops at uniform distance less than the injectivity radius are freely homotopic by pointwise minimizing geodesics. These classes form a countable Borel partition. Hence

$$b_c(t) \geq 0, \qquad \sum_{c\in C(\Pi)} b_c(t) = \Theta_X(t) < \infty. \tag{14.7}$$

Let $\widetilde{X}$ be the universal Riemannian cover of X, $dv_{\widetilde{X}}$ its Riemannian volume, and $\widetilde{K}$ its heat kernel. Let $F \subset \widetilde{X}$ be a measurable fundamental domain for its deck group Π. Lifting the bridges gives

$$b_c(t) = \int_F \sum_{\gamma\in c} \widetilde{K}(t,x,\gamma x)\, dv_{\widetilde{X}}(x).$$

All terms are nonnegative, so Tonelli's theorem applies. For closed oriented hyperbolic surfaces and flat two-tori, the integrals $\int_0^\infty b_c(t)\,dt/t$ for nontrivial c are computed in [113, arXiv version 3, Lemmas 3.2 and 3.3].

Let m_K be Haar probability on K. For $c \in C(\Pi)$, choose $\gamma \in c$, and let m_c be the pushforward of m_K under $k \mapsto k\iota(\gamma)k^{-1}$. Invariance of Haar measure makes m_c independent of γ and gives $m_c = m_d$ whenever $c \approx d$. For $B \in C(\Pi)/\!\approx$, denote this common measure by m_B and put $B_X(B,t) = \sum_{c\in B} b_c(t)$ for $t>0$.

Proposition 14.6. *One has the total-variation convergent positive decomposition*

$$\lambda_{X,t} = \sum_c b_c(t) m_c = \sum_{B\in C(\Pi)/\approx} B_X(B,t) m_B. \tag{14.8}$$

The measure $\lambda_{X,t}$ gives full measure to $\bigcup_{c\in C(\Pi)} O_c$. The complete marked finite-image flat spectra are equivalent to all $B_X(B,t)$; if c is conjugacy distinguished, they recover $b_c(t)$.

Proof. Let ρ have finite image. Under (5.1), backward parallel transport along a loop whose lift starts at $x \in \widetilde{X}$ and ends at γx is $\rho(\gamma)$. Feynman–Kac and flatness therefore give

$$\Theta_{X,\rho}(t) = \sum_c b_c(t)\chi_\rho(c) = \sum_c b_c(t)\overline{\chi_\rho(c)}.$$

The second equality follows from time reversal, $b_c(t) = b_{c^{-1}}(t)$, and unitarity of ρ. Both sums are absolute by $|\chi_\rho| \leq \dim\rho$ and (14.7). The sum $\sum_c b_c(t)m_c$ converges in total variation by (14.7). Its character moments equal those of $\lambda_{X,t}$, so Theorem 14.4 gives (14.8). Distinct compact conjugacy orbits are disjoint; thus $\lambda_{X,t}(O_c) = B_X([c]_\approx, t)$. Conversely, these masses determine $\lambda_{X,t}$ by (14.8). A conjugacy-distinguished class has a singleton fiber. □

Proposition 14.7. *For $c \in C(\Pi)$, let $q(c) \subset Q$ be its image conjugacy class and put $A_{q,c}(t) = |Q|^{-1} \sum_{g \in q(c)} H_{q,g}(t)$. Let $(q_n)_{n\geq 1}$ be finite quotients with $\ker q_{n+1} \subseteq \ker q_n$ such that every finite quotient factors through some q_n. Then $A_{q_n,c}(t) \downarrow B_X([c]_{\approx}, t)$ for every $t > 0$. For every finite $R \geq 0$, there is a finite quotient q such that, for every c with $\ell_X(c) \leq R$,*

$$0 \leq A_{q,c}(t) - B_X([c]_{\approx}, t) \leq \sum_{\ell_X(d)>R} b_d(t) \qquad (t > 0). \tag{14.9}$$

Proof. An orbital probability pushes forward uniformly to its finite conjugacy orbit. Hence $A_{q,c}(t) = \sum_{q(d)=q(c)} b_d(t)$. Refinement decreases these sets to $[c]_{\approx}$, by profinite conjugacy; continuity from above for the finite measure (14.7) proves the limit. Such sequences exist because a finitely generated group has finitely many subgroups of each bounded index. To prove the tail bound, separate every pair of classes of minimum at most R lying in different profinite fibers in one common finite quotient. Finiteness of these classes leaves only terms of length greater than R in the difference. □

14.3. **The geodesic rate and its functoriality.**

Theorem 14.8 (The geodesic rate). *As $t \downarrow 0$, the probabilities $\nu_{X,t}$ satisfy a large-deviation principle on K with speed $1/t$ and good rate $I_X(z) = \widehat{\ell}_X(z)^2/4$. For open $O \subseteq K$ and closed $F \subseteq K$, with $\log 0 = -\infty$ and $\inf \emptyset = \infty$,*

$$\liminf_{t\downarrow 0} t \log \nu_{X,t}(O) \geq -\inf_O I_X, \qquad \limsup_{t\downarrow 0} t \log \nu_{X,t}(F) \leq -\inf_F I_X.$$

The set $\{z \in K : I_X(z) < \infty\}$ is $\bigcup_{c\in C(\Pi)} O_c$.

Proof. Put $\nu_{q,t} = (\widehat{q})_* \nu_{X,t}$ and $I_q(g) = \ell_q(g)^2/4$. Lemma 5.2, applied to the identity on X, gives $t \log \Theta_X(t) \to 0$. Equations (14.4) and (5.5) therefore give

$$\lim_{t\downarrow 0} t \log \nu_{q,t}(\{g\}) = -I_q(g).$$

For every nonempty $A \subseteq Q$,

$$\max_{g\in A} \nu_{q,t}(\{g\}) \leq \nu_{q,t}(A) \leq |A| \max_{g\in A} \nu_{q,t}(\{g\}).$$

Hence $\nu_{q,t}$ satisfies the large-deviation principle on the finite discrete space Q with good rate I_q. The Dawson–Gärtner projective-limit theorem [32, Theorem 3.3 and §3.1, pp. 257–258], applied to the compact Hausdorff inverse limit K, gives the good rate

$$\sup_q I_q(\widehat{q}(z)) = \frac{1}{4}\left(\sup_q \ell_q(\widehat{q}(z))\right)^2 = I_X(z).$$

Its finite locus is given by Theorem 14.2. □

14.3.1. *Epimorphisms.* For a one-point manifold X, set $\Pi = K = \{1\}$, $\Theta_X(t) = 1$, $\lambda_{X,t} = \nu_{X,t} = \delta_1$, and $\widehat{\ell}_X = I_X = 0$.

Theorem 14.9. *Let $p : \pi_1(M) \twoheadrightarrow \pi_1(N)$ be an epimorphism for closed connected smooth Riemannian manifolds. For a unitary representation ρ of $\pi_1(N)$, write $p^*\rho = \rho \circ p$. Suppose that, for every such ρ with finite image,*

$$\operatorname{Spec} \Delta_{M,p^*\rho} = \operatorname{Spec} \Delta_{N,\rho} \tag{14.10}$$

with multiplicities. The induced homomorphism $\widehat{p} : \widehat{\pi_1(M)} \to \widehat{\pi_1(N)}$ is onto. For $t > 0$ and $z \in \widehat{\pi_1(N)}$, one has

$$(\widehat{p})_* \nu_{M,t} = \nu_{N,t}, \qquad I_N(z) = \min_{\widehat{p}(w)=z} I_M(w), \qquad \widehat{\ell}_N(z) = \min_{\widehat{p}(w)=z} \widehat{\ell}_M(w). \tag{14.11}$$

If the target group is conjugacy separable, then, for every $\gamma \in \pi_1(N)$,

$$\ell_N(\gamma) = \min_{p(e)=\gamma} \ell_M(e), \tag{14.12}$$

and the minimum is attained.

Proof. If either manifold is a point, equality of the scalar heat traces and the small-time heat asymptotic imply that both are points; the conclusions follow from the stated conventions. Suppose henceforth that both have positive dimension. Write ι_M and ι_N for the canonical maps to their respective profinite completions. The image of $\widehat{p}$ is compact and contains the dense image of $\pi_1(N)$, so $\widehat{p}$ is onto. The trivial representation in (14.10) gives $\Theta_M = \Theta_N$. Since $\widehat{p}$ is onto, $(\widehat{p})_* \nu_{M,t}$ is central. Equation (14.5) and finite-group character inversion therefore give $(\widehat{p})_* \nu_{M,t} = \nu_{N,t}$. For every finite quotient $q : \pi_1(N) \twoheadrightarrow Q$, let superscripts M and N specify the manifold in H and ℓ. The same inversion and (5.5) give, for $g \in Q$,

$$H^M_{q\circ p,g}(t) = H^N_{q,g}(t), \qquad \ell^M_{q\circ p}(g) = \ell^N_q(g). \tag{14.13}$$

By the contraction principle [32, Theorem 3.2, p. 258], the image probabilities have good rate $J(z) = \min_{\widehat{p}(w)=z} I_M(w)$. The map $\widehat{p}$ is continuous between compact Hausdorff spaces, and I_M is good by Theorem 14.8. Uniqueness of the rate function [32, §3.1, p. 257] gives $J = I_N$. Every fiber is nonempty and compact; lower semicontinuity gives attainment. Taking nonnegative square roots gives the length identity in (14.11).

Suppose $\pi_1(N)$ is conjugacy separable, and let $\gamma \in \pi_1(N)$. Corollary 14.3 gives $\widehat{\ell}_N(\iota_N(\gamma)) = \ell_N(\gamma) < \infty$. Choose a minimizing w above $\iota_N(\gamma)$. By (14.2), there is $e \in \pi_1(M)$ such that w is conjugate to $\iota_M(e)$ and $\ell_M(e) = \widehat{\ell}_M(w) = \ell_N(\gamma)$. Thus $p(e)$ is profinitely conjugate to γ, hence conjugate to it in $\pi_1(N)$. Choose $v \in \pi_1(N)$ with $vp(e)v^{-1} = \gamma$, and choose $u \in \pi_1(M)$ with $p(u) = v$. Then $p(ueu^{-1}) = \gamma$ and $\ell_M(ueu^{-1}) = \ell_N(\gamma)$. For every e' with $p(e') = \gamma$, (14.11) gives

$$\ell_N(\gamma) \le \widehat{\ell}_M(\iota_M(e')) \le \ell_M(e').$$

This proves (14.12). □

Corollary 14.10. *In Theorem 10.4, put $M_* = M_{[r]}$. For each prescribed $p_I : E/A_I \twoheadrightarrow \Gamma$ and every $\gamma \in \Gamma$,*

$$\ell_{M_*}(\gamma) = \min_{p_I(u)=\gamma} \ell_{M_I}(u).$$

The minimum is attained, even when E/A_I is infinite. The same holds on the corresponding covers for each $L \le \Gamma$ and $\gamma \in L$, with source group $p^{-1}(L)/A_I$ and target L.

Proof. Theorem 10.4(4) gives the required pulled-back flat spectral equalities for p_I and for its restrictions to the indicated covering groups. Their targets Γ and L are finite, so Theorem 5.4 applies. □

14.4. **Limits and a circle example.** As $t \downarrow 0$, $\nu_{X,t}$ converges weakly to δ_1; as $t \to \infty$, it converges weakly to m_K. In a fixed finite quotient, the nonidentity masses vanish exponentially at zero by (5.5). At infinity, the constant eigenfunction on connected Y_q gives $H_{q,g}(t) \to 1$, while $\Theta_X(t) \to 1$; the finite pushforward therefore tends to uniform probability. The functions factoring through finite quotients are uniformly dense in the continuous functions on K, so these finite-quotient limits give both weak limits on K.

For Borel probabilities μ, η on K, write $d_{\mathrm{TV}}(\mu,\eta) = \sup_A |\mu(A) - \eta(A)|$, where A ranges over Borel subsets of K.

Proposition 14.11. *For a circle of circumference $L > 0$, with $\Pi = \mathbb{Z} \subset \widehat{\mathbb{Z}}$,*

$$\nu_{X,t} = \frac{\sum_{k\in\mathbb{Z}} e^{-L^2k^2/(4t)}\delta_k}{\sum_{k\in\mathbb{Z}} e^{-L^2k^2/(4t)}}. \tag{14.14}$$

Its rate is $L^2k^2/4$ on integers and infinite elsewhere. For every $t > 0$, $d_{\mathrm{TV}}(\nu_{X,t}, m_K) = 1$, although $\nu_{X,t}$ converges weakly to m_K as $t \to \infty$. For the quotient $q_m : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{Z}_{\ge 1}$, put $\ell_m(k) = \ell_{q_m}(q_m(k))$ for $k \in \mathbb{Z}$. Then

$$\ell_m(k) = L \min_{r\in\mathbb{Z}} |k + rm|.$$

For every $N \in \mathbb{Z}_{\ge 1}$,

$$\big(\ell_m(k) = L|k| \text{ for all } |k| \le N\big) \quad \Longleftrightarrow \quad m \ge 2N.$$

No finite quotient has $\ell_m(k) = L|k|$ for all $k \in \mathbb{Z}$.

Proof. For $j \in \mathbb{Z}/m\mathbb{Z}$, put $H_{m,j} = H_{q_m,j}$. Periodization on the covering circle of length mL gives

$$H_{m,j}(t) = \frac{mL}{\sqrt{4\pi t}} \sum_{k \equiv j\,(m)} e^{-L^2k^2/(4t)}.$$

Divide by $m\Theta_X(t)$ to obtain every finite pushforward of (14.14). The envelope theorem gives its rate. The measure in (14.14) is concentrated on the countable subset $\mathbb{Z}$, which is Haar-null in the infinite compact group $\widehat{\mathbb{Z}}$. Hence $d_{\mathrm{TV}}(\nu_{X,t}, m_K) = 1$.

The deck formula is the least distance between translates on the covering circle. It equals $L|k|$ for every $|k| \le N$ when $m \ge 2N$. If $m < 2N$, a shortest representative of N modulo m has absolute value at most $m/2 < N$. Since every finite quotient of $\mathbb{Z}$ is cyclic, the least possible degree for simultaneous attainment when $|k| \le N$ is $2N$. □

Remark 14.12. The first positive scalar eigenvalue is $(2\pi/L)^2$, so the scalar spectrum determines L and all marked lengths $L|k|$.

15. Class minima and equivariant wave singularities

For a marked closed connected smooth manifold X, put $\Pi = \pi_1(X)$, and let $\iota : \Pi \to \widehat{\Pi}$ be the canonical map. For each Riemannian metric g, retain the functions $\ell_{(X,g)}$ and $\widehat{\ell}_{(X,g)}$ from Section 14.

15.1. **A criterion valid for every metric.**

Theorem 15.1 (Separation of unoriented classes). *For a fixed closed connected smooth manifold X of dimension at least three, the following are equivalent:*

(1) for all $c, d \in \Pi$, if $\iota(c)$ and $\iota(d)$ are conjugate in $\widehat{\Pi}$, then d is conjugate in Π to c or c^{-1};

(2) for every smooth metric g and every $c \in \Pi$, $\widehat{\ell}_{(X,g)}(\iota(c)) = \ell_{(X,g)}(c)$.

Proof. Suppose (1) holds, and fix $c \in \Pi$ and a smooth metric g. Every $d \in \Pi$ whose image is conjugate to $\iota(c)$ is conjugate to c or c^{-1}, and hence $\ell_{(X,g)}(d) = \ell_{(X,g)}(c)$. Equation (14.3) gives (2).

Conversely, suppose (2) holds. If $c \neq 1$ belongs to $\ker \iota$, then $\widehat{\ell}_{(X,g)}(\iota(c)) = 0 < \ell_{(X,g)}(c)$ by Lemma 5.3, a contradiction. Thus ι is injective. If (1) fails, choose nonidentity $c, d \in \Pi$ whose images are conjugate but whose unoriented conjugacy classes are distinct. Put $S = \{c, d\}$, $m(c) = 1$ and $m(d) = 3/2$. The prescription theorem [36, Theorem 5.1] applies when

$$m(x) \le |k| m(y) \qquad (x, y \in S,\ k \in \mathbb{Z},\ x \sim_\Pi y^k).$$

For $x \neq y$, such a relation implies $|k| \ge 2$; for $x = y$, it implies $|k| \ge 1$. The displayed inequalities therefore hold. There is a smooth metric g with $\ell_{(X,g)}(c) = 1$ and $\ell_{(X,g)}(d) = 3/2$. Equation (14.3) then gives $\widehat{\ell}_{(X,g)}(\iota(d)) \le 1$, contrary to (2). □

Remark 15.2 (Failure of a triangle inequality). Take $X = \mathbb{RP}^2 \times \mathbb{RP}^3$ and write $\Pi = \pi_1(X) = \{1, a, b, ab\} \simeq (\mathbb{Z}/2)^2$. Identify $\widehat{\Pi}$ with Π by ι. Distinct nonidentity elements are not powers of one another. The prescription theorem [36, Theorem 5.1] therefore gives a smooth metric with $\ell_X(a) = \ell_X(b) = 1$ and $\ell_X(ab) = 3$. Since Π is finite, (14.3) gives $\widehat{\ell}_X(a) = \widehat{\ell}_X(b) = 1$ and $\widehat{\ell}_X(ab) = 3$. Thus $\widehat{\ell}_X(ab) > \widehat{\ell}_X(a) + \widehat{\ell}_X(b)$.

15.2. **The first singularity of a deck cosine trace.**

Theorem 15.3 (Nonidentity deck minima). *Let a finite group Γ act freely by isometries on a closed connected Riemannian manifold Z of positive dimension. Put $X = \Gamma\backslash Z$ and $R_\gamma u(z) = u(\gamma^{-1}z)$. For $\gamma \in \Gamma \setminus \{1\}$, set $\ell = \ell_Z(\gamma) = \min_{x \in Z} d_Z(x, \gamma x) > 0$. It is the attained minimum of the lengths of loops on X whose deck monodromy is conjugate to γ, and is represented by a smoothly closed geodesic. With the spectral coefficients $a_\lambda(\gamma)$ of Section 5, define the tempered distribution*

$$C_\gamma(t) = \sum_\lambda a_\lambda(\gamma) \cos(t\sqrt{\lambda}) = \mathrm{Tr}\big(R_\gamma \cos(t\sqrt{\Delta_Z})\big).$$

Then C_γ is singular at both ℓ and $-\ell$. The deck half-wave trace D_γ and some W_ρ, $\rho \in \mathrm{Irr}(\Gamma)$, are singular at ℓ. For $s > 0$, the heat trace $H_\gamma(s) = \mathrm{Tr}(R_\gamma e^{-s\Delta_Z})$ has exponential rate (5.5)*, also for $\gamma = 1$ with $\ell_Z(1) = 0$.*

Proof. Lemmas 5.3 and 5.2 identify ℓ with the minimum of the corresponding free homotopy lengths and give the heat bounds (5.4). Finite propagation for the cosine operator [74, §3, preceding equation (3.6)] gives

$$C_\gamma = 0 \quad \text{on } (-\ell, \ell). \tag{15.1}$$

For $\phi \in C_c^\infty((-\ell, \ell))$, the operator $\int \phi(t)\cos(t\sqrt{\Delta_Z})\,dt$ is smoothing. Finite propagation makes its kernel zero on the deck graph, since $d_Z(x, \gamma x) \geq \ell$. Hence $\langle C_\gamma, \phi\rangle = 0$. The Gaussian formula [74, Theorem 2.1 and Lemma 2.4], applied to the nonnegative scalar Laplacian on the compact manifold Z, gives

$$H_\gamma(s) = (4\pi s)^{-1/2}\langle C_\gamma(t), e^{-t^2/(4s)}\rangle. \tag{15.2}$$

The Weyl bound in Section 5 justifies pairing the spectral series with the Gaussian.

Suppose C_γ were smooth near ℓ. Its spectral coefficients are real, and the cosine is even. Hence C_γ is real and even. Choose $\epsilon > 0$ such that it is smooth on $(-\ell - 2\epsilon, \ell + 2\epsilon)$. By (15.1), all its derivatives vanish at ℓ and $-\ell$. Choose an even $\eta \in C_c^\infty(\mathbb{R})$, equal to one on $[-\ell - \epsilon, \ell + \epsilon]$, with support in $(-\ell - 2\epsilon, \ell + 2\epsilon)$. For every integer $m \geq 0$, there is $A_m > 0$ such that $|\eta C_\gamma(t)| \leq A_m(|t| - \ell)^m$ for $|t| \geq \ell$. For some constants $A'_m, A''_m > 0$, independent of $0 < s \leq 1$, its contribution to (15.2) is bounded by

$$A'_m s^{-1/2} e^{-\ell^2/(4s)} \int_0^{2\epsilon} u^m e^{-\ell u/(2s)}\,du \leq A''_m s^{m+1/2} e^{-\ell^2/(4s)}.$$

For $T = (1 - \eta)C_\gamma$, choose $\beta \in C^\infty(\mathbb{R})$ with $\beta = 0$ for $|t| \leq \ell + \epsilon/2$ and $\beta = 1$ for $|t| \geq \ell + 3\epsilon/4$. Then $\beta = 1$ on a neighborhood of $\mathrm{supp}\, T$, so

$$\langle T, e^{-t^2/(4s)}\rangle = \langle T, \beta e^{-t^2/(4s)}\rangle.$$

Write $\mathcal{S}(\mathbb{R})$ for the Schwartz space. Temperedness gives fixed nonnegative integers a, b and $B > 0$ with

$$|\langle T, \phi\rangle| \leq B \sum_{j=0}^{b} \sup_{t\in\mathbb{R}} (1 + |t|)^a |\phi^{(j)}(t)| \qquad (\phi \in \mathcal{S}(\mathbb{R})).$$

Fix c' with $\ell^2 < c' < (\ell + \epsilon/2)^2$. There are nonnegative integers a', b' and constants $C, C' > 0$ such that, for $\phi = \beta e^{-t^2/(4s)}$ and $0 < s \leq 1$,

$$\sum_{j=0}^{b} \sup_{t\in\mathbb{R}} (1 + |t|)^a |\phi^{(j)}(t)| \leq C s^{-b'} \sup_{|t|\geq \ell+\epsilon/2} (1 + |t|)^{a'} e^{-t^2/(4s)} \leq C' s^{-b'} e^{-c'/(4s)}.$$

Consequently,

$$s^{-1/2}|\langle T, e^{-t^2/(4s)}\rangle| = O(s^N e^{-\ell^2/(4s)}) \qquad (N \in \mathbb{Z}_{\geq 0}).$$

Put $n = \dim Z$ and choose m, N with $m + 1/2 > n/2$ and $N > n/2$. The two estimates give $H_\gamma(s) = o(s^{n/2} e^{-\ell^2/(4s)})$, contrary to the lower bound $c s^{n/2} e^{-\ell^2/(4s)}$ in (5.4). Thus C_γ is singular at both endpoints. Finally $C_\gamma = (D_\gamma + \overline{D_\gamma})/2$, so smoothness of D_γ would imply smoothness of C_γ. Equation (5.3) then gives an irreducible representation ρ of Γ for which $\mathrm{Tr}(e^{it\sqrt{\Delta_{X,\rho}}})$ is singular at ℓ. □

Remark 15.4. If Z is the universal cover and Γ is finite, $\ell_Z(\gamma)$ is the minimum in the free homotopy class of $\gamma \in \pi_1(X)$. For a general finite regular cover, it is the minimum over the free homotopy classes whose deck monodromy is conjugate to γ, by (5.6). The identity deck class has minimum zero. For every nonidentity deck class, Theorem 15.3 gives a singularity at its positive minimum.

15.3. **Class minima and clean periods.**

Proposition 15.5. *If X is a closed connected Riemannian manifold and $\pi_1(X)$ is conjugacy separable, its marked finite-image flat spectra determine the minimum of every nontrivial free homotopy class. Each such minimum is a singularity of some irreducible finite-image flat half-wave trace.*

Proof. Let $c \in C(\Pi)$ be nontrivial, choose $\gamma \in c$, and put $L = \ell_X(c) > 0$. Corollary 14.3 and Theorem 14.2 give

$$L = \sup_{q:\Pi \twoheadrightarrow Q} \ell_q(q(\gamma)),$$

with attainment in some finite quotient q. Since $\ell_q(1) = 0$, one has $q(\gamma) \neq 1$. Theorem 15.3 gives an irreducible flat half-wave trace singular at L. Each term of the supremum is determined by (5.2) and (5.5). □

Proposition 15.6. *Let $Z \to X$ be a finite regular Riemannian cover of closed connected manifolds, with deck group Γ. Let Φ_t be the geodesic flow on SX. Suppose $L > 0$ is an isolated period and $\mathrm{Fix}(\Phi_L)$ is clean. Fix a conjugacy class $a \subseteq \Gamma$ occurring as the deck monodromy of a closed geodesic of length L. Let $\mathcal{F}_a$ be the set of connected components of $\mathrm{Fix}(\Phi_L)$ with deck class a, and put $d_a = \max_{F \in \mathcal{F}_a} \dim F$. For $F \in \mathcal{F}_a$, let $d\mu_F$ be its positive Duistermaat–Guillemin density and let σ_F be the Morse index in the free loop space of its period-L geodesics, which is constant on F. Put*

$$m_F = \int_F d\mu_F > 0, \qquad \zeta_F = i^{\sigma_F}.$$

For the convention $e^{it\sqrt{\Delta_X}}$, the Maslov factor ζ_F is the conjugate of the factor for $e^{-it\sqrt{\Delta_X}}$ in [37, Theorem 4.5 and pp. 69–70]. *If*

$$\sum_{\substack{F \in \mathcal{F}_a \\ \dim F = d_a}} \zeta_F m_F \neq 0,$$

then some W_ρ, $\rho \in \mathrm{Irr}(\Gamma)$, is singular at L. In particular, this holds when the phases of all components of dimension d_a are equal.

Proof. Deck monodromy is locally constant on $\mathrm{Fix}(\Phi_L)$. The clean fixed set is a finite disjoint union of closed submanifolds. Choose a scalar pseudodifferential operator A_a of order zero such that $A_a - \mathrm{Id}$ is microlocally smoothing near the components in $\mathcal{F}_a$ and A_a is microlocally smoothing near all the other components. The metric identifies unit tangent and unit cotangent bundles; the neighborhoods are conic in $T^*X \setminus 0$.

Lift A_a in evenly covered coordinate neighborhoods and average over Γ, obtaining a Γ-invariant operator $\widetilde{A}_a$ on Z, defined modulo smoothing operators. The finite-cover trace identity gives

$$\mathrm{Tr}(A_a e^{it\sqrt{\Delta_X}}) = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \mathrm{Tr}(\widetilde{A}_a R_h e^{it\sqrt{\Delta_Z}}) \pmod{C^\infty}.$$

Write $\widetilde{\Phi}_t$ for the lifted geodesic flow and $h_* : SZ \to SZ$ for the tangent map induced by $h \in \Gamma$. For $v \in SZ$, the fixed-point equation for $R_h e^{iL\sqrt{\Delta_Z}}$ is $h_* \widetilde{\Phi}_{-L}(v) = v$, equivalently $\widetilde{\Phi}_L(v) = h_* v$. Thus, near $t = L$, the summands with $h \notin a$ are smooth, while those with $h \in a$ equal D_h modulo smooth functions. This is the localization in the proof of Proposition 7.6. Since D_h is constant on conjugacy classes, for $\gamma \in a$,

$$\mathrm{Tr}(A_a e^{it\sqrt{\Delta_X}}) = \frac{|a|}{|\Gamma|} D_\gamma(t) \pmod{C^\infty \text{ near } L}.$$

Let Π_0 be the orthogonal projection onto the constant functions. The operator $\sqrt{\Delta_X} + \Pi_0$ is positive elliptic of order one, with zero subprincipal symbol on half-densities. Since

$$e^{it(\sqrt{\Delta_X} + \Pi_0)} - e^{it\sqrt{\Delta_X}} = (e^{it} - 1)\Pi_0,$$

this change adds a smooth function to the localized trace.

Choose an interval $I \subset (0,\infty)$ containing L and no other period, and $\chi \in C_c^\infty(I)$ equal to one near L. Put $T_a(t) = \operatorname{Tr}(A_a e^{it\sqrt{\Delta_X}})$ and define

$$\widehat{\chi T_a}(\tau) = \langle \chi(t) T_a(t), e^{-it\tau} \rangle.$$

By the localized clean trace formula [37, Theorem 4.5, equations (4.6)–(4.8)] and [102, §1.2, equations (1.5)–(1.6), arXiv v1], there is a nonzero constant κ_{d_a}, depending only on d_a and the Fourier convention, such that

$$e^{iL\tau}\widehat{\chi T_a}(\tau) = \kappa_{d_a}\tau^{(d_a-1)/2}\left(\sum_{\substack{F\in\mathcal{F}_a \\ \dim F = d_a}} \zeta_F m_F + o(1)\right) \qquad (\tau \to +\infty).$$

Components of smaller dimension contribute terms of smaller order. The sum is nonzero, so $\widehat{\chi T_a}$ is not rapidly decreasing. The localized trace formula also gives $T_a \in C^\infty(I \setminus \{L\})$. Thus T_a, and hence D_γ, is singular at L. Equation (5.3) gives the conclusion. □

15.4. **Equal flat spectra and unequal marked minima.**

Lemma 15.7. *There are a closed connected oriented smooth four-manifold X, a diffeomorphism $F : X \to X$, a smooth metric g, and $c \in \pi_1(X)$ such that $\pi_1(X)$ is infinite and finitely presented with no nontrivial finite-dimensional unitary representation, while*

$$\ell_{(X,g)}(c) = 1, \qquad \ell_{(X,F^*g)}(c) = 3/2.$$

The two metrics are isometric.

Proof. Let

$$H = \left\langle a_1, a_2, a_3, a_4 \;\middle|\; \begin{matrix} a_1^{-1}a_2a_1 = a_2^2, & a_2^{-1}a_3a_2 = a_3^2, \\ a_3^{-1}a_4a_3 = a_4^2, & a_4^{-1}a_1a_4 = a_1^2 \end{matrix} \right\rangle.$$

Higman's group H is infinite and has no nontrivial finite quotient [57, §2, pp. 62–63]. By [79, Theorem 6], there is a closed connected smooth stably parallelizable four-manifold Y with $\pi_1(Y) = H$. Choose an orientation and set $X = Y\#Y$. Van Kampen gives $\pi_1(X) = H * H$. This group is infinite and finitely presented. Every homomorphism from $H * H$ to a finite group is trivial on both factors, hence trivial. The image of a finite-dimensional complex representation of $H * H$ is finitely generated and linear, so is residually finite by Malcev's theorem [78, Theorem VII, p. 416]; see [87, p. 1]. Since it has no nontrivial finite quotient, it is trivial. For Higman's group itself, see also [71, Remark 2].

Remove corresponding balls from the two copies of Y. Identify the ends of $S^3 \times [-1,1]$ with the boundary spheres by $(x,-1) \mapsto x$ and $(x,1) \mapsto r(x)$, where $r : S^3 \to S^3$ is an involutive reflection. The exchange of the punctured copies extends over the neck by $(x,t) \mapsto (r(x),-t)$. The maps agree on the attaching spheres and on their product collars, and define a diffeomorphism F. Choose a base point $(x_0,0)$ with $r(x_0) = x_0$, and paths from it to the two punctured copies which are exchanged by F. Then F_* interchanges the two free factors.

