SPECTRAL RESOLUTION AND RIGIDITY OF SPHERICAL SPACE FORMS

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ABSTRACT. Let $n \geq 5$ be odd. We prove quantitative rigidity from scalar Laplace spectra for two classes of round n-manifolds: homogeneous spherical space forms of even order and scalar lens spaces of odd order at least five. At fixed sufficiently small normalized frequency tolerance, an explicit strict bound on the lower limit of the least total unmatched multiplicity divided by $R^{(n+1)/2}$, along prescribed cutoffs R, forces a closed connected Riemannian manifold to be isometric to the reference space. Nonisometric lens spaces show that this exponent is sharp in general. A Besse homotopy n-sphere is round if it admits a constant-displacement isometry of finite odd order at least five.

Let P be a self-adjoint classical elliptic operator of order one with positive scalar principal symbol, acting on a Hermitian bundle of positive rank over a closed connected d-manifold, $d \geq 2$. If its spectrum concentrates modulo one near s fixed phases, with vanishing tolerance and $o(R^d)$ exceptional eigenvalue copies along a sequence of cutoffs, then the time-2π Hamiltonian map has order at most s, and there is a commuting quantizer A. Finer spectral resolutions give sharp upper bounds on the order of $P - A$. If $P - A$ is not smoothing, signed displacement profiles and Weyl mass determine the Liouville eigenvalue distribution of its normalized leading symbol.

For finite free isometric quotients of closed connected Zoll manifolds of dimension at least two, the Poisson relation is an equality, and the scalar spectrum determines the maximal dimension of returning components at each length. Among closed connected Riemannian n-manifolds whose distinct scalar eigenvalues have multiplicities $m_j = cj^{n-1} + O(j^{n-2})$, the possible positive coefficients are exactly $c = 2^n/((n-1)!N)$, $N \in \mathbf{Z}_{>0}$. The largest value characterizes round real projective space up to scale.

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Introduction

Let $m \in \mathbb{Z}_{\geq 2}$, put $n = 2m+1$, and let $r > 0$. Let $S^n(r)$ be the round sphere of radius r, and let $\Gamma_0 \subset O(n+1)$ be a finite subgroup acting freely on it. Put $Q = S^n(r)/\Gamma_0$ and $q = |\Gamma_0|$. Let M be a smooth closed connected Riemannian manifold of positive dimension. For a closed Riemannian manifold Z, write Δ_Z for its nonnegative scalar Laplacian and $\lambda_j(Z)$, $j \geq 0$, for its eigenvalues in nondecreasing order, repeated according to multiplicity. Put

$$\mu_j = \sqrt{r^2\lambda_j(M) + m^2}, \qquad \beta_j = \sqrt{r^2\lambda_j(Q) + m^2}.$$

For $\eta \geq 0$ and $R > 0$, let $E_\eta(R)$ be the least total number of unpaired copies in the two frequency multisets restricted to $(-\infty, R]$, when a pair (μ, β) is allowed if $|\mu - \beta| \leq \eta$ (Definition 1.1). Exact scalar isospectrality is equivalent to $E_0(R) = 0$ for every R.

Rigidity at a fixed frequency tolerance. The reference frequencies belong to $m + \mathbb{Z}_{\geq 0}$; if q is even, they belong to $m + 2\mathbb{Z}_{\geq 0}$. For Riemannian manifolds, $\cong$ denotes isometry. Suppose first that q is even, and put $R_K = m + 2K + 1$ for $K \in \mathbb{Z}_{\geq 0}$. For $0 \leq \eta < 1/2$, the inequality

$$\liminf_{K\to\infty} \frac{E_\eta(R_K)}{R_K^{m+1}} < \frac{1}{q\, 2^{m-1}(m+1)!}$$

implies that there is a finite subgroup $\Gamma \subset O(2m+2)$ of order q, acting freely, such that $M \cong S^{2m+1}(r)/\Gamma$. If Q is homogeneous, then M is isometric to Q (Theorem 2.2).

Suppose instead that

$$Q = S^{2m+1}(r)/\langle e^{2\pi i/q} I_{\mathbb{C}^{m+1}} \rangle, \qquad q \geq 5 \text{ odd},$$

and put $R_K = m + K + 1/2$. For $0 \leq \eta < 1/4$,

$$\liminf_{K\to\infty} \frac{E_\eta(R_K)}{R_K^{m+1}} < c_{m,q} \quad \Longrightarrow \quad M \cong Q, \qquad c_{m,q} = \begin{cases} (48q)^{-1}, & m = 2, \\ \dfrac{2}{q|1 - e^{4\pi i/q}|^m (m+1)!}, & m \geq 3. \end{cases}$$

This is Theorem 2.4. Nonisometric round quotients with $E_\eta(R_K) \asymp R_K^{m+1}$ show that the exponent $(n+1)/2$ in the isometry conclusions cannot be increased (Theorem 2.7 and Proposition 22.2).

Ikeda's Problem (0.1) concerns determination by the scalar spectrum among spherical space forms. His Theorem III gives isometry when one of the two space forms is homogeneous. In dimension five, Theorem IV determines a homogeneous lens space among arbitrary closed Riemannian manifolds with the same scalar spectrum [45, Theorems III–IV]. Prüfer proves finite spectral determination in the round class: two unit round space forms of dimension $2m+1$ and fundamental-group order q are isospectral if their eigenvalues $\lambda \le m^2 q(q+2)$, with multiplicities, agree [69, Corollary 3, p. 293]. For $q \ge 3$, his Corollaries 4–6 characterize homogeneous lens spaces in this round class by the multiplicity $(m+1)^2 - 1$ of the eigenvalue $4(m+1)$ [69, pp. 308–310]. For finite subgroups of a fixed compact connected semisimple group G, a closed subgroup $K \subset G$, and a finite-dimensional representation τ of K, Lauret–Miatello determine all τ-spherical representation multiplicities from cofinite data. When the τ-spherical representations form a finite union of strings, suitable finite data suffice; sufficiently sparse exceptional indices are also allowed within each string [56, Theorems 1.1–1.2 and Corollary 3.2]. For an arbitrary comparison manifold, Theorems 2.2 and 2.4 deduce rigidity from a fixed frequency tolerance and a bound on unmatched eigenvalue copies, counted with multiplicity, along a sequence of cutoffs.

For homogeneous Q in the two classes, put $\varepsilon_Q = 1/2$ in the even case and $\varepsilon_Q = 1/4$ in the odd case. Corollary 2.5 gives

$$\limsup_{j\to\infty} r\left|\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)}\right| < \varepsilon_Q \quad \Longrightarrow \quad M \cong Q.$$

Write Spec T for the spectrum of an operator T; spectral comparisons retain multiplicities. Hence

$$\mathrm{Spec}(\Delta_M + B_M) = \mathrm{Spec}(\Delta_Q + B_Q) \quad \Longrightarrow \quad M \cong Q$$

for arbitrary bounded self-adjoint operators B_M, B_Q on the respective scalar L^2 spaces (Corollary 2.6).

For a Hermitian bundle E over a closed manifold Z, let $\Omega_Z^{1/2}$ be the half-density bundle. Write $H^s(Z; E \otimes \Omega_Z^{1/2})$ for the Sobolev space of order s of E-valued half-densities, and $\Psi_{\mathrm{cl}}^\alpha(Z; E)$ for the classical pseudodifferential operators of order α on these half-densities. Their principal symbol is denoted by σ_α. In the scalar case, write $H^s(Z)$ and $\Psi_{\mathrm{cl}}^\alpha(Z)$; a Riemannian volume density identifies functions with half-densities. The same isometry conclusion holds for the self-adjoint form sums $\Delta_Z + b_Z$, $Z \in \{M, Q\}$, where b_Z is a symmetric sesquilinear form on $H^1(Z)$. Put $b_Z[u] = b_Z(u,u)$ and suppose that, for some $0 \le \alpha < 1/2$ and constants $C_Z \ge 0$,

$$|b_Z[u]| \le C_Z \langle (1+\Delta_Z)^\alpha u, u\rangle, \qquad u \in H^1(Z)$$

(Proposition 19.1). For symmetric scalar $B_Z \in \Psi_{\mathrm{cl}}^1(Z)$, let $|\xi|_Q$ be the dual Riemannian norm and put $S^*Q = \{(x,\xi) : |\xi|_Q = 1\}$. It suffices that the geodesic average of $\sigma_1(B_Q)/(2|\xi|_Q)$ over a common period be constant on S^*Q (Theorem 20.8). There are nonhomeomorphic manifolds whose scalar Laplacians become isospectral after symmetric first-order differential perturbations (Theorem 21.1). Isospectral connection Laplacians on line bundles over nonhomeomorphic manifolds also occur in the magnetic examples of Gordon–Kirwin–Schueth–Webb [29, Proposition 3.4 and Theorem 3.6].