Choose corresponding nonidentity elements c_1, c_2 in the two factors. Under the retraction onto the first factor, c_1 remains nonidentity whereas every conjugate of every power of c_2 maps to 1. The retraction onto the second factor excludes conjugacy of c_2 to any power of c_1. Thus neither c_i is conjugate to a power of the other. Put $m(c_1) = 1$ and $m(c_2) = 3/2$. For $k \in \mathbb{Z}$, if c_i is conjugate to c_i^k, then $k \neq 0$ and $m(c_i) \leq |k| m(c_i)$. Thus m satisfies the hypotheses of [36, Theorem 5.1], which gives a smooth metric g with these minima. The isometry $F : (X, F^*g) \to (X, g)$ gives

$$\ell_{(X,F^*g)}(c_1) = \ell_{(X,g)}(F_*c_1) = \ell_{(X,g)}(c_2) = 3/2.$$

Take $c = c_1$ and compare the two metrics under the identity marking. □

Theorem 15.8. *Suppose A, B are closed connected simply connected scalar-isospectral Riemannian d-manifolds with $b_2(A) \neq b_2(B)$, where b_2 denotes the second rational Betti number. There are closed connected Riemannian $(d+4)$-manifolds M, N with the same marked fundamental group, equal flat spectra for every finite-dimensional unitary representation of that group, distinct minimum marked length functions, and different homeomorphism types. The seven-dimensional pair $A = N_{1,-1}$, $B = Q^{1,1,1}$ in Theorem 12.1, with second Betti numbers* $1, 2$*, gives dimension* 11.

Proof. Take X, F, g from Lemma 15.7 and put $M = (X, g) \times A$, $N = (X, F^*g) \times B$, with the identity marking on $G = \pi_1(X)$. Simple connectivity of A and B identifies both fundamental groups with G. For $s > 0$, the product heat formula gives

$$\Theta_M(s) = \Theta_{(X,g)}(s)\Theta_A(s) = \Theta_{(X,F^*g)}(s)\Theta_B(s) = \Theta_N(s).$$

Every rank-r unitary representation of G is trivial, so its flat Laplacian is the direct sum of r scalar Laplacians. The heat identity and Lemma 2.3 give equality of all these flat spectra.

Equation (12.1), applied to the simply connected factors A and B, gives, for every $e \in G$,

$$\ell_M(e) = \ell_{(X,g)}(e), \qquad \ell_N(e) = \ell_{(X,F^*g)}(e).$$

For the element c of Lemma 15.7, these values are 1 and $3/2$. Since $H^1(A;\mathbb{Q}) = H^1(B;\mathbb{Q}) = 0$, the rational Künneth formula gives

$$b_2(M) = b_2(X) + b_2(A), \qquad b_2(N) = b_2(X) + b_2(B).$$

These Betti numbers differ, so M and N are not homeomorphic. □

Remark 15.9. For $G = \pi_1(X)$ in the preceding construction,

$$\widehat{G} = \{1\}, \qquad \nu_{M,t} = \nu_{N,t} = \delta_1 \quad (t > 0), \qquad I_M(1) = I_N(1) = 0.$$

The functions ℓ_M and ℓ_N differ under the identity marking on G, whereas $\ell_N = \ell_M \circ F_*$ for the automorphism induced by F.

Part V. Inverse theorems for marked groups

16. Metric and adjoint recovery from equivariant heat data

Let P and J be compact connected Lie groups, let Q be a positive bi-invariant metric on J, and let $\iota : P \to J$ be a smooth homomorphism. Write $\mathfrak{p} = \mathrm{Lie}(P)$, $\mathfrak{j} = \mathrm{Lie}(J)$, $d = \dim J$, and $d\iota = d\iota_e : \mathfrak{p} \to \mathfrak{j}$. For $Y \in \mathfrak{j}$, write $|Y|_Q^2 = Q(Y, Y)$. The complex scalar eigenspaces

$$E_\lambda(J) = \ker(\Delta_J - \lambda), \qquad \lambda \geq 0,$$

carry the unitary right-P action $R_p f(g) = f(g\iota(p))$. Their eigenvalues and P-module structures constitute the graded right-P scalar data. Equality of these data means that the corresponding P modules E_λ are isomorphic for every $\lambda \geq 0$; $E_\lambda = 0$ when λ is not an eigenvalue.

Let dv_Q be Riemannian volume, put $v = \mathrm{Vol}_Q(J)$, and, for $t > 0$, let K_t be the scalar heat kernel with respect to dv_Q. Set $k_t(g) = K_t(e, g)$. Left invariance and symmetry give, for $t > 0$ and $p \in P$,

$$\mathrm{Tr}(R_p e^{-t\Delta_J}) = \int_J K_t(g\iota(p), g)\, dv_Q(g) = v\, k_t(\iota(p)). \tag{16.1}$$

For $X \in \mathfrak{p}$, $\iota(\exp_P X) = \exp_J(d\iota X)$. The function

$$H_J(t, X) = \frac{\mathrm{Tr}(R_{\exp_p X} e^{-t\Delta_J})}{\mathrm{Tr}(e^{-t\Delta_J})} = \frac{k_t(\exp_J(d\iota X))}{k_t(e)}, \qquad t > 0, \quad X \in \mathfrak{p},$$

is positive. For $Y \in \mathfrak{j}$, put

$$j_J(Y) = \det_{\mathfrak{j}} \left(\int_0^1 e^{-u \operatorname{ad} Y}\, du \right). \tag{16.2}$$

Lemma 16.1. *For X in a neighborhood of zero in $\mathfrak{p}$, the limits*

$$I_J(X) = \lim_{t\downarrow 0} -4t \log H_J(t,X), \qquad A_J(X) = \lim_{t\downarrow 0} e^{I_J(X)/(4t)} H_J(t,X) \tag{16.3}$$

exist and satisfy

$$I_J(X) = Q(d\imath X, d\imath X), \qquad A_J(X) = j_J(d\imath X)^{-1/2} > 0. \tag{16.4}$$

The square root is positive. In this neighborhood define $\Phi_J(X) = \log A_J(X)$. Then I_J is quadratic and Φ_J is even and real analytic, with $\Phi_J(0) = 0$.

Proof. If $d = 0$, then $J = \{e\}$, $H_J = A_J = j_J = 1$, and $I_J = \Phi_J = 0$. Assume $d > 0$. The left-translated differential of $\exp_J$ at Y is $\int_0^1 e^{-u \operatorname{ad} Y}\, du$ [103, equations (3.C.15)–(3.C.16)]. Since left translation is an isometry, j_J is the Riemannian Jacobian of $\exp_J$ near zero. It is real analytic and $j_J(0) = 1$. Moreover, $\operatorname{ad} Y$ is Q-skew-adjoint. Replacing Y by $-Y$ in (16.2) takes the adjoint of the operator under the determinant, so $j_J(-Y) = j_J(Y)$.

The Lie exponential is the Riemannian exponential at e. Choose a normal ball about zero in $\mathfrak{j}$ on which $j_J > 0$. The relative heat asymptotic on this ball is

$$k_t(\exp_J Y) = (4\pi t)^{-d/2} e^{-|Y|_Q^2/(4t)} \big(j_J(Y)^{-1/2} + O(t) \big), \tag{16.5}$$

uniformly on compact subsets as $t \downarrow 0$. Indeed, [74, Theorem 1.1] applies to the scalar Laplacian on the closed manifold J, away from the cut locus. Its leading coefficient is the inverse square root of the exponential Jacobian [73, arXiv version 3, equations (1.13)–(1.14)]; parallel transport in the scalar bundle is the identity. For X such that $Y = d\imath X$ lies in this ball, division by the diagonal expansion gives

$$H_J(t,X) = e^{-|Y|_Q^2/(4t)} \frac{j_J(Y)^{-1/2} + O(t)}{1 + O(t)}.$$

The positive function $j_J^{-1/2}$ is bounded away from zero on each compact subset. Hence

$$-4t \log H_J(t,X) = |Y|_Q^2 + O(t), \qquad e^{|Y|_Q^2/(4t)} H_J(t,X) = j_J(Y)^{-1/2} + O(t).$$

These limits give (16.4), and $\Phi_J = -\frac{1}{2} \log j_J \circ d\imath$ is even and real analytic, with value zero at zero. □

For $X \in \mathfrak{p}$, put $m_X = \frac{1}{2} \operatorname{rank} \operatorname{ad}(d\imath X)$. Let $\omega_1(X), \dots, \omega_{m_X}(X) > 0$ be the numbers for which the nonzero complex eigenvalues of $\operatorname{ad}(d\imath X)$ are $\pm i\omega_j(X)$, each pair counted with its multiplicity. Set $\sin z/z = 1$ at $z = 0$. Each pair contributes a squared sine factor to (16.2), and each zero eigenvalue contributes one. Thus, for $s \in \mathbb{R}$,

$$j_J(s\, d\imath X) = \prod_{j=1}^{m_X} \left(\frac{\sin(s\omega_j(X)/2)}{s\omega_j(X)/2} \right)^2. \tag{16.6}$$

For the Riemann zeta function ζ, the logarithmic sine expansion [85, equation (25.8.8)] is

$$-\log \frac{\sin z}{z} = \sum_{k\geq 1} \frac{\zeta(2k)}{k\pi^{2k}} z^{2k}, \qquad z \in \mathbb{R}, \quad |z| < \pi.$$

Consequently,

$$\Phi_J(sX) = \sum_{k\geq 1} \frac{\zeta(2k)}{k(2\pi)^{2k}} s^{2k} \sum_{j=1}^{m_X} \omega_j(X)^{2k} \tag{16.7}$$

for sufficiently small real s. If $m_X = 0$, then $\Phi_J(sX) = 0$.

The adjoint P action on $\mathfrak{j}$ is $p \cdot Y = \operatorname{Ad}(\imath(p))Y$. The pullback form on $\mathfrak{p}$ is $(\imath^* Q)(X,Z) = Q(d\imath X, d\imath Z)$.

Theorem 16.2. *Suppose (J_1, Q_1, ι_1) and (J_2, Q_2, ι_2) have equal graded right-P scalar data, and put $\mathfrak{j}_i = \mathrm{Lie}(J_i)$. Then*

$$\ker \iota_1 = \ker \iota_2, \qquad \iota_1^* Q_1 = \iota_2^* Q_2,$$

and there is a real orthogonal P-intertwiner

$$F : \mathfrak{j}_1 \longrightarrow \mathfrak{j}_2, \qquad F \circ d\iota_1 = d\iota_2.$$

Proof. An element p belongs to $\ker \iota_i$ if and only if R_p is the identity on every $E_\lambda(J_i)$. For the converse, completeness of the scalar eigenspaces gives $R_p = 1$ on $L^2(J_i)$. If $\iota_i(p) \neq e$, choose a continuous function f such that $f(e) \neq f(\iota_i(p))$. Then $R_p f - f$ is nonzero on an open set of positive volume, a contradiction. The kernels therefore agree.

For $t > 0$ and $p \in P$, the character series

$$\mathrm{Tr}(R_p e^{-t\Delta_{J_i}}) = \sum_{\lambda \geq 0} e^{-t\lambda} \mathrm{Tr}(R_p|_{E_\lambda(J_i)})$$

converges absolutely, since $|\mathrm{Tr}(R_p|_{E_\lambda(J_i)})| \leq \dim E_\lambda(J_i)$. Equality of the graded data gives equal equivariant heat traces. The ordinary heat trace determines a common dimension d and volume v. For the scalar heat kernel $K_{i,t}$ of (J_i, Q_i), put $k_{i,t}(g) = K_{i,t}(e, g)$ and $j_i = j_{J_i}$. By (16.1),

$$k_{1,t}(\iota_1(p)) = k_{2,t}(\iota_2(p)). \tag{16.8}$$

If $d = 0$, both connected groups are points and F is the unique map between their zero Lie algebras. Suppose $d > 0$. Lemma 16.1 gives equality of the pullback quadratic forms near zero, hence everywhere by homogeneity. Polarization gives $\iota_1^* Q_1 = \iota_2^* Q_2$. The same lemma gives

$$j_1(s\, d\iota_1 X) = j_2(s\, d\iota_2 X) \tag{16.9}$$

for every $X \in \mathfrak{p}$ and all sufficiently small real s.

Put $m = \lfloor d/2 \rfloor$. For each i, augment the multiset of squared positive frequencies of $\mathrm{ad}(d\iota_i X)$ by zeros to obtain m numbers. By (16.7), their power sums of orders $1, \dots, m$ agree: the coefficients $\zeta(2k)/(k(2\pi)^{2k})$ are nonzero. Newton's identities, in the form given in the proof of Theorem 3.4, identify these two multisets. Their positive elements determine the positive frequencies with multiplicities. The common dimension determines the multiplicity of the zero eigenvalue. This also covers $m = 0$, when both frequency lists are empty. Write $\omega_1(X), \dots, \omega_{m_X}(X)$ for the common positive frequencies. Since $\mathrm{Ad}(\iota_i(\exp_P X)) = \exp(\mathrm{ad}(d\iota_i X))$, the adjoint characters at $\exp_P X$ are both

$$d - 2m_X + 2 \sum_{j=1}^{m_X} \cos \omega_j(X).$$

Every element of a compact connected Lie group lies in a maximal torus [41, Theorem 43.11], and is therefore an exponential. The complexified adjoint P modules have equal characters and are equivalent by complete reducibility and character orthogonality [41, Corollaries 35.6 and 35.12].

Put $V_i = d\iota_i(\mathfrak{p})$ and $N_i = V_i^{\perp_{Q_i}}$. Since

$$\mathrm{Ad}(\iota_i(p)) d\iota_i X = d\iota_i(\mathrm{Ad}_P(p) X),$$

the spaces V_i and N_i are P-invariant. The kernel of $d\iota_i$ is the radical of the common pullback form. Thus

$$T : V_1 \longrightarrow V_2, \qquad d\iota_1 X \longmapsto d\iota_2 X,$$

is a well-defined P-equivariant isometry. Subtracting the characters of V_i gives equal characters on $N_i \otimes_{\mathbb{R}} \mathbb{C}$. Their equivalence implies the existence of a real P-isomorphism $C : N_1 \to N_2$ [110, §3.6, Theorem 5, p. 39]. Writing C^* for its adjoint with respect to Q_1 and Q_2, the operator C^*C commutes with P. Hence the orthogonal factor $C(C^*C)^{-1/2}$ is P-equivariant. Then $F = T \oplus C(C^*C)^{-1/2}$ is orthogonal and P-equivariant, and $F \circ d\iota_1 = d\iota_2$. □

Corollary 16.3. *Let K be a compact Lie group and, for $i = 1, 2$, let $\iota_i : K \hookrightarrow J_i$ be embeddings into compact connected Lie groups with positive bi-invariant metrics Q_i. Put $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{m}_i = (d\iota_i\mathfrak{k})^{\perp_{Q_i}}$. The K action on $\mathfrak{m}_i$ is induced by $\mathrm{Ad} \circ \iota_i$. Suppose $\iota_1^* Q_1 = \iota_2^* Q_2$ and the graded right-K scalar data agree. If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonally equivalent K modules, identify them by such an equivalence. For every common finite-dimensional equivariant Hermitian Clifford module W, the cubic Dirac operators on $J_i/\iota_i(K)$ of Theorem 4.9 have equal square spectra. A common K-invariant grading for which Clifford multiplication is odd gives equal signed spectra.*

If K is connected, equality of the graded right-K scalar data implies equality of the induced metrics and orthogonal equivalence of the tangent modules. It therefore implies equality of all common canonical rough Laplace spectra, cubic-square spectra, and, under the grading hypothesis, signed cubic spectra. These spectral equalities also hold for every closed subgroup in the product construction of Corollary 4.6.

Proof. Let Scal_J denote the scalar curvature of (J, Q), and let (e_a) be a Q-orthonormal basis of $\mathfrak{j}$. For $\rho_{\mathfrak{j}}$ as in Theorem 4.9, Kostant's identity [65, Proposition 1.84], applied with invariant form $-Q$, and (4.16) with trivial isotropy give

$$\|\rho_{\mathfrak{j}}\|^2 = \frac{1}{24} \sum_{a,b,c} Q([e_a, e_b], e_c)^2 = \frac{\mathrm{Scal}_J}{6}.$$

The scalar heat coefficients determine $\mathrm{Vol}_Q(J)$ and $\mathrm{Vol}_Q(J)\,\mathrm{Scal}_J/6$ [47, Theorem A.1(a),(c)]; scalar curvature is constant. Hence $\|\rho_{\mathfrak{j}_1}\|^2 = \|\rho_{\mathfrak{j}_2}\|^2$. The common induced metric on $\mathfrak{k}$ gives equal isotropy root norms. Thus the constants $b_{J_i} = \|\rho_{\mathfrak{j}_i}\|^2 - \|\rho_{\mathfrak{k}}\|^2$ in Theorem 4.9 agree. The identified tangent K modules and the common module W give the same endomorphism C_κ in that theorem. The scalar K-equivariant unitary of Theorem 4.4, tensored with 1_W, therefore intertwines the cubic squares. With a common grading, the signed spectrum is determined by the square spectrum, as in the last paragraph of the proof of Theorem 4.9.

For connected K, Theorem 16.2 gives the equality of induced metrics and an orthogonal K-intertwiner carrying $d\iota_1\mathfrak{k}$ to $d\iota_2\mathfrak{k}$, hence also $\mathfrak{m}_1$ to $\mathfrak{m}_2$. Theorem 4.4 gives the canonical rough Laplace spectra. In the product construction, the tangent identifications and rough Laplace spectra are those of Corollary 4.6; the cubic spectra are those of Theorem 4.9. □

17. Finite moments, compact relative embeddings, and generated ideals

Fix $n \geq 1$. Let $V = \mathbb{C}^n$ be the defining $\mathrm{U}(n)$ module and z the character of the final $\mathrm{U}(1)$ factor of P_n. Write

$$\mathfrak{p}_n = \{(X + iaI_n, ib) : X \in \mathfrak{su}(n),\ a, b \in \mathbb{R}\}, \qquad A_n = (iI_n, 0), \quad B_n = (0, i),$$

and put $Q_{\mathrm{U}(n)}(Y, Z) = -\mathrm{Re}\,\mathrm{Tr}_{\mathbb{C}}(YZ)$ for $Y, Z \in \mathfrak{u}(n)$. A marking of a Lie algebra $\mathfrak{a}$ is a homomorphism $\phi : \mathfrak{p}_n \to \mathfrak{a}$. An isomorphism of marked Lie algebras intertwines these homomorphisms; for marked metric Lie algebras it is also an isometry.

Identify the central torus of P_n with $\mathbb{T}^2$ by $\tau(a, b) = (e^{ia} I_n, e^{ib})$. For the functions of Section 16, write $I_J(a, b) = I_J(aA_n + bB_n)$ and $\Phi_J(a, b) = \Phi_J(aA_n + bB_n)$. By (4.4), the nonzero weights of $\rho_{\mathrm{ref}} = \mathrm{Ad}_{G_n} \circ i_n \circ \tau$ on $(\mathfrak{g}_n)_{\mathbb{C}}$ are

$$(\pm 2, 0) \text{ with multiplicity } n^2 \text{ each}, \qquad (1, 1), (1, -1), (-1, 1), (-1, -1) \text{ with multiplicity } n \text{ each}. \tag{17.1}$$

The zero-weight multiplicity is $2n^2 + 1$.

For a finite-dimensional real orthogonal representation $\rho : \mathbb{T}^2 \to \mathrm{O}(W)$, a weight $(\alpha, \beta) \in \mathbb{Z}^2$ of $W_{\mathbb{C}}$ denotes the character $(a, b) \mapsto e^{i(\alpha a + \beta b)}$. For $u, v \in \mathbb{Z}_{\geq 0}$, set $M_{uv}(\rho) = \sum_{(\alpha,\beta)} \alpha^u \beta^v$, where all weights are counted with multiplicity and a zero exponent denotes the constant monomial 1. For $x = (a, b) \in \mathbb{R}^2 = \mathrm{Lie}(\mathbb{T}^2)$ near zero, put

$$\Phi_\rho(x) = -\frac{1}{2} \log \det_W \left(\int_0^1 e^{-u\, d\rho_e(x)}\, du \right).$$

Here $d\rho_e$ is the differential at the identity; the real determinant is positive near zero. In particular, $\Phi_{\rho_{\mathrm{ref}}}(a,b) = \Phi_{G_n}(a,b)$.

Lemma 17.1. *Suppose* $\dim_{\mathbb{R}} W = (2n+1)^2$ *and the nine moments*

$$M_{02}, M_{04}, M_{06}, M_{20}, M_{40}, M_{60}, M_{22}, M_{42}, M_{11} \tag{17.2}$$

of ρ *equal those of* ρ_{ref}. *Then* $W_{\mathbb{C}}$ *has the weights* (17.1), *with zero-weight multiplicity* $2n^2+1$. *In particular, equality of the sixth Taylor polynomials at zero of* Φ_ρ *and* Φ_{G_n} *suffices.*

Proof. Write $\Phi = \Phi_\rho$ and $M_{uv} = M_{uv}(\rho)$. The logarithmic sine expansion in (16.7) gives

$$\Phi(a,b) = \frac{1}{2}\sum_{k\ge 1} \frac{\zeta(2k)}{k(2\pi)^{2k}} \sum_{(\alpha,\beta)} (\alpha a + \beta b)^{2k}.$$

The factor $1/2$ compensates for counting both conjugate complex weights of each real rotation plane. Thus the coefficient of $a^u b^v$, $u+v=2k$, is $\frac{1}{2}\frac{\zeta(2k)}{k(2\pi)^{2k}}\binom{2k}{u} M_{uv}$, with nonzero multiplier. The reference moments give

$$\sum \beta^2(\beta^2-1)^2 = M_{06} - 2M_{04} + M_{02} = 0.$$

Each summand is nonnegative, so $\beta \in \{0,1,-1\}$. Next,

$$\sum \beta^2(\alpha^2-1)^2 = M_{42} - 2M_{22} + M_{02} = 0$$

gives $\alpha = \pm 1$ when $\beta \neq 0$. Let r, s be the multiplicities of $(1,1),(1,-1)$. Complex conjugation gives the same multiplicities at their negatives. Since $2(r+s) = M_{02} = 4n$ and $2(r-s) = M_{11} = 0$, one has $r = s = n$.

The remaining weights have $\beta = 0$, and their pure second, fourth and sixth moments are $8n^2, 32n^2, 128n^2$. Therefore

$$\sum_{\beta=0} \alpha^2(\alpha^2-4)^2 = 128n^2 - 8(32n^2) + 16(8n^2) = 0.$$

Thus $\alpha \in \{0,2,-2\}$ there. The second moment and complex conjugation give multiplicity n^2 at each nonzero weight. Since $\dim_{\mathbb{C}} W_{\mathbb{C}} = \dim_{\mathbb{R}} W = (2n+1)^2$, the zero-weight multiplicity is $2n^2+1$. □

Lemma 17.2. *For* $n \ge 2$, *a complex* n*-dimensional unitary* $\mathrm{U}(n)$ *module on which* $e^{ia} I_n$ *acts by* e^{ia} *is the defining module* V. *With central weight* -1 *it is* V^*.

Proof. If the $\mathrm{SU}(n)$ action is trivial, the module is a sum of characters $\det^k$ [103, Proposition 4.4.5], whose central weights are multiples of n. Thus its $\mathrm{SU}(n)$ restriction has a nontrivial constituent. By Lemma B.1, this restriction is irreducible and is V or V^*. By [103, Proposition 4.5.2], the $\mathrm{U}(n)$ module is $V \otimes \det^k$ or $V^* \otimes \det^k$ for some $k \in \mathbb{Z}$. Central weight 1 gives $nk = 0$ in the first case and $nk = 2$ in the second. The latter occurs only for $n = 2$, when $V^* \otimes \det \simeq V$. Duality gives the assertion at weight -1. □

17.1. **The semisimple ideal generated by the nontrivial module.** For a Lie algebra $\mathfrak{a}$ and $S \subset \mathfrak{a}$, write $\mathfrak{c}_{\mathfrak{a}}(S) = \{Y \in \mathfrak{a} : [Y,s] = 0 \text{ for every } s \in S\}$. For $B \in \mathfrak{a}$, abbreviate $\mathfrak{c}_{\mathfrak{a}}(\{B\})$ to $\mathfrak{c}_{\mathfrak{a}}(B)$. Suppose $\mathfrak{a}$ is compact and $B \in \mathfrak{a}$ satisfies $\mathrm{spec}(\mathrm{ad}\, B) \subset \{0, i, -i\}$ on $\mathfrak{a}_{\mathbb{C}}$. Its three-grading is

$$\mathfrak{a}_{\mathbb{C}} = \mathfrak{a}_- \oplus (\mathfrak{a}_0)_{\mathbb{C}} \oplus \mathfrak{a}_+, \qquad \mathfrak{a}_0 = \mathfrak{c}_{\mathfrak{a}}(B), \quad \mathfrak{a}_\pm = \ker(\mathrm{ad}\, B \mp i\,\mathrm{Id}).$$

Its positive tangent module is $\mathfrak{a}_+$. For a marking ϕ, put $A = \phi(A_n)$ and $B = \phi(B_n)$. A joint weight (α,β) denotes the simultaneous eigenvalues $i\alpha, i\beta$ of $\mathrm{ad}\, A, \mathrm{ad}\, B$ on the complexification. When a positive invariant inner product is fixed, denote their orthogonal projections to an ideal $\mathfrak{q}$ by $A_{\mathfrak{q}}, B_{\mathfrak{q}}$.

Let $V_n = (\mathfrak{g}_n^{P_n})^{\perp}$, with orthogonality taken for Q_G. By (4.4), $\dim \mathfrak{g}_n^{P_n} = 3$.

Lemma 17.3. *Let $\mathfrak{s}$ be compact semisimple and let $\phi : \mathfrak{p}_n \to \mathfrak{s}$ be a homomorphism whose adjoint action integrates to P_n. With respect to the negative Killing form, put $\mathfrak{v} = (\mathfrak{s}^{P_n})^{\perp}$. Suppose $\mathfrak{v} \simeq V_n$ as real P_n modules and $\mathfrak{v}$ generates $\mathfrak{s}$ as an ideal. Then $(\mathfrak{s}, \phi)$ is isomorphic to exactly one of*

$$\mathfrak{su}(2n+1), \qquad \mathfrak{sp}(n) \oplus \mathfrak{so}(2n+2),$$

marked respectively by the traceless projection of di_n and by dj_n. Their fixed subalgebras have dimensions two and three.

Proof. An element $(X + iaI_n, ib)$ acting trivially on $Vz \oplus V^*z$ satisfies

$$X + i(a+b)I_n = 0, \qquad -X^T + i(-a+b)I_n = 0.$$

Since $\operatorname{Tr} X = 0$, it follows that $X = 0$ and $a = b = 0$. Thus the action on $\mathfrak{v}$ is faithful and ϕ is injective. No nonzero ideal of $\mathfrak{s}$ is P_n-trivial: its projection of $\mathfrak{v}$ would vanish, contrary to generation.

Assume $n \geq 2$ and put $\mathfrak{h} = \mathfrak{su}(n)$. The positive-$B$ space is $(V \oplus V^*)z$, with two inequivalent P_n summands distinguished by their A weights. Each simple ideal $\mathfrak{q}$ with $B_{\mathfrak{q}} \neq 0$ has the three-grading of (B.1); its positive tangent module is irreducible under its zero-grade algebra. Its contribution is a nonzero P_n submodule of $(V \oplus V^*)z$, so at most two such ideals occur.

If there are two, each has tangent module V or V^* under $\mathfrak{h}$. By Lemma B.2, both are projective $\mathfrak{su}(n+1)$ ideals. Each zero grade is the direct sum of the projected copy of $\mathfrak{h}$ and its one-dimensional center; the projected A lies in that center. They therefore contain the two adjoint $\mathfrak{h}_{\mathbb{C}}$ summands of (4.4) at joint weight $(0,0)$ and have no weight $(2,0)$. A nonzero $\mathfrak{h}$ projection to a remaining simple ideal would contribute a third adjoint summand at $(0,0)$. Hence $\mathfrak{h}$ acts trivially on the remaining module, contrary to its nontrivial action on $\operatorname{Sym}^2 V$ at weight $(2,0)$. There is consequently one simple ideal with nonzero B projection. By Lemma B.2, it is the projective $\mathfrak{su}(2n+1)$ or the quadric $\mathfrak{so}(2n+2)$, with marked projection determined by the tangent action.

The projective alternative contains all of $\mathfrak{v}$, so generation makes it all of $\mathfrak{s}$. In the quadric alternative, the complexification of the remaining nontrivial module is

$$\mathfrak{h}_{\mathbb{C}} \oplus \operatorname{Sym}^2 V \oplus \operatorname{Sym}^2 V^*.$$

Its positive-A space $\operatorname{Sym}^2 V$ is irreducible and belongs to one simple ideal $\mathfrak{q}$. The element $A_{\mathfrak{q}}/2$ defines a three-grading. The nontrivial $\mathfrak{h}$ action on $\operatorname{Sym}^2 V$ puts the remaining adjoint copy of $\mathfrak{h}$ in $\mathfrak{q}$. Every further ideal has zero A and B weights and zero $\mathfrak{h}$ projection, so is P_n-trivial and vanishes.

The zero grade $\mathfrak{q}_0$ consists of the embedded $\mathfrak{h}$ and an $\mathfrak{h}$-trivial orthogonal complement. The complement centralizes $\mathfrak{h}$ and, by Schur's lemma and compactness, acts by imaginary scalars on $\operatorname{Sym}^2 V$. The faithful zero-grade action on the tangent module (Appendix B) bounds the real dimension of this complement by one. The complement contains $A_{\mathfrak{q}}/2$, so $\mathfrak{q}_0 \simeq \mathfrak{u}(n)$. In the projective row of (B.1), the simple derived algebra forces block dimensions $1, n$, with tangent dimension $n \neq n(n+1)/2$. In the alternating row it forces $m = n$, with tangent dimension $n(n-1)/2$. In the vector row, $\mathfrak{so}(d) \simeq \mathfrak{su}(n)$ gives $(d,n) = (3,2)$ or $(6,4)$. The former is $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$; the latter has tangent dimension $6 \neq 10$. The exceptional derived algebras have dimensions $45, 78$, neither of the form $n^2 - 1$. Thus $\mathfrak{q} = \mathfrak{sp}(n)$. Its $\mathfrak{h}$ embedding is determined by $\operatorname{Sym}^2 V$: an outer automorphism would replace this module by its inequivalent dual for $n \geq 3$, and all automorphisms are inner for $n = 2$ [119, Proposition 4.43]. These inner automorphisms are realized in $\mathrm{SU}(n) \subset \mathrm{Sp}(n)$. The central weight two determines $A_{\mathfrak{q}}$; $B_{\mathfrak{q}} = 0$ because its adjoint action vanishes. This gives the marked symplectic projection.

For $n = 1$, put $\mathfrak{w} = \mathfrak{c}_{\mathfrak{s}}(\phi(\mathfrak{p}_1))$. Its orthogonal complement $\mathfrak{v}$ is the sum of three pairwise inequivalent real rotation planes, of weights $2a, a+b, a-b$. The action of $\mathfrak{w}$ on these planes has image in $\mathfrak{so}(2)^3$, so $[\mathfrak{w}, \mathfrak{w}]$ annihilates $\mathfrak{v}$. Thus $[\mathfrak{w}, \mathfrak{w}]$ is a P_1-trivial ideal of $\mathfrak{s} = \mathfrak{w} \oplus \mathfrak{v}$ and vanishes. Since $\phi(\mathfrak{p}_1) \subset \mathfrak{w}$, the abelian algebra $\mathfrak{w}$ is self-centralizing, hence a Cartan algebra. Its rank is at least two and its root system has six roots. The root-system classification gives A_2 or A_1^3 [41, Theorems 19.10 and 22.1]. In type A_2, the roots restricting to $a+b, a-b$ form a simple system, with sum $2a$. In type A_1^3, label the positive roots by their restrictions $2a, a+b, a-b$. Since the roots span the Cartan dual, these labelled restrictions determine the marking. Root-labelled compact isomorphisms identify the two cases with the stated models. The fixed dimensions in both cases follow from (4.4). □

Theorem 17.4. *On the reference Euclidean P_n module $\mathfrak{g}_n$, with its distinguished embedded $\mathfrak{p}_n$, there are exactly two classes of Lie brackets, up to orthogonal Lie isomorphisms fixing $\mathfrak{p}_n$ pointwise, for which the metric is invariant, the distinguished subalgebra has its prescribed bracket, and its adjoint action is the prescribed P_n action. They are*

$$(\mathfrak{u}(2n+1), Q_G, di_n), \qquad (\mathfrak{sp}(n) \oplus \mathfrak{so}(2n+2), Q_C \oplus Q_D, dj_n).$$

Proof. The positive invariant form makes the algebra the orthogonal sum of its center and compact simple ideals [119, Exercise 3.30(c), Proposition 3.25(b), and Theorem 3.21(a)]. Let $\mathfrak{a}$ be the ideal generated by V_n. It lies in the semisimple part. The orthogonal projection $\mathrm{pr}_{\mathfrak{a}}$ is a Lie homomorphism, and $\phi = \mathrm{pr}_{\mathfrak{a}} \circ di_n$ induces the prescribed P_n action on $\mathfrak{a}$. Its nontrivial module is V_n, which generates $\mathfrak{a}$ also as an ideal of $\mathfrak{a}$, since $[\mathfrak{a}, \mathfrak{a}^{\perp}] = 0$. Lemma 17.3 applies. In the unitary alternative, $\mathfrak{a}$ has codimension one and its complement is central; in the other alternative it has the full dimension.