Geodesic return and constant displacement. The matching bounds determine the dimension and volume by Weyl's law. Put $t = \pi$ in the even case and $t = 2\pi$ in the odd case. Along a subsequence of the stated cutoffs with $E_\eta(R) = O(R^{m+1})$, among the frequencies $\mu_j \le R$, all but $o(R^n)$ of the phases $e^{it\mu_j}$, counted with multiplicity, lie in a fixed closed subarc of the open semicircle centered at e^{itm}. Positivity of localized traces gives the common geodesic period tr. For the scalar Laplacian, the principal return symbol is a fourth root of unity; the reference semicircle contains exactly one such root. A quantizer for a discrete set is a self-adjoint elliptic operator of order one whose spectrum is contained in that set. For $P = \sqrt{r^2\Delta_M + m^2}$, a commuting logarithm gives a quantizer A for $m + 2\mathbb{Z}_{\ge 0}$ in the even case and $m + \mathbb{Z}_{\ge 0}$ in the odd case, with $P - A \in \Psi_{\mathrm{cl}}^{-1}$. The matching bound controls the difference between its cluster multiplicities and the reference multiplicities (Proposition 10.1). The

return symbol is given by Duistermaat–Guillemin [21, proof of Theorem 3.1, pp. 53–54]; Colin de Verdière gives the scalar commuting normal form [15, Theorem 1.1, pp. 510–511].

A metric is Besse if all its geodesics are closed, and Zoll here if all unit tangent vectors have the same prime period. Under either matching hypothesis, M is Besse. Its universal Riemannian cover is a compact smooth homotopy sphere [5, Theorem 1.1 and the following paragraph]. All prime geodesics on that cover have the same length [70, Theorem A]. If the covering degree is even, the metric has constant sectional curvature [59, Theorem 2].

An isometry h has constant displacement if the Riemannian distance $d(x, hx)$ is independent of x; such an isometry is called a Clifford translation. For odd covering degree, the following criterion applies: *a Besse homotopy sphere of odd dimension at least five is round if it admits a constant-displacement isometry of finite odd order at least five* (Theorem 14.2 and Corollary 14.3). Berestovskii–Nikonorov classify simply connected Clifford–Wolf homogeneous manifolds, in which any two points are related by a constant-displacement isometry. On a compact homogeneous manifold, a sufficiently small such isometry is generated by a constant-length Killing field [8, Theorems 1 and 6]. The criterion above assumes a single isometry of the indicated finite order on a Besse homotopy sphere.

Let X be a closed connected Zoll manifold of prime period L, let a finite group Γ act freely by isometries, and put $Y = X/\Gamma$. Write SY for the unit tangent bundle of Y, G_t for its geodesic flow, and $\operatorname{Fix} G_t$ for the fixed point set of its time-t map. For every integer $s \geq 2$,

$$\operatorname{Fix} G_{L/s} \neq \emptyset \quad \Longleftrightarrow \quad \Gamma \text{ contains an element of order } s.$$

Every constant-speed closed geodesic of length L/s minimizes energy in its free homotopy class on a fixed parameter interval; its periodic Morse index is therefore zero (Lemma 15.1). At each nonempty reciprocal return L/s, the components of $\operatorname{Fix} G_{L/s}$ of maximal dimension have a common leading trace phase and positive densities (Lemma 15.2). For the odd scalar references, the trace bound gives an n-dimensional returning family on the cover whose evaluation map $(x, v) \mapsto x$ has nonzero degree (Proposition 13.6). Let L be the prime period on the cover. Surjectivity of evaluation gives $d(x, hx) \leq L/q$, while Lemma 15.1 gives $d(x, hx) \geq L/q$ for its deck transformation h of order q. Thus h has constant displacement. Since $q \geq 5$, evaluation is a diffeomorphism and defines a geodesic circle action. The contact-volume inequality of Proposition 14.1, derived from [2, Theorem J.1(i)], and the volume identity of Proposition 12.3 make this action isometric. Its half-period is a free isometric involution; Lin–Schmidt's theorem gives roundness.

For an arbitrary odd-order reference $Q = S^n(r)/\Gamma_0$, put $R_K = m + K + 1/2$. If $E_\eta(R_{K_\nu}) = o(R_{K_\nu})$ along $K_\nu \to \infty$, with fixed $0 \leq \eta < 1/4$, the universal Riemannian cover X of M is a Zoll homotopy sphere with prime period $2\pi r$, and

$$|\pi_1(M)| = q, \qquad \operatorname{Vol}(X) = \operatorname{Vol}(S^n(r)).$$

The groups $\pi_1(M)$ and Γ_0 have the same sets of element orders (Theorem 15.6). If Γ_0 contains a Clifford translation of order at least five, then M is a round spherical space form of radius r and has the same complete scalar spectrum as Q (Theorem 15.7). For round space forms, the determination of the set of element orders is already contained in Ikeda's Corollary 2.8 [45], and its relation with the length set is given by Prüfer's Corollary 1 [68]. Colantonio–Lauret construct isospectral spherical space forms with nonisomorphic fundamental groups [17, Theorem 1.5].

Resolution and correction symbols. For a periodic quantizer A commuting with P, the eigenvalues of $P - A$ give the displacements within each A-cluster. Zelditch's Problem 1 asks for the relation between the geometry of a metric with periodic geodesic flow and the widths of its eigenvalue clusters [88].

Let E be a Hermitian bundle of positive rank over a closed connected n-manifold M, $n \geq 2$, and let $P \in \Psi^1_{\mathrm{cl}}(M; E)$ be self-adjoint and elliptic, with positive scalar principal symbol p. Write $H^s = H^s(M; E \otimes \Omega_M^{1/2})$. For an integer $d \geq 1$ and real numbers $c_1, \dots, c_d$ distinct modulo $\mathbb{Z}$, put

$$\Lambda = \bigcup_{j=1}^{d} (c_j + \mathbb{Z}), \qquad \mathcal{S} = \{e^{2\pi i c_1}, \dots, e^{2\pi i c_d}\}.$$

Here $E_\eta(R)$ compares Spec P with a multiset supported on Λ, bounded below and finite on bounded intervals. If, along $R_\nu \to \infty$,

$$\eta_\nu \longrightarrow 0, \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^n),$$

then the time-2π map of the Hamiltonian flow of p has finite order $L \le d$. Thus $2\pi L$ is a common period; individual orbits may have smaller prime periods. The operator $e^{2\pi i L P}$ belongs to Ψ^0_{cl} and has unitary principal symbol H on $T^*M \setminus 0$. For every eigenvalue h of $H(\rho)$, all roots of $z^L = h$ belong to $\mathcal{S}$. There is a self-adjoint $A \in \Psi^1_{\mathrm{cl}}(M;E)$, with domain H^1 and strongly commuting with P, such that

$$\operatorname{Spec} A \subset \Lambda, \qquad P - A \in \Psi^{-1}_{\mathrm{cl}}$$

(Theorem 4.1).

If $s > 0$, $0 < \beta \le n$, and

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^\beta),$$

then

$$P - A \in \Psi^{-\lfloor ns/\beta \rfloor - 1}_{\mathrm{cl}}.$$

This upper bound on the order is sharp, even under the stronger hypothesis $\eta_\nu = o(R_\nu^{-s})$ (Theorem 5.1 and Proposition 5.3).

For positive self-adjoint scalar classical elliptic operators of order one on a closed connected manifold of dimension at least two, Duistermaat–Guillemin deduce periodicity from concentration near one arithmetic progression: there is a fixed interval length such that, for every $\varepsilon > 0$, at least a proportion $1 - \varepsilon$ of the frequencies in each sufficiently high interval of that length lie within distance ε of the progression; frequencies are counted with multiplicity [21, Theorem 3.2, p. 55]. Helton relates orbit periods to asymptotic eigenvalue differences. In particular, concentration of all sufficiently high scalar frequencies near one arithmetic progression, with errors tending to zero, implies periodicity [40, Propositions 4.1–4.2 and Theorem 4.4]. Humbert–Privat–Trélat obtain Besse periodicity from a uniform bound on the number of distinct scalar frequencies in intervals of a fixed length [44, Theorem 2(6)]. The finite-phase hypothesis above is imposed only along a sequence of cumulative cutoffs. It permits arbitrarily many distinct frequencies in a shrinking cluster and an exceptional set counted with multiplicity.

Let $p \ge 5$ be prime, let $a \in \mathbb{Z}$ satisfy $a \not\equiv 0, \pm 1 \pmod p$, and put $\zeta = e^{2\pi i/p}$. The unit round lens space

$$Q = L(p;1,1,a) := S^5/\langle \operatorname{diag}(\zeta,\zeta,\zeta^a)\rangle, \qquad S^5 \subset \mathbb{C}^3,$$

has dimension five. Here $E_\eta(R)$ compares the spectra of $P = \sqrt{\Delta_M + 4}$ and $\sqrt{\Delta_Q + 4}$. The bounds $\eta_\nu = O(R_\nu^{-3/5})$ and $E_{\eta_\nu}(R_\nu) = o(R_\nu)$ imply $\dim M = 5$ by Weyl's law. Theorem 4.1 gives a commuting quantizer A with integer spectrum, and Theorem 5.1 gives $P - A \in \Psi^{-4}_{\mathrm{cl}}$. The first three heat coefficients and the leading trace coefficient at time $2\pi/p$ then give isometry to Q (Theorem 23.2).

At a fixed negative order, the limiting displacement profiles are expressed by the Liouville eigenvalue measure of the leading correction symbol. Let $A \ge 1$ be a self-adjoint classical elliptic operator of order one on E, with positive scalar principal symbol and integer spectrum. Let $B = B^* \in \Psi^{-k}_{\mathrm{cl}}$, $k > 0$, strongly commute with A, and put $P = A + B$ on H^1. Write p_A for the positive scalar principal symbol of A, and put $c = \sigma_0(A^k B)$. Let $d\rho$ be the canonical symplectic volume density on T^*M and tr_E the trace on a fiber of E. For a Borel set $J \subset \mathbb{R}$, let $\mathbf{1}_J(c(\rho))$ be the spectral projection of the Hermitian endomorphism $c(\rho)$. Define the Liouville eigenvalue measure of c by

$$\nu_c(J) = \frac{1}{(2\pi)^n} \int_{0 < p_A(\rho) \le 1} \operatorname{tr}_E\big(\mathbf{1}_J(c(\rho))\big)\, d\rho.$$

This is the normalization of (6.1). Its total mass is

$$\nu_c(\mathbb{R}) = \lim_{R\to\infty} R^{-n} \#\big(\operatorname{Spec} A \cap (-\infty, R]\big),$$

with multiplicities retained. For joint eigenvalues (a_j, b_j) of (A, B), counted with multiplicity, put

$$D_{\pm}(R, \eta) = \#\{j \,:\, a_j \le R,\ \pm b_j > \eta\}.$$

Put $y_+ = \max\{y, 0\}$ for $y \in \mathbb{R}$, and let $E_\eta(R; P, A)$ compare the spectra of P and A. For $t > 0$ and $R_N = N + 1/2$, $N \in \mathbb{Z}_{\ge 1}$, Theorem 6.3 gives

$$\lim_{N\to\infty} R_N^{-n} D_{\pm}(R_N, t R_N^{-k}) = \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^{n/k}\}\, d\nu_c(x),$$
$$\lim_{N\to\infty} R_N^{-n} E_{tR_N^{-k}}(R_N; P, A) = 2\int_{\mathbb{R}} \min\{1, (|x|/t)^{n/k}\}\, d\nu_c(x).$$

The second profile and the total mass $\nu_c(\mathbb{R})$ determine the pushforward of ν_c under $x \mapsto |x|$. The two signed profiles and that mass determine ν_c, including its atom at zero (Corollary 6.4). In the finite-phase setting, suppose $P - A \in \Psi_{\mathrm{cl}}^{-k}$. For the given return order L, the signed profiles restricted to each spectral phase, together with the leading coefficient of its counting function, determine the Liouville eigenvalue measure of the correction symbol on each eigensubbundle of H (Theorem 7.1). For scalar periodic operators, trace formulas for cluster distributions are given in [15, Theorem 3.1, pp. 518–519]. The scalar cumulative moment formula is the weak Szegő law [33, §2, (2.1), pp. 510–511]. For self-adjoint elliptic operators of order one with spectrum $\mathbb{Z}_{>0}$ and common prime Hamiltonian period 2π, Guillemin–Okikiolu give Szegő trace asymptotics for spectral compressions, including operators on bundles with scalar principal symbol [32, Theorem 2; p. 452, item 3]. At coarser scales $\eta_N \downarrow 0$ with $\eta_N R_N^k \to \infty$, Theorem 6.5 is the Birman–Solomyak eigenvalue asymptotic for negative-order operators [67, Theorem 6.1 and Section 6.2].

Length spectra and maximal multiplicities. The cluster ranks determine the trace of the periodic quantizer. On a finite free quotient of a Zoll manifold, the scalar square root differs from a lattice quantizer by a constant and a commuting operator of order -1. The order -1 correction preserves the principal and subprincipal symbols, and hence the leading clean-trace coefficients [21, Theorem 4.5, equations (4.7)–(4.8)]. Sutton asks whether the scalar spectrum determines the lengths of all smoothly closed geodesics [77, Problem 1]. Equality in the Poisson relation is sufficient [77, Problem 2]; this is the noncancellation question in [16, Section 13]. Write $\mathscr{L}(M)$ for the positive length set, including iterates. If M is a finite free isometric quotient of a closed connected Zoll manifold of dimension at least two, then

$$\operatorname{sing\,supp} \operatorname{Tr} e^{-it\sqrt{\Delta_M}} = \{0\} \cup \mathscr{L}(M) \cup (-\mathscr{L}(M)).$$

Write SM for the unit tangent bundle of M and G_t^M for its geodesic flow. For every $\tau \in \mathscr{L}(M)$, the scalar spectrum also determines

$$D(\tau) = \max\{\dim C \,:\, C \text{ is a connected component of } \operatorname{Fix} G_\tau^M\}.$$

The highest local frequency order of the wave trace at τ is $(D(\tau) - 1)/2$ (Theorem 24.1).

For unit round spherical space forms, Prüfer [68, Theorems 2–3 and Corollary 4, pp. 142, 150, 152] proves the corresponding equality for the shifted cosine trace

$$\operatorname{Tr} \cos\bigl(t\sqrt{\Delta + (n-1)^2/4}\bigr).$$

In the round case, the highest nonzero trace order also gives $D(\tau)$. In odd dimension, his proof combines a same-sign calculation at reciprocal periods with cyclotomic conjugacy, following Ikeda's argument for the poles of the harmonic generating function [45, Proposition 2.7, p. 83]. At lengths outside $\pi\mathbb{Z}$, Prüfer constructs local isometric immersions of spherical space forms carrying the returning geodesics, with totally geodesic image. He expresses their leading Poisson coefficients through volumes and transverse Poincaré determinants [69, Theorems 1–2, pp. 291–292]. His calculation gives the round Poincaré maps and their eigenvalues [69, Lemma 6 and its proof, p. 299]. For a unit round form of dimension $2m + 1$, the multiplicity of $k(k + 2m)$ is an exact quasipolynomial in the harmonic degree k for every $k \ge 0$ [69, Lemma 7, pp. 302–304]. Cianci gives direct clean-trace proofs

for homogeneous lens spaces and three-dimensional lens spaces of prime order [14, Corollary 3 and Theorem 5].