On a compact simple ideal, the invariant metric is unique up to a positive scalar [119, Lemma 6.15]. In the unitary case, write the simple metric as sQ_G, $s > 0$. For $n \geq 2$, its restriction to $\mathfrak{su}(n)$ is $2sQ_{\mathrm{U}(n)}$, so $s = 1$. For $Y \in \mathfrak{u}(n)$, the traceless projection of $di_n(Y, ib)$ has squared norm

$$2Q_{\mathrm{U}(n)}(Y, Y) + \frac{2n}{2n+1} b^2.$$

The pullback of the metric on the central line is therefore $b^2/(2n+1)$; a reflection aligns its sign with the reference central component. For $n = 1$, the simple pullback is $2sa^2 + (2s/3)b^2$. The residual form $2(1-s)a^2 + (1-2s/3)b^2$ is nonnegative and has rank at most one. Its determinant vanishes only at $s = 1$ or $s = 3/2$; the latter has a negative a^2 coefficient. Hence $s = 1$ again, with the same central pullback.

For $n \geq 2$ in the symplectic–orthogonal case, write the metric as $sQ_C \oplus tQ_D$, $s, t > 0$. Its restriction is $(s+t)Q_{\mathrm{U}(n)} + tb^2$, so $s = t = 1$. For $n = 1$, the three simple rank-one ideals have positive coefficients q_1, q_2, q_3 satisfying

$$q_1(2a)^2 + q_2(a+b)^2 + q_3(a-b)^2 = 2a^2 + b^2.$$

Comparison of coefficients gives $q_1 = 1/4$ and $q_2 = q_3 = 1/2$, the coefficients of $Q_C \oplus Q_D$. Existence on the fixed Euclidean module follows from Proposition 4.5. The center dimensions distinguish the two classes. An isomorphism fixing $\mathfrak{p}_n$ intertwines its adjoint action and hence, since P_n is connected, its group action. □

17.2. **Relative embeddings from central moments.**

Proposition 17.5. *Let J be a Lie group of dimension $(2n+1)^2$ with a positive bi-invariant metric Q_J, and let $\iota : P_n \to J$ be a homomorphism with $(d\iota_e)^* Q_J = Q_P$. Suppose the nine moments* (17.2) *of $\rho_\iota = \mathrm{Ad}_J \circ \iota \circ \tau$ equal those of ρ_{ref}. Then* $(\mathrm{Lie}(J), Q_J, d\iota_e)$ *is isomorphic to one of the marked metric Lie algebras of Theorem 17.4, except that for $n = 4$ its symplectic component may instead be*

$$(X + iaI_4, ib) \longmapsto -X^T + iaI_4, \qquad X \in \mathfrak{su}(4),$$

in the zero-grade $\mathfrak{u}(4) \subset \mathfrak{sp}(4)$. The $\mathfrak{so}(10)$ component is then the corresponding component of dj_4.

Proof. Put $\mathfrak{j} = \mathrm{Lie}(J)$, $\phi = d\iota_e$, $A = \phi(A_n)$, and $B = \phi(B_n)$. The positive invariant form makes $\mathfrak{j}$ compact. Lemma 17.1 determines its central torus module. For $n = 1$, this is the whole P_1 module, and $\phi(\mathfrak{p}_1)$ lies in its fixed subspace. The orthogonal extension in the last paragraph of the proof of Theorem 16.2 identifies this module with $\mathfrak{g}_1$ and intertwines ϕ with di_1. Theorem 17.4 applies.

Assume $n \geq 2$ and put $\mathfrak{h} = \mathfrak{su}(n)$. The positive-$B$ spaces have A weights $1, -1$ and complex dimension n each. By Lemma 17.2, their $\mathrm{U}(n)$ modules are V, V^*. At most two simple ideals have nonzero B projection. If there are two, each is projective by Lemma B.2, with joint zero-weight dimension n^2. Their orthogonal complement therefore has zero-A dimension one and positive-A complex dimension n^2, of weight two. Each simple ideal there with nonzero A projection contributes a central grading direction to its zero grade, after division of the grading element by two. Thus there

is exactly one such ideal and its zero grade has dimension one. By (B.1), it is $\mathfrak{su}(2)$, with positive tangent dimension one, contrary to $n^2 > 1$.

There is one simple ideal with nonzero B projection, projective or quadric by Lemma B.2, with the marked projection specified there. In the projective case it has codimension one, and the metric calculation in Theorem 17.4 gives the reference $\mathfrak{u}(2n+1)$ metric and marking.

In the quadric case, its orthogonal complementary ideal $\mathfrak{l}$ satisfies

$$\dim_{\mathbb{R}} \mathfrak{l} = 2n^2 + n, \qquad \dim_{\mathbb{R}} \mathfrak{l}_0 = n^2, \qquad \dim_{\mathbb{C}} \mathfrak{l}_+ = n(n+1)/2, \tag{17.3}$$

where the subscripts refer to the three-grading defined by $A_{\mathfrak{l}}/2$. The B action on $\mathfrak{l}$ is zero. The projection of $\mathfrak{h}$ to $\mathfrak{l}$ is nonzero: otherwise the metric tQ_D on $\mathfrak{so}(2n+2)$ would restrict to $2Q_{\mathrm{U}(n)}$ on $\mathfrak{h}$, giving $t = 2$. Its B projection would then have squared norm two, greater than $Q_P(B_n, B_n) = 1$.

The injective $\mathfrak{h}$ projection and $A_{\mathfrak{l}} \neq 0$ commute and are linearly independent. Their dimensions sum to n^2, so, identifying $\mathfrak{h}$ with its image, $\mathfrak{l}_0 = \mathfrak{h} \oplus \mathbb{R}A_{\mathfrak{l}}$. Its center has dimension one. Hence exactly one simple ideal has nonzero A projection, and $\mathfrak{l}$ has zero center. A simple ideal contained in $\mathfrak{l}_0$ would equal its whole $\mathfrak{h}$ summand; the ideal with nonzero A projection would then have one-dimensional zero grade and positive tangent dimension one, contrary to (17.3). Thus $\mathfrak{l}$ is simple, with zero grade $\mathfrak{u}(n)$ and positive tangent dimension $n(n+1)/2$. The row comparison in Lemma 17.3 gives $\mathfrak{l} = \mathfrak{sp}(n)$.

By [119, Proposition 4.43], its zero-grade $\mathfrak{h}$ embedding is conjugate under $\mathrm{SU}(n) \subset \mathrm{Sp}(n)$ to the standard embedding or its composition with complex conjugation. The central weight two determines $A_{\mathfrak{l}}$. The element $B_{\mathfrak{l}}$ acts trivially on every graded summand of $\mathfrak{l}$; simplicity gives $B_{\mathfrak{l}} = 0$. The prescribed B norm therefore makes the orthogonal metric scale one, and the $\mathfrak{h}$ norm makes the symplectic metric scale one. In the inner case the marking is dj_n.

In the outer case the positive-A module restricts to $\mathrm{Sym}^2 V^*$ on $\mathrm{SU}(n)$ and has central weight two. Put $\xi = e^{2\pi i/n}$. The element ξI_n acts by ξ^{-2} through $\mathrm{SU}(n)$ and by ξ^2 through the central circle of $\mathrm{U}(n)$. Since these actions agree, $\xi^4 = 1$, so $n \mid 4$. For $n \geq 3$ this gives $n = 4$; for $n = 2$ the automorphism is inner. The differential at $n = 4$ is the one displayed. □

18. Global periods and exceptional forms

Lemma 18.1. *Let P be a connected Lie group and let (J_0, Q_0) and (J, Q) be compact connected Lie groups with positive bi-invariant metrics. Put $\mathfrak{j}_0 = \mathrm{Lie}(J_0)$ and $\mathfrak{j} = \mathrm{Lie}(J)$. Let $\iota_0 : P \to J_0$ and $\iota : P \to J$ be homomorphisms. Suppose $(\iota_0)_* : \pi_1(P) \to \pi_1(J_0)$ is surjective and $\psi : \mathfrak{j}_0 \to \mathfrak{j}$ is an isometric Lie algebra isomorphism satisfying $\psi \circ d\iota_0 = d\iota$. Then ψ integrates to a surjective local isometry $q : J_0 \to J$ with $q \circ \iota_0 = \iota$ and finite central kernel.*

Proof. By [41, Theorem 8.2 and Corollary 8.4], ψ identifies the simply connected covers of J_0 and J. Denote this common covering group by S, with covering maps $p_0 : S \to J_0$ and $p : S \to J$, and put $\Omega_0 = \ker p_0$, $\Omega = \ker p$. Let $p_P : \widetilde{P} \to P$ be the universal covering and put $\Lambda = \ker p_P$. These kernels are discrete and central [41, Proposition 2.11]. The homomorphic lifts to S of $\iota_0 \circ p_P$ and $\iota \circ p_P$ have the same differential, hence agree by [41, Theorem 8.2]. Write $\widetilde{\iota} : \widetilde{P} \to S$ for this common lift. The fundamental groups identify with the covering kernels, so surjectivity of $(\iota_0)_*$ gives

$$\Omega_0 = \widetilde{\iota}(\Lambda) \subseteq \Omega.$$

Thus the identity of S descends to a surjective homomorphism $q : S/\Omega_0 \to S/\Omega$, with $q \circ \iota_0 = \iota$ and $\ker q = \Omega/\Omega_0 \subset Z(J_0)$. Its differential is ψ, so q is a local isometry. Its kernel is closed and discrete in compact J_0, hence finite. □

Fix $n \geq 1$. For each integer $m \geq 1$, let $C_m \leq Z(\mathrm{U}(2n+1))$ be the scalar subgroup of order m, and put

$$G_{n,m} = \mathrm{U}(2n+1)/C_m, \qquad H_n^{\mathrm{S}} = (\mathrm{Sp}(n)/\{\pm 1\}) \times \mathrm{SO}(2n+2),$$

$$H_n^{\mathrm{O}} = \mathrm{Sp}(n) \times (\mathrm{SO}(2n+2)/\{\pm I\}).$$

Each quotient carries the induced reference metric and the reference marking composed with the quotient homomorphism. Write $v_n = \mathrm{Vol}(G_n) = \mathrm{Vol}(H_n)$; the equality follows from Theorem 3.3 and the leading heat coefficient.

Theorem 18.2. *The compact connected Lie groups J with Lie algebra $\mathfrak{j}$, a positive bi-invariant metric Q, a faithful marking $\iota : P_n \to J$, and an orthogonal isomorphism*

$$F : \mathfrak{j} \longrightarrow \mathfrak{g}_n, \qquad F \circ d\iota = di_n,$$

intertwining the adjoint P_n actions are exactly

$$G_{n,m} \quad (m \geq 1 \text{ odd}), \qquad H_n, \quad H_n^{\mathrm{S}}, \quad H_n^{\mathrm{O}},$$

up to an isometric Lie group isomorphism intertwining the marking.

Proof. Transporting the bracket by F gives the hypotheses of Theorem 17.4. Choose $(J_0, \iota_0) = (G_n, i_n)$ or (H_n, j_n) according to the resulting marked metric Lie algebra. By (2.7), $(\iota_0)_*$ is surjective. Lemma 18.1 gives $J = J_0/\Gamma$ for a finite central subgroup Γ, with the induced metric and marking. Since ι_0 is injective, the quotient marking is faithful exactly when $\Gamma \cap \iota_0(P_n) = \{1\}$. The intersections are (B.2). Every finite subgroup of the scalar circle is C_m; it avoids $-I$ exactly when m is odd. In $Z(H_n) = \{\pm 1\} \times \{\pm I\}$, the subgroups avoiding $(-1, -I)$ are the trivial subgroup and the two coordinate subgroups. These give $H_n, H_n^{\mathrm{S}}, H_n^{\mathrm{O}}$.

Conversely, each listed quotient has a faithful marking by the same intersection calculation. Its metric Lie algebra and adjoint P_n module are unchanged, so it has the required intertwiner. The center dimensions distinguish the unitary and semisimple cases. Within each case, the volume and eigenspace calculations in the proof of Theorem 18.3 distinguish the listed forms. □

Theorem 18.3. *For fixed n, the unique pair of distinct marked groups in Theorem 18.2 with equal scalar spectra is G_n, H_n. Groups with different values of n have different dimensions.*

Proof. The quotient volumes are

$$\mathrm{Vol}(G_{n,m}) = v_n/m, \qquad \mathrm{Vol}(H_n^{\mathrm{S}}) = \mathrm{Vol}(H_n^{\mathrm{O}}) = v_n/2.$$

The scalar spectrum determines volume. Thus distinct values of m give distinct spectra. Equality of the volume of $G_{n,m}$ with that of H_n requires $m = 1$, whereas equality with that of H_n^{S} or H_n^{O} requires $m = 2$, excluded by faithfulness. The volume also separates H_n from these two quotients.

Put $N = 2n + 1$. By Proposition 3.8, $E_N(H_n)$ is the direct sum of the defining coefficient spaces of $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n + 2)$. For a finite central subgroup Γ, pullback identifies $E_N(H_n/\Gamma)$ with $E_N(H_n)^\Gamma$. The elements $-1 \in \mathrm{Sp}(n)$ and $-I \in \mathrm{SO}(2n + 2)$ act by minus the identity on their respective defining representations and trivially on the other coefficient space. Hence

$$\dim E_N(H_n^{\mathrm{S}}) = (2n + 2)^2, \qquad \dim E_N(H_n^{\mathrm{O}}) = (2n)^2.$$

These dimensions differ. The pair G_n, H_n has equal scalar spectra by Theorem 3.3 and different Lie algebra center dimensions. Finally, $(2n + 1)^2$ determines n. □

If faithfulness is omitted from Theorem 18.2 and its other hypotheses are retained, the resulting marked groups are precisely the finite central quotients of G_n or H_n, with their induced metrics and markings.

18.1. **The exceptional marking.** For $n = 4$, put

$$J_* = H_4^{\mathrm{S}} = (\mathrm{Sp}(4)/\{\pm 1\}) \times \mathrm{SO}(10),$$

and let Q_* be its quotient reference metric. Brackets in the first factor denote classes modulo $\{\pm 1\}$. For $(A, z) \in P_4$, choose $\eta \in \mathrm{U}(1)$ with $\eta^2 = \det A$ and define

$$\iota_*(A, z) = \big([c(\eta \bar{A})],\ r(A \oplus z)\big). \tag{18.1}$$

Replacing η by $-\eta$ multiplies $c(\eta\bar{A})$ by $-I_8$, so the quotient class is unchanged. If $\eta_j^2 = \det A_j$ for $j = 1, 2$, then

$$(\eta_1\eta_2)^2 = \det(A_1A_2), \qquad c(\eta_1\bar{A}_1)c(\eta_2\bar{A}_2) = c(\eta_1\eta_2\overline{A_1A_2}).$$

Thus ι_* is a homomorphism. Local smooth square roots give the same quotient-valued map on overlaps, so it is smooth. Identify $\mathrm{Lie}(\mathrm{Sp}(4)/\{\pm1\})$ with $\mathfrak{sp}(4)$ by the differential of the quotient map. Writing dc, dr for the differentials at the identity, one obtains, for $X \in \mathfrak{su}(4)$ and $a, b \in \mathbb{R}$,

$$d\iota_*(X + iaI_4, ib) = \big(dc(-X^T + iaI_4),\ dr((X + iaI_4) \oplus ib)\big).$$

Theorem 18.4. *Let J be a compact connected Lie group with a positive bi-invariant metric Q, and let $\iota : P_n \to J$ be a homomorphism. Suppose* $\dim J = (2n+1)^2$, *$\iota^*Q = Q_P$, and the nine moments* (17.2) *of $\rho_\iota = \mathrm{Ad}_J \circ \iota \circ \tau$ equal those of ρ_{ref}. Then (J, ι) is a finite central quotient, with induced metric and marking, of (G_n, i_n) or (H_n, j_n), or, when $n = 4$, of (J_*, ι_*). Conversely, each such quotient satisfies these hypotheses. The exceptional forms have volume at most $v_4/2$, with equality precisely for (J_*, ι_*) up to marked isometry.*

Proof. Proposition 17.5 gives the marked metric Lie algebras. For the two reference differentials, (2.7) and Lemma 18.1 give the stated quotients.

For the exceptional differential, the universal covering

$$p_* : \widetilde{J}_* = \mathrm{Sp}(4) \times \mathrm{Spin}(10) \longrightarrow J_*$$

has kernel $\Omega_* = \{\pm1\} \times \{\pm1\}$ [103, Proposition 7.4.2, Corollary 7.6.5, and Proposition 10.6.9]. Let $A(\theta) = \mathrm{diag}(e^{i\theta}, 1, 1, 1)$, $0 \le \theta \le 2\pi$. Along the loop $(A(\theta), 1)$, the symplectic lift of ι_* is $c(e^{i\theta/2}\overline{A(\theta)})$ and ends at -1. The lift of its orthogonal coordinate-plane rotation ends at $-1 \in \mathrm{Spin}(10)$ [103, equation (7.4.23)]. The lifted endpoint is therefore $(-1, -1)$. The loop $(I_4, e^{i\theta})$ has lifted endpoint $(1, -1)$. In the winding coordinates of (2.7), this gives

$$(\iota_*)_* : \pi_1(P_4) \simeq \mathbb{Z}^2 \longrightarrow \pi_1(J_*) \simeq (\mathbb{Z}/2)^2, \qquad (m, l) \longmapsto (m \bmod 2, m + l \bmod 2).$$

This map is surjective. Lemma 18.1 therefore gives a finite central quotient of (J_*, ι_*).

A finite central quotient with the induced metric preserves the marked metric Lie algebra and hence the functions I_J, Φ_J near zero, by (16.4). For $X \in \mathfrak{su}(4)$ and $a, b \in \mathbb{R}$,

$$\|d\iota_*(X + iaI_4, ib)\|^2_{Q_*} = 2Q_{\mathrm{U}(4)}(X, X) + 8a^2 + b^2.$$

Thus $\iota_*^*Q_* = Q_P$. On the central two-torus, its adjoint weights are (17.1). Equations (16.4) and (16.7) give the reference restrictions of I_J, Φ_J to $\mathfrak{z}(\mathfrak{p}_4)$. Finally, $\mathrm{Vol}(J_*) = v_4/2$; a quotient by a finite central subgroup F has volume $v_4/(2|F|)$. □

Let

$$\mathfrak{t}_n = \{(i\,\mathrm{diag}(y_1, \dots, y_n), ib) : y_1, \dots, y_n, b \in \mathbb{R}\} \subset \mathfrak{p}_n.$$

This is the Lie algebra of the diagonal maximal torus of P_n. The reference functions I_{G_n}, Φ_{G_n} correspond to the marking i_n.

Corollary 18.5. *Let J be a compact connected Lie group with a positive bi-invariant metric Q, and let $\iota : P_n \to J$ be a homomorphism. Suppose* $\dim J = (2n+1)^2$ *and*

$$\mathrm{jet}^2_0(I_J|_{\mathfrak{t}_n}) = \mathrm{jet}^2_0(I_{G_n}|_{\mathfrak{t}_n}),$$
$$\mathrm{jet}^6_0(\Phi_J|_{\mathfrak{t}_n}) = \mathrm{jet}^6_0(\Phi_{G_n}|_{\mathfrak{t}_n}).$$

Then (J, ι) is a finite central quotient of (G_n, i_n) or (H_n, j_n), with induced metric and marking. Every such quotient has these Taylor polynomials. If moreover $\mathrm{Vol}(J) \ge v_n$, *then (J, ι) is isometrically isomorphic, with its marking, to one of the two reference groups.*

Proof. The functions I_J, I_{G_n} are invariant quadratic forms. By unitary diagonalization, their equality on $\mathfrak{t}_n$ gives equality on $\mathfrak{p}_n$. Polarization gives $\iota^* Q = Q_P$. The sixth Taylor polynomials restrict equally to $\mathfrak{z}(\mathfrak{p}_n) \subset \mathfrak{t}_n$ and determine the nine moments by the coefficient formula in the proof of Lemma 17.1. Theorem 18.4 applies. Proposition 4.5 and (16.4) give $I_{G_n} = I_{H_n}$ and $\Phi_{G_n} = \Phi_{H_n}$ near zero for every $n \geq 1$.

Suppose the exceptional alternative occurs. Then $n = 4$. Take $X = i\,\mathrm{diag}(x_1, x_2, x_3, x_4)$, with $x_i \in \mathbb{R}$ and $\sum_{i=1}^4 x_i = 0$, and evaluate the weights at $(X + iaI_4, 0)$, $a \in \mathbb{R}$. By the symplectic decomposition in the proof of Proposition 4.5, the weights with a-coefficient 2 are $2a + x_i + x_j$ for the reference marking and $2a - x_i - x_j$ for the exceptional one, $1 \leq i \leq j \leq 4$. Their negatives contribute the same fourth powers; the remaining weights are independent of a. The coefficients of a in the two sums of fourth powers over all complex symplectic weights are therefore

$$128 \sum_{i=1}^4 x_i^3 \quad \text{and} \quad -128 \sum_{i=1}^4 x_i^3, \tag{18.2}$$

since

$$\sum_{1 \leq i \leq j \leq 4} (x_i + x_j)^3 = \tfrac{1}{2} \left(\sum_{i,j=1}^4 (x_i + x_j)^3 + \sum_{i=1}^4 (2x_i)^3 \right) = 8 \sum_{i=1}^4 x_i^3.$$

The orthogonal components of the two markings coincide. Put

$$X_0 = i\,\mathrm{diag}(1, 1, -2, 0), \qquad Y(a, t) = (tX_0 + iaI_4, 0), \qquad a, t \in \mathbb{R}.$$

The degree-four term of each logarithmic amplitude is $1/5760$ times its fourth-power sum over complex adjoint weights, by (16.7). The symplectic contributions to the coefficient of at^3 are $-2/15$ and $2/15$, respectively. Thus the fourth-degree terms along $Y(a, t)$ differ for the exceptional and reference markings, contrary to the sixth-jet equality. The exceptional marking is excluded.

Finite central quotients of the reference groups preserve I_J, Φ_J, which proves the converse. For a quotient by a finite central subgroup F, the volume is $v_n/|F|$. The lower bound gives $|F| = 1$. □

19. Marked recognition and product classification

Fix an integer $n \geq 1$. Write $M_{uv} = M_{uv}(\rho_{\mathrm{ref}})$ for the reference moments. The weights (17.1) give

$$\begin{gathered} M_{0,2k} = 4n, \qquad M_{2k,0} = 2n^2\, 2^{2k} + 4n \quad (k = 1, 2, 3), \\ M_{22} = M_{42} = 4n, \qquad M_{11} = 0. \end{gathered} \tag{19.1}$$

For integers $k \geq 1$ and $u, v \geq 0$ with $u + v = 2k$, the coefficient of $a^u b^v$ in $\Phi_{G_n}(a, b)$ is

$$\frac{1}{2} \frac{\zeta(2k)}{k(2\pi)^{2k}} \binom{2k}{u} M_{uv},$$

by the coefficient formula in the proof of Lemma 17.1.

Theorem 19.1 (Marked recognition). *Let J be a compact connected Lie group with a positive bi-invariant metric Q and a homomorphism $\iota : P_n \to J$. Suppose*

(1) $\dim J = (2n + 1)^2$*;*

(2) $I_J(a, b) = 2na^2 + b^2$ *in the central coordinates near zero;*

(3) for $n \geq 2$, the coefficient of s^2 in $I_J(s(H_0, 0))$ is $2Q_{\mathrm{U}(n)}(H_0, H_0)$ for one fixed nonzero $H_0 \in \mathfrak{su}(n)$;

(4) the coefficients of

$$b^2,\ b^4,\ b^6,\ a^2,\ a^4,\ a^6,\ a^2b^2,\ a^4b^2,\ ab$$

in $\Phi_J(a, b)$ equal those in $\Phi_{G_n}(a, b)$;

(5) $\mathrm{Vol}_Q(J) \geq v_n$.

Then (J, Q, ι) is isometrically isomorphic, with its marking, to (G_n, Q_G, i_n) or (H_n, Q_H, j_n). Conversely, both reference groups satisfy these conditions. In particular, ι is faithful and its graded right-P_n scalar data equal those of (G_n, i_n).

Proof. The pullback ι^*Q is an invariant symmetric bilinear form on $\mathfrak{p}_n$. For $n \geq 2$, its complexification is invariant for the adjoint P_n module $\mathfrak{p}_{n,\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}^2$. The first summand is nontrivial and irreducible, and the second is trivial. Schur's lemma [41, Lemma 9.9 and Corollary 9.12] makes the mixed term zero and the restriction to $\mathfrak{su}(n)$ a scalar multiple of $Q_{\mathrm{U}(n)}$. Conditions (2)–(3) fix this scalar as two and the central form as $2na^2 + b^2$. Thus $\iota^*Q = Q_P$. For $n = 1$, condition (2) determines the whole form.

The coefficients in (4) determine the nine moments (17.2), so Theorem 18.4 applies. A quotient of either reference group by a finite central subgroup F has volume $v_n/|F|$. The exceptional alternative occurs only at $n = 4$ and has volume at most $v_4/2 < v_4$. Condition (5) excludes that alternative and gives $|F| = 1$ for a reference quotient. The reference markings are embeddings, and Theorem 3.3 identifies their graded right-P_n scalar data. Their common metric adjoint module gives the local conditions, and both have volume v_n. □

Conditions (2)–(4) specify three central coefficients of I_J, nine coefficients of Φ_J, and, for $n \geq 2$, one coefficient along $s(H_0, 0)$. Thus thirteen real coefficients suffice, or twelve when $n = 1$. They depend only on the second Taylor polynomial of I_J and the sixth Taylor polynomial of Φ_J. Dimension and volume are separate conditions.

Corollary 19.2. *The graded right-P_n scalar data of (G_n, i_n) have exactly two compact connected bi-invariant marked realizations, (G_n, i_n) and (H_n, j_n), up to isometric Lie group isomorphisms intertwining the marking.*

Proof. By the heat-trace argument in the proof of Theorem 16.2, the graded data determine the dimension, volume, and functions I_J, Φ_J near zero. Theorem 19.1 gives the two reference groups. Their graded data agree by Theorem 3.3; their Lie algebra centers have dimensions one and zero, so they are distinct. □

The groups H_n^{S} and H_n^{O} have the same marked metric adjoint module and volume, but their scalar multiplicities at $2n + 1$ differ by Theorem 18.3.

19.1. **Equal local germs with different kernels.** For each integer $m \geq 1$, let $\mu_m = \{\zeta \in \mathrm{U}(1) : \zeta^m = 1\}$.

Example 19.3. Identify $J = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ with $\mathrm{U}(1)^2$, give it the Euclidean metric, and let $P = \mathrm{U}(1)$. For $s \in \mathbb{R}$, put

$$\iota_1(e^{is}) = (e^{5is}, 1), \qquad \iota_2(e^{is}) = (e^{3is}, e^{4is}),$$

$$Y_1(s) = (5s, 0), \qquad Y_2(s) = (3s, 4s).$$

Both markings have the same ambient spectrum, volume and dimension. Near $s = 0$, their common rate and amplitude are $I(s) = 25s^2$ and $A(s) = 1$. The kernels are μ_5 and $\{1\}$, respectively: $3s, 4s \in 2\pi\mathbb{Z}$ implies $s \in 2\pi\mathbb{Z}$.

For the heat kernel relative to Riemannian volume,

$$k_t([Y]) = (4\pi t)^{-1} \sum_{k \in \mathbb{Z}^2} e^{-|Y + 2\pi k|^2/(4t)}, \qquad Y \in \mathbb{R}^2, \quad t > 0.$$

For $|Y| \leq \pi/2$ and $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$|Y + 2\pi k|^2 - |Y|^2 \geq 4\pi^2|k|^2 - 2\pi^2|k| \geq 2\pi^2|k|^2.$$

The sum of the nonzero lattice terms, divided by the zero term, is therefore $O(e^{-\pi^2/(2t)})$, uniformly on this ball. Indeed, for fixed $t_0 > 0$ and $0 < t \leq t_0$,

$$\sum_{k \neq 0} e^{-\pi^2|k|^2/(2t)} \leq e^{-\pi^2/(2t)} \sum_{k \neq 0} e^{-\pi^2(|k|^2 - 1)/(2t_0)},$$

and the last sum is finite. Let $H_j(t,s) = k_t([Y_j(s)])/k_t(0)$ for $j = 1, 2$. Since $|Y_j(s)| = 5|s|$, it follows that

$$e^{25s^2/(4t)} H_j(t,s) = 1 + O(e^{-\pi^2/(2t)}) \qquad (|s| \le \pi/10).$$

Thus the two normalized traces have the same complete formal local expansion after removal of their common Gaussian factor.

For every integer $m \ge 1$, give the same torus the metric with matrix $Q_m = \mathrm{diag}(m^{-2}, m^2)$ in the covering coordinates and the marking $\iota_m(e^{is}) = (e^{ims}, 1)$. Let I_m, A_m be the local rate and amplitude for this metric and marking. Then

$$\dim J = 2, \qquad \mathrm{Vol}_{Q_m}(J) = 4\pi^2, \qquad I_m(s) = s^2, \qquad A_m(s) = 1, \qquad \ker \iota_m = \mu_m.$$

For each fixed m, the same Gaussian calculation applies on a sufficiently small neighborhood of $s = 0$. The kernel orders are therefore unbounded at fixed dimension, volume, and local germs.

19.2. **Equal complete local heat expansions on a two-torus.** Let (K, Q) be a compact connected Lie group with a positive bi-invariant metric. Put $\mathfrak{k} = \mathrm{Lie}(K)$ and $d = \dim K$, and, for $t > 0$, let k_t^K be its scalar heat kernel at the identity, relative to Riemannian volume. Write Scal_K for its constant scalar curvature and j_K for (16.2). On a sufficiently small normal ball about zero in $\mathfrak{k}$, for every integer $M \ge 0$,

$$(4\pi t)^{d/2} e^{|Y|_Q^2/(4t)} k_t^K(\exp_K Y) = j_K(Y)^{-1/2} \sum_{\ell=0}^{M} \frac{t^\ell}{\ell!} \left(\frac{\mathrm{Scal}_K}{6}\right)^\ell + O(t^{M+1}), \tag{19.2}$$

uniformly for Y in compact subsets as $t \downarrow 0$. The coefficients follow from [59, Theorem 3.3 and Proposition 3.5], with its time parameter $2t$ and its Jacobian factor $j_K^{1/2}$; the scalar shift is $\mathrm{Scal}_K/6$ by the identity in the proof of Corollary 16.3. The relative remainder follows from [74, Theorem 1.1], applied to the scalar Laplacian and pairs $(e, \exp_K Y)$ in this normal ball. For the negative Killing metric on a simply connected compact semisimple group, the coefficient formula appears in [108, proof of Theorem 3, pp. 290–291].