At each nonempty reciprocal return of a finite free Zoll quotient, energy minimization gives a nonzero leading coefficient. Normalize the upstairs period to 2π. The eventual quasipolynomial multiplicities of the periodic quantizer give a generating function in $\mathbb{Q}(z)$ [15, Theorems 1.1 and 1.4, pp. 510–513]. The leading Weyl coefficient gives a pole of order $\dim M$ at 1. Cyclotomic conjugacy gives the same pole order at every primitive root of a given order. For $(a, s) = 1$, the fixed sets at $2\pi a/s$ and $2\pi/s$ coincide. Restoring the scalar phase gives a nonzero leading wave-trace coefficient at every length.

Fix an odd dimension $n \geq 5$, and let m_j denote the multiplicity of the jth distinct scalar eigenvalue. The asymptotic

$$m_j = cj^{n-1} + O(j^{n-2}), \qquad c > 0,$$

makes the geodesic flow periodic by Zelditch's Theorem A [88]. Each sufficiently high cluster then contains the same number of distinct eigenvalues (Lemma 25.1); the volume and covering identities restrict the leading coefficient. Among all closed connected Riemannian n-manifolds, its possible values are exactly

$$c = \frac{2^n}{(n-1)!\,N}, \qquad N \in \mathbb{Z}_{>0}.$$

Every value is realized by the round scalar lens space of order $2N$. The largest value occurs precisely on round real projective space, up to scale (Theorem 25.3 and Corollary 25.5). Within the round class, Prüfer's multiplicity formula and eventual spectral support already give this coefficient set and its maximal value [69, Lemma 7 and Corollary 1, pp. 302–305]. Zelditch's Theorem F(b) determines the round metric on an already specified real projective space when the complete multiplicity sequence, least geodesic-flow period, and volume agree with the round values [88].

Cyclic actions. For a finite free isometric quotient X/Γ, pullback identifies a scalar spectral cluster with the Γ-invariant subspace of the cluster on X in the same frequency interval. On a fixed closed manifold, Zoll contact forms with the same prime Reeb period are strictly contactomorphic when sufficiently close in C^2 [1, Proposition 3.2]. Let g_0 be the unit round metric on S^{2m+1}, $m \geq 2$. There is a C^3 neighborhood of g_0, independent of the finite group $H \subset O(2m+2)$, such that every H-invariant Zoll metric g in this neighborhood with prime period 2π has the following property. Put $P_g = \sqrt{\Delta_g + m^2}$, where Δ_g is the scalar Laplacian of g. For every sufficiently large integer k, there is an isomorphism of H-modules

$$\operatorname{ran} \mathbf{1}_{(k-1/4,k+1/4)}(P_g) \simeq \ker(P_{g_0} - k)$$

(Theorem 28.1).

Let h generate a free isometric action of order three on a Zoll homotopy sphere of odd dimension at least five, with prime period L. Write G_t for its geodesic flow on the unit tangent bundle and $\widehat{h}$ for the tangent lift of h. The two families $\operatorname{Fix}(\widehat{h}^{-1}G_{L/3})$ and $\operatorname{Fix}(\widehat{h}^{-1}G_{-L/3})$ are homotopy spheres, and their evaluation maps are homotopy equivalences (Theorem 29.6). Roundness in this case is Question 29.9.

Let h generate a free isometric action of prime order $p \geq 5$ on a Zoll homotopy n-sphere. Assume the hypotheses of Theorem 33.1, with four return families of dimensions $n-2, n-2, 1, 1$. Their relative return maps are symplectically conjugate to the corresponding round maps. If $p \geq 11$, the common evaluation image of the two large families and that of the two small families are disjoint embedded submanifolds with integral linking number of absolute value one (Theorem 34.1). Recognition of five-dimensional prime two-weight lens spaces at fixed positive tolerance is Question 23.3.

1. Spectral matching and operator conventions

For a finite multiset X in $\mathbb{R}$, let $|X|$ denote its cardinality, counted with multiplicity. For finite multisets X, Y and $\eta \geq 0$, an η-matching is a collection of pairs of labelled copies $(x, y) \in X \times Y$ such that each copy occurs in at most one pair and $|x - y| \leq \eta$ for every pair.

Definition 1.1. Let $b_\eta(X, Y)$ be the maximum cardinality of an η-matching, and put

$$E_\eta(X, Y) = |X| + |Y| - 2b_\eta(X, Y).$$

For multisets bounded below and finite on bounded intervals, and for $R \in \mathbb{R}$, put

$$E_\eta(R; X, Y) = E_\eta(X \cap (-\infty, R], Y \cap (-\infty, R]).$$

Intersections retain multiplicities. For self-adjoint operators A, B with compact resolvent and spectra bounded below, $E_\eta(R; A, B)$ refers to their eigenvalue multisets, counted with multiplicity. Limits of matching defects are taken along the specified cutoff sequences.

Lemma 1.2. *For finite multisets, $E_\eta(X, Y) \geq \big||X| - |Y|\big|$. Inserting or deleting one copy in either multiset changes E_η by at most one. Fix nondecreasing enumerations of the labelled copies in each multiset. A maximum matching can be chosen so that the matched right-hand indices increase with the matched left-hand indices.*

Proof. A matching has at most $\min(|X|, |Y|)$ pairs. After one copy is inserted, every old matching remains admissible; removing the pair containing the new copy, if any, from a new matching leaves an old matching. Thus b_η increases by zero or one, and E_η changes by one in absolute value. Deletion is the reverse operation. If $x \leq x'$ and $y \leq y'$ form the crossed pairs (x, y') and (x', y), then

$$x - \eta \leq x' - \eta \leq y \leq y' \leq x + \eta \leq x' + \eta.$$

Thus (x, y) and (x', y') are admissible. Among maximum matchings, choose one with the fewest inversions in its sequence of matched right-hand indices. If this sequence is not increasing, two consecutive terms form an inversion. Exchanging their right-hand copies removes that inversion and leaves the other contributions to the inversion count unchanged, a contradiction. □

Unless stated otherwise, M is a smooth closed connected manifold of dimension $n \geq 2$, and $E \to M$ is a smooth Hermitian vector bundle of positive rank. Let $\Omega_M^{1/2}$ be the half-density bundle. Operators act on $E \otimes \Omega_M^{1/2}$. For $s \in \mathbb{R}$, write $H^s = H^s(M; E \otimes \Omega_M^{1/2})$ and $L^2 = H^0$. For $a \in \mathbb{R}$, the notation $\Psi^a_{\mathrm{cl}}(M; E)$ denotes the classical pseudodifferential operators of order a, and σ_a denotes their principal symbol. For $T \in \Psi^a_{\mathrm{cl}}(M; E)$, $\sigma_a(T)(x, \xi)$ is an endomorphism of E_x for $\xi \neq 0$; its dependence on ξ is homogeneous of degree a. Write $\operatorname{Spec} T$ for the spectrum of an operator T, I for the operator identity, I_E for the fiber identity, Tr for the operator trace, tr_E for the fiber trace, and $\|\cdot\|$ for the operator norm on the Hilbert space in question. For a Riemannian metric g, the map

$$L^2(M, d\,\mathrm{vol}_g) \longrightarrow L^2(M; \Omega_M^{1/2}), \qquad f \longmapsto f\,(d\,\mathrm{vol}_g)^{1/2},$$

is unitary.

Let $q : T^*M \setminus 0 \to (0, \infty)$ be smooth and satisfy $q(x, s\xi) = s\,q(x, \xi)$ for $s > 0$. A self-adjoint elliptic operator $P \in \Psi^1_{\mathrm{cl}}(M; E)$ with principal symbol qI_E has domain H^1, is bounded below, and has compact resolvent with a complete orthonormal basis of smooth eigensections [28, Lemmas 1.3.5, 1.6.3 and 1.6.4, pp. 30, 43 and 46]. Two normal operators strongly commute if every spectral projection of one commutes with every spectral projection of the other. A self-adjoint elliptic operator of order one is a quantizer for a discrete set $\Lambda \subset \mathbb{R}$ if its spectrum is contained in Λ.

Put $\Sigma_q = \{q = 1\}$, and let $\delta_s(x, \xi) = (x, s\xi)$ be fiberwise dilation. Write $dx\,d\xi$ for the canonical symplectic volume density on T^*M. The radial Liouville measure $d\Sigma_q$ is characterized by

$$dx\,d\xi = s^{n-1}\,ds\,d\Sigma_q(\rho), \qquad (x, \xi) = \delta_s(\rho), \quad s > 0, \quad \rho \in \Sigma_q.$$

Let $A \in \Psi^1_{\mathrm{cl}}(M;E)$ be self-adjoint with principal symbol qI_E, and let $C \in \Psi^0_{\mathrm{cl}}(M;E)$ have principal symbol c. Let $\chi \in C_c^\infty((0,\infty))$, extended by zero to $\mathbb{R}$. For $R > 0$, the operator $C\chi(A/R)$ has finite rank, and, as $R \to +\infty$,

$$\operatorname{Tr}\big(C\chi(A/R)\big) = \frac{R^n}{(2\pi)^n} \int_{T^*M} \chi(q)\, \operatorname{tr}_E c \, dx\, d\xi + o(R^n). \tag{1.1}$$

Here c is homogeneous of degree zero, and the integrand is extended by zero at the zero section.

Under the half-density identification for an auxiliary Riemannian metric, the trace expansion [25, Theorem 2.13, p. 2170; vector bundles, pp. 2161 and 2182] applies with weight C of order zero, elliptic operator A of order one, test function χ, and parameter R^{-1}. Its leading coefficient is the coefficient of R^n in (1.1) [25, §4.2, p. 2186]. In local coordinates x, y, the Fourier phase $e^{i(x-y)\cdot\xi}$ gives the factor $(2\pi)^{-n}$. The remainder is $O(R^{n-1})$.

Let Log denote the holomorphic logarithm on $\mathbb{C} \setminus (-\infty, 0]$ with $\operatorname{Log} 1 = 0$. For a bounded operator T whose spectrum lies in this domain, write $\log T = \operatorname{Log}(T)$, defined by the holomorphic functional calculus.

Lemma 1.3. *Let $P \in \Psi^1_{\mathrm{cl}}(M;E)$ be self-adjoint and elliptic with positive scalar principal symbol. Suppose that Z is unitary, strongly commutes with P, and satisfies $Z - I \in \Psi^{-1}_{\mathrm{cl}}(M;E)$ and $\|Z - I\| < 1$. Then the principal logarithm $\log Z \in \Psi^{-1}_{\mathrm{cl}}(M;E)$ is skew-adjoint and strongly commutes with P.*

Proof. Put $K = Z - I$ and define

$$f(z) = \frac{\operatorname{Log}(1+z)}{z} \quad (0 < |z| < 1), \qquad f(0) = 1.$$

The function f is holomorphic on $|z| < 1$. Choose ρ with $\|K\| < \rho < 1$, and give the circle $\Gamma = \{z \in \mathbb{C} : |z| = \rho\}$ its positive orientation. For $z \in \Gamma$, the operator $zI - K$ is elliptic, with principal symbol zI_E, and invertible on L^2. Its inverse belongs to Ψ^0_{cl} and depends continuously on z in the classical Fréchet topology [6, Example 2, pp. 7–8; Theorem 5.4, pp. 15–16; proof of Theorem 7.6, p. 22, arXiv version]. For a nontrivial E, Proposition 1.4 of [38, p. 13], with complex trivializations and a smooth partition of unity, gives a smooth complementary bundle F with $E \oplus F$ trivial. Give F a Hermitian metric. In a smooth trivialization of $E \oplus F$, these results apply to $(zI - K) \oplus I$; restriction of its inverse to E gives the asserted inverse. Integration in the Fréchet space Ψ^0_{cl} gives

$$f(K) = \frac{1}{2\pi i} \int_\Gamma f(z)(zI - K)^{-1}\, dz \in \Psi^0_{\mathrm{cl}}, \qquad \log Z = K f(K) \in \Psi^{-1}_{\mathrm{cl}}.$$

The spectral theorem gives $(\log Z)^* = -\log Z$, since $\operatorname{Log}(\operatorname{Spec} Z) \subset i\mathbb{R}$. The spectral projections of $\log Z$ are spectral projections of Z corresponding to inverse images under Log; they therefore commute with the spectral projections of P. □

Lemma 1.4. *Let a finite group H act unitarily on a Hilbert space, and let Π_t, $0 \le t \le 1$, be a norm-continuous family of finite-rank orthogonal projections commuting with H. The representations $\operatorname{ran} \Pi_t$ are mutually isomorphic.*

Proof. Let $s, t \in [0,1]$, and put $P = \Pi_s$ and $Q = \Pi_t$. If $\|P - Q\| < 1$, Kato's formula [49, Chapter I, Section 4.6, equations (4.36)–(4.42), pp. 33–34; Chapter III, Problem 3.21, p. 156] gives an invertible operator

$$U = \big(QP + (I-Q)(I-P)\big)\big(I - (P-Q)^2\big)^{-1/2}, \qquad UP = QU.$$

The inverse square root is given by its norm-convergent binomial series. Each factor commutes with H, so U induces an isomorphism $\operatorname{ran} P \to \operatorname{ran} Q$ of H-modules. Norm continuity makes the isomorphism class locally constant, hence constant on $[0,1]$. □

2. Recognition of spherical space forms

Let M be a smooth closed connected Riemannian manifold of positive dimension. For such a manifold (Z, g_Z), let $\Delta_Z = -\operatorname{div}_{g_Z} \nabla_{g_Z}$ be its scalar Laplacian. Denote its eigenvalues, counted with multiplicity, by

$$0 = \lambda_0(Z) < \lambda_1(Z) \le \lambda_2(Z) \le \cdots .$$

All spectral equalities in this section include multiplicities, and $Z_1 \cong Z_2$ means that Z_1 and Z_2 are isometric.

For $d \in \mathbb{Z}_{\ge 1}$ and $r > 0$, let $\mathbb{S}^d(r) \subset \mathbb{R}^{d+1}$ be the Euclidean sphere of radius r, with its round metric. A spherical space form of radius r is a quotient $Q = \mathbb{S}^d(r)/\Gamma_0$, where Γ_0 is a finite subgroup of $\mathrm{O}(d+1)$ acting freely on the sphere. Define

$$\delta_r(M, Q) = \limsup_{j \to \infty} r \left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right|. \tag{2.1}$$

2.1. Reference spaces and matching conditions.

Fix

$$m \in \mathbb{Z}_{\ge 2}, \qquad n = 2m + 1, \qquad V_n = \operatorname{Vol}(\mathbb{S}^n(1)).$$

A spherical space form is homogeneous if its isometry group acts transitively. Identify $\mathbb{R}^{2m+2}$ with $\mathbb{C}^{m+1}$.