For $a, b \in \mathbb{R}$, identify $\mathbb{T}^2$ with $Z(P_2)$ by $(e^{ia}, e^{ib}) \mapsto (e^{ia} I_2, e^{ib})$.

Proposition 19.4. *There is a compact connected Lie group J with a positive bi-invariant metric and a faithful marking $\iota_J : \mathbb{T}^2 \to J$ having the same dimension 25, volume, scalar curvature 60, and complete formal local equivariant heat expansions as $(G_2, i_2|_{Z(P_2)})$, but a different scalar spectrum.*

Proof. Let $G = \mathrm{U}(5)$ with its reference metric Q_G and marking

$$\iota_G(e^{ia}, e^{ib}) = \mathrm{diag}(e^{ia}, e^{ia}, e^{-ia}, e^{-ia}, e^{ib}).$$

For $L > 0$, let S_L^1 be $\mathrm{U}(1)$ with its translation-invariant metric of total length L, and put

$$J = (\mathrm{U}(3), Q_G/2)^2 \times (\mathrm{Sp}(1), Q_C)^2 \times S_L^1,$$

with the trace metrics of (2.2) in the indicated matrix sizes. The marking ι_J has component $\mathrm{diag}(e^{ia}, e^{-ia}, e^{ib})$ in each unitary factor, $c(e^{ia})$ in each symplectic factor, and 1 in S_L^1. The diagonal entries make both markings faithful, and $\dim J = 2 \cdot 9 + 2 \cdot 3 + 1 = 25$. Their induced quadratic forms are

$$I_G(a,b) = 4a^2 + b^2, \qquad I_J(a,b) = 2(a^2 + b^2/2) + 2a^2 = 4a^2 + b^2.$$

Both adjoint modules have four real rotation planes of weight $2a$, two of weight $a + b$, two of weight $a - b$, and a fixed space of dimension nine. For $K = G, J$, put $Y_K(a,b) = d\iota_K(a,b)$ in angular Lie algebra coordinates. Their common restricted exponential Jacobian is

$$\begin{aligned} j(a,b) &= j_G(Y_G(a,b)) = j_J(Y_J(a,b)) \\ &= \left(\frac{\sin a}{a}\right)^8 \left(\frac{\sin((a+b)/2)}{(a+b)/2}\right)^4 \left(\frac{\sin((a-b)/2)}{(a-b)/2}\right)^4, \end{aligned}$$

with each quotient equal to 1 at zero.

For $\mathrm{U}(N)$ with Q_G, the Killing form is $-2NQ_G$ on $\mathfrak{su}(N)$ and zero on the center. The curvature formula (4.16) with trivial isotropy gives $\mathrm{Scal} = N(N^2-1)/2$. Each scaled unitary factor therefore has scalar curvature 24. The metric Q_C on $\mathrm{Sp}(1)$ is that of the unit round three-sphere, of scalar curvature 6. Hence $\mathrm{Scal}_J = 48 + 12 = 60 = \mathrm{Scal}_G$. Choose

$$L = \frac{\mathrm{Vol}(G)}{\mathrm{Vol}(\mathrm{U}(3), Q_G/2)^2 \,\mathrm{Vol}(\mathrm{Sp}(1), Q_C)^2} > 0.$$

Then $v = \mathrm{Vol}(G) = \mathrm{Vol}(J)$.

For $K = G, J$ and $t > 0$, write $T_K(t,a,b) = \mathrm{Tr}(R_{(e^{ia},e^{ib})}e^{-t\Delta_K})$ for the marked heat trace and $H_K(t,a,b) = T_K(t,a,b)/T_K(t,0,0)$. Equations (16.1) and (19.2) give, for every integer $M \geq 0$,

$$\frac{(4\pi t)^{25/2}}{v} e^{(4a^2+b^2)/(4t)} T_K(t,a,b) = j(a,b)^{-1/2} \sum_{\ell=0}^{M} \frac{(10t)^\ell}{\ell!} + O(t^{M+1}),$$

uniformly on compact subsets of a common neighborhood of $(0,0)$. Division by the diagonal expansion gives

$$e^{(4a^2+b^2)/(4t)} H_K(t,a,b) = j(a,b)^{-1/2} + O(t^{M+1}).$$

The coefficient functions agree for $K = G$ and $K = J$ in both displayed expansions.

An eigenfunction on a symplectic factor with eigenvalue three pulls back to J with that eigenvalue. Proposition 3.8 at $n = 2$ gives first positive eigenvalue five for $\mathrm{U}(5)$. Their scalar spectra are therefore different. □

The centralizer of $\iota_J(\mathbb{T}^2)$ in J is a torus. Each unitary factor has three distinct one-dimensional character spaces; each symplectic centralizer is a circle, and the final circle factor is central. The marking ι_J does not extend to a homomorphism $P_2 \to J$. Indeed, the differential of an extension would send $\mathfrak{su}(2)$ into the abelian Lie algebra of this centralizer. Since $\mathfrak{su}(2)$ is perfect, its image would be zero. The extension would therefore be trivial on the connected subgroup $\mathrm{SU}(2) \times \{1\}$. But $(-I_2, 1)$ belongs to this subgroup and to $Z(P_2)$, and its prescribed image has component $\mathrm{diag}(-1,-1,1)$ in each unitary factor, a contradiction.

19.3. **Orthogonal splitting and products.** Fix integers $r \geq 1$ and $n_i \geq 1$, and real numbers $c_i > 0$, for $1 \leq i \leq r$. Put $P = \prod_{i=1}^r P_{n_i}$ and $\mathfrak{p} = \mathrm{Lie}(P)$. Let (J_0, Q_0, ι_0) be the product of $(G_{n_i}, c_i Q_G, i_{n_i})$, and put $\mathfrak{g}_0 = \mathrm{Lie}(J_0)$. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \{G, H\}^r$, let $(J_\varepsilon, Q_\varepsilon, \iota_\varepsilon)$ be the product whose ith marked factor is $(G_{n_i}, c_i Q_G, i_{n_i})$ if $\varepsilon_i = G$, and $(H_{n_i}, c_i Q_H, j_{n_i})$ if $\varepsilon_i = H$. A marked isomorphism intertwines the given maps from the fixed group P.

Lemma 19.5. *Let $\mathfrak{g}$ be a compact Lie algebra with a positive invariant form Q, and let $\phi : \mathfrak{p} \to \mathfrak{g}$ be a homomorphism whose adjoint action integrates to P. Suppose there is an orthogonal P-isomorphism*

$$F : \mathfrak{g} \longrightarrow \mathfrak{g}_0, \qquad F \circ \phi = d\iota_0.$$

Then $(\mathfrak{g}, Q, \phi)$ is isomorphic to the marked metric Lie algebra of one J_ε.

Proof. Let $W = \mathfrak{g}^P$, and let V_i be the sum of the nontrivial isotypic components on which every factor other than P_{n_i} acts trivially. The prescribed module gives

$$\mathfrak{g} = W \perp \bigoplus_{i=1}^{r} V_i, \qquad \dim W = 3r.$$

For $i \neq j$, every complex constituent of $V_i \otimes V_j$ is nontrivial on both factors, whereas no constituent of $\mathfrak{g}$ is. Equivariance of the bracket gives $[V_i, V_j] = 0$ and $[W, V_i] \subset V_i$. The subalgebra generated by V_i is preserved by W and, by the derivation identity, commutes with each V_j, $j \neq i$. It is therefore an ideal; denote it by $\mathfrak{a}_i$.

The center of $\mathfrak{g}$ is P-fixed, so V_i has zero central projection. The compact reductive decomposition [119, Proposition 3.25(b) and Theorem 3.21(a),(b)] places $\mathfrak{a}_i$ in $[\mathfrak{g},\mathfrak{g}]$ as a sum of simple ideals. The ideals $\mathfrak{a}_i$ commute, so these sums have disjoint simple factors and are orthogonal. Put $\mathfrak{a}_0 = (\bigoplus_i \mathfrak{a}_i)^{\perp}$. Then

$$\mathfrak{g} = \mathfrak{a}_0 \perp \bigoplus_{i=1}^{r} \mathfrak{a}_i, \tag{19.3}$$

an orthogonal sum of ideals, with $\mathfrak{a}_0$ P-trivial.

For $j \neq i$, $\phi(\mathfrak{p}_{n_j})$ centralizes V_i and hence $\mathfrak{a}_i$. Its projection into $\mathfrak{a}_i$ is central in this semisimple ideal and is zero. Put $\phi_i = \mathrm{pr}_{\mathfrak{a}_i} \circ \phi|_{\mathfrak{p}_{n_i}}$. The ideal projection is a Lie homomorphism, and the adjoint action of ϕ_i integrates to the prescribed P_{n_i} action. Its nontrivial part is exactly V_i, which is also the orthogonal complement of the fixed space for the negative Killing form. Moreover, V_i generates $\mathfrak{a}_i$ as a Lie algebra and hence as an ideal of itself. Lemma 17.3 applies. Let u be the number of unitary alternatives. Counting fixed dimensions in (19.3) gives

$$\dim \mathfrak{a}_0 = 3r - 2u - 3(r-u) = u. \tag{19.4}$$

For $1 \le i \le r$, put $L_i = \mathrm{pr}_{\mathfrak{a}_0} \circ \phi|_{\mathfrak{p}_{n_i}}$. Since the P action on $\mathfrak{a}_0$ is trivial, $[L_i X, Y] = 0$ for $X \in \mathfrak{p}_{n_i}$ and $Y \in \mathfrak{a}_0$. Thus $\operatorname{im} L_i$ is central and L_i vanishes on $\mathfrak{su}(n_i)$. The images of distinct L_i are orthogonal: the marked factors are orthogonal and their semisimple projections have disjoint support. Each residual form $L_i^* Q$ is nonnegative and has rank $\dim \operatorname{im} L_i$.

For a unitary alternative, write its simple metric as $s_i Q_G$, $s_i > 0$. The unitary metric calculation in the proof of Theorem 17.4, with marked form $c_i Q_P$, gives the following restrictions. If $n_i \ge 2$, the norm on $\mathfrak{su}(n_i)$ gives $s_i = c_i$, and

$$\|L_i(X + iaI_{n_i}, ib)\|_Q^2 = \frac{c_i b^2}{2n_i + 1}, \qquad X \in \mathfrak{su}(n_i), \quad a, b \in \mathbb{R}.$$

If $n_i = 1$, the residual form is

$$\|L_i(ia, ib)\|_Q^2 = 2(c_i - s_i)a^2 + (c_i - 2s_i/3)b^2. \tag{19.5}$$

Its nonnegativity gives $s_i \le c_i$, so the coefficient of b^2 is at least $c_i/3 > 0$. Thus every unitary alternative has nonzero residual image.

There are u such mutually orthogonal images in the u-dimensional space (19.4). Each has dimension one, their sum is $\mathfrak{a}_0$, and every other residual image is zero. All these images are central, so $\mathfrak{a}_0$ is abelian. For $n_i = 1$ in a unitary alternative, rank one in (19.5) and the positive coefficient of b^2 give $s_i = c_i$.

Adjoin $\operatorname{im} L_i$ to each unitary $\mathfrak{a}_i$, and retain the other $\mathfrak{a}_i$. These orthogonal ideals have the reference P_{n_i} modules, marked subalgebras, and induced forms $c_i Q_P$. Their marked subspaces are isometric. The real representation and polar-decomposition argument in the proof of Proposition 4.5 extends these isometries to orthogonal module isomorphisms. After division of each metric by c_i, Theorem 17.4 identifies the marked metric Lie algebra with its reference factor. Their orthogonal sum is the required marked metric Lie algebra. □

Theorem 19.6 (The marked product classification). *Let J be a compact connected Lie group with a positive bi-invariant metric Q and a homomorphism $\iota : P \to J$. Its graded right-P scalar data agree with those of J_0 if and only if (J, Q, ι) is isometrically isomorphic, with its marking, to one of the 2^r groups $(J_\varepsilon, Q_\varepsilon, \iota_\varepsilon)$. These marked classes are distinct, including repeated values of n_i or c_i.*

Proof. Theorem 16.2 gives an orthogonal P-isomorphism of the metric adjoint modules respecting the differential markings, and gives $\ker \iota = \ker \iota_0 = \{1\}$. Lemma 19.5 identifies the marked metric Lie algebra of J with that of one J_ε. The factorwise maps (2.7) give a surjection $\pi_1(P) \to \pi_1(J_\varepsilon)$. Lemma 18.1 gives a marked surjective local isometry $J_\varepsilon \to J$ with finite central kernel F.

Theorem 3.3 and the leading heat coefficient give $\operatorname{Vol}(G_{n_i}) = \operatorname{Vol}(H_{n_i}) = v_{n_i}$. Hence

$$\operatorname{Vol}(J_\varepsilon) = \prod_{i=1}^{r} c_i^{(2n_i+1)^2/2} v_{n_i} = \operatorname{Vol}(J_0).$$

The assumed scalar data give $\operatorname{Vol}(J) = \operatorname{Vol}(J_0)$. Since $\operatorname{Vol}(J) = \operatorname{Vol}(J_\varepsilon)/|F|$, one has $|F| = 1$.

Conversely, the finite tensor-product assertion of Lemma 2.4 applies to the factor intertwiners of Theorem 3.3 and the diagonal function

$$f : [0,\infty)^r \longrightarrow [0,\infty), \qquad f(\lambda_1, \dots, \lambda_r) = \sum_{i=1}^{r} c_i^{-1} \lambda_i,$$

where λ_i is an eigenvalue for the unscaled reference metric. It gives the graded right-P equivalence. A marked isomorphism preserves each V_i, defined by the action of the fixed factor P_{n_i}, and hence the semisimple ideal it generates. The two alternatives in Lemma 17.3 have dimensions $(2n_i + 1)^2 - 1$ and $(2n_i + 1)^2$, respectively. Thus the isomorphism preserves every ε_i. □

For the empty product $r = 0$, equality with the scalar data of the point forces $\dim J = 0$. A connected realization is therefore the point.

Part VI. Metrics on compact simple groups

For $n \geq 1$, the groups G_n and H_n embed in $L = \mathrm{SU}(4n + 2)$, equipped with its trace metric. Yu's weighted dimension-data identities [116], combined with finite support separation and the leading Weyl-denominator term, give equal spectra for their induced Casimirs in every finite-dimensional L-module (Theorem 20.1). The scalar Laplacian formula for subgroup scaling then gives scalar-isospectral left-invariant metrics on L for every $s > 0$; for $0 < s < 1$, they are normal homogeneous and not locally isometric (Corollary 22.4). For $m \geq 6r \geq 6$, commuting block embeddings give 2^r scalar-isospectral metrics on $\mathrm{SU}(m)$ that are pairwise not locally isometric (Theorem 22.5).

20. Casimir operators in an ambient representation

Fix an integer $n \geq 1$, and put $N = 2n + 1$, $J = \mathrm{SU}(2N)$, and $\mathfrak{j} = \mathrm{Lie}(J)$. For $X, Y \in \mathfrak{j}$, set $Q(X, Y) = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(XY)$. Consider the connected compact subgroups

$$K_A = \{\operatorname{diag}(A, \overline{A}) : A \in \mathrm{U}(N)\},$$
$$K_B = \{\operatorname{diag}(S, B) : S \in \mathrm{Sp}(n),\ B \in \mathrm{SO}(2n + 2)\}.$$

Both embeddings are injective, and both groups have dimension N^2. For $R \in \{A, B\}$, put $\mathfrak{k}_R = \mathrm{Lie}(K_R)$. Identify K_A with $\mathrm{U}(N)$ and K_B with $\mathrm{Sp}(n) \times \mathrm{SO}(2n + 2)$ through the displayed embeddings. For $X, Y \in \mathfrak{u}(N)$, $X_1, Y_1 \in \mathfrak{sp}(n)$, and $X_2, Y_2 \in \mathfrak{so}(2n + 2)$, define the reference forms by

$$Q_A(X, Y) = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(XY),$$
$$Q_B((X_1, X_2), (Y_1, Y_2)) = -\tfrac{1}{2}\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(X_1 Y_1) - \tfrac{1}{2}\operatorname{Tr}_{\mathbb{R}}(X_2 Y_2).$$

Under these embeddings,

$$Q|_{\mathfrak{k}_A} = 2Q_A, \qquad Q|_{\mathfrak{k}_B} = 2Q_B. \tag{20.1}$$

Put $\mathfrak{j}_{\mathbb{C}} = \mathfrak{j} \otimes_{\mathbb{R}} \mathbb{C}$. For a Q_R-orthonormal basis $(Z_{R,a})_{a=1}^{N^2}$ of $\mathfrak{k}_R$, define in the universal enveloping algebra of $\mathfrak{j}_{\mathbb{C}}$

$$C_R^{\mathrm{ref}} = -\sum_{a=1}^{N^2} Z_{R,a}^2.$$

The same notation denotes their induced endomorphisms on a J-module. Every finite-dimensional complex J-module admits an invariant Hermitian form [41, Proposition 35.5]; the operators C_R^{ref} are then nonnegative and self-adjoint. Put

$$D_R = \tfrac{1}{2} C_R^{\mathrm{ref}}, \qquad R \in \{A, B\}. \tag{20.2}$$

By (20.1), D_R is the positive Casimir for $Q|_{\mathfrak{k}_R}$.

Yu's weighted dimension-data identities [116, Theorem 2.1 and Proposition 2.2] (see also [117, arXiv version 2, Proposition 2.5(i)]) imply the following equality.

Theorem 20.1 (The ambient Casimir identity). *For every finite-dimensional complex representation V of $\mathrm{SU}(4n+2)$, the endomorphisms C_A^{ref} and C_B^{ref} on V have the same eigenvalues with the same multiplicities. Equivalently,*

$$\mathrm{Tr}_V(e^{-tC_A^{\mathrm{ref}}}) = \mathrm{Tr}_V(e^{-tC_B^{\mathrm{ref}}}) \qquad (t \geq 0).$$

Equip $\mathbb{R}^N$ with its Euclidean inner product and norm. For $\theta \in \mathbb{R}^N$, put

$$\tau(\theta) = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}).$$

The torus $T = \{\tau(\theta) : \theta \in \mathbb{R}^N\}$ is maximal in K_A. Choose a maximal torus $T_B \subset K_B$ and $g_0 \in J$ such that $g_0 T_B g_0^{-1} = T$, with the first n coordinates corresponding to $\mathrm{Sp}(n)$ and the remaining $n+1$ to $\mathrm{SO}(2n+2)$. For a character χ of K_B and $a \in T$, write $\chi|_T(a) = \chi(g_0^{-1} a g_0)$. Under this identification, the character lattice is $\mathbb{Z}^N$; for $\nu \in \mathbb{Z}^N$, write $e^\nu(\tau(\theta)) = \exp(i\langle \nu, \theta\rangle)$. The reference forms induce the Euclidean dual metric, as in (2.2). Let S_N be the symmetric group on the N coordinates and put $W = \{\pm 1\}^N \rtimes S_N$. The normalizer of T in J induces precisely W on its character lattice [116, p. 5].

The group algebra $\mathbb{Q}[\mathbb{Z}^N]$ has basis $(e^\nu)_{\nu \in \mathbb{Z}^N}$; for $\mu, \nu \in \mathbb{Z}^N$, its multiplication is $e^\mu e^\nu = e^{\mu+\nu}$. For $\gamma \in W$, $\nu \in \mathbb{Z}^N$, $X \in \mathbb{R}^N$, and a polynomial p on $\mathbb{R}^N$, define

$$\gamma e^\nu = e^{\gamma\nu}, \qquad (\gamma p)(X) = p(\gamma^{-1} X).$$

For a subgroup $\Gamma \subset W$, set $\mathrm{Av}_\Gamma = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \gamma$. Write $\epsilon(\gamma) = \det \gamma$.

Write $X = (x, y, z) \in \mathbb{R}^N$, with $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and put $w = (y_1, \dots, y_n, z)$. Define

$$p_A(X) = \prod_{1 \leq i < j \leq N} (X_i - X_j),$$

$$p_C(x) = \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2),$$

$$p_D(y, z) = \prod_{1 \leq i < j \leq n+1} (w_i^2 - w_j^2), \qquad p_B(X) = p_C(x) p_D(y, z).$$

Let $W(C_n)$ consist of all signed permutations of x, and $W(D_{n+1})$ of permutations of (y, z) with an even number of sign changes. The Weyl groups are $W_A = S_N$ and $W_B = W(C_n) \times W(D_{n+1})$. Choose positive roots whose half-sums are

$$\rho_C = (n, n-1, \dots, 1), \qquad \rho_D = (n, n-1, \dots, 1, 0),$$
$$\rho_A = (n, n-1, \dots, 1, 0, -1, \dots, -n), \qquad \rho_B = \rho = (\rho_C, \rho_D).$$

Put $\kappa = 2\sum_{j=1}^n j^2$. Then $|\rho_A|^2 = |\rho_B|^2 = \kappa$. Let $\mathcal{S} \subset W$ be generated by the independent exchanges $x_i \leftrightarrow y_i$ and by $z \mapsto -z$. For $s \in \mathbb{R}$, set $v(s) = (1, s, \dots, s^{2n})^T$ and

$$D(X) = \det[v(x_1), v(-y_1), \dots, v(x_n), v(-y_n), v(z)].$$

Lemma 20.2. *The denominators in the following formula are positive, and*

$$\mathrm{Av}_{\mathcal{S}} \frac{D(X)}{D(\rho)} = \mathrm{Av}_{\mathcal{S}} \frac{p_C(x) p_D(y, z)}{p_C(\rho_C) p_D(\rho_D)}. \tag{20.3}$$

Proof. Define $L \in W$ by

$$LX = (x_1, \dots, x_n, z, -y_n, \dots, -y_1). \tag{20.4}$$

Then $L\rho = \rho_A$. In $\mathbb{Q}[\mathbb{Z}^N]$, put

$$b_A = \sum_{u \in W_A} \epsilon(u) e^{-L^{-1} u \rho_A}, \qquad b_B = \sum_{u \in W_B} \epsilon(u) e^{-u\rho}.$$

For every integer $m \geq 1$, the weights $(m-1)\rho_A$ and $(m-1)\rho$ are dominant integral. Since $(\rho_A)_i + (\rho_A)_{N+1-i} = 0$ for $1 \leq i \leq N$, and the first n and first $n+1$ coordinates of ρ_A are ρ_C and ρ_D, these weights form the pair in [116, Theorem 2.1]. The orthogonal weight $(m-1)\rho_D$ is integral and has last coordinate zero. By [116, Proposition 2.2], the normalized W-averages of the corresponding shifted Weyl denominators are equal. Since $\mathrm{Av}_W L^{-1} = \mathrm{Av}_W$, applying L^{-1} inside the unitary average gives

$$\mathrm{Av}_W(e^{m\rho} b_A) = \mathrm{Av}_W(e^{m\rho} b_B).$$

Every exponent of b_A and b_B has norm $\sqrt{\kappa}$. Distinct points of $W\rho$ are integral and have distance at least one. For $m = 2\kappa + 1 > 2\sqrt{\kappa}$, the closed balls of radius $\sqrt{\kappa}$ centered at distinct points of $mW\rho$ are disjoint. Each positive coordinate of ρ occurs twice, and zero occurs once; hence $\mathrm{Stab}_W(\rho) = \mathcal{S}$. Restriction of the preceding equality to the coefficients whose exponents lie in the ball centered at $m\rho$ gives

$$\frac{|\mathcal{S}|}{|W|} e^{m\rho} \mathrm{Av}_{\mathcal{S}}\, b_A = \frac{|\mathcal{S}|}{|W|} e^{m\rho} \mathrm{Av}_{\mathcal{S}}\, b_B.$$

Cancellation yields

$$\mathrm{Av}_{\mathcal{S}}\, b_A = \mathrm{Av}_{\mathcal{S}}\, b_B. \tag{20.5}$$

Put $q = n(2n+1)$, the common number of positive roots of K_A and K_B. For $R = A, B$ and $t \in \mathbb{R}$, let $b_R(tX)$ denote the image of b_R under $e^{\mu} \mapsto \exp(t\langle \mu, X\rangle)$. The Weyl denominator formula [41, Corollary 25.5] gives, for each fixed X as $t \to 0$,

$$\begin{aligned} b_A(tX) &= (-t)^q p_A(LX) + O(t^{q+2}), \\ b_B(tX) &= (-t)^q 2^n p_C(x) p_D(y,z) + O(t^{q+2}). \end{aligned}$$

The factor 2^n comes from the n positive roots $x \mapsto 2x_i$ of C_n. The column permutation from the coordinate order in LX to that in D has n^2 inversions. The Vandermonde determinant in the former order is $(-1)^q p_A(LX)$. Since $q \equiv n^2 \pmod 2$,

$$p_A(LX) = D(X). \tag{20.6}$$

Comparison of the coefficients of t^q in (20.5) gives

$$\mathrm{Av}_{\mathcal{S}}\, D = 2^n \mathrm{Av}_{\mathcal{S}}\, p_B. \tag{20.7}$$

Evaluation at the $\mathcal{S}$-fixed point ρ yields

$$D(\rho) = p_A(\rho_A) = 2^n p_C(\rho_C) p_D(\rho_D) > 0.$$

Division proves (20.3). □

Proof of Theorem 20.1. Fix a finite-dimensional complex J-module V, with character χ_V. For $\nu \in \mathbb{Z}^N$, let f_ν be the multiplicity of e^ν in $V|_T$. Its character is the finite sum

$$f = \chi_V|_T = \sum_{\nu \in \mathbb{Z}^N} f_\nu e^\nu.$$

Since characters are invariant under conjugation and the normalizer of T induces W, one has $f_{\gamma\nu} = f_\nu$ for $\gamma \in W$.

For $R \in \{A, B\}$, let $\Lambda_R^+ \subset \mathbb{Z}^N$ be the set of dominant integral highest weights of K_R. For $\lambda \in \Lambda_R^+$, denote the irreducible K_R-module of highest weight λ by $V_{R,\lambda}$, the restriction of its character to T by $\chi_{R,\lambda}$, and its multiplicity in $V|_{K_R}$ by $m_{R,\lambda}$. Complete reducibility [41, Corollary 35.6] gives

$$V|_{K_R} \simeq \bigoplus_{\lambda \in \Lambda_R^+} m_{R,\lambda} V_{R,\lambda}, \qquad f = \sum_{\lambda \in \Lambda_R^+} m_{R,\lambda} \chi_{R,\lambda},$$

with only finitely many nonzero integers $m_{R,\lambda} \geq 0$. For $\mathrm{U}(N)$, apply the semisimple character and dimension formulas to the $\mathrm{SU}(N)$ factor of the finite cover $\mathrm{SU}(N)\times\mathrm{U}(1)$. The central circle contributes a single character, whose value at the identity is one. Thus [41, Proposition 28.1] gives

$$\dim V_{R,\lambda} = \frac{p_R(\lambda+\rho_R)}{p_R(\rho_R)}.$$

By Lemma 2.2, the reference Casimir eigenvalue is $|\lambda+\rho_R|^2 - |\rho_R|^2$. The norm is that of all N coordinates, including the central direction of $\mathrm{U}(N)$.

For $t \geq 0$, set $\mathcal{H}_R(f;t) = \mathrm{Tr}_V(e^{-tC_R^{\mathrm{ref}}})$. Then

$$\begin{aligned}\mathcal{H}_R(f;t) &= \sum_{\lambda\in\Lambda_R^+} m_{R,\lambda}\dim V_{R,\lambda}\, e^{-t(|\lambda+\rho_R|^2-|\rho_R|^2)} \\ &= \frac{e^{\kappa t}}{p_R(\rho_R)}\sum_{\nu\in\mathbb{Z}^N} f_\nu\, p_R(\nu+\rho_R)e^{-t|\nu+\rho_R|^2}.\end{aligned} \tag{20.8}$$

Indeed, put $\delta_R = \sum_{u\in W_R}\epsilon(u)e^{u\rho_R}$. For $\mu\in\mathbb{Z}^N$ and $h\in\mathbb{Q}[\mathbb{Z}^N]$, denote the coefficient of e^μ in h by $[e^\mu]h$. Weyl's character formula [41, Theorem 25.4] gives

$$\begin{aligned}&\sum_{\mu\in\mathbb{Z}^N}[e^\mu](f\delta_R)p_R(\mu)e^{-t|\mu|^2} \\ &\quad= \sum_{\lambda\in\Lambda_R^+} m_{R,\lambda}\sum_{u\in W_R}\epsilon(u)p_R\big(u(\lambda+\rho_R)\big)e^{-t|u(\lambda+\rho_R)|^2} \\ &\quad= |W_R|\sum_{\lambda\in\Lambda_R^+} m_{R,\lambda}p_R(\lambda+\rho_R)e^{-t|\lambda+\rho_R|^2}.\end{aligned}$$

Here $p_R(uX) = \epsilon(u)p_R(X)$ and u is orthogonal. Expansion of δ_R gives another expression for the same sum:

$$\begin{aligned}&\sum_{u\in W_R}\sum_{\nu\in\mathbb{Z}^N} f_\nu\,\epsilon(u)p_R(\nu+u\rho_R)e^{-t|\nu+u\rho_R|^2} \\ &\qquad= |W_R|\sum_{\nu\in\mathbb{Z}^N} f_\nu\, p_R(\nu+\rho_R)e^{-t|\nu+\rho_R|^2}.\end{aligned}$$

The equality follows by replacing ν with $u\nu$ and applying $f_{u\nu} = f_\nu$ and $\epsilon(u)^2 = 1$. All sums are finite. Cancellation of $|W_R|$ and multiplication by $e^{\kappa t}/p_R(\rho_R)$ prove (20.8).

Let $L\in W$ be given by (20.4). By (20.6),

$$L\rho = \rho_A, \qquad \frac{p_A(LX)}{p_A(\rho_A)} = \frac{D(X)}{D(\rho)}.$$

Changing variables by L in (20.8) and applying $f_{L\nu} = f_\nu$ gives

$$\mathcal{H}_A(f;t) = e^{\kappa t}\sum_{\nu\in\mathbb{Z}^N} f_\nu\,\frac{D(\nu+\rho)}{D(\rho)}\,e^{-t|\nu+\rho|^2}.$$

The expression for $\mathcal{H}_B(f;t)$ has $p_B(\nu+\rho)/p_B(\rho)$ in place of the determinant ratio. Since S fixes ρ, both f_ν and $e^{-t|\nu+\rho|^2}$ are invariant under $\nu\mapsto\gamma\nu$ for $\gamma\in S$. Averaging the polynomial factor and applying Lemma 20.2 gives $\mathcal{H}_A(f;t) = \mathcal{H}_B(f;t)$ for $t\geq 0$. Lemma 2.3, with $t_0 = 0$ and the finite union of the two Casimir spectra, equates their multiplicities. □

21. Subgroup contractions on a compact simple group

Let L be a compact connected Lie group with simple Lie algebra $\mathfrak{l}$, and let Q be a positive $\mathrm{Ad}(L)$-invariant inner product on $\mathfrak{l}$. For a Lie algebra $\mathfrak{a}$, denote its Killing form by $B_{\mathfrak{a}}$. Write

$$B_{\mathfrak{l}} = -4cQ, \qquad c > 0.$$

For a closed connected subgroup $K \leq L$, put $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{m} = \mathfrak{k}^{\perp_Q}$. For $0 < s < 1$, define the left-invariant metric

$$g_{K,s}|_{\mathfrak{m}} = Q|_{\mathfrak{m}}, \qquad g_{K,s}|_{\mathfrak{k}} = sQ|_{\mathfrak{k}}, \qquad \mathfrak{m} \perp_{g_{K,s}} \mathfrak{k}. \tag{21.1}$$

Lemma 21.1. *The metric $g_{K,s}$ is normal homogeneous for $L \times K$ and has nonnegative sectional curvature. For $\Delta K = \{(k,k) : k \in K\} \leq L \times K$, its normal presentation is*

$$(L, g_{K,s}) \cong \big((L,Q) \times (K, \tfrac{s}{1-s} Q|_{\mathfrak{k}})\big)/\Delta K.$$

The diffeomorphism is induced by $(x,k) \mapsto xk^{-1}$.