For integers $q \ge 2$ and $a_0, \dots, a_m$ with $\gcd(a_j, q) = 1$ for every j, put $\zeta = e^{2\pi i/q}$ and

$$L(q; a_0, \dots, a_m) = \mathbb{S}^{2m+1}(1)/\langle \operatorname{diag}(\zeta^{a_0}, \dots, \zeta^{a_m}) \rangle.$$

The metric is induced by the unit sphere. Multiplication of all weights by a unit modulo q, permutation of coordinates, and replacement of individual weights by their negatives preserve its isometry class [45, Lemma 1.2, p. 78].

Let Q belong to one of the following classes:

$$\begin{aligned} &\text{(E)} \quad Q = \mathbb{S}^{2m+1}(r)/\Gamma_0, \qquad q = |\Gamma_0| \text{ even}; \\ &\text{(O)} \quad Q = Q_{q,n}(r) := \mathbb{S}^{2m+1}(r)/\langle z \mapsto e^{2\pi i/q} z \rangle, \qquad q \ge 5 \text{ odd}. \end{aligned} \tag{2.2}$$

The unitary group $\mathrm{U}(m+1)$ commutes with the scalar deck group and acts transitively on $\mathbb{S}^{2m+1}(r)$. Its induced action on $Q_{q,n}(r)$ is transitive, so every space in (O) is homogeneous. Here $n = \dim Q$; the dimension of M is arbitrary. Put

$$\mu_j = \sqrt{r^2 \lambda_j(M) + m^2}, \qquad \beta_j = \sqrt{r^2 \lambda_j(Q) + m^2}, \qquad j \ge 0. \tag{2.3}$$

Let $a = 2$ in (E) and $a = 1$ in (O), and set

$$R_K = m + aK + \frac{a}{2}, \qquad K \in \mathbb{Z}_{\ge 0}. \tag{2.4}$$

By Lemmas 8.1 and 8.3, $\beta_j \in m + a\mathbb{Z}_{\ge 0}$ for every j.

Definition 2.1. With the notation of Definition 1.1, put

$$\mathcal{E}_\eta(K) = E_\eta(R_K; \{\mu_j\}_{j \ge 0}, \{\beta_j\}_{j \ge 0}), \qquad 0 \le \eta < a/4. \tag{2.5}$$

Both multisets are restricted to frequencies at most R_K. Thus $\mathcal{E}_\eta(K)$ is the least total number of unmatched copies in the two multisets.

The radius r, the space Q, and the tolerance η are fixed as $K \to \infty$.

2.2. **Recognition theorems.**

Theorem 2.2 (Rigidity for even-order spherical space forms). *Let Q belong to (E). If*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{E}}(n,q) := \frac{1}{q2^{m-1}(m+1)!}, \qquad 0 \le \eta < \frac{1}{2}, \tag{2.6}$$

then $\dim M = n$ *and M has constant sectional curvature r^{-2}. There is a finite subgroup $\Gamma < \mathrm{O}(n+1)$, acting freely on $\mathbb{S}^n(r)$, such that*

$$M \cong \mathbb{S}^n(r)/\Gamma, \qquad |\Gamma| = q.$$

If Q is homogeneous, then $M \cong Q$.

Theorem 2.3. *Let Q belong to (O). If*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K} = 0, \qquad 0 \le \eta < \frac{1}{4}, \tag{2.7}$$

then $M \cong Q$.

Theorem 2.4 (Rigidity for odd-order scalar lens spaces). *Let Q belong to (O). Put*

$$c_{\mathrm{O}}(n,q) = \begin{cases} \dfrac{1}{48q}, & n = 5, \\[2ex] \dfrac{2}{q|1-e^{4\pi i/q}|^m (m+1)!}, & n \ge 7. \end{cases} \tag{2.8}$$

If

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{O}}(n,q), \qquad 0 \le \eta < \frac{1}{4}, \tag{2.9}$$

then $M \cong Q$.

Theorem 2.2 is proved in Sections 16–17, and Theorem 2.4 in Section 18. Theorem 2.3 follows from Theorem 2.4: along any sequence on which $\mathcal{E}_\eta(K)/R_K \to 0$,

$$\frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} = \frac{\mathcal{E}_\eta(K)}{R_K}\, R_K^{-m} \longrightarrow 0.$$

Exact scalar isospectrality gives $\mathcal{E}_0(K) = 0$ for every K. Ikeda proved that a homogeneous spherical space form is determined by its scalar spectrum among spherical space forms, and that a homogeneous five-dimensional lens space is determined by its scalar spectrum among closed Riemannian manifolds [45, Theorems III and IV, p. 76].

2.3. **Ordered spectra and perturbations.** For $s \in \mathbb{R}$, write $H^s(Z)$ for the scalar Sobolev space on Z, and put $L^2(Z) = H^0(Z)$, with the Riemannian density.

Corollary 2.5. *Set $\varepsilon_Q = 1/2$ in (E) and $\varepsilon_Q = 1/4$ in (O). If*

$$\delta_r(M,Q) < \varepsilon_Q, \tag{2.10}$$

then M is a spherical space form of radius r and is scalar-isospectral to Q. If Q is homogeneous, then $M \cong Q$. In particular, $M \cong Q$ in (O).

For homogeneous Q, the contrapositive is

$$M \ncong Q \quad\Longrightarrow\quad \delta_r(M,Q) \ge \varepsilon_Q. \tag{2.11}$$

Moreover,

$$\lambda_j(M) - \lambda_j(Q) = O(1) \quad\Longrightarrow\quad \delta_r(M,Q) = 0.$$

For $\dim M = 5$ and $Q = \mathbb{RP}^5(r) = \mathbb{S}^5(r)/\{\pm I\}$, Wu proved that $|\lambda_j(M) - \lambda_j(Q)| = O(j^{-4/5})$ implies $M \cong Q$ [86, Theorem 1, p. 1374].

Corollary 2.6. *Let Q be homogeneous in (E), or let Q belong to (O). For $Z \in \{M, Q\}$, let B_Z be a bounded self-adjoint operator on $L^2(Z)$. The operator $\Delta_Z + B_Z$ is self-adjoint on $H^2(Z)$* [49, Chapter V, Theorem 4.3, p. 287]*, and*

$$\operatorname{Spec}(\Delta_M + B_M) = \operatorname{Spec}(\Delta_Q + B_Q) \quad \Longrightarrow \quad M \cong Q. \tag{2.12}$$

In particular, this applies to $B_Z u = V_Z u$ with $V_Z \in C^\infty(Z; \mathbb{R})$.

Let b_Z be a symmetric sesquilinear form on $H^1(Z)$. Suppose that, for constants $C_Z \geq 0$ and $0 \leq \alpha < 1/2$,

$$|b_Z(u,u)| \leq C_Z \langle (1+\Delta_Z)^\alpha u, u\rangle_{L^2(Z)}, \qquad u \in H^1(Z), \quad Z \in \{M, Q\}.$$

The form

$$\mathfrak{h}_Z(u,v) = \langle \Delta_Z^{1/2} u, \Delta_Z^{1/2} v\rangle_{L^2(Z)} + b_Z(u,v), \qquad u, v \in H^1(Z),$$

is closed and bounded below. Its associated self-adjoint operator H_Z has compact resolvent [49, Chapter VI, Theorems 2.6 and 3.4, pp. 323 and 338]*, and*

$$\operatorname{Spec}(H_M) = \operatorname{Spec}(H_Q) \quad \Longrightarrow \quad M \cong Q.$$

This includes self-adjoint classical pseudodifferential perturbations of order $\rho < 1$.

Corollaries 2.5 and 2.6 are proved in Section 19; the form assertion is Proposition 19.1.

2.4. **Optimality of the exponent.**

Theorem 2.7 (Sharpness of the multiplicity exponent). *Let $s \in \mathbb{Z}_{\geq 3}$, and put*

$$Q = L(8; \underbrace{1, \dots, 1}_{s}), \qquad Q' = L(8; \underbrace{1, \dots, 1}_{s-1}, 3).$$

These spaces have dimension $2s-1$ and carry the unit round quotient metrics. The space Q is homogeneous, and $Q' \ncong Q$. In Definition 2.1, take $M = Q'$, $r = 1$, and $m = s-1$, so that $R_K = 2K + s$. For every $0 \leq \eta < 1/2$,

$$\mathcal{E}_\eta(K) = C_s R_K^s + O(R_K^{s-1}), \qquad C_s > 0. \tag{2.13}$$

In dimension seven, $C_4 = 1/192 = 4c_{\mathrm{E}}(7,8)$. For every $p > m+1$, these spaces satisfy

$$\lim_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^p} = 0, \qquad Q' \ncong Q.$$

Hence the exponent $m+1$ in Theorem 2.2 cannot be increased.