Proof. Apply [3, Theorem 2.21] with trivial initial isotropy, one deformation subgroup K, and

$$t_0 = r_0 = 1, \qquad t_1 = s, \qquad r_1 = \frac{s}{1-s}.$$

The parameter relation is $r_1^{-1} = t_1^{-1} - t_0^{-1}$. Since $0 < s < 1$, both factors have positive bi-invariant metrics. The horizontal lifts of $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$ are, respectively,

$$(X, 0), \qquad (sY, -(1-s)Y).$$

Nonnegative sectional curvature follows from (4.16). □

Let $\mathrm{Ric}_{K,s}$ denote the Ricci tensor of $g_{K,s}$. Ricci eigenvalues are taken with respect to $g_{K,s}$.

Lemma 21.2. *Let $(E_i)_{i=1}^{\dim K}$ be a Q-orthonormal basis of $\mathfrak{k}$. The splitting $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{m}$ is orthogonal for $\mathrm{Ric}_{K,s}$, whose restrictions are*

$$\mathrm{Ric}_{K,s}(X,X) = cs^2 Q(X,X) - \frac{1-s^2}{4} B_{\mathfrak{k}}(X,X), \qquad X \in \mathfrak{k}, \tag{21.2}$$

$$\mathrm{Ric}_{K,s}(X,X) = cQ(X,X) + \frac{1-s}{2} \sum_{i=1}^{\dim K} Q([E_i,X],[E_i,X]), \qquad X \in \mathfrak{m}, \tag{21.3}$$

Every $g_{K,s}$ has positive Ricci curvature. If $\mathfrak{z}(\mathfrak{k}) \neq 0$, the smallest eigenvalue of the Ricci endomorphism is cs. If $\mathfrak{k}$ is semisimple, the smallest eigenvalue of the Ricci endomorphism is strictly larger than cs.

Proof. Suppose first that $0 < \dim K < \dim L$. Write $\mathfrak{k} = \bigoplus_{i=1}^{r} \mathfrak{k}_i$ orthogonally for Q, where each $\mathfrak{k}_i$ is simple or one-dimensional and central. Apply the D'Atri–Ziller Ricci formulas in [7, Proposition 2.2] with background form $Q_0 = -B_{\mathfrak{l}} = 4cQ$ and parameters

$$\alpha = \frac{1}{4c}, \qquad \alpha_i = \frac{s}{4c} \quad (1 \leq i \leq r).$$

Then $\alpha_i/\alpha = s$. In the notation of that proposition, $\kappa_i Q_0|_{\mathfrak{k}_i} = -B_{\mathfrak{k}_i}$. For a Q-orthonormal basis $(E_{i,j})$ of $\mathfrak{k}_i$, the bilinear form A_i is

$$A_i(X,Y) = -\sum_j Q([X,E_{i,j}],[Y,E_{i,j}]), \qquad X, Y \in \mathfrak{m}.$$

The three formulas of that proposition give the vanishing mixed terms and (21.2)–(21.3). For $K = \{1\}$ and $K = L$, the metrics are Q and sQ; their Ricci tensors are both cQ, as follows from (4.17) with trivial isotropy. The same formulas therefore hold in these cases.

Define $T_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{k}$ by

$$Q(T_{\mathfrak{k}}X, Y) = -\tfrac{1}{4} B_{\mathfrak{k}}(X,Y), \qquad X, Y \in \mathfrak{k}.$$

This operator is nonnegative and Q-self-adjoint. On $\mathfrak{k}$ the Ricci endomorphism is

$$csI + (s^{-1} - s)T_{\mathfrak{k}}.$$

For $X \in \mathfrak{k}$, the identity $-B_{\mathfrak{k}}(X,X) = \sum_i \|[X,E_i]\|_Q^2$ identifies the kernel of $T_{\mathfrak{k}}$ with $\mathfrak{z}(\mathfrak{k})$. On $\mathfrak{m}$, every Ricci eigenvalue is at least $c > cs > 0$. Thus Ricci curvature is positive, with least eigenvalue cs exactly when $\mathfrak{z}(\mathfrak{k}) \neq 0$. For nonzero semisimple $\mathfrak{k}$, the operator $T_{\mathfrak{k}}$ is positive definite, so the least Ricci eigenvalue exceeds cs. If $\mathfrak{k} = 0$, then $\mathrm{Ric} = cQ$ and the same strict inequality holds. □

Corollary 21.3. *Let K_1, K_2 be closed connected subgroups of the same compact simple group (L, Q). If $\mathfrak{z}(\mathfrak{k}_1) \neq 0$ and $\mathfrak{k}_2$ is semisimple, then for every $0 < s < 1$ the metrics $g_{K_1,s}$ and $g_{K_2,s}$ are not locally isometric.*

Proof. By Lemma 21.2, the two least Ricci eigenvalues are respectively cs and a number strictly greater than cs. A local isometry intertwines the Ricci endomorphisms. □

Lemma 21.4. *Let J be a compact connected Lie group, K a closed subgroup, and Q a positive $\mathrm{Ad}(J)$-invariant form on $\mathfrak{j}$. Give J/K the normal homogeneous metric induced by Q. Suppose that $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{j}$. Every local Riemannian product decomposition of J/K induces a Q-orthogonal decomposition of $\mathfrak{j}$ into commuting ideals, and $\mathfrak{k}$ decomposes as its intersections with those ideals. In particular, if $\mathfrak{j}$ is simple, the normal homogeneous space is locally irreducible.*

Proof. Pass to the complete simply connected Riemannian cover, lift the action to the universal covering group of J, and quotient by its discrete kernel. The Lie algebra pair $(\mathfrak{j}, \mathfrak{k})$ is unchanged. Homogeneity makes the dimension of local holonomy constant. By [64, Vol. I, Chapter II, Theorem 10.3], local and restricted holonomy agree. Thus a local parallel splitting extends to the cover by parallel transport. Every orthogonal parallel splitting is obtained by grouping the nonflat de Rham factors and orthogonally splitting the Euclidean factor [64, Vol. I, Chapter IV, Theorem 6.2].

Put $\mathfrak{m} = \mathfrak{k}^{\perp_Q}$, and let $\mathfrak{m} = \mathfrak{m}_0 \oplus \cdots \oplus \mathfrak{m}_r$ be the de Rham tangent decomposition, with $\mathfrak{m}_0$ tangent to the Euclidean factor. Invariance of Q makes this presentation naturally reductive. By [64, Vol. II, Chapter X, Theorem 5.2 and Corollary 5.3], the subspaces

$$\mathfrak{j}_i = \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i], \qquad 0 \le i \le r,$$

are commuting ideals and $\mathfrak{j} = \bigoplus_{i=0}^{r} \mathfrak{j}_i$. For $i \neq j$, the ideals commute, and invariance gives

$$Q([\mathfrak{j}_i, \mathfrak{j}_i], \mathfrak{j}_j) = Q(\mathfrak{j}_i, [\mathfrak{j}_j, \mathfrak{j}_j]) = 0.$$

Since $\mathfrak{j}_i = \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i]$ and $Q(\mathfrak{m}_i, \mathfrak{m}_j) = 0$, it follows that $\mathfrak{j}_i \perp_Q \mathfrak{j}_j$. Hence $\mathfrak{k} = \bigoplus_i (\mathfrak{k} \cap \mathfrak{j}_i)$. For the Euclidean factor, (4.16) gives $[\mathfrak{m}_0, \mathfrak{m}] = 0$. The $\mathfrak{m}_i$ generate $\mathfrak{j}$; hence $\mathfrak{m}_0$ is central by the Jacobi identity. Its orthogonal subspaces are therefore commuting ideals as well. □

Lemma 21.5. *For every positive $\mathrm{Ad}(K)$-invariant inner product q_K on $\mathfrak{k}$, the normal homogeneous metric on*

$$((L, Q) \times (K, q_K))/\Delta K \cong L$$

is locally irreducible. In particular, this holds for every metric $g_{K,s}$ in (21.1).

Proof. Write $\mathfrak{j} = \mathfrak{l} \oplus \mathfrak{k}$ and $q = Q \oplus q_K$, and put $\mathfrak{d} = \Delta\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{d}^{\perp_q}$. If an ideal contained in $\mathfrak{d}$ contains (X, X), then $([Y, X], 0) \in \mathfrak{d}$ for all $Y \in \mathfrak{l}$. Hence $X \in \mathfrak{z}(\mathfrak{l}) = 0$.

By Lemma 21.4, a nontrivial local product would give $\mathfrak{j} = \mathfrak{j}_1 \perp \mathfrak{j}_2$ with $\mathfrak{d} = \bigoplus_i (\mathfrak{d} \cap \mathfrak{j}_i)$. The projections of the $\mathfrak{j}_i$ to $\mathfrak{l}$ are commuting ideals with sum $\mathfrak{l}$. Since $\mathfrak{l}$ is simple, one projection vanishes; say $\mathfrak{j}_2 \subset 0 \oplus \mathfrak{k}$. Then $\mathfrak{d} \cap \mathfrak{j}_2 = 0$, so $\mathfrak{d} \subset \mathfrak{j}_1$ and $\mathfrak{j}_2 \subset \mathfrak{h}$. But $(0, X) \in \mathfrak{h}$ implies $q_K(X, Y) = 0$ for every $Y \in \mathfrak{k}$, and hence $X = 0$. This contradicts $\mathfrak{j}_2 \neq 0$. □

21.1. **The two subgroups of** $\mathrm{SU}(4n+2)$. For $n \ge 1$, take $L = \mathrm{SU}(4n+2)$, Q, K_A, and K_B as in Section 20. Put $N = 2n + 1$ and $\mathfrak{m}_R = \mathfrak{k}_R^{\perp_Q}$ for $R \in \{A, B\}$. The Lie algebra of K_A has one-dimensional center; that of K_B is semisimple, including $n = 1$. The ambient Killing form is $B_{\mathfrak{l}} = -4NQ$. Write $\rho_{\min}(g)$ for the least eigenvalue of the Ricci endomorphism of a homogeneous metric g.

Proposition 21.6. *For every $n \ge 1$ and $0 < s < 1$, the two metrics $g_{A,s} = g_{K_A,s}$ and $g_{B,s} = g_{K_B,s}$ are locally irreducible, left invariant and normal homogeneous on the fixed simply connected simple group $\mathrm{SU}(4n+2)$. They have positive Ricci curvature and nonnegative sectional curvature, but they are not locally isometric. Their smallest Ricci eigenvalues are*

$$\rho_{\min}(g_{A,s}) = Ns, \qquad \rho_{\min}(g_{B,s}) = \min\left\{N, \frac{(3n+2)s + ns^{-1}}{2}\right\} > Ns. \tag{21.4}$$

Their volumes and scalar curvatures agree and are

$$\operatorname{Vol}(g_{A,s}) = \operatorname{Vol}(g_{B,s}) = s^{N^2/2}\operatorname{Vol}(Q), \tag{21.5}$$

$$\begin{aligned}\operatorname{Scal}(g_{A,s}) &= \operatorname{Scal}(g_{B,s})\\ &= N\big(4N^2 - 1 + (1-s)N^2\big) + \frac{N(N^2-1)(1-s)^2}{4s}.\end{aligned} \tag{21.6}$$

Neither metric is Einstein for $0 < s < 1$. *For* $n = 1$, *the common manifold is* $\mathrm{SU}(6)$ *of dimension* 35, *and the scalar curvature is* $120 - 21s + 6/s$.

Proof. Normal homogeneity, local irreducibility, both curvature inequalities, and the first Ricci minimum follow from Lemmas 21.1, 21.2, and 21.5, with $c = N$. By (20.1) and the classical Killing form formulas [14, Remark 6.15, p. 88],

$$\begin{aligned}B_{\mathfrak{su}(N)} &= -NQ|_{\mathfrak{su}(N)},\\ B_{\mathfrak{sp}(n)} &= -2(n+1)Q|_{\mathfrak{sp}(n)},\\ B_{\mathfrak{so}(2n+2)} &= -2nQ|_{\mathfrak{so}(2n+2)}.\end{aligned}$$

The last identity also holds for $\mathfrak{so}(4)$. Thus the least Ricci eigenvalue of $g_{B,s}$ on $\mathfrak{k}_B$ is

$$Ns + \frac{n}{2}(s^{-1} - s) = \frac{(3n+2)s + ns^{-1}}{2}.$$

The element $\operatorname{diag}((n+1)iI_{2n}, -niI_{2n+2})$ lies in $\mathfrak{m}_B$, commutes with $\mathfrak{k}_B$, and is nonzero. By (21.3), the least Ricci eigenvalue of $g_{B,s}$ on $\mathfrak{m}_B$ is therefore N. This proves (21.4).

Since $\dim K_A = \dim K_B = N^2$, the volume density of either metric is $s^{N^2/2}$ times that of Q. For $K \in \{K_A, K_B\}$, put $b_K = -\operatorname{tr}_{Q|_{\mathfrak{k}}} B_{\mathfrak{k}}$. Relative to the background form $4NQ$, the metric coefficients on $\mathfrak{m}$ and $\mathfrak{k}$ are $1/(4N)$ and $s/(4N)$. Thus [7, Proposition 2.2 and Corollary 2.3] give

$$\operatorname{Scal}(g_{K,s}) = N\big(\dim L + (1-s)\dim K\big) + \frac{(1-s)^2}{4s}b_K.$$

For K_A, the semisimple summand has dimension $N^2 - 1$, so $b_{K_A} = N(N^2-1)$. The symplectic and orthogonal summands of $\mathfrak{k}_B$ have dimensions nN and $(n+1)N$, respectively. Hence

$$b_{K_B} = 2(n+1)nN + 2n(n+1)N = N(N^2-1) = b_{K_A}.$$

Together with $\dim L = 4N^2 - 1$ and $\dim K = N^2$, this gives (21.6).

The metric $g_{A,s}$ has Ricci eigenvalue $Ns < N$ while its Ricci eigenvalues on $\mathfrak{m}_A$ are at least N. The metric $g_{B,s}$ has different Ricci eigenvalues on the symplectic and orthogonal summands of $\mathfrak{k}_B$, their difference being $(s^{-1} - s)/2 > 0$. Thus neither metric is Einstein. □

22. Isospectral metrics on compact simple groups

Schueth constructed nontrivial continuous isospectral families of left-invariant metrics on $\mathrm{SU}(m)$, $m \geq 6$, arbitrarily close to a bi-invariant metric. These families have sectional curvatures of both signs [94, Example 3.9 and Remarks 3.12(i)–(ii), author version]. The metrics below with contraction parameters in $(0, 1)$ are normal homogeneous and have nonnegative sectional curvature.

Let $m \geq 6$ be an integer, put $L = \mathrm{SU}(m)$, and let $Q(X, Y) = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}}(XY)$ for $X, Y \in \mathfrak{su}(m)$. In a fixed six-dimensional block define

$$K_A^{(m)} = \{\operatorname{diag}(A, \overline{A}, I_{m-6}) \,:\, A \in \mathrm{U}(3)\}, \tag{22.1}$$

$$K_B^{(m)} = \{\operatorname{diag}(S, R, I_{m-6}) \,:\, S \in \mathrm{Sp}(1),\ R \in \mathrm{SO}(4)\}. \tag{22.2}$$

Blocks of size zero are omitted. Put $\mathfrak{k}_R^{(m)} = \operatorname{Lie}(K_R^{(m)})$ for $R \in \{A, B\}$. For either subgroup K, and $s > 0$, define a left-invariant metric by

$$g_{K,s} = Q|_{\mathfrak{k}^{\perp_Q}} \oplus sQ|_{\mathfrak{k}}. \tag{22.3}$$

Write $g_{A,s}$ and $g_{B,s}$ for these two metrics.

Theorem 22.1 (Isospectral metrics on SU(m)). *Let $m \geq 6$. The metrics $g_{A,s}$ and $g_{B,s}$ on* SU(m) *are scalar-isospectral for every $s > 0$. Their spectral equivalence is induced by a single unitary operator, independent of s, which commutes with the left* SU(m)*-action. For $0 < s < 1$, both metrics are normal homogeneous and locally irreducible, with positive Ricci curvature and nonnegative sectional curvature. They are not locally isometric for $0 < s < 1$. In this range, their least Ricci endomorphism eigenvalues are*

$$\rho_{\min}(g_{A,s}) = \frac{ms}{2}, \qquad \rho_{\min}(g_{B,s}) = \min\left\{\frac{m}{2}, \frac{(m-1)s + s^{-1}}{2}\right\} > \frac{ms}{2}. \tag{22.4}$$

Within the union of the two curves with $0 < s < 1$ and fixed Q, each spectral fiber has exactly two isometry classes.

22.1. **Subgroup Casimirs and the Laplacian.** For a compact connected Lie group L with a positive Ad(L)-invariant inner product Q on $\mathfrak{l}$, write C_L for its positive Casimir and D_K for the positive Casimir defined by $Q|_{\mathfrak{k}}$, where $K \leq L$ is a connected closed subgroup. For $s > 0$, define $g_{K,s}$ by (22.3). Write $\Delta_{K,s} = \Delta_{g_{K,s}}$ for its scalar Laplacian. For a compact group H, let $\mathrm{Irr}(H)$ be a set of representatives of its irreducible finite-dimensional unitary representations, and write V_π for the representation space of $\pi \in \mathrm{Irr}(H)$.

Lemma 22.2. *Let L be a compact connected Lie group containing an injective copy of* SU(6)*, and let Q_L be a positive invariant form on $\mathfrak{l}$. Take inside that copy the subgroups K_A, K_B of Section 20 with $n = 1$. Then their Q_L-induced Casimirs have the same eigenvalues, with the same multiplicities, on every finite-dimensional complex representation of L.*

Proof. The restriction of Q_L to the simple algebra $\mathfrak{su}(6)$ is aQ for some $a > 0$, where $Q = -\mathrm{Re}\,\mathrm{Tr}$. Both subgroup Casimirs are therefore scaled by a^{-1}. Restrict an L-representation to SU(6) and apply Theorem 20.1, with (20.2), to that finite-dimensional representation. □

Proposition 22.3. *Suppose K_1, K_2 are connected closed subgroups of (L, Q) whose induced Casimirs have the same eigenvalues, with the same multiplicities, on every finite-dimensional complex L-representation. Then the metrics $g_{K_1,s}$ and $g_{K_2,s}$ in* (22.3) *are scalar-isospectral for every $s > 0$. There is one unitary U on $L^2(L)$, for normalized Haar measure, which commutes with the left L-action and Δ_Q, preserves smooth functions, and satisfies*

$$U\Delta_{K_1,s} = \Delta_{K_2,s}U \qquad (s > 0)$$

with $U\mathcal{D}(\Delta_{K_1,s}) = \mathcal{D}(\Delta_{K_2,s})$. For $0 < s < 1$, both metrics are normal homogeneous.

Proof. The normalized volume measure of each $g_{K,s}$ is Haar probability measure. The scalar Laplacian is the negative sum of squares in a left-invariant orthonormal frame [49, proof of Lemma 2.2, p. 1621]. Splitting a Q-orthonormal basis into subgroup and complementary directions gives

$$\Delta_{K,s} = C_L + (s^{-1} - 1)D_K. \tag{22.5}$$

For simple L and semisimple K, compare [50, Section 5]; the branching formula for commuting subgroups is given in [3, Theorem 2.23]. The difference $C_L - D_K$ is the negative sum of squares in the complementary directions and is nonnegative. Hence, as quadratic forms on every finite-dimensional unitary representation,

$$\min\{1, s^{-1}\}C_L \leq \Delta_{K,s} \leq \max\{1, s^{-1}\}C_L.$$

In the Peter–Weyl decomposition [41, Theorem 35.9], the left L-action is on V_π and the right L-action is on V_π^* in $V_\pi \otimes V_\pi^*$, for $\pi \in \mathrm{Irr}(L)$. Let $c_L(\pi)$ be the eigenvalue of C_L on V_π. The operator in (22.5) is

$$I \otimes \left(c_L(\pi)I + (s^{-1} - 1)D_K|_{V_\pi^*}\right).$$

By hypothesis, choose a unitary u_π on V_π^* intertwining the two subgroup Casimirs themselves. These choices are independent of s. The Hilbert direct sum of $I \otimes u_\pi$ is the required unitary U. It commutes

with Δ_Q, which is scalar on each ambient type, and it intertwines the displayed Laplacians. For $i \in \{1, 2\}$ and each subgroup-Casimir eigenvalue μ, put

$$E_{i,\pi,\mu} = V_\pi \otimes \ker\bigl(D_{K_i}|_{V_\pi^*} - \mu I\bigr).$$

The maps $I \otimes u_\pi$ identify these finite-dimensional L-modules. Under the Peter–Weyl identification, the spaces $E_{i,\pi,\mu}$ consist of smooth eigenfunctions. Orthonormal bases in these spaces give a complete orthonormal eigenbasis for each i. Thus the self-adjoint Laplacian is the maximal diagonal operator with eigenvalues $c_L(\pi) + (s^{-1} - 1)\mu$. Lemma 2.4, applied to the labels (π, μ), gives the equality of operator domains and all eigenvalue multiplicities. For its Sobolev assertion, take both manifolds to be (L, Q) and the eigenvalue function to be $\nu(\pi, \mu) = c_L(\pi)$. Hence U preserves every Sobolev space defined by Δ_Q and their intersection $C^\infty(L)$.

For $0 < s < 1$, [3, Theorem 2.21], with trivial initial isotropy, deformation subgroup K, and $t_0 = r_0 = 1$, $t_1 = s$, $r_1 = s/(1 - s)$, gives the normal presentation of Lemma 21.1. □

Proof of Theorem 22.1. The six-dimensional block embedding of SU(6) in SU(m) is injective, and the ambient trace form restricts to the same trace form. Lemma 22.2 and Proposition 22.3 give the common spectra and a single left-equivariant unitary for all $s > 0$. Lemmas 21.1, 21.2, and 21.5 give normal homogeneity, nonnegative sectional curvature, positive Ricci curvature, and local irreducibility.

For $L = \mathrm{SU}(m)$, $B_{\mathfrak{l}} = -2mQ$, so $c = m/2$. The subgroup $K_A^{(m)}$ has a one-dimensional center and $K_B^{(m)}$ is semisimple. Lemma 21.2 gives $\rho_{\min}(g_{A,s}) = ms/2$ and a strictly larger minimum for $g_{B,s}$. The eigenvalues of $T_{\mathfrak{k}_B^{(m)}}$ are 1 on $\mathfrak{sp}(1)$ and $1/2$ on $\mathfrak{so}(4)$, as computed in Proposition 21.6. Thus the least vertical eigenvalue for $g_{B,s}$ is $((m - 1)s + s^{-1})/2$. The nonzero vector

$$Z = \mathrm{diag}(2iI_2, -iI_4, 0_{m-6})$$

lies in $(\mathfrak{k}_B^{(m)})^{\perp_Q}$ and commutes with $\mathfrak{k}_B^{(m)}$. It attains the horizontal lower bound $m/2$. This proves (22.4). The strict inequality excludes every local isometry, since the Ricci eigenvalues are constant on each homogeneous manifold and are preserved by local isometries.

Both subgroups have dimension nine. Hence

$$\mathrm{Vol}(g_{A,s}) = \mathrm{Vol}(g_{B,s}) = s^{9/2}\,\mathrm{Vol}(Q). \tag{22.6}$$

Weyl's law [38, Theorem 17.7] makes volume a scalar spectral invariant. The expression in (22.6) is strictly increasing in s. Within the two stated curves, isospectrality therefore forces the same parameter, and exactly the two already distinguished isometry classes occur at each parameter. □

Corollary 22.4. *The subgroups of Section 20 give two isospectral metrics $g_{A,s}, g_{B,s}$ on $\mathrm{SU}(4n + 2)$ for every $n \geq 1$ and $s > 0$. For $0 < s < 1$ their least Ricci eigenvalues, volumes, and scalar curvatures are those of Proposition 21.6; in particular, they are not locally isometric. With Q fixed, the spectral fibers in these two curves have exactly two isometry classes for $0 < s < 1$.*

Proof. Theorem 20.1, with $D_R = C_R^{\mathrm{ref}}/2$, and Proposition 22.3 give scalar isospectrality. The remaining statements follow from Proposition 21.6 and the strict monotonicity of $s^{(2n+1)^2/2}\,\mathrm{Vol}(Q)$. □

22.2. **Binary families on a fixed simple group.** Retain the trace form Q and put $K_R = K_R^{(6)}$ for $R \in \{A, B\}$, with $K_R^{(6)}$ as in (22.1)–(22.2).

Fix integers m, r with $m \geq 6r \geq 6$ and embed $\mathrm{SU}(6)^r$ block diagonally in $L = \mathrm{SU}(m)$, with the identity on the remaining $m - 6r$ coordinates. Let $\iota_j : \mathrm{SU}(6) \longrightarrow L$ be the inclusion in the jth block. For $\omega = (\omega_1, \dots, \omega_r) \in \{A, B\}^r$, put

$$K_{\omega,j} = \iota_j(K_{\omega_j}), \qquad \mathfrak{k}_{\omega,j} = \mathrm{Lie}(K_{\omega,j}),$$

$$K_\omega = \prod_{j=1}^{r} K_{\omega,j}, \qquad \mathfrak{k}_\omega = \mathrm{Lie}(K_\omega) = \bigoplus_{j=1}^{r} \mathfrak{k}_{\omega,j},$$

$$\mathfrak{m}_\omega = \mathfrak{k}_\omega^{\perp_Q}.$$

For $0 < \varepsilon < 1$, set

$$s_j(\varepsilon) = 1 - \varepsilon + \frac{\varepsilon j}{100mr}, \qquad j = 1, \dots, r. \tag{22.7}$$

These parameters are distinct and belong to $(0, 1)$. Write $s_j = s_j(\varepsilon)$ when ε is fixed. Let $g_{\omega,\varepsilon}$ be the left-invariant metric defined by

$$g_{\omega,\varepsilon}|_{\mathfrak{m}_\omega} = Q, \qquad g_{\omega,\varepsilon}|_{\mathfrak{k}_{\omega,j}} = s_j Q, \tag{22.8}$$

with all the displayed summands orthogonal. Write $\mathrm{Spec}(g) = \mathrm{Spec}(\Delta_g)$, with eigenvalues counted with multiplicity.

Theorem 22.5 (Exact binary families). *Fix integers m, r with $m \geq 6r \geq 6$. For every $\varepsilon \in (0, 1)$, the 2^r metrics in* (22.8) *have the same scalar Laplace spectrum, including multiplicities, and are pairwise not locally isometric. Every metric is normal homogeneous and locally irreducible, with positive Ricci curvature and nonnegative sectional curvature.*

For all $\omega, \omega' \in \{A, B\}^r$ and $\varepsilon, \varepsilon' \in (0, 1)$,

$$\mathrm{Spec}(g_{\omega,\varepsilon}) = \mathrm{Spec}(g_{\omega',\varepsilon'}) \quad \Longleftrightarrow \quad \varepsilon = \varepsilon'. \tag{22.9}$$

Thus each spectral fiber within this family contains exactly 2^r isometry classes. Every C^∞ neighborhood of Q contains one of these 2^r-member isospectral sets. In particular, every fixed $\mathrm{SU}(m)$ with $m \geq 6$ admits such a set of at least $2^{\lfloor m/6 \rfloor}$ isometry classes.

Proof. The connected block subgroups $K_{\omega,j}$ commute and have Q-orthogonal Lie algebras. Their product map is injective, since the blocks are disjoint. Apply [3, Theorem 2.21] with trivial initial isotropy and

$$t_0 = r_0 = 1, \qquad t_j = s_j, \qquad r_j = \frac{s_j}{1 - s_j}.$$

The theorem identifies (22.8) with the metric on $(L \times K_\omega)/\Delta K_\omega$ induced by

$$Q \oplus \bigoplus_{j=1}^{r} \frac{s_j}{1 - s_j} Q|_{\mathfrak{k}_{\omega,j}}. \tag{22.10}$$

This form is positive and invariant. Thus the metric is normal homogeneous and has nonnegative sectional curvature by (4.16). Lemma 21.5, applied to the second-factor form in (22.10), gives local irreducibility.

Under the jth block embedding, $D_{K_{\omega,j}}$ is the image of $\frac{1}{2} C^{\mathrm{ref}}_{\omega_j}$ from (20.2): the ambient trace form restricts to the same trace form on $\mathfrak{su}(6)$. Write $\Delta_{\omega,\varepsilon} = \Delta_{g_{\omega,\varepsilon}}$. Then

$$\Delta_{\omega,\varepsilon} = C_L + \sum_{j=1}^{r} (s_j^{-1} - 1) D_{K_{\omega,j}} \tag{22.11}$$

under the right regular representation.

Fix $\omega, \omega' \in \{A, B\}^r$ and $\pi \in \mathrm{Irr}(L)$. For $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r) \in \mathrm{Irr}(\mathrm{SU}(6))^r$, let $M_{\pi,\boldsymbol{\tau}}$ be the multiplicity space of $V_{\tau_1} \otimes \cdots \otimes V_{\tau_r}$ in the restriction of V_π^* to $\mathrm{SU}(6)^r$. The irreducible representations of a product [50, §4.6(ii)] give an orthogonal decomposition

$$V_\pi^*|_{\mathrm{SU}(6)^r} \cong \bigoplus_{\boldsymbol{\tau}} M_{\pi,\boldsymbol{\tau}} \otimes V_{\tau_1} \otimes \cdots \otimes V_{\tau_r}.$$

Theorem 20.1, with $n = 1$, gives, for every j, a unitary $u_{\boldsymbol{\tau},j}$ on V_{τ_j} satisfying

$$u_{\boldsymbol{\tau},j} C^{\mathrm{ref}}_{\omega_j} = C^{\mathrm{ref}}_{\omega_j'} u_{\boldsymbol{\tau},j}.$$

Consequently

$$u_\pi = \bigoplus_{\boldsymbol{\tau}} I_{M_{\pi,\boldsymbol{\tau}}} \otimes u_{\boldsymbol{\tau},1} \otimes \cdots \otimes u_{\boldsymbol{\tau},r}$$

intertwines all r subgroup Casimirs simultaneously. In the Peter–Weyl decomposition, apply Lemma 2.4 to the corresponding joint eigenspaces. Here μ_j is the eigenvalue of $D_{K_{\omega,j}}$; the labels are $(\pi, \mu_1, \dots, \mu_r)$ and the diagonal value is

$$c_L(\pi) + \sum_{j=1}^{r} (s_j^{-1} - 1)\mu_j.$$

This gives unitary equivalence of $\Delta_{\omega,\varepsilon}$ and $\Delta_{\omega',\varepsilon}$, including their self-adjoint domains and all eigenvalue multiplicities.

Put $c = m/2$. The subgroup K_ω is nontrivial and proper, since $0 < 9r < m^2 - 1$. Decompose each $\mathfrak{k}_{\omega,j}$ into simple ideals and Q-orthogonal central lines; in particular, $\mathfrak{so}(4)$ has two simple ideals. Apply the Ricci formula of D'Atri and Ziller, as stated in [7, Proposition 2.2], with background form $Q_0 = -B_{\mathfrak{su}(m)} = 4cQ$, coefficient $1/(4c)$ on $\mathfrak{m}_\omega$, and coefficient $s_j/(4c)$ on every simple ideal or central line of $\mathfrak{k}_{\omega,j}$. The normalization is that in the proof of Lemma 21.2. Define $T_j : \mathfrak{k}_{\omega,j} \longrightarrow \mathfrak{k}_{\omega,j}$ by

$$Q(T_j X, Y) = -\tfrac{1}{4} B_{\mathfrak{k}_{\omega,j}}(X, Y), \qquad X, Y \in \mathfrak{k}_{\omega,j}.$$

The Ricci endomorphism on $\mathfrak{k}_{\omega,j}$ is

$$c s_j I + (s_j^{-1} - s_j) T_j. \tag{22.12}$$

For each j, let $(E_{j,i})_{i=1}^{9}$ be a Q-orthonormal basis of $\mathfrak{k}_{\omega,j}$. Since the metric on $\mathfrak{m}_\omega$ is Q, its Ricci quadratic form is

$$cQ(X, X) + \frac{1}{2} \sum_{j=1}^{r} (1 - s_j) \sum_{i=1}^{9} Q([E_{j,i}, X], [E_{j,i}, X]), \qquad X \in \mathfrak{m}_\omega. \tag{22.13}$$

The same proposition gives zero mixed Ricci terms between distinct vertical blocks and between the vertical and horizontal spaces. Since $T_j \geq 0$ and $0 < s_j < 1$, both displayed restrictions are positive definite.

By Proposition 21.6 with $n = 1$, the subgroup Killing ratios are unchanged under the block embeddings. If $\omega_j = A$, then T_j has a one-dimensional kernel and eigenvalue $3/4$ on the eight-dimensional $\mathfrak{su}(3)$ summand. If $\omega_j = B$, its eigenvalues are 1 on $\mathfrak{sp}(1)$ and $1/2$ on $\mathfrak{so}(4)$. Set

$$R_\varepsilon = c(1 - \varepsilon) + \frac{\varepsilon}{4}.$$

If $\omega_j = A$, the central line gives the Ricci eigenvalue $c s_j$, and

$$c s_j = c(1 - \varepsilon) + \frac{\varepsilon j}{200r} \leq c(1 - \varepsilon) + \frac{\varepsilon}{200} < R_\varepsilon. \tag{22.14}$$

Every other Ricci eigenvalue λ on $\mathfrak{k}_{\omega,j}$ satisfies

$$\begin{aligned} \lambda &\geq c s_j + \tfrac{1}{2}(s_j^{-1} - s_j) \\ &\geq c(1 - \varepsilon) + (1 - s_j) \\ &\geq c(1 - \varepsilon) + \left(1 - \frac{1}{100m}\right) \varepsilon \\ &\geq c(1 - \varepsilon) + \frac{599}{600} \varepsilon > R_\varepsilon. \end{aligned} \tag{22.15}$$

Here $(s^{-1} - s)/2 \geq 1 - s$ for $0 < s < 1$ and $m \geq 6$. Every horizontal Ricci eigenvalue is at least $c > R_\varepsilon$. Consequently the Ricci multiset below R_ε is exactly

$$\{c s_j \,:\, \omega_j = A\}, \tag{22.16}$$

with multiplicity one at each value. Since the s_j are distinct, this multiset determines ω. The 2^r metrics are therefore pairwise not locally isometric.

Finally, their common volume is

$$V(\varepsilon) = \mathrm{Vol}(Q) \prod_{j=1}^{r} s_j(\varepsilon)^{9/2}. \tag{22.17}$$

Each positive factor $s_j(\varepsilon)$ is strictly decreasing in ε, so $V(\varepsilon)$ is strictly decreasing. By Weyl's law, equality of spectra implies equality of volume, and hence $\varepsilon = \varepsilon'$. The converse was proved above. This proves (22.9) and the exact fiber count. For each ω, the tensor $g_{\omega,\varepsilon} - Q$ is ε times a fixed smooth left-invariant symmetric tensor. Hence $g_{\omega,\varepsilon} \to Q$ in C^∞ as $\varepsilon \downarrow 0$. The set of words is finite, so every prescribed neighborhood of Q contains all 2^r metrics for sufficiently small positive ε. Taking $r = \lfloor m/6 \rfloor$ gives $2^{\lfloor m/6 \rfloor}$ classes in each of these spectral fibers.

□

23. Further determination problems

Question 23.1. What is the least dimension of a closed simply connected real-analytic manifold admitting two real-analytic metrics with positive Ricci curvature, equal scalar spectra, and different sets of closed-geodesic periods? Does such a pair exist if both metrics are also required to have nonnegative sectional curvature, and if so, what is the least dimension?

Theorem 8.8 gives the upper bound eighteen for the first part.

In the bi-invariant classification of Theorem 19.6, the scalar eigenvalues and the right action of the marking group on each eigenspace are those of a fixed reference product. Gordon, Schueth and Sutton [49, p. 1620] conjectured an affirmative answer to the following question.

Question 23.2. Let J be a compact connected Lie group and g_0 a bi-invariant metric on J. Is there a neighborhood $\mathcal{U}$ of g_0 in the space of smooth Riemannian metrics on J, endowed with the C^∞ topology, such that every $g \in \mathcal{U}$ satisfying $\mathrm{Spec}(\Delta_g) = \mathrm{Spec}(\Delta_{g_0})$ is isometric to g_0?

Fix integers m, r with $m \geq 6r \geq 6$. For $\omega \in \{A, B\}^r$ and $0 < \varepsilon < 1$, let $g_{\omega,\varepsilon}$ and Q be as in Theorem 22.5. Equation (22.17) gives

$$\mathrm{Vol}(g_{\omega,\varepsilon}) < \mathrm{Vol}(Q).$$

By Weyl's law, $\mathrm{Spec}(\Delta_{g_{\omega,\varepsilon}}) \neq \mathrm{Spec}(\Delta_Q)$. The spectra for distinct values of ε are different by (22.9).

Appendix A. Clean quotients and analytic wave traces

A.1. **Clean geodesic flows.** Let (M, g) be a closed Riemannian manifold, let SM be its unit tangent bundle, and let Φ_t be its geodesic flow. Cleanliness is as in Section 1; the empty fixed set is allowed. All tangent kernels below are taken in SM.

Lemma A.1. *If a finite group Γ acts freely by isometries on a closed clean Riemannian manifold (M, g), the quotient $N = M/\Gamma$, with its quotient metric, is clean.*

Proof. Let $p : SM \to SN$ be the unit-tangent covering, and let $\overline{\Phi}_t$ be the geodesic flow on SN. Each $\gamma \in \Gamma$ also denotes its induced action on SM. Fix $T > 0$ and choose a positive integer q divisible by the orders of all elements of Γ. Put

$$S = \mathrm{Fix}(\Phi_{qT}), \qquad F_\gamma = \gamma^{-1}\Phi_T, \qquad B_\gamma = \mathrm{Fix}(F_\gamma).$$

Deck transformations commute with the flow. Hence $F_\gamma^q = \Phi_{qT}$, $F_\gamma(S) = S$, and $B_\gamma \subset S$. If $S = \emptyset$, then $\mathrm{Fix}(\overline{\Phi}_T) = \emptyset$. Suppose $S \neq \emptyset$. Cleanliness makes S a finite disjoint union of compact embedded manifolds. Since $(F_\gamma|_S)^q = I$, averaging a Riemannian metric over $\langle F_\gamma|_S \rangle$ makes $F_\gamma|_S$ an isometry. The fixed-locus theorem and equivariance of the exponential map [69, Problem 8-32, p. 262, and Proposition 5.20, pp. 130–131] give embedded fixed components with

$$T_u B_\gamma = \ker(D_u F_\gamma - I) \cap T_u S,$$

and compactness gives finitely many such components. If $D_uF_\gamma v = v$, then $D_u\Phi_{qT}v = (D_uF_\gamma)^q v = v$. Cleanliness gives $v \in T_uS$, and therefore $T_uB_\gamma = \ker(D_uF_\gamma - I)$.

The covering identity gives

$$p^{-1}\operatorname{Fix}(\overline{\Phi}_T) = \coprod_{\gamma\in\Gamma} B_\gamma.$$

The union is disjoint by freeness. Its connected components are finitely many closed embedded submanifolds. The identity $\delta B_\gamma = B_{\delta\gamma\delta^{-1}}$ shows that Γ permutes these components. Distinct orbits of components have disjoint images under p. These images are closed by compactness, and covering charts identify them locally with the components upstairs. They are therefore embedded submanifolds. Differentiating $pF_\gamma = \overline{\Phi}_T p$ at $u \in B_\gamma$ gives

$$(D_{p(u)}\overline{\Phi}_T - I)D_up = D_up(D_uF_\gamma - I).$$

Since D_up is an isomorphism, the tangent-kernel equality upstairs descends to the required equality downstairs. □

Cianci proved the lens-space case of the following corollary [25, Theorem 2].

Corollary A.2. *Every round spherical space form is clean.*

Proof. On the unit sphere,

$$\Phi_t(x,v) = (\cos(t)x + \sin(t)v, -\sin(t)x + \cos(t)v), \qquad |x| = |v| = 1, \quad \langle x, v\rangle = 0.$$

Its fixed set is empty for $t \notin 2\pi\mathbb{Z}$ and is the whole unit tangent bundle for $t \in 2\pi\mathbb{Z}$. In the latter case $D\Phi_t = I$. Thus the round sphere is clean. Rescaling preserves this property, and Lemma A.1 applies to every finite free isometric quotient. □

A.2. **The analytic Poisson inclusion.** Let (X, g) be a closed real-analytic Riemannian manifold of dimension d. Write $\mathcal{L}(X, g) = \operatorname{Lsp}(X, g)$ and

$$S^*X = \{(x,\xi) \in T^*X \,:\, |\xi|_x = 1\}.$$

Let Φ^s be the Hamiltonian flow of $q(x,\xi) = |\xi|_x$ on $T^*X \setminus 0$; its restriction to S^*X is the geodesic flow. If $d = 0$, all traces below are finite sums of entire functions. Assume henceforth $d > 0$, and let $0 \le \lambda_0 \le \lambda_1 \le \cdots$ be the square roots of the scalar Laplace eigenvalues, repeated according to multiplicity. Put

$$w_g(t) = \sum_j e^{it\lambda_j}, \qquad c_g(t) = \sum_j \cos(t\lambda_j).$$

Weyl's law [38, Theorem 17.7, p. 236] gives $\#\{j \,:\, \lambda_j \le R\} = O((1+R)^d)$, hence $\sum_j (1+\lambda_j)^{-m} < \infty$ for every $m > d$. Thus both series converge in $\mathcal{S}'(\mathbb{R})$. The Fourier convention is $\widehat{u}(\tau) = \int e^{-it\tau}u(t)\,dt$. Denote the smooth and analytic wavefront sets by $\mathrm{WF}(u)$ and $\mathrm{WF}_A(u)$, respectively. The projection of $\mathrm{WF}_A(u)$ to the base is denoted by $\operatorname{singsupp}_A(u)$; its complement is the largest open set on which u is real analytic [60, Theorem 8.4.5, p. 284].

Proposition A.3 (The analytic Poisson inclusion). *One has*

$$\operatorname{singsupp}_A w_g \subseteq \{0\} \cup \mathcal{L}(X,g) \cup (-\mathcal{L}(X,g)). \tag{A.1}$$

The same inclusion holds for the cosine trace and for both traces of the connection Laplacian on every finite-rank unitary flat bundle over X.

Proof. For $\lambda \ge 0$, set

$$s_\lambda(t) = \begin{cases} \sin(t\lambda)/\lambda, & \lambda > 0, \\ t, & \lambda = 0. \end{cases}$$

Let H be the Heaviside function, and choose an orthonormal basis (e_j) of $L^2(X, dV_g)$ with $\Delta_X e_j = \lambda_j^2 e_j$. Relative to $dV_g(y)$, put

$$E_+(t, x, y) = \sum_j H(t) s_{\lambda_j}(t) e_j(x) \overline{e_j(y)}.$$

The series converges as a distribution. For a test function ψ supported in $[-T, T] \times X \times X$, integration by parts in x bounds its jth pairing by

$$T^2 (1 + \lambda_j^2)^{-m} \sup_{|t| \le T} \|(1 + \Delta_x)^m \psi(t, \cdot, \cdot)\|_{L^2}.$$

Here $|s_\lambda(t)| \le |t|$ and $\|e_j \otimes \overline{e}_j\|_2 = 1$. The bound is summable when $2m > d$. Let δ_0 be the Dirac distribution at 0, and let δ_{diag} be the kernel of the identity operator relative to $dV_g(y)$. Put $P_x = \partial_t^2 + \Delta_x$ and $P_y = \partial_t^2 + \Delta_y$. The identities $s_\lambda(0) = 0$ and $s_\lambda'(0) = 1$ give

$$P_x E_+ = P_y E_+ = \delta_0(t) \delta_{\mathrm{diag}}(x, y), \qquad E_+|_{t<0} = 0. \tag{A.2}$$

The equality for P_y follows from $\Delta_y \overline{e_j(y)} = \lambda_j^2 \overline{e_j(y)}$. The source is the analytic density on the analytic submanifold $\{(0, x, x) : x \in X\}$, regarded as a distribution. Its analytic wavefront set is contained in its conormal bundle [60, Theorem 8.5.2 and the following paragraph, p. 297],

$$\{(0, x, x; \tau, \xi, -\xi) : (\tau, \xi) \neq 0\}. \tag{A.3}$$

Over $t > 0$, analytic microlocal elliptic regularity [60, Theorem 8.6.1, p. 306], applied separately to both equations in (A.2), gives

$$(t, x, y; \tau, \xi, \eta) \in \mathrm{WF}_A(E_+) \quad \Longrightarrow \quad \tau^2 = |\xi|_x^2 = |\eta|_y^2. \tag{A.4}$$

Such a covector is nonzero, hence $\tau \neq 0$. In particular $p_x = |\xi|_x^2 - \tau^2$ satisfies $\partial_\tau p_x = -2\tau \neq 0$ there. On every conic neighborhood with $\tau \neq 0$, the characteristic roots are simple and the slices $t =$ constant are noncharacteristic, since $p_x(dt) = -1$. Analytic singularities therefore propagate along the p_x-bicharacteristics in these neighborhoods [63, Theorem 3.3′, p. 387].

Consider the p_x-bicharacteristic through a covector in (A.4). Its variables (y, η) and its temporal covector τ are constant. Since $dt/ds = -2\tau$, it crosses $t = 0$ once. Completeness of the geodesic flow extends it across this crossing. Its temporal covector remains nonzero. Each compact segment is covered by finitely many of the preceding coordinate and conic neighborhoods, so the local propagation statements apply successively along it. If the covector at $t = 0$ were absent from (A.3), the entire segment extending to a negative time would avoid $\mathrm{WF}_A(P_x E_+)$. Propagation would then carry membership in $\mathrm{WF}_A(E_+)$ to that negative time, contradicting $E_+|_{t<0} = 0$. Thus the covector at $t = 0$ is $(0, y, y; \tau, -\eta, \eta)$.

On $p_x = 0$ one has $q = |\tau|$, and Hamilton's equations, with t as parameter, give

$$\frac{d(x, \xi)}{dt} = -\frac{1}{2\tau} H_{q^2} = -\operatorname{sgn}(\tau) H_q.$$

Consequently

$$(t, x, y; \tau, \xi, \eta) \in \mathrm{WF}_A(E_+), \quad t > 0 \quad \Longrightarrow \quad (x, \xi) = \Phi^{-\operatorname{sgn}(\tau) t}(y, -\eta). \tag{A.5}$$

Define $\iota(t, x) = (t, x, x)$ and $\pi(t, x) = t$ for $t > 0$. The conormal bundle of ι consists of $(t, x, x; 0, \xi, -\xi)$ with $\xi \neq 0$, and is disjoint from $\mathrm{WF}_A(E_+)$ by (A.4). Since the smooth wavefront set satisfies $\mathrm{WF}(E_+) \subseteq \mathrm{WF}_A(E_+)$, the pullback $\iota^* E_+$ exists by [60, Theorem 8.2.4, p. 263]. The analytic pullback estimate [60, Theorem 8.5.1, p. 296] gives

$$\mathrm{WF}_A(\iota^* E_+) \subseteq \big\{(t, x; \tau, \xi + \eta) : (t, x, x; \tau, \xi, \eta) \in \mathrm{WF}_A(E_+)\big\}.$$

The map π is a proper analytic submersion. Denote by π_* integration over X against the analytic density dV_g. Analytic integration along the fiber [63, Lemma 2.2, pp. 375–376] implies that

$$(t; \tau) \in \mathrm{WF}_A(\pi_* \iota^* E_+)$$

requires a covector in (A.5) with $y = x$ and $\eta = -\xi$. It follows that

$$(x, \xi) = \Phi^{-\operatorname{sgn}(\tau)t}(x, \xi), \qquad \xi \neq 0.$$

The flow of q commutes with positive dilation in the cotangent fibers. A fixed point of Φ^{-t} is also a fixed point of Φ^{t}. Replacing ξ by $\xi/|\xi|_x$ therefore gives a closed unit-speed geodesic of period t. Hence $\pi_* \iota^* E_+$ is real analytic on $(0, \infty) \setminus \mathcal{L}(X, g)$.

For $\varphi \in C_c^\infty((0, \infty))$, set

$$a_\varphi(\lambda) = \int \varphi(t) s_\lambda(t)\, dt.$$

Repeated integration by parts gives, for every $m \geq 0$, $|a_\varphi(\lambda)| \leq C_{\varphi,m}(1 + \lambda)^{-m}$. The elliptic estimate and Sobolev embedding [38, Theorems 16.7 and 12.16, pp. 216 and 147] bound every C^k norm of e_j by a fixed power of $1 + \lambda_j$. Thus the series

$$A_\varphi(x, y) = \sum_j a_\varphi(\lambda_j) e_j(x) \overline{e_j(y)}$$

converges in $C^\infty(X \times X)$, and $\sum_j a_\varphi(\lambda_j)$ converges absolutely. In local coordinates (t, x, z) with $y = x + z$, every nonzero conormal to $z = 0$ has $(\tau, \xi, \eta) = (0, -\zeta, \zeta)$ in the original (t, x, y) cotangent coordinates. Such covectors are absent from $\mathrm{WF}(E_+)$ by (A.4). For χ smooth and compactly supported in the coordinate chart, the restriction formula [60, Theorem 8.2.4 and (8.2.5), pp. 263–264], paired against $\varphi(t)\chi(x)$ after compact localization, is an absolutely convergent Fourier integral. Fubini's theorem, with integration first in the variable dual to t, gives

$$\langle \iota^* E_+, \varphi \otimes \chi \rangle = \int_X \chi(x) A_\varphi(x, x)\, dV_g(x).$$

A finite partition of unity yields

$$\left\langle \pi_* \iota^* E_+, \varphi \right\rangle = \int_X A_\varphi(x, x)\, dV_g(x) = \sum_j a_\varphi(\lambda_j).$$

Since $s_\lambda' = \cos(t\lambda)$, differentiation in positive time gives $c_g = \partial_t(\pi_* \iota^* E_+)$. Therefore c_g is real analytic off $\mathcal{L}(X, g)$ in positive time. Since $c_g(-t) = c_g(t)$, it is real analytic off $-\mathcal{L}(X, g)$ in negative time.

The series

$$W_g(z) = \sum_j e^{iz\lambda_j}, \qquad \operatorname{Im} z > 0,$$

is holomorphic, with all derivatives locally uniformly convergent, because $\sum_j (1 + \lambda_j)^m e^{-\varepsilon\lambda_j} < \infty$ for every m and $\varepsilon > 0$. For $0 < \varepsilon \leq 1$, polynomial counting gives

$$|W_g(t + i\varepsilon)| \leq \varepsilon \int_0^\infty e^{-\varepsilon R} \#\{j \,:\, \lambda_j \leq R\}\, dR \leq C\varepsilon^{-d}.$$

For a compactly supported smooth φ, rapid decay of $\widehat{\varphi}(-\lambda_j)$ permits termwise passage to the boundary and gives w_g in distributions. The analytic boundary-value theorem [96, Corollary 2.9], applied to the upper half-plane, yields

$$\mathrm{WF}_A(w_g) \subset \{(t; \tau) \,:\, \tau > 0\}, \qquad \mathrm{WF}_A(w_g(-\cdot)) \subset \{(t; \tau) \,:\, \tau < 0\}. \tag{A.6}$$

If c_g is analytic on an interval I, the identity $w_g = 2c_g - w_g(-\cdot)$ implies

$$\mathrm{WF}_A(w_g)|_I \subseteq \mathrm{WF}_A(w_g(-\cdot))|_I.$$

Since a nonzero real number cannot be both positive and negative, (A.6) gives $\mathrm{WF}_A(w_g)|_I = \emptyset$. This proves (A.1).

Let $E \to X$ be a finite-rank unitary flat bundle, with connection Laplacian $\Delta_E = \nabla^* \nabla$. Applied to the underlying real bundle, the heat-trace expansion [47, p. 233] gives $\operatorname{Tr}(e^{-s\Delta_E}) = O(s^{-d/2})$ as $s \downarrow 0$;

its real heat trace is twice its complex heat trace. Let $N_E(R)$ count, with multiplicity, the eigenvalues of $\sqrt{\Delta_E}$ not exceeding R. Then

$$N_E(R) \le e\ \mathrm{Tr}(e^{-\Delta_E/R^2}) = O(R^d), \qquad R \ge 1.$$

Local parallel unitary frames have constant transition matrices and define an analytic bundle structure. In these frames Δ_E is the scalar analytic Laplacian on each component. Its retarded kernel is a distributional section of $E \boxtimes E^*$ and satisfies (A.2), with source $\delta_0(t)\delta_{\mathrm{diag}}\,\mathrm{Id}_E$ and with the y operator acting on E^*. Equations (A.4)–(A.5) hold componentwise. Diagonal restriction followed by the contraction $E_x \otimes E_x^* \to \mathbb{C}$ is analytic, so the same proper pushforward gives (A.1) for the cosine trace. The polynomial counting bound and [96, Corollary 2.9] give the one-sided analytic wavefront inclusions in (A.6) for the connection spectrum, and hence the half-wave inclusion. □

Remark A.4. For every closed Riemannian manifold, $\mathcal{L}(X,g)$ is closed in $(0,\infty)$. If $t_j \to t > 0$ and $\Phi^{t_j}(v_j) = v_j$ in S^*X, compactness gives a subsequence with $v_j \to v$. Continuity of the flow then gives $\Phi^t(v) = v$. Thus every positive number outside $\mathcal{L}(X,g)$ has an open interval disjoint from that set.

Corollary A.5. *Let $E_X \to (X,g)$ and $E_Y \to (Y,h)$ be finite-rank unitary flat bundles over closed real-analytic Riemannian manifolds. Suppose their connection Laplacians are isospectral, and write*

$$w(t) = \mathrm{Tr}(e^{it\sqrt{\Delta_{E_X}}}) = \mathrm{Tr}(e^{it\sqrt{\Delta_{E_Y}}}).$$

The distribution w is real analytic on every open interval $I \subset (0,\infty)$ disjoint from $\mathcal{L}(X,g)$. If $\ell \in \mathcal{L}(Y,h) \setminus \mathcal{L}(X,g)$, then w is real analytic on an open interval about ℓ, and

$$\ell \notin \mathrm{singsupp}_A\, w, \qquad \mathrm{singsupp}_A\, w \cap (0,\infty) \subsetneq \mathcal{L}(Y,h).$$

The trivial line bundles give the scalar case.

Proof. Proposition A.3, applied to E_X, makes w analytic on I. If $\ell \in \mathcal{L}(Y,h) \setminus \mathcal{L}(X,g)$, Remark A.4 gives such an interval containing ℓ. The same proposition, applied to E_Y, gives the displayed inclusion, which is strict because it omits ℓ. □

A.3. **Spherical space forms.** Prüfer's shifted Poisson formula [89, Corollary 4] gives the ordinary scalar formula of Corollary A.8 by Lemma A.7. Spectral determination of the spherical period set is already contained in [89, Corollary 5, p. 153]; for odd-dimensional space forms, it also follows from [61, Proposition 2.7 and Corollary 2.8, pp. 83–84].

Let $\Gamma \le \mathrm{O}(n+1)$ be finite and act freely on the unit sphere, where $n \ge 2$. Endow $M = \mathbb{S}^n/\Gamma$ with its quotient metric, and put

$$\rho = \frac{n-1}{2}, \qquad \mathcal{O}(\Gamma) = \{\mathrm{ord}(\gamma) \,:\, \gamma \in \Gamma\}, \qquad C_\rho(t) = \mathrm{Tr}\cos\!\left(t\sqrt{\Delta_M + \rho^2}\right).$$

Theorem A.6 (Prüfer). *One has*

$$\mathrm{sing\,supp}\, C_\rho = \bigcup_{q \in \mathcal{O}(\Gamma)} \frac{2\pi}{q}\mathbb{Z} = \{0\} \cup \mathrm{Lsp}(M) \cup (-\mathrm{Lsp}(M)). \tag{A.7}$$

Proof. Put $L_\pm(M) = \mathrm{Lsp}(M) \cup (-\mathrm{Lsp}(M))$. By [89, Corollary 1, p. 135],

$$L_\pm(M) = \left\{ \frac{2\pi k}{q} \,:\, q \in \mathcal{O}(\Gamma), k \in \mathbb{Z} \setminus \{0\},\ \gcd(k,q) = 1 \right\}.$$

If $d \mid q$, an element of order q has a power of order d. Reduction of k/q to lowest terms therefore gives

$$\mathrm{Lsp}(M) = \bigcup_{q \in \mathcal{O}(\Gamma)} \frac{2\pi}{q}\mathbb{N}_{>0}. \tag{A.8}$$

By [89, Corollary 4, p. 152], $\mathrm{sing\,supp}\, C_\rho = \{0\} \cup L_\pm(M)$. □

Lemma A.7. *Put*

$$V_\rho(t) = \operatorname{Tr} e^{it\sqrt{\Delta_M+\rho^2}}, \qquad W_M(t) = \operatorname{Tr} e^{it\sqrt{\Delta_M}}.$$

Then

$$\operatorname{sing\,supp} C_\rho = \operatorname{sing\,supp} V_\rho = \operatorname{sing\,supp} W_M.$$

Proof. Let μ_j be the Laplace eigenvalues, repeated according to multiplicity, and put $\nu_j = \sqrt{\mu_j + \rho^2}$. The polynomial counting bound for (ν_j) and the boundary-value theorem [96, Corollary 2.9] give, as in (A.6),

$$\mathrm{WF}_A(V_\rho) \subset \{(t;\tau) \,:\, \tau > 0\}.$$

Thus V_ρ and $\overline{V_\rho}$ have disjoint smooth wavefront sets. Since $2C_\rho = V_\rho + \overline{V_\rho}$ and complex conjugation preserves singular support,

$$\operatorname{sing\,supp} C_\rho = \operatorname{sing\,supp} V_\rho. \tag{A.9}$$

For $m \in \mathbb{R}$, let $S^m_{1,0}(\mathbb{R}^2)$ be the class of smooth symbols $a(t,\xi)$ satisfying

$$|\partial_t^k \partial_\xi^j a(t,\xi)| \le C_{kj}\langle\xi\rangle^{m-j} \quad (k,j \ge 0), \qquad \langle\xi\rangle = (1+\xi^2)^{1/2}.$$

Fix a bounded open interval $I \subset \mathbb{R}$ and choose $\theta \in C^\infty_c(\mathbb{R};\mathbb{R})$ equal to t near $\overline{I}$. Choose a real smooth function b, zero for sufficiently negative ξ, such that

$$b(\xi) = \sqrt{\xi^2-\rho^2} - \xi = -\frac{\rho^2}{\sqrt{\xi^2-\rho^2}+\xi}$$

for all sufficiently large positive ξ. Then $\partial_\xi^j b = O(\langle\xi\rangle^{-1-j})$ for every $j \ge 0$. The symbol

$$a(t,\xi) = e^{i\theta(t)b(\xi)}$$

belongs to $S^0_{1,0}(\mathbb{R}^2)$, $a-1 \in S^{-1}_{1,0}(\mathbb{R}^2)$, and $|a| = 1$. Let A be its left quantization,

$$(Af)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi} a(t,\xi)\widehat{f}(\xi)\,d\xi, \qquad f \in \mathcal{S}(\mathbb{R}).$$

The operator A is elliptic. By [75, Lemma 2.14 and Theorem 5.7, pp. 16 and 47], it acts continuously on $\mathcal{S}'(\mathbb{R})$. Left quantization gives

$$Ae^{it\nu_j} = e^{it\nu_j + i\theta(t)b(\nu_j)}.$$

On I this is $e^{it\sqrt{\mu_j}}$ for every sufficiently large ν_j. Applying A to the convergent series in $\mathcal{S}'(\mathbb{R})$ therefore gives

$$(AV_\rho - W_M)|_I \in C^\infty(I), \tag{A.10}$$

since only finitely many frequencies are excluded.

By [75, Lemma 2.22, pp. 19–20], write $A = P + R$, where the Schwartz kernel K_P of P satisfies $\operatorname{supp} K_P \subset \{|t-u| \le \varepsilon\}$ for some $\varepsilon > 0$ and R has a symbol in $S^{-\infty}_{1,0} := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(\mathbb{R}^2)$. The kernel of R belongs to $C^\infty(\mathbb{R}_t;\mathcal{S}(\mathbb{R}_u))$, so R maps tempered distributions to smooth functions. Choose $\psi \in C^\infty_c(\mathbb{R})$ equal to one on a neighborhood of $\overline{I} + [-\varepsilon,\varepsilon]$. Then

$$\big(AV_\rho - A(\psi V_\rho)\big)|_I = \big(R((1-\psi)V_\rho)\big)|_I \in C^\infty(I).$$

The distribution ψV_ρ has compact support. Since A is elliptic, [75, Lemma 6.9, p. 52] gives

$$\operatorname{sing\,supp}(AV_\rho) \cap I = \operatorname{sing\,supp} V_\rho \cap I.$$

Together with (A.10), this proves the second equality, since I is arbitrary. □

Corollary A.8. *For every spherical space form M as above,*

$$\operatorname{sing\,supp} W_M = \operatorname{singsupp}_A W_M = \bigcup_{q\in\mathcal{O}(\Gamma)} \frac{2\pi}{q}\mathbb{Z}. \tag{A.11}$$

For a sphere of radius R, the numerator is $2\pi R$. Two scalar-isospectral spherical space forms have the same ordinary length set.

Proof. The smooth singular-support equality follows from Theorem A.6 and Lemma A.7. The union in (A.11) is $\{0\}\cup\operatorname{Lsp}(M)\cup(-\operatorname{Lsp}(M))$, by (A.8). It contains the analytic singular support by Proposition A.3, and the analytic singular support contains the smooth singular support. This proves the analytic equality. Multiplication of the metric by R^2 changes $W_M(t)$ to $W_M(t/R)$ and every geodesic period T to RT. Equality of the scalar spectra implies equality of the half-wave traces and hence equality of their positive singular supports. □

Remark A.9. Let $\Gamma \neq \{1\}$, let reg be its regular unitary representation, and put

$$W_{\mathrm{reg}}(t) = \operatorname{Tr} e^{it\sqrt{\Delta_{M,\mathrm{reg}}}}.$$

For $\sigma \in \operatorname{Irr}(\Gamma)$, let V_σ be its representation space and let $W_\sigma(t) = \operatorname{Tr} e^{it\sqrt{\Delta_{M,\sigma}}}$. The decomposition of the regular representation and (5.3) at the identity give

$$W_{\mathrm{reg}} = \sum_{\sigma\in\operatorname{Irr}(\Gamma)} (\dim V_\sigma) W_\sigma = D_1 = W_{\mathbb{S}^n},$$

where D_1 is the scalar half-wave trace of the covering sphere. The positive periods of the covering sphere are $2\pi\mathbb{N}_{>0}$. Proposition A.3, applied to that sphere, makes this trace real analytic on $\mathbb{R}\setminus 2\pi\mathbb{Z}$. For every $q > 1$ in $\mathcal{O}(\Gamma)$, one therefore has

$$\frac{2\pi}{q} \in \operatorname{Lsp}(M) \setminus \operatorname{singsupp}_A W_{\mathrm{reg}}.$$

Appendix B. Compact three-gradings and central intersections

B.1. **Simple compact three-gradings.** Let $\mathfrak{s}$ be a compact simple Lie algebra and let $0 \neq B \in \mathfrak{s}$. Suppose that the eigenvalues of $\operatorname{ad} B$ on the complexification $\mathfrak{s}_{\mathbb{C}}$ belong to $\{0, i, -i\}$. Let τ be conjugation with fixed algebra $\mathfrak{s}$, let κ be the complex Killing form, and put

$$h(u,v) = -\kappa(\tau u, v), \qquad \mathfrak{k} = \{X \in \mathfrak{s} : [X,B] = 0\}, \qquad T = \ker(\operatorname{ad} B - i\operatorname{Id}).$$

The form h is positive definite and Hermitian, with conjugate-linear first argument. Since $\operatorname{ad} B$ is h-skew-adjoint,

$$\mathfrak{s}_{\mathbb{C}} = \tau(T) \oplus \mathfrak{k}_{\mathbb{C}} \oplus T.$$

Moreover, $[T,T] = 0$, since $2i$ is not an adjoint eigenvalue. The graded pair is $(\mathfrak{s}, B)$; an isomorphism $(\mathfrak{s}, B) \to (\mathfrak{s}', B')$ is a real Lie algebra isomorphism carrying B to B'. We also write $\mathfrak{s}/\mathfrak{k}$, with positive tangent module T.