The proof is given in Section 22.

Part 1. Spectral concentration and correction symbols

Let E be a positive-rank Hermitian bundle over a closed connected Riemannian n-manifold, $n \geq 2$. For $P = (\nabla^*\nabla + W)^{1/2} > 0$, with ∇ a unitary connection and W a smooth Hermitian endomorphism, the fixed-window hypothesis of Theorem 3.1 gives a common geodesic period and a pointwise constraint on the return symbol. For a general self-adjoint classical elliptic operator P of order one on E with positive scalar principal symbol, matching to a multiset supported on finitely many translates of $\mathbb{Z}$, with tolerance tending to zero and $o(R_\nu^n)$ unmatched copies along $R_\nu \to \infty$, gives a common Hamiltonian period and a strongly commuting quantizer A (Theorem 4.1). Polynomial bounds on the tolerance and unmatched count give an upper bound for the order of $P - A$ (Corollary 5.2). When $P - A$ has a first nonzero homogeneous term, signed phase profiles and phase Weyl masses determine the Liouville eigenvalue measures of its normalized leading symbol on each principal return eigensubbundle (Theorem 7.1).

3. Spectral phase concentration and bundle holonomy

Let (M, g) be a smooth closed connected Riemannian manifold of dimension $n \geq 2$, and let $E \to M$ be a smooth Hermitian vector bundle of positive rank with unitary connection ∇. Write dV_g for the Riemannian density and Δ_g for the nonnegative scalar Laplacian. Identify sections with half-densities by $s \mapsto s\, dV_g^{1/2}$. For a smooth Hermitian endomorphism W such that $D = \nabla^*\nabla + W > 0$, put $P = D^{1/2}$. Then $P \in \Psi^1_{\mathrm{cl}}(M; E)$ has principal symbol $|\xi|_g I_E$ [48, Section 3.1, p. 10, arXiv version]. Let G_t be the Hamiltonian flow of $|\xi|_g$ on $T^*M \setminus 0$, let $\pi : T^*M \to M$ be the projection, and put $S^*M = \{(x, \xi) : |\xi|_g = 1\}$. For $R > 0$, let

$$\Pi_R = \mathbf{1}_{(0,R]}(P), \qquad N_P(R) = \operatorname{Tr} \Pi_R.$$

All eigenvalues are counted with multiplicity. The canonical transformation of e^{itP} is G_{-t}. For $v \in S^*M$ and $t \geq 0$, put $\gamma_v(t) = \pi G_{-t}(v)$, and let

$$H_t(v) : E_{\pi(v)} \longrightarrow E_{\gamma_v(t)}$$

be parallel transport along γ_v from 0 to t. For a matrix or bounded operator B, write $\operatorname{Re} B = (B + B^*)/2$.

For scalar operators, concentration in shrinking neighborhoods of an arithmetic progression, with asymptotically full relative density uniformly in translated intervals of a fixed length, implies a common bicharacteristic period [21, Theorem 3.2, p. 55]. All geodesics are also closed if the number of distinct eigenvalues of $\sqrt{\Delta_g}$ in each interval of some fixed positive length is uniformly bounded [44, Theorem 2(6), pp. 9–10]. For connection Laplacians on manifolds whose geodesics have a common least period, the return-symbol formula and the criterion for spectral clustering are given in [53, Proposition 2.1, p. 245, and Theorem 5.1, p. 255]. For line bundles, see also [54, Proposition 3.1, p. 86].

Theorem 3.1. *Fix $T > 0$, $\beta \in \mathbb{R}$, and $0 < \kappa \leq 1$. Suppose that along some $R_j \to \infty$, all but $o(N_P(R_j))$ eigenvalues $\mu \leq R_j$ of P, counted with multiplicity, satisfy*

$$\operatorname{Re}(e^{-i\beta} e^{iT\mu}) \geq \kappa. \tag{3.1}$$

Then $G_T = I$. The unitary propagator $U = e^{iTP}$ belongs to $\Psi^0_{\mathrm{cl}}(M; E)$ and has unitary principal symbol u_T satisfying

$$\operatorname{Re}(e^{-i\beta} u_T(v)) \geq \kappa I_E \qquad (v \in S^*M). \tag{3.2}$$

Moreover,

$$u_T(v) = \rho_T H_T(v), \qquad \rho_T \in \{1, -1, i, -i\}, \tag{3.3}$$

where ρ_T is the constant scalar Maslov factor for the period T. The principal symbol u_T is independent of W.

Proof. Let $\mathcal{H} = \operatorname{Re}(e^{-i\beta} U)$, and let E_{exc} be the spectral projection onto the eigenvalues violating (3.1). Spectral calculus gives

$$\mathcal{H} \geq \kappa I - (1 + \kappa) E_{\mathrm{exc}}. \tag{3.4}$$

Weyl's law [67, Corollary 6.9, pp. 749–750] gives $N_P(R) \asymp R^n$. Hence, along $h = R_j^{-1}$,

$$\operatorname{Tr}(E_{\mathrm{exc}} \Pi_{1/h}) = o(h^{-n}).$$

Fix $0 \neq \chi \in C_c^\infty((0, 1))$ with $0 \leq \chi \leq 1$, and put $\psi = \chi^2$. For $B \in \Psi^0_{\mathrm{cl}}(M; E)$ with principal symbol b, set

$$A_h = \chi(hP) B^* B \chi(hP).$$

Then

$$0 \leq A_h \leq \|B\|^2 \psi(hP) \leq \|B\|^2 \Pi_{1/h}.$$

Cyclicity of the finite-rank trace and (1.1) give

$$\operatorname{Tr} A_h = \operatorname{Tr}\big(B^* B \psi(hP)\big) = c_b h^{-n} + o(h^{-n}), \qquad c_b = \frac{1}{(2\pi)^n} \int_{T^*M} \psi(|\xi|_g) \operatorname{tr}_E(b^* b)\, dx\, d\xi.$$

If $b \neq 0$, then $c_b > 0$. Taking the trace of (3.4) against A_h yields

$$\operatorname{Tr}(\boldsymbol{\mathcal{H}} A_h) \geq \kappa \operatorname{Tr} A_h - (1+\kappa)\|B\|^2 \operatorname{Tr}(E_{\mathrm{exc}}\Pi_{1/h}) = \kappa c_b h^{-n} + o(h^{-n}). \tag{3.5}$$

Suppose that $G_T(v_0) \neq v_0$ for some $v_0 \in S^*M$. Choose an open set $\Omega \subset S^*M$ containing v_0 such that $\overline{\Omega} \cap G_T(\overline{\Omega}) = \emptyset$, and choose B with $0 \neq b|_{S^*M}$ supported in Ω. The bundle Egorov theorem [48, Proposition 3.3, pp. 11–12, arXiv version] gives $UB^*U^* \in \Psi^0_{\mathrm{cl}}$, with principal symbol supported in $G_{-T}(\operatorname{supp} b)$ on S^*M. Consequently,

$$K_B := B(UB^*U^*) \in \Psi^{-1}_{\mathrm{cl}}, \qquad BUB^* = K_B U.$$

In particular, $K_B(1+P)$ is bounded. The operator B^* preserves H^1, and $[P, B^*] \in \Psi^0_{\mathrm{cl}}$ is bounded, since the principal symbol of P is scalar. Duhamel's formula gives

$$\|[e^{itP}, B^*]\| \leq |t|\,\|[P, B^*]\| \qquad (t \in \mathbb{R}).$$

Put $\widehat{\psi}(t) = \int_{\mathbb{R}} e^{-its}\psi(s)\,ds$. Fourier inversion gives

$$\|[\psi(hP), B^*]\| \leq \frac{h}{2\pi}\|[P, B^*]\| \int_{\mathbb{R}} |t\widehat{\psi}(t)|\,dt = O(h).$$