Choose a Cartan subalgebra containing B and a positive system, with simple roots $\alpha_1, \dots, \alpha_r$, for which $-iB$ is dominant. Then $\mathfrak{p} = \mathfrak{k}_{\mathbb{C}} \oplus T$ is the standard parabolic whose Levi roots vanish on B, with nilradical T. The classification of cominuscule parabolics [68, Proposition 2.1, Table 1 and Lemma 2.3, pp. 35–36] gives a single omitted simple root α_t,

$$\alpha_t(-iB) = 1, \qquad \alpha_j(-iB) = 0 \quad (j \neq t), \qquad \mathfrak{z}(\mathfrak{k}) = \mathbb{R}B,$$

the irreducibility of T, and the following table. The third column is the $[\mathfrak{k},\mathfrak{k}]$ module; B acts by i.

$\mathfrak{s}$	$[\mathfrak{k},\mathfrak{k}]$	T	$\dim_{\mathbb{C}} T$
$\mathfrak{su}(p+q)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(q)$	$\mathbb{C}^p\otimes(\mathbb{C}^q)^*$	pq
$\mathfrak{sp}(m)$	$\mathfrak{su}(m)$	$\mathrm{Sym}^2\,\mathbb{C}^m$	$m(m+1)/2$
$\mathfrak{so}(2m)$	$\mathfrak{su}(m)$	$\Lambda^2\mathbb{C}^m$	$m(m-1)/2$
$\mathfrak{so}(d+2)$	$\mathfrak{so}(d)$	$\mathbb{C}^d$	d
$\mathfrak{e}_6$	$\mathfrak{so}(10)$	half-spin	16
$\mathfrak{e}_7$	$\mathfrak{e}_6$	minuscule	27.

(B.1)

The parameters are integers, with $p,q\geq 1$, $m\geq 1$ in the symplectic row, $m\geq 3$ in the alternating row, and $d=1$ or $d\geq 3$ in the vector row. The exceptional modules and their dimensions are given in [41, §31.3, pp. 141–142]. The low-rank identities $D_2=A_1\times A_1$, $D_3=A_3$ and $B_2=C_2$ are as in [41, Remark 22.11, p. 101].

Let $\mathfrak{u}(T,h)$ be the real Lie algebra of h-skew-adjoint complex endomorphisms of T. The isotropy homomorphism

$$r_T:\mathfrak{k}\longrightarrow\mathfrak{u}(T,h),\qquad r_T(X)u=[X,u],$$

is injective. Indeed, its kernel $\mathfrak{a}$ is an ideal of $\mathfrak{k}$, and $[\mathfrak{a}_{\mathbb{C}},T]=[\mathfrak{a}_{\mathbb{C}},\tau(T)]=0$. Thus $\mathfrak{a}_{\mathbb{C}}$ is an ideal of $\mathfrak{s}_{\mathbb{C}}$. Since $B\neq 0$ and $\mathfrak{z}(\mathfrak{s})=0$, one has $\mathfrak{k}\neq\mathfrak{s}$; simplicity gives $\mathfrak{a}=0$.

Lemma B.1. *Let $n\geq 2$ and let V be the defining complex $\mathfrak{su}(n)$ module. Every nontrivial finite-dimensional complex $\mathfrak{su}(n)$ module has dimension at least n. An irreducible module of dimension n is V or V^*. A nontrivial four-dimensional complex $\mathfrak{su}(3)$ module is $V\oplus\mathbb{C}$ or $V^*\oplus\mathbb{C}$.*

Proof. Extend the action to $\mathfrak{sl}_n(\mathbb{C})$. By complete reducibility [41, Theorem 16.4, p. 77], it suffices to consider irreducible modules. Let $\varpi_1,\dots,\varpi_{n-1}$ be the standard fundamental weights and let L_λ be the irreducible module of highest weight $\lambda=\sum_{j=1}^{n-1}a_j\varpi_j$, where $a_j\in\mathbb{Z}_{\geq 0}$ and $\lambda\neq 0$. If $a_j>0$, the Weyl dimension formula and the fundamental representations of type A [41, Proposition 28.1 and §28.3, pp. 125–127] give

$$\dim L_\lambda\geq\dim L_{\varpi_j}=\binom{n}{j}\geq n.$$

Every Weyl factor is nondecreasing when a dominant weight is added, and a simple-root factor increases strictly unless that weight is zero. Equality in the first inequality therefore holds only for $\lambda=\varpi_j$, and in the second only for $j=1,n-1$. For $n=3$, put $a=a_1$ and $b=a_2$. The same formula is

$$\dim L_{a\varpi_1+b\varpi_2}=\frac{(a+1)(b+1)(a+b+2)}{2}.$$

It is at least six when $a+b\geq 2$. Complete reducibility gives the last assertion. □

Lemma B.2. *Let $n\geq 2$, let V be the defining complex $\mathfrak{su}(n)$ module, and let $\phi:\mathfrak{su}(n)\to\mathfrak{k}$ be a homomorphism.*

(1) If $T\simeq V$ or V^* as a $\mathfrak{su}(n)$ module, then the graded pair is $\mathfrak{su}(n+1)/\mathfrak{u}(n)$.

(2) Suppose $T=T_+\oplus T_-$ with $T_+\simeq V$ and $T_-\simeq V^*$. Let $A_T\in\mathrm{End}_{\mathbb{C}}(T)$ be h-skew-adjoint, commute with $r_T\circ\phi$, and act by i on T_+ and by $-i$ on T_-. Then the graded pair is

$$\mathfrak{su}(2n+1)/\mathfrak{u}(2n)\quad\text{or}\quad\mathfrak{so}(2n+2)/(\mathfrak{so}(2n)\oplus\mathfrak{so}(2)).$$

Under (2), there is a unique $A\in\mathfrak{k}$ with $r_T(A)=A_T$. For each of the two graded pairs, the homomorphisms

$$\mathfrak{su}(n)\oplus\mathbb{R}^2\longrightarrow\mathfrak{s},\qquad(X,a,b)\longmapsto\phi(X)+aA+bB,$$

arising in this way are conjugate by an automorphism of $\mathfrak{s}$ fixing B.

Proof. Since the action on T is nontrivial, simplicity and perfectness of $\mathfrak{su}(n)$ give

$$\ker \phi = 0, \qquad \phi(\mathfrak{su}(n)) \subset [\mathfrak{k}, \mathfrak{k}].$$

In the first row of (B.1), a nonzero projection to either defining factor makes its dimension at least n, by Lemma B.1. Order the factors so that $p \le q$. If $pq = n$, then $(p, q) = (1, n)$. If $pq = 2n$, then $(p, q) = (1, 2n)$ or $(2, n)$. For $n \ge 3$, one has $V \ne V^*$, since their highest weights ϖ_1, ϖ_{n-1} are distinct. The second possibility has tangent module $2V$ or $2V^*$ and is excluded. For $n = 2$ and $(p, q) = (2, 2)$, nonzero projections to both factors give $V \otimes V \simeq \operatorname{Sym}^2 V \oplus \mathbb{C}$, which is excluded. If only one projection is nonzero, the tangent module is $2V$, and the pair is the quadric pair under $A_3 = D_3$.

In the symplectic row, the defining m-dimensional module is nontrivial, so $m \ge n$. The equation $m(m+1)/2 = n$ has no solution with $n \ge 2$. The equation $m(m+1)/2 = 2n$ implies $m \le 3$ and leaves only $(m, n) = (3, 3)$. Its tangent module is $\operatorname{Sym}^2 V$ or $\operatorname{Sym}^2 V^*$, hence is irreducible [41, Example 28.3, p. 127], and is excluded.

In the alternating row, again $m \ge n$. The equation $m(m-1)/2 = n$ leaves only $(m, n) = (3, 3)$, the projective pair under $D_3 = A_3$. The equation $m(m-1)/2 = 2n$ implies $m \le 5$ and leaves $(m, n) = (4, 3), (5, 5)$. The second has irreducible tangent module $\Lambda^2 V$ or $\Lambda^2 V^*$ [41, §28.3, pp. 126–127], and is excluded. For $(m, n) = (4, 3)$, Lemma B.1 identifies the defining module with $V \oplus \mathbb{C}$ or $V^* \oplus \mathbb{C}$; its exterior square is $V \oplus V^*$. The three outer nodes of D_4 are permuted by diagram automorphisms, whose lifts preserve the compact real form [41, §§39.1, 39.4, pp. 180–185]. The values of the simple roots on $-iB$ show that these lifts carry the normalized grading elements to one another. Thus this pair is the quadric pair with $n = 3$.

In the vector row, $d = n$ is impossible because $\phi(\mathfrak{su}(n)) \subset \mathfrak{so}(n)$ would imply

$$n^2 - 1 \le \frac{n(n-1)}{2},$$

contrary to $n \ge 2$. The case $d = 2n$ is the quadric pair. For $\mathfrak{e}_6$, the tangent dimensions force $n = 16$ or 8, whereas $n^2 - 1 > 45 = \dim \mathfrak{so}(10)$. For $\mathfrak{e}_7$, the first case would give $n = 27$ and $n^2 - 1 > 78 = \dim \mathfrak{e}_6$, while $2n = 27$ is impossible.

Under (2), the eigenspaces T_+, T_- are h-orthogonal. In the projective pair, take

$$B = \frac{i}{2n+1} \operatorname{diag}(I_{2n}, -2n)$$

and identify T with the upper-right column in $\mathfrak{sl}_{2n+1}(\mathbb{C})$. In these coordinates, h is a positive multiple of the standard Hermitian form. The faithful map r_T identifies $\mathfrak{k}$ with $\mathfrak{u}(T, h)$, since both have real dimension $4n^2$. Thus $A_T = r_T(A)$ for a unique A. For a second choice (ϕ', A'_T), let $T'_\pm$ and A' denote the corresponding eigenspaces and element of $\mathfrak{k}$. The corresponding unitary $\mathrm{SU}(n)$ modules admit unitary intertwiners $T_\pm \to T'_\pm$ [103, Lemma 2.1.3, p. 29]. Their direct sum is a unitary map U on T intertwining both the $\mathfrak{su}(n)$ actions and A_T, A'_T. The automorphism $\Psi = \mathrm{Ad}(\operatorname{diag}(U, 1))$ of $\mathfrak{su}(2n+1)$ fixes B and induces U on T.

In the quadric pair, write $\mathbb{R}^{2n+2} = \mathbb{R}^{2n} \oplus \mathbb{R}^2$ and take

$$B = \operatorname{diag}(0_{2n}, J), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Identify T with $\mathbb{C}^{2n}$ by assigning x to the off-diagonal skew-symmetric matrix with upper-right block $(x, -ix)$. Put $q(x, y) = x^t y$, where ^{t} is transpose. The form h is a positive multiple of $\bar{x}^t y$; * denotes the adjoint with respect to h. The form q is $\phi(\mathfrak{su}(n))$-invariant. Its restrictions to T_+ and T_- vanish, since $\operatorname{Sym}^2 V$ and $\operatorname{Sym}^2 V^*$ are nontrivial irreducible modules [41, Example 28.3, p. 127]. Therefore

$$q(A_T x, y) + q(x, A_T y) = 0 \qquad (x, y \in T).$$

Thus $A_T^t = A_T^* = -A_T$. Hence A_T is real and belongs to $\mathfrak{so}(2n)$, and is induced by a unique $A \in \mathfrak{so}(2n) \subset \mathfrak{k}$.

Let $\sigma x = \bar{x}$ on $\mathbb{C}^{2n}$. Since q is a positive multiple of $h(\sigma x, y)$, isotropy and orthogonality give $\sigma(T_+) = T_-$. For a second choice (ϕ', A'_T), use primes as above and choose a unitary intertwiner $U_+ : T_+ \to T'_+$. Define

$$U|_{T_+} = U_+, \qquad U|_{T_-} = \sigma U_+ \sigma.$$

Then U is unitary, commutes with σ, and intertwines both the $\mathfrak{su}(n)$ actions and A_T, A'_T. It is the complexification of an orthogonal map $Q \in \mathrm{O}(2n)$. The automorphism $\Psi = \mathrm{Ad}(\mathrm{diag}(Q, I_2))$ of $\mathfrak{so}(2n+2)$ fixes B and induces U on T. No condition on $\det Q$ is required.

In either pair, injectivity of r_T gives, for $X \in \mathfrak{su}(n)$,

$$[A, \phi(X)] = [B, \phi(X)] = [A, B] = 0, \qquad \Psi \circ \phi = \phi', \quad \Psi(A) = A', \quad \Psi(B) = B.$$

These identities prove the assertions about the homomorphisms. □

B.2. **Central intersections.** Let $n \geq 1$. For the groups of Section 2, the standard center formulas [41, Corollary 9.10 and Exercise 39.10(ii),(iii), pp. 53 and 185], with the SO(4) case given in [103, §4.1, Exercise 8, p. 153], are

$$Z(G_n) = \{\zeta I_{2n+1} : \zeta \in \mathrm{U}(1)\},$$
$$Z(H_n) = \{(\varepsilon I_{2n}, \delta I_{2n+2}) : \varepsilon, \delta \in \{1, -1\}\}.$$

The embeddings of (2.1) satisfy

$$\begin{aligned} Z(G_n) \cap i_n(P_n) &= \{\varepsilon I_{2n+1} : \varepsilon \in \{1, -1\}\}, \\ Z(H_n) \cap j_n(P_n) &= \{(\varepsilon I_{2n}, \varepsilon I_{2n+2}) : \varepsilon \in \{1, -1\}\}. \end{aligned} \tag{B.2}$$

Indeed, $i_n(A, z) = \zeta I_{2n+1}$ implies $A = \zeta I_n = \bar{A}$ and $z = \zeta$, hence $\zeta = \pm 1$. If $j_n(A, z)$ is central, write $c(A) = \varepsilon I_{2n}$ and $r(A \oplus z) = \delta I_{2n+2}$ with $\varepsilon, \delta \in \{1, -1\}$. The first equality gives $A = \varepsilon I_n$; the second gives $A = \delta I_n$ and $z = \delta$. Thus $\varepsilon = \delta$ and $z = \varepsilon$. Conversely, $(A, z) = (\varepsilon I_n, \varepsilon)$, for $\varepsilon = \pm 1$, has the two central images in (B.2).

Appendix C. Central quotients and characteristic classes

C.1. **The subgroup lattice and its factor count.** For a finite abelian group C, let $\mathrm{Sub}(C)$ be its subgroup lattice, ordered by inclusion, with join $D \vee D' = \langle D, D' \rangle$. Let $\mathcal{J}(C)$ be the set of its nontrivial cyclic subgroups of prime-power order, and put $j(C) = |\mathcal{J}(C)|$.

Lemma C.1. *The nontrivial join-irreducible elements of* $\mathrm{Sub}(C)$ *are exactly the members of* $\mathcal{J}(C)$*. Choose a generator* c_H *of each such* H*. Every* $D \leq C$ *is generated by the* c_H *with* $H \leq D$*.*

Proof. The join-irreducible subgroups of a finite group are its nontrivial cyclic subgroups of prime-power order [1, §1.3, p. 3]. Since $\mathrm{Sub}(C)$ is finite, every $D \leq C$ is the join of its join-irreducibles [84, Lemma 2.8, p. 19]. Thus $D = \langle c_H : H \in \mathcal{J}(C),\ H \leq D \rangle$. □

Theorem C.2. *Let* E *be finite and* $1 \neq C \leq Z(E)$*, and put* $I(D) = \{H \in \mathcal{J}(C) : H \leq D\}$ *for* $D \leq C$*. There is a family* $M_I = J_I/K$*, indexed by* $I \subseteq \mathcal{J}(C)$*, with* $r = j(C)$ *factors as in Theorem 10.4, such that*

$$\pi_1(M_\emptyset) \simeq E, \qquad \pi_1(M_{\mathcal{J}(C)}) \simeq E/C, \qquad \pi_1(M_{I(D)}) \simeq E/D \quad (D \leq C).$$

The spectral equalities, curvature, coverings and flat-twist spectra are those of Theorem 10.4. If $D \leq D'$*, the induced homomorphism on lifted-stabilizer component groups is the quotient map* $E/D \to E/D'$*.*

Proof. Apply Theorem 10.4 to the tuple $(c_H)_{H \in \mathcal{J}(C)}$, and put $C_I = \langle c_H : H \in I \rangle$ for $I \subseteq \mathcal{J}(C)$. Lemma C.1 gives $C_{I(D)} = D$, and $D \leq D'$ gives $I(D) \subseteq I(D')$. □

The map $I : \mathrm{Sub}(C) \to 2^{\mathcal{J}(C)}$ is injective and satisfies

$$I(D \cap D') = I(D) \cap I(D'), \qquad I(D \vee D') = I\big(\langle c_H : H \in I(D) \cup I(D') \rangle\big).$$

Its image need not be closed under unions. Taking $C = Z(E)$ realizes all central quotients of E; for $C = 1$, take one identity entry.

Proposition C.3. *If every subgroup of C is generated by a subtuple of a fixed tuple in C, then that tuple has length at least $j(C)$. Hence Theorem C.2 has the least number of factors among realizations with this property. Write $C_{(p)} = \prod_{a=1}^{d_p} C_{p^{\lambda_a}}$ for the Sylow p-subgroup of C, with $\lambda_a \geq 1$, and set $S_p(k) = \sum_a \min(k, \lambda_a)$, with $S_p(0) = 0$. Then*

$$j(C) = \sum_{p \| C|} \sum_{k=1}^{\max_a \lambda_a} \frac{p^{S_p(k)} - p^{S_p(k-1)}}{p^{k-1}(p-1)}. \tag{C.1}$$

In particular $j((\mathbb{Z}/p)^d) = (p^d - 1)/(p-1)$ and $j(C_m) = \sum_{p|m} v_p(m)$, where $v_p(m)$ is the exponent of p in m.

Proof. A subtuple generating a cyclic p-group H must contain a generator of H: otherwise all its entries lie in the unique maximal subgroup of H. A coordinate generates only one subgroup, so there must be at least $j(C)$ coordinates. Apply [107, Corollary 2, equation (9)] to each Sylow subgroup of C and sum over the nontrivial prime-power orders. This gives (C.1); the two specializations follow by substituting $\lambda_a = 1$ and $d_p = 1$, respectively. □

Proposition C.4. *Let $r, n_1, \dots, n_r \geq 1$ be integers and $K \leq \prod_{j=1}^r P_{n_j}$ a closed subgroup. For $I \subseteq [r]$, put*

$$J_I = \prod_{j \notin I} G_{n_j} \times \prod_{j \in I} H_{n_j}, \qquad M_I = J_I / K.$$

Suppose $E = \pi_1(M_\emptyset)$ is finite. Put $\Gamma = \pi_0 K$ and let C be the kernel of the boundary homomorphism $E \to \Gamma$, with the convention of Lemma 10.2. Let $z : K \to \mathbb{T}^r$ be the product of the ambient unitary determinants, and let $z_{I^c} : K \to \mathbb{T}^{I^c}$ be its coordinate projection. There are central generators $c_1, \dots, c_r$ of C such that

$$\pi_0(\ker z_{I^c}) \simeq E / \langle c_j : j \in I \rangle.$$

If the boundary homomorphism $\pi_1(M_I) \to \pi_0(\ker z_{I^c})$ is an isomorphism for every I, and every subgroup of C is generated by a subtuple of $(c_1, \dots, c_r)$, then $r \geq j(C)$. The markings are the quotient homomorphisms from E.

Proof. The homotopy sequence makes $\operatorname{im}(\pi_1(K^0) \to \mathbb{Z}^r)$ a finite-index subgroup. Thus $z(K^0) = \mathbb{T}^r$. Put $L = \ker z$. The simply connected product of the special unitary groups acts transitively on the all-unitary quotient, with stabilizer L, so the boundary homomorphism gives $E \simeq \pi_0 L$. The homotopy sequence of $L \to K \to \mathbb{T}^r$ gives

$$\mathbb{Z}^r \xrightarrow{\partial} E \longrightarrow \Gamma \longrightarrow 1.$$

For $l \in L$, conjugation by l induces the identity on $\mathbb{T}^r$. Naturality of ∂ gives $l \partial(a) l^{-1} = \partial(a)$ in $\pi_0 L$ for every $a \in \mathbb{Z}^r$. Hence $C = \operatorname{im} \partial$ is central, with generators $c_j = \partial e_j$, where $(e_j)_{j=1}^r$ is the standard basis of $\mathbb{Z}^r$.

For $I \subseteq [r]$, the sequence $L \to \ker z_{I^c} \to \mathbb{T}^I$ is onto on the right because $z(K^0) = \mathbb{T}^r$. Its boundary homomorphism is $\partial|_{\mathbb{Z}^I}$, and therefore

$$\pi_0(\ker z_{I^c}) \simeq E / \langle c_j : j \in I \rangle.$$

The group

$$J_I' = \prod_{j \notin I} SU(2n_j + 1) \times \prod_{j \in I} H_{n_j}$$

acts transitively on M_I with stabilizer $\ker z_{I^c}$ and has fundamental group $(C_2)^I$. Its boundary homomorphism is an isomorphism whenever

$$\{0\} \times (C_2)^I \subseteq \operatorname{im}\big(\pi_1(K^0) \longrightarrow \mathbb{Z}^{I^c} \times (C_2)^I\big).$$

Indeed, a loop with zero unitary winding represents a class from $\pi_1((\ker z_{I^c})^0)$, by the homotopy sequence over $\mathbb{T}^{I^c}$; the displayed inclusion kills every remaining orthogonal fundamental-group generator. Proposition C.3 gives the lower bound. □

C.2. **Compatible quotient diagrams.** Let $\mathcal{D}$ be a finite directed multigraph with nonempty vertex set $\mathcal{V}$ and arrow set $\mathcal{A}$, and let Γ be a finite group. For each $v \in \mathcal{V}$, let $p_v : E_v \twoheadrightarrow \Gamma$ have finitely generated central kernel. For each arrow $a : v \to w$, let $f_a : E_v \to E_w$ satisfy $p_w f_a = p_v$. Set

$$L = \left\{ (e_v) \in \prod_{v \in \mathcal{V}} E_v \ : \ \begin{array}{l} p_v(e_v) \text{ is independent of } v, \\ f_a(e_v) = e_w \text{ for every } a : v \to w \end{array} \right\}. \tag{C.2}$$

Theorem C.5. *There is a central extension $E \twoheadrightarrow \Gamma$ with finitely generated kernel and compatible epimorphisms over Γ $E \twoheadrightarrow E_v$ exactly when every projection $L \to E_v$ is onto. One may then take $E = L$, and the diagram is realized on the marked component groups of one family in Theorem 10.4, with its spectral equalities, curvature, covers and flat-twist spectra.*

Proof. Compatible epimorphisms give $E \to L$ whose coordinate images are all E_v, proving necessity. Conversely, surjectivity of one projection makes the common-value map $L \to \Gamma$ onto. Its kernel $C \le \prod_v \ker p_v$ is central and finitely generated abelian. Let $D_v = \ker(L \to E_v) \le C$. Then $L/D_v \simeq E_v$ with the prescribed marking, and an arrow gives $D_v \le D_w$ and the required quotient map. Concatenate generating tuples for C and all D_v. The indices of all entries lying in D_v generate D_v and are nested along arrows. Apply Theorem 10.4; for $C = 1$, take one identity entry. Repeated vertices may correspond to the same manifold. For nontrivial finite C, Theorem C.2 gives the sharper full lattice factor count. Ricci curvature is positive precisely at finite fundamental groups. □

Corollary C.6. *If every f_a is onto and the underlying undirected multigraph of $\mathcal{D}$ is a forest, every projection in* (C.2) *is onto.*

Proof. Fix a vertex v_0 and $e_{v_0} \in E_{v_0}$, and put $\gamma = p_{v_0}(e_{v_0})$. Root its tree at v_0. Inductively, let w have parent v, with e_v already chosen. If their edge is $a : v \to w$, set $e_w = f_a(e_v)$; if it is $a : w \to v$, choose $e_w \in f_a^{-1}(e_v)$. In both cases $p_w(e_w) = \gamma$. For each other tree, choose a root element above γ and apply the same induction. Every edge joins a vertex to its parent, so all arrow equations hold. The resulting element of L projects to e_{v_0}. □

Example C.7. Take four copies A, B, C, D of C_3 over $\Gamma = 1$, with arrows $A \to B, A \to D, C \to B$ equal to the identity and $C \to D$ equal to minus the identity. The directed graph is acyclic, but a coherent tuple satisfies $a = b = c$ and $d = a = -c$, hence is zero. None of the limit projections is onto. Separate realizability of the arrows does not give a compatible realization from one source.

C.3. **Central characters and projective flatness.** Let J be a compact connected Lie group with bi-invariant metric Q, and let $K \le J$ be closed with $T = K^0 \le Z(K)$. Then T is a torus and $\Gamma = K/T$ is finite. Write $\mathfrak{j} = \mathrm{Lie}(J)$, $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{m} = \mathfrak{t}^{\perp_Q}$. The projection $\mathrm{pr}_{\mathfrak{t}}$ is Q-orthogonal, and the canonical principal connection is $\mathrm{pr}_{\mathfrak{t}}(g^{-1}dg)$. Write Q^* for the dual inner product on $\mathfrak{t}^*$. All representations are continuous, finite-dimensional and unitary. For a K-representation ρ on V_ρ, let $E_\rho = J \times_K V_\rho$ have its canonical connection, and let $\mathbb{P}(E_\rho)$ be its bundle of complex lines with the induced projective connection. Write $B_{\mathfrak{j}}$ for the Killing form. If (e_a) is a Q-orthonormal basis of $\mathfrak{m}$ and (e^a) its dual basis, the forms $e^a \wedge e^b$, $a < b$, are orthonormal in $\Lambda^2\mathfrak{m}^*$.

Proposition C.8. *An irreducible $\rho : K \to U(d)$ restricts to $\rho(t) = \chi(t) I_d$ on T, for a character $\chi : T \to U(1)$. Write $d\chi = i\ell_\chi$ and let $Z_\chi \in \mathfrak{t}$ be the Q-dual of the real linear form ℓ_χ. Let β_χ be the invariant real two-form whose value at eK is the expression below for $X, Y \in \mathfrak{m}$. The canonical curvature and the vertical Casimir of* (4.3) *are*

$$F_\rho = i\beta_\chi I_d, \quad \beta_\chi(X,Y) = -\ell_\chi(\mathrm{pr}_{\mathfrak{t}}[X,Y]) = -Q([Z_\chi, X], Y), \quad C_\rho = \|\ell_\chi\|^2_{Q^*} I_d.$$

The connections on $\mathbb{P}(E_\rho)$ and $\mathrm{End}\, E_\rho$ are flat, with holonomy factoring through $\pi_1(J/K) \to \Gamma$. If τ is another irreducible unitary K-representation with the same central character, the connection on $\mathrm{Hom}(E_\rho, E_\tau)$ is flat. Moreover,

$$\beta_\chi = 0 \iff Z_\chi \in \mathfrak{z}(\mathfrak{j}), \qquad \|\beta_\chi\|^2 = -\tfrac{1}{2} B_{\mathfrak{j}}(Z_\chi, Z_\chi).$$

If $J = J_I$ and $K \le P = \prod_j P_{n_j}$ under the embeddings of Theorem 4.3, put $\mathfrak{p} = \mathrm{Lie}(P)$. Then $\mathfrak{p} \cap \mathfrak{z}(\mathfrak{j}) = 0$, and the canonical connection on E_ρ is flat if and only if $\chi = 1$.

Proof. Schur's lemma [41, Corollary 9.10, p. 53] gives the scalar T action. The curvature Ω of the canonical principal connection satisfies $\Omega(X,Y) = -\mathrm{pr}_{\mathfrak{t}}[X,Y]$ by the Maurer–Cartan equation. Applying $d\rho = i\ell_\chi I_d$ gives the formula for F_ρ. Since $\mathrm{Ad}(K)$ is trivial on $\mathfrak{t}$, the displayed two-form is K-invariant. For a Q-orthonormal basis (u_r) of $\mathfrak{t}$,

$$C_\rho = -\sum_r d\rho(u_r)^2 = \sum_r \ell_\chi(u_r)^2 I_d = \|\ell_\chi\|_{Q^*}^2 I_d.$$

The representations on projective space, endomorphisms and equal-character Hom spaces are trivial on T; they factor through Γ. Their connections are induced from the covering $J/T \to J/K$ with discrete structure group Γ. Since $[Z_\chi, \mathfrak{t}] = 0$ and $[Z_\chi, \mathfrak{m}] \subseteq \mathfrak{m}$, the curvature vanishes exactly when $\operatorname{ad} Z_\chi = 0$. Since $\operatorname{ad} Z_\chi$ is Q-skew-adjoint and vanishes on $\mathfrak{t}$,

$$\|\beta_\chi\|^2 = \tfrac{1}{2}\sum_{a,b} Q([Z_\chi, e_a], e_b)^2 = -\tfrac{1}{2}\operatorname{tr}_{\mathfrak{j}}((\operatorname{ad} Z_\chi)^2) = -\tfrac{1}{2}B_{\mathfrak{j}}(Z_\chi, Z_\chi).$$

The Lie algebra of H_n is semisimple. For $X \in \mathfrak{u}(n)$ and $t, c \in \mathbb{R}$, the equality $\operatorname{diag}(X, \bar{X}, it) = icI$ in the G_n embedding gives $X = icI_n$ and $\bar{X} = icI_n$. Conjugation gives $c = 0$, and then $t = 0$. Thus $\mathfrak{p} \cap \mathfrak{z}(\mathfrak{j}) = 0$ in every ambient product, and a character of connected T has zero differential exactly when it is trivial. □

Corollary C.9. *Let $K \le \prod_j P_{n_j}$ be closed with $K^0 \le Z(K)$. For each irreducible unitary K-representation, C_ρ and $\|\beta_\chi\|^2$ are independent of the ambient choices and agree for matched positive factor scales.*

Proof. The isotropy metric is common, and Proposition 4.5 identifies the ambient adjoint representations restricted to P, hence the restrictions of their Killing forms. For $X, Y \in \mathfrak{u}(n)$ and $t, s \in \mathbb{R}$,

$$B_{\mathfrak{g}_n}|_{\mathfrak{p}_n}((X,it),(Y,is)) = B_{\mathfrak{h}_n}|_{\mathfrak{p}_n}((X,it),(Y,is)) = 4(2n+1)\operatorname{Re}\operatorname{Tr}(XY) - 4n\,ts.$$

The common dual vector Z_χ and the preceding proposition give the same curvature norm, while the common isotropy metric gives the same Casimir. Matched scales change the dual vector equally on both sides. □

C.4. **Projective multipliers.**

Proposition C.10. *Fix a character $\chi : T \to U(1)$ and a section $s : \Gamma \to K$ with $s(1) = 1$. Set $c(\gamma,\delta) = s(\gamma)s(\delta)s(\gamma\delta)^{-1} \in T$, and put $\alpha_\chi = \chi \circ c$. This is a normalized 2-cocycle for the trivial Γ action on $U(1)$. Representations with central character χ correspond to α_χ-projective representations of Γ, by*

$$R_\gamma = \rho(s(\gamma)), \quad R_\gamma R_\delta = \alpha_\chi(\gamma,\delta)R_{\gamma\delta}, \qquad \rho(ts(\gamma)) = \chi(t)R_\gamma.$$

Tensor products multiply the multipliers and duals invert them.

Proof. Form the central extension

$$K_\chi = (K \times U(1))/\{(t, \chi(t)^{-1}) : t \in T\}.$$

The representations of K with central character χ identify with the representations of K_χ of central weight one by $\widehat{\rho}([k,\lambda]) = \lambda\rho(k)$. In the coordinates determined by $[s(\gamma), 1]$, multiplication is

$$(\lambda,\gamma)(\mu,\delta) = (\lambda\mu\alpha_\chi(\gamma,\delta), \gamma\delta).$$

Mackey's correspondence [76, Theorem 2.1, p. 270], applied to this extension of the finite group Γ, gives the displayed formulas and preserves intertwiners and irreducibility. His convention is $L_{\gamma\delta} = \sigma(\gamma,\delta)L_\gamma L_\delta$, so $\sigma = \alpha_\chi^{-1}$ [76, §1, p. 267]. Tensor products and duals have the stated multipliers by these formulas. □

Proposition C.11. *Let E be finite, $C \le Z(E)$, and $i : C \hookrightarrow T$ an embedding in a torus. Set $K = (E \times T)/\{(c, i(c)^{-1}) : c \in C\}$ and identify T with $\{[1,t] : t \in T\} = K^0$; then $K/T \simeq E/C$. Let σ be an irreducible unitary representation of E and write $\sigma(c) = \eta(c)I$ for $c \in C$. The character $\eta : C \to U(1)$ extends along i to a character χ of T. Then $\rho([e,t]) = \sigma(e)\chi(t)$ is irreducible, and its projective E/C multiplier is the push-forward of the extension cocycle by η.*

Proof. The subgroup $i(C)$ is closed in the compact abelian group T. The character-extension theorem [58, Corollary 2.33(ii), p. 44] therefore gives $\chi : T \to U(1)$ with $\chi \circ i = \eta$. The diagonal relation acts by $\eta(c)\chi(i(c))^{-1} = 1$, so ρ descends to K. Its restriction to the embedded subgroup E is σ; hence ρ is irreducible. A section of $E \to E/C$ has cocycle c_E, whose image in K is $i(c_E)$. Proposition C.10 gives the multiplier $\eta(c_E)$. □

Remark C.12. Write $\mathrm{Hom}_{\mathrm{cts}}$ for continuous homomorphisms. Pullback along $T \to T/i(C)$ identifies $\mathrm{Hom}_{\mathrm{cts}}(T/i(C), U(1))$ with a subgroup of $\mathrm{Hom}_{\mathrm{cts}}(T, U(1))$. The extensions of η form a coset of this subgroup, since the quotient of two extensions is trivial on $i(C)$. If $\eta \neq 1$, every extension is nontrivial. If, in addition, $K \leq \prod_j P_{n_j}$ is a closed embedding, the canonical connections in the ambient products of Theorem 4.3 are nonflat and projectively flat by Proposition C.8. For $\eta = 1$ one may choose $\chi = 1$. The representation then factors through E/C, and its canonical connection is flat.

C.5. **Examples and algebraic limits.**

Example C.13 (A non-direct-factor kernel). Let E be finite and $C = \langle a \rangle \simeq C_4 \leq Z(E)$. For $I = \emptyset, \{1\}, \{2\}, \{1,2\}$, the tuple (a, a^2) has coordinate kernels $1, C, \langle a^2 \rangle, C$, respectively, and relation lattice $\{(m,n) \in \mathbb{Z}^2 : m + 2n \equiv 0 \pmod 4\}$. The corresponding family has fundamental groups $E, E/C, E/\langle a^2 \rangle, E/C$. Choose distinct odd ranks as in Theorem 10.4; Theorem 12.8 then makes the four manifolds pairwise nonhomeomorphic. For C_8, all four marked subgroups require three coordinates, since $j(C_8) = 3$; counting vertices alone would give only two.

Proposition C.14. *Let p be an odd prime, $d \geq 2$, $V = \mathbb{F}_p^d$, and $W = \Lambda^2 V$. The multiplication on $V \oplus W$*

$$(v, w)(v', w') = (v + v', w + w' + \tfrac{1}{2} v \wedge v') \tag{C.3}$$

defines a group F of order $p^{d+\binom{d}{2}}$ and exponent p, with $Z(F) = [F,F] = \{0\} \oplus W$, identified with W. Every finite exponent-p group of nilpotency class at most two and abelianization dimension d is F/U for a subspace $U \leq W$. One family from Theorem C.2 realizes this entire marked quotient diagram with $r = (p^{\binom{d}{2}} - 1)/(p-1)$ factors. This is the least number of coordinates for which every subgroup of W is generated by a subtuple.

Proof. The alternating bilinear map $(v, v') \mapsto \frac{1}{2} v \wedge v'$ is a 2-cocycle on the additive group V, so (C.3) defines a group. For the commutator convention $[x, y] = x^{-1} y^{-1} x y$ and every $a \in \mathbb{Z}$,

$$(v, w)^a = (av, aw), \qquad [(v, w), (v', w')] = (0, v \wedge v').$$

These identities give the exponent and $[F, F] = W$. If $v \neq 0$, there is v' independent of v, since $d \geq 2$; thus $Z(F) = W$.

Let $(e_i)_{i=1}^d$ be the standard basis of V, and let N be the 2-nilpotent product of d cyclic groups of order p. Its universal property and normal form [77, Definition 1.3, Theorem 1.4 and the following paragraphs, pp. 2–3] give a surjection $N \to F$ sending its generators to $(e_i, 0)$. The commutators span W, so these images generate F. Both groups have order $p^{d+\binom{d}{2}}$; hence this map is an isomorphism. If G has exponent p, nilpotency class at most two and $\dim_{\mathbb{F}_p} G/[G,G] = d$, lifts of an abelianization basis minimally generate G [77, p. 18, before Corollary 5.8]. Consequently [77, §2, pp. 3–4] gives a surjection $N \to G$ inducing an isomorphism on abelianizations; its kernel lies in $[N, N]$. Under $N \simeq F$, this subgroup is a subspace of W. Theorem C.2 and Proposition C.3 give the simultaneous realization and its factor count. □

For $p = d = 3$, the center has 28 subspaces: 0, thirteen lines, thirteen planes, and W. Thirteen coordinates are necessary and sufficient. The resulting family has $2^{13} = 8192$ members. For distinct odd ranks, these are pairwise nonhomeomorphic by Theorem 12.8. Among the twenty-eight selected marked quotients, some may be isomorphic as abstract groups.

Example C.15 (Nontransitivity). For $D_{16} = \langle r, s : r^8 = s^2 = 1,\ srs = r^{-1} \rangle$, the central maps $D_{16} \to D_8 \to C_2^2$ have kernels $\langle r^4 \rangle$ and $Z(D_8)$, respectively. Their composite kernel $\langle r^2 \rangle$ is not central.

Since $Z(D_{16}) = \langle r^4 \rangle$, every central quotient of D_{16} has order sixteen or eight, whereas every quotient of C_2^2 has order at most four. Thus these endpoints have no common central quotient and cannot occur in one common-isotropy homogeneous family by Theorem 11.1, although each adjacent pair can.

C.6. **Finite covers and centerless groups.**

Proposition C.16. *Scalar-isospectral closed connected manifolds with finite fundamental groups and an isometric universal Riemannian cover have fundamental groups of equal order. In particular, groups E and E/A, with E finite and $1 \neq A \lhd E$, cannot occur in such a pair.*

Proof. Let M, N be such manifolds and X their common universal Riemannian cover. Fibre integration gives $\operatorname{Vol} X = |\pi_1(M)| \operatorname{Vol} M = |\pi_1(N)| \operatorname{Vol} N$. Weyl's law [38, Theorem 17.7, p. 236] gives $\operatorname{Vol} M = \operatorname{Vol} N > 0$. □

Proposition C.17. *There are strongly isospectral compact normal homogeneous manifolds with positive Ricci curvature and nonisomorphic centerless fundamental groups. Those groups admit no common-isotropy homogeneous realization.*

Proof. Put $V = \mathbb{F}_7^2$ and $E_+ = V \rtimes_{2I} C_3$, $E_- = V \rtimes_{\mathrm{diag}(2,4)} C_3$, where a chosen generator of C_3 acts by the indicated matrix. Both actions have order three and no nonzero fixed vector. A central element in V must be fixed; one outside V would act nontrivially on V. Hence both centers vanish. The Sylow seven subgroup is unique, since its number divides three and is 1 modulo seven. An isomorphism would conjugate $2I$ to $\mathrm{diag}(2,4)$ or its square, which is impossible. Each group has one identity, 48 elements of order seven, and 98 of order three: for $A = 2I$ or $\mathrm{diag}(2,4)$, the identity $I + A + A^2 = 0$ makes every element outside V have order three. Their regular embeddings in S_{147} are therefore almost conjugate by the Todd–Komatsu construction [36, Example 7.7]. Embed S_{147} in $J = SU(148)$ by $P \mapsto \mathrm{diag}(P, \det(P)^{-1})$. Give J one bi-invariant metric and $J/E_\pm$ the corresponding quotient metrics. These manifolds have dimension 21903, the prescribed fundamental groups, nonnegative sectional curvature and positive Ricci curvature. Let $\mathcal{E} \to J$ be a finite-rank Hermitian vector bundle with an S_{147} action by fibrewise unitary maps covering the right action on J. Let D be an S_{147}-equivariant self-adjoint elliptic differential operator of positive order on $\mathcal{E}$, and let $D_\pm$ be its descents to $\mathcal{E}/E_\pm$. For $\lambda \in \mathbb{R}$,

$$\dim\ker(D_\pm - \lambda) = \dim\ker(D - \lambda)^{E_\pm} = \frac{1}{147} \sum_{g \in E_\pm} \operatorname{Tr}\big(g|_{\ker(D-\lambda)}\big).$$

The last two expressions are equal for E_+ and E_- by almost conjugacy [30, §3.3, p. 21; §4.2, p. 33]. Thus the quotient operators are isospectral. This includes the Hodge and canonical tensor operators, and the cubic operators on equivariant Clifford bundles; for spinor bundles, the S_{147} action must lift to the spin structure. A common-isotropy realization would give central epimorphisms $E_\pm \to \Gamma$ by Theorem 11.1. Their kernels would be trivial, contradicting $E_+ \not\cong E_-$. □

C.7. **Finite central extensions and parity.**

Proposition C.18. *Let E be finite and $C = \prod_{j=1}^{r} C_{q_j} \leq Z(E)$ a specified cyclic decomposition, with $C_I = \prod_{j \in I} C_{q_j}$. Set $s = \#\{j : 2 \mid q_j\}$ and $d_0 = |E| + r$. Let $n_j = 2d_0 + 1$ when q_j is even and $n_j = 2d_0$ when q_j is odd. There is a common-isotropy realization of all E/C_I in dimension $\sum_j (2n_j + 1)^2 - r - s$. For $C = 1$, take $r = 1$, $q_1 = 1$.*

Proof. Let $i : C \hookrightarrow \mathbb{T}^r$ identify the jth cyclic factor with $\{z \in U(1) : z^{q_j} = 1\}$ in the jth coordinate. Put

$$\overline{K} = (E \times \mathbb{T}^r)/\{(c, i(c)^{-1}) : c \in C\}, \qquad z_j([e,t]) = t_j^{q_j}.$$

The joint kernel of z is E. Let L_e denote left translation by e on $\mathbb{C}[E]$, and write $t^a = \prod_j t_j^{a_j}$ for $a \in \mathbb{Z}^r$. Let $\widehat{C} = \mathrm{Hom}(C, U(1))$ and decompose $\mathbb{C}[E] = \bigoplus_{\eta \in \widehat{C}} V_\eta$ into its C-character spaces. Choose $a_\eta \in \mathbb{Z}^r$ with $i(c)^{a_\eta} = \eta(c)$, taking $a_1 = 0$. Define $R([e,t])|_{V_\eta} = t^{a_\eta} L_e$ and $\rho = R \oplus z_1 \oplus \cdots \oplus z_r$. The diagonal

relations act trivially. If $\rho([e,t]) = 1$, then $z_j([e,t]) = 1$ for every j, so $t = i(c')$ for some $c' \in C$. On every V_η, one has $t^{a_\eta} L_e = \eta(c') L_e = L_{ec'}$. Hence $R([e,t]) = L_{ec'} = 1$. Faithfulness of the regular representation gives $ec' = 1$, so $[e,t] = 1$. Thus ρ is faithful of dimension d_0. Put $K = \overline{K} \times \mathbb{T}^s$, with one auxiliary circle w_j for each even q_j, and extend ρ, z_j trivially over $\mathbb{T}^s$. Set $A_j = \rho \oplus \rho \oplus w_j$ for even q_j and $A_j = \rho \oplus \rho$ for odd q_j. The map $k \mapsto ((A_j(k), z_j(k)))_j$ embeds K faithfully in $\prod_j P_{n_j}$. At a vertex I, fix $j \in I$. If q_j is odd, the t_j loop has zero unitary winding and orthogonal winding $q_j e_j = e_j$ in $(C_2)^I$, since each ρ occurs twice. If q_j is even, the w_j loop has these windings. Thus the image of $\pi_1(K^0)$ contains $\{0\} \times (C_2)^I$. Each coordinate boundary element generates the specified factor C_{q_j}, so Proposition C.4 gives $\pi_1(J_I/K) \simeq E/C_I$. Finally $\dim K = r + s$, which gives the stated dimension. □

Proposition C.19. *Let $C = \langle c \rangle \le Z(E)$ have order q, with E finite. Put $\zeta_q = e^{2\pi i/q}$ and $\overline{K} = (E \times S^1)/\langle (c, \zeta_q^{-1}) \rangle$, $z([e,t]) = t^q$. A faithful $A : \overline{K} \to U(n)$ for some n can make the H_n boundary map an isomorphism onto E/C, without an auxiliary circle, exactly when q is odd or $c^{q/2} \notin [E,E]$ for even q.*

Proof. The group $\overline{K}$ is a compact Lie group and has a faithful unitary representation [58, Corollary 2.40, pp. 47–48]. For any such representation A, its determinant pulls back to $\chi(e)t^d$, where $\chi : E \to U(1)$ is a character and $d \in \mathbb{Z}$. It descends exactly when $\chi(c) = \zeta_q^d$. The identity component is the circle $t \mapsto [1,t]$; its image in $\pi_1(H_n) = C_2$ is $d + q \pmod 2$. Thus the required surjectivity is equivalent to $d + q$ being odd. For odd q, double a faithful representation to make d even. For even q, odd d forces $\chi(c^{q/2}) = -1$. Conversely, suppose $c^{q/2}$ has nontrivial image in $E/[E,E]$. The nontrivial character of the subgroup it generates extends to $E/[E,E]$ by [58, Corollary 2.33(ii), p. 44]. Its pullback $\chi : E \to U(1)$ satisfies $\chi(c^{q/2}) = -1$. Write $\chi(c) = \zeta_q^d$; then d is odd. Append the descending character $\chi(e)t^d$ to a doubled faithful representation. This realizes the required odd winding. The auxiliary circle in Theorem 10.4 supplies it when this criterion fails. □

Example C.20 (Quaternion quotient in dimension 47). Embed $Q_8 \le SU(2)$ and take

$$K = ((Q_8 \times S^1)/\langle (-I_2, -1) \rangle) \times S^1_w, \quad A([e,t], w) = \operatorname{diag}(te, w), \quad z = t^2.$$

If $te = I_2$, then $(e,t) = (I_2, 1)$ or $(-I_2, -1)$; the w line then proves faithfulness. The determinant is $t^2 w$, so the w loop supplies the odd orthogonal winding. The quotients of $U(7)$ and $Sp(3) \times SO(8)$ have dimension 47, groups Q_8, C_2^2, and the spectral and curvature properties of Theorem 10.4. Since $-I_2 \in [Q_8, Q_8]$, Proposition C.19 excludes this boundary isomorphism for $(Q_8 \times S^1)/\langle (-I_2, -1) \rangle$ without the factor S^1_w.

Example C.21 (Odd Heisenberg quotients). Let p be an odd prime, put $\zeta_p = e^{2\pi i/p}$, and let $(v_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ be the standard basis of $\mathbb{C}^p$. Define $Sv_j = v_{j+1}$ and $Tv_j = \zeta_p^j v_j$. For $a, b, c \in \mathbb{Z}/p\mathbb{Z}$, the matrices $\zeta_p^a S^b T^c$ are distinct: the shift determines b, and its nonzero entries determine a, c. They form a group E of order p^3, with center C_p and quotient C_p^2, since $TS = \zeta_p ST$. The defining representation σ is faithful and has determinant one: this holds for S, T and $\zeta_p I$. On $K = (E \times S^1)/\langle (\zeta_p I, \zeta_p^{-1}) \rangle$ set $A([e,t]) = t^{p+1}\sigma(e)$ and $z = t^p$. These maps descend. If $z = 1$ and $A = I$, then $t = \zeta_p^k$, $e = \zeta_p^{-k} I$, precisely the diagonal relation, so (A, z) is faithful. Its orthogonal winding is $p(p+1) + p = p(p+2)$, which is odd. The resulting quotients of $U(2p+1)$ and $Sp(p) \times SO(2p+2)$ have dimension $4p(p+1)$ and groups E, C_p^2; for $p = 3$ the dimension is 48. The representation A alone need not be injective.

C.8. **Circle metrics and Chern classes.**

Proposition C.22. *Let q be a positive integer. For the circle $(a,b) = (q-1, q)$ of Corollary 11.12, put $D = 2a^2 + b^2$. Its length is $2\pi\sqrt{D}$, and $\operatorname{Vol}(X_q) = \operatorname{Vol}(Y_q) = 16\pi^5/\sqrt{D}$. The covering $SU(2)^3 \to H_1$ has metric $Q_0 \oplus 2Q_0 \oplus 2Q_0$, where $Q_0(X,Y) = -\frac{1}{2}\operatorname{Re}\operatorname{Tr}_{\mathbb{C}^2}(XY)$ for $X, Y \in \mathfrak{su}(2)$. Its primitive circle has length $4\pi\sqrt{D}$. For $q = 1$, the metric on $X_1 \cong SU(3)$ is left invariant. Let E_{ab} denote the standard matrix units and put $Q(X,Y) = -\operatorname{Re}\operatorname{Tr}_{\mathbb{C}^3}(XY)$ on $\mathfrak{u}(3)$. At the identity,*

$$g(X,Y) = Q(X,Y) - Q(X, iE_{33})Q(Y, iE_{33}), \qquad X, Y \in \mathfrak{su}(3).$$

Proof. Equation (2.3) gives the circle length. The ambient-volume calculation preceding (6.14) gives $\mathrm{Vol}(G_1) = \mathrm{Vol}(H_1) = 32\pi^6$ for these metrics. Division by $2\pi\sqrt{D}$ gives the quotient volumes. The Spin covering has the metric scales given by (6.7). Its primitive circle maps twice onto K_q by (11.5), so its length is $4\pi\sqrt{D}$. For $q = 1$, orthogonal projection away from the unit vector iE_{33} gives the displayed metric. □

Proposition C.23. *Let q be a positive integer, put $D = 2(q-1)^2 + q^2$, and let L_X, L_Y over X_q, Y_q correspond to the weight-one character of K_q. Write c_1 for the first integral Chern class. Then*

$$H^2(X_q;\mathbb{Z}) = \mathbb{Z}/q, \quad c_1(L_X) \text{ generates } \mathbb{Z}/q, \qquad H^2(Y_q;\mathbb{Z}) = \mathbb{Z}u, \quad c_1(L_Y) = \pm 2u.$$

For every $m \in \mathbb{Z}$, their mth tensor powers have equal canonical rough spectra; negative powers denote tensor powers of the dual bundle. The bundle $L_X^{\otimes m}$ is trivial exactly when $q \mid m$, and $L_Y^{\otimes m}$ is nontrivial for $m \neq 0$. For $q = 1$, both bases are simply connected and L_X is trivial.

Proof. The integral Gysin sequence [53, §4.D, p. 438] gives

$$H^1(U(3);\mathbb{Z}) \longrightarrow H^0(X_q;\mathbb{Z}) \xrightarrow{\smile c_1(L_X)} H^2(X_q;\mathbb{Z}) \longrightarrow H^2(U(3);\mathbb{Z}).$$

The first map is multiplication by q, since the determinant restricts to the weight-q character of K_q. The diffeomorphism $(S, z) \mapsto S\,\mathrm{diag}(z, 1, 1)$ from $SU(3) \times S^1$ to $U(3)$ and $\pi_1 SU(3) = \pi_2 SU(3) = 0$ give $H^2(U(3);\mathbb{Z}) = 0$. Thus $c_1(L_X)$ generates $H^2(X_q;\mathbb{Z}) \simeq \mathbb{Z}/q$.

For Y_q, the boundary homomorphism identifies $\pi_2(Y_q)$ with $2\mathbb{Z} \leq \pi_1(K_q) = \mathbb{Z}$, by Corollary 11.12 and its proof. Since Y_q is simply connected, Hurewicz [53, Theorem 4.32] identifies this group with $H_2(Y_q;\mathbb{Z})$. The first Chern class evaluates as the clutching winding, so $H^2(Y_q;\mathbb{Z}) = \mathbb{Z}u$ and $c_1(L_Y) = \pm 2u$. The weight-m character has vertical Casimir m^2/D; Theorem 4.3 gives the equality of rough spectra. Finally $c_1(L^{\otimes m}) = mc_1(L)$, and complex line bundles on these compact manifolds are classified by c_1 [54, Proposition 3.10]. Therefore $L_X^{\otimes m}$ is trivial if and only if $q \mid m$, and $L_Y^{\otimes m}$ is trivial if and only if $m = 0$. The case $q = 1$ of Corollary 11.12 gives simple connectivity of both bases. □

Put $a = q - 1$, $b = q$, and $U = i\,\mathrm{diag}(a, -a, b) \in \mathfrak{u}(3)$, so that U generates K_q and $Q(U, U) = D$. The Q-orthonormal horizontal vectors

$$X = (E_{13} - E_{31})/\sqrt{2}, \qquad Y = i(E_{13} + E_{31})/\sqrt{2}$$

satisfy $[X, Y] = i(E_{11} - E_{33})$ and $Q([X, Y], U) = a - b = -1$. Since the weight-one character has differential $U \mapsto i$, Proposition C.8 gives

$$F_{L_X}(X, Y) = \frac{i}{D}, \qquad F_{L_X^{\otimes q}}(X, Y) = \frac{qi}{D} \neq 0.$$

Thus $L_X^{\otimes q}$ is topologically trivial and its canonical connection is nonflat.

C.9. **Integral cohomology in dimension seven.**

Proposition C.24. *For $K = P_1$, the simply connected isospectral seven-manifolds $A_7 = U(3)/T^2 \cong N_{1,-1}$ and $B_7 = (Sp(1) \times SO(4))/T^2 \cong Q^{1,1,1}$ satisfy*

$$b_2(A_7) = 1, \quad b_2(B_7) = 2, \qquad H^4(A_7;\mathbb{Z}) = 0, \quad H^4(B_7;\mathbb{Z}) = \mathbb{Z}/2.$$

The character $z : P_1 = U(1) \times U(1) \to U(1)$ given by the second projection gives a trivial line bundle on A_7 and a primitive nonzero Chern class on B_7, with equal canonical spectra for every corresponding tensor power.

Proof. The homogeneous identifications and simple connectivity are those at the beginning of Section 12. The kernels of the two winding maps are respectively $\mathbb{Z}(1, 0)$ and $\Lambda = \{(m, l) \in \mathbb{Z}^2 : m + l \in 2\mathbb{Z}\}$. The homotopy sequence and Hurewicz give $b_2(A_7) = 1$ and $b_2(B_7) = 2$.

In the notation of [24, arXiv v1, §1], inversion identifies A_7 with $E_{k,l}$ for $k = (1, -1, 0)$ and $l = (0, 0, 0)$. The coordinate sums agree and every freeness condition is $\gcd(1, -1) = 1$. Moreover

$|\sigma_2(k)-\sigma_2(l)|=1$, where σ_2 is the second elementary symmetric function. Eschenburg's computation [40, §4, Example 3, pp. 163–164] therefore gives $H^4(A_7;\mathbb{Z})=0$; see also [24, arXiv v1, equation (1.6)]. The space $B_7=Q^{1,1,1}$ is the principal circle bundle over $(S^2)^3$ with first Chern class $x_1+x_2+x_3$, where x_j is the pullback of the positive generator of $H^2(S^2;\mathbb{Z})$ from the jth factor [10, arXiv v1, §2.1]. Its fourth integral cohomology is $\mathbb{Z}/2$ [10, arXiv v1, §2.2].

The z character extends to the determinant on $U(3)$ and trivializes the first line bundle by $[g,w]\mapsto(gK,\det(g)w)$. On B_7, the Chern functional on Λ is $(m,l)\mapsto\pm l$, with sign determined by the clutching convention. Its value at $(1,1)$ is ±1, so the class is primitive. Theorem 4.3 applies to every tensor power. □

Put $Q(X,Y)=-\operatorname{Re}\operatorname{Tr}_{\mathbb{C}^3}(XY)$ on $\mathfrak{u}(3)$, $U_0=i\operatorname{diag}(1,-1,0)$ and $W_0=iE_{33}$. At the identity coset of $N_{1,-1}$,

$$g(X,Y)=Q(X,Y)-Q(X,W_0)Q(Y,W_0),\qquad X,Y\in(\mathbb{R}U_0)^{\perp_Q}\cap\mathfrak{su}(3).$$

For $V_0=i\operatorname{diag}(1,1,-2)$, one has $Q(V_0,V_0)=6$ and $g(V_0,V_0)=2$. On $B_7=Q^{1,1,1}$, the normal metric is the quotient of $Q_0\oplus2Q_0\oplus2Q_0$, by Proposition C.22.

C.10. **Central-character Chern and lifting classes.** Retain J,K,T,Γ and χ from Proposition C.8. For a complex vector bundle F, write $c_k(F)$ for its kth integral Chern class and $\operatorname{ch}(F)$ for its rational Chern character.

Proposition C.25. *In Proposition C.8, put $m=|\Gamma|$ and $E_\rho=J\times_K\mathbb{C}^d$. In integral cohomology,*

$$\begin{aligned} m\big(d^k c_k(E_\rho)-\tbinom{d}{k}c_1(E_\rho)^k\big)&=0 && (1\le k\le d),\\ m\big(2d\,c_2(E_\rho)-(d-1)c_1(E_\rho)^2\big)&=0 && (d\ge2),\\ m\,c_k(\operatorname{End}E_\rho)&=0 && (k>0).\end{aligned}$$

Rationally, $c_k(E_\rho)=\binom{d}{k}(c_1(E_\rho)/d)^k$ and $\operatorname{ch}(E_\rho)=d\exp(c_1(E_\rho)/d)$.

Proof. On the degree-m cover $p:J/T\to J/K$, let $L_\chi=J\times_T\mathbb{C}_\chi$ be the line bundle associated to χ. Then $p^*E_\rho\simeq L_\chi^{\oplus d}$. For $x=c_1(L_\chi)$,

$$p^*c_k(E_\rho)=\binom{d}{k}x^k,\qquad p^*c_1(E_\rho)=dx,$$

while $p^*\operatorname{End}E_\rho$ is trivial. Each displayed integral expression pulls back to zero. The transfer identity $\operatorname{tr}p^*=m\,\mathrm{id}$ [53, §3.G, pp. 321–322] gives the integral annihilators. It also makes rational pullback injective, giving the remaining formulas. □

Proposition C.26. *Put $M=J/K$. Let $B\Gamma$ be a classifying space of Γ and let $f_M:M\to B\Gamma$ classify the principal covering $J/T\to M$. For a trivial Γ-module A, identify $H^r(\Gamma;A)$ with $H^r(B\Gamma;A)$. Let*

$$\beta:H^2(\Gamma;U(1))\longrightarrow H^3(B\Gamma;\mathbb{Z})$$

be the connecting homomorphism for

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{R}\xrightarrow{t\mapsto e^{2\pi it}}U(1)\longrightarrow0,$$

with trivial Γ action on the coefficient groups. For the multiplier of Proposition C.10,

$$f_M^*\big(\beta[\alpha_\chi]\big)=0\quad\text{in }H^3(M;\mathbb{Z}).$$

Fix a representation $\rho:K\to U(d)$ with central character χ, and put $R_\gamma=\rho(s(\gamma))$. On a good cover (U_a) of M, let $\gamma_{ab}:U_a\cap U_b\to\Gamma$ be the transition functions of $J/T\to M$. The induced flat projective connection admits a flat $U(d)$ lift if and only if there are locally constant functions $\lambda_{ab}:U_a\cap U_b\to U(1)$ such that

$$\lambda_{ab}\lambda_{bc}\alpha_\chi(\gamma_{ab},\gamma_{bc})=\lambda_{ac}\tag{C.4}$$

on every triple intersection.

Proof. On each contractible U_a, choose a smooth local section of $J \to M$ lifting the local section of $J/T \to M$ that defines γ_{ab}. Write its transition functions as $k_{ab} = t_{ab}\, s(\gamma_{ab})$, with $t_{ab} : U_a \cap U_b \to T$. The γ_{ab} are locally constant. The identity $k_{ab}k_{bc} = k_{ac}$ gives

$$\chi(t_{ab})\chi(t_{bc})\alpha_\chi(\gamma_{ab}, \gamma_{bc}) = \chi(t_{ac}).$$

Thus the image of the pulled-back multiplier in the sheaf of smooth $U(1)$-valued functions is a coboundary. The inclusion of locally constant functions into smooth functions gives a morphism of exponential sequences. Write $\underline{\mathbb{Z}}$ for the sheaf of locally constant integer-valued functions and $\mathcal{C}^\infty_M(A)$ for the sheaf of smooth maps to A, for $A = \mathbb{R}$ or $U(1)$. The smooth sequence is

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{C}^\infty_M(\mathbb{R}) \xrightarrow{h \mapsto e^{2\pi i h}} \mathcal{C}^\infty_M(U(1)) \longrightarrow 0.$$

Naturality of the connecting homomorphisms identifies its connecting class with $f_M^*(\beta[\alpha_\chi])$, which is therefore zero; this is the central-extension obstruction of [81, arXiv v1, §4, equations (4.2)–(4.3), p. 7].

The projective transition matrices are $[R_{\gamma_{ab}}]$ in $PU(d) = U(d)/U(1)$. If (C.4) holds, the locally constant matrices $\lambda_{ab}R_{\gamma_{ab}}$ satisfy the ordinary transition identity and define a flat unitary lift. Conversely, local parallel unitary frames for a flat lift can be chosen to project to these parallel projective frames. Their transition matrices then have the form $\lambda_{ab}R_{\gamma_{ab}}$, with locally constant λ_{ab}, and their transition identity is (C.4). □

Example C.27. Embed $Q_8 \leq SU(2)$ and put

$$K = (Q_8 \times U(1))/\langle(-I_2, -1)\rangle, \qquad T = \{[1, t] : t \in U(1)\}, \qquad \Gamma = K/T \simeq C_2^2.$$

The representation $\rho([e, t]) = te$ is faithful and irreducible, and its central character is $\chi([1, t]) = t$. The matrices of two quaternion generators anticommute. If α_χ were a coboundary, scalar multiples of these matrices would define an ordinary representation of the abelian group C_2^2 and would commute, a contradiction. Hence $[\alpha_\chi] \neq 0$. For any closed embedding of K into a compact connected Lie group J, Proposition C.26 nevertheless gives $f_{J/K}^*(\beta[\alpha_\chi]) = 0$.

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