Moreover,

$$\operatorname{rank}[\psi(hP), B^*] \leq 2 \operatorname{rank} \psi(hP) \leq 2N_P(1/h) = O(h^{-n}).$$

Since U commutes with $\chi(hP)$, cyclicity gives

$$\operatorname{Tr}(UA_h) = \operatorname{Tr}\big(BU\psi(hP)B^*\big) = \operatorname{Tr}\big(K_B U\psi(hP)\big) + \operatorname{Tr}\big(BU[\psi(hP), B^*]\big).$$

The second term is $O(h^{1-n})$ by the rank and norm bounds. Choose $a > 0$ such that $\operatorname{supp}\psi \subset [a, 1)$. Then

$$\big|\operatorname{Tr}\big(K_B U\psi(hP)\big)\big| \leq \|K_B(1+P)\|\,\operatorname{Tr}\big((1+P)^{-1}\psi(hP)\big) \leq \frac{h}{a}\|K_B(1+P)\|\,N_P(1/h).$$

Thus $\operatorname{Tr}(UA_h) = O(h^{1-n}) = o(h^{-n})$, contrary to (3.5), since

$$\operatorname{Tr}(\boldsymbol{\mathcal{H}} A_h) = \operatorname{Re}\big(e^{-i\beta}\operatorname{Tr}(UA_h)\big).$$

Therefore $G_T = I$. The wave parametrix in [53, Section 2, pp. 243–244] has canonical graph G_{-T}, which is now the diagonal. Hence $U \in \Psi^0_{\mathrm{cl}}$; unitarity gives $u_T^* u_T = I_E$.

Put $C = \boldsymbol{\mathcal{H}} - \kappa I$ and $F = \sigma_0(C) = \operatorname{Re}(e^{-i\beta}u_T) - \kappa I_E$. The operator C commutes with $\chi(hP)$. For every fixed $B \in \Psi^0_{\mathrm{cl}}$, cyclicity and (1.1) give

$$\operatorname{Tr}(CA_h) = \operatorname{Tr}\big(CB^*B\psi(hP)\big) = \frac{h^{-n}}{(2\pi)^n}\int_{T^*M} \psi(|\xi|_g)\operatorname{tr}_E(Fb^*b)\,dx\,d\xi + o(h^{-n}).$$

By (3.4), $\liminf_{j\to\infty} h^n \operatorname{Tr}(CA_h) \geq 0$ along $h = R_j^{-1}$. Hence

$$\int_{T^*M} \psi(|\xi|_g)\operatorname{tr}_E(Fb^*b)\,dx\,d\xi \geq 0.$$

If $F(v_0)$ had a negative eigenvalue for some $v_0 \in S^*M$, choose a local unitary frame, a constant unit vector w, and an open neighborhood $\Omega \subset S^*M$ of v_0 such that $w^*F(v)w < 0$ on Ω. For $0 \neq \varphi \in C_c^\infty(\Omega)$, take $b = \varphi w w^*$ on S^*M, extended homogeneously of degree zero. Then

$$\operatorname{tr}_E(Fb^*b) = |\varphi|^2 w^* F w,$$

and the preceding integral is negative. This proves (3.2).

Put $P_0 = (\Delta_g + 1)^{1/2}$ on scalar half-densities. The principal symbol of P_0 is $|\xi|_g$, and its subprincipal symbol is zero. The wave parametrices for e^{itP} and e^{itP_0} have the same canonical graph and may be written with the same local phase [53, Section 2, pp. 242–244]. On this graph, divide the principal

amplitude of e^{itP} by the nonvanishing scalar amplitude of e^{itP_0}. The common Maslov and half-density factors cancel on overlaps, so the quotient defines

$$K_t(v) \in \mathrm{Hom}\big(E_{\pi(v)}, E_{\gamma_v(t)}\big), \qquad K_0(v) = I_E.$$

The subprincipal partial connection of P is the lift of ∇ [48, Proposition 3.2, p. 11, arXiv version]. Write $\nabla = d + \omega$ in a local unitary frame, where ω is a skew-Hermitian matrix-valued one-form. The first transport equation is

$$\frac{d}{dt}K_t(v) + \omega_{\gamma_v(t)}(\dot\gamma_v(t))K_t(v) = 0;$$

see [53, equation (2.5) and p. 245], with t replaced by $-t$. Thus $K_t(v) = H_t(v)$.

The scalar operator $TP_0/(2\pi)$ has zero subprincipal symbol, and its Hamiltonian flow at time 2π is the identity. Condition (C) and Theorem 1.1 of [15, pp. 508, 510] give

$$\sigma_0(e^{iTP_0}) = \rho_T \in \{1, -1, i, -i\}.$$

The Maslov factor is locally constant on S^*M. Since M is connected and $n \geq 2$, so is S^*M; hence ρ_T is constant. Therefore

$$u_T(v) = \rho_T H_T(v).$$

The principal symbol and subprincipal partial connection of P are independent of W, so the same is true of u_T. □

For $v \in S^*M$, $a \in \mathrm{End}(E_{\pi(v)})$, and $z \in \mathbb{S}^1$,

$$(zH_T(v))a(zH_T(v))^{-1} = H_T(v)aH_T(v)^{-1}.$$

Scalar multiplication leaves the induced transport on $\mathrm{End}\, E$ unchanged.

Theorem 3.2. *Fix $T > 0$, $\beta \in \mathbb{R}$, $0 < \kappa \leq 1$, and an integer $K \geq 1$. For $k = 0, \dots, K$, let $E_k = \mathrm{Sym}^k E$ carry the induced connection ∇_k, and put*

$$D_k = \nabla_k^*\nabla_k + W_k > 0, \qquad P_k = D_k^{1/2},$$

where W_k is a smooth Hermitian endomorphism of E_k. Here $E_0 = M \times \mathbb{C}$ has the trivial connection. Suppose each P_k satisfies the hypotheses of Theorem 3.1 with these T, β, κ; the subsequences may depend on k. Then $G_T = I$. Let ρ_T be the scalar factor in (3.3)*. Write* $\mathrm{Arg}\, z \in (-\pi, \pi]$ *for the principal argument of $z \in \mathbb{S}^1$, and put*

$$a = \arccos \kappa \in [0, \pi/2), \qquad x_0 = \mathrm{Arg}(e^{-i\beta}\rho_T) \in [-a, a].$$

For every $v \in S^*M$, each eigenvalue $e^{i\theta}$ of $H_T(v)$, with $\theta \in (-\pi, \pi]$, satisfies

$$|\theta| \leq \frac{a - |x_0|}{K} \leq \frac{a}{K}. \tag{3.6}$$

If the hypotheses hold for every $k \geq 0$, with the same T, β, κ, then $H_T(v) = I_E$ for every $v \in S^*M$. In this case, for the complexified tangent or cotangent bundle with its Levi-Civita connection, the orthonormal frame flow has period T. Here the frame flow transports a frame along the geodesic whose initial velocity is its first vector.

Proof. The case $k = 0$ gives $G_T = I$ and $\mathrm{Re}(e^{-i\beta}\rho_T) \geq \kappa$, hence $x_0 \in [-a, a]$. For each k, the scalar Hamiltonian is $|\xi|_g$, so the Maslov factor is ρ_T; the bundle factor is $\mathrm{Sym}^k H_T$. Fix $v \in S^*M$ and $0 \neq w \in E_{\pi(v)}$ such that $H_T(v)w = e^{i\theta}w$. For $1 \leq k \leq K$, the nonzero symmetric tensor $w^{\otimes k}$ is an eigenvector of $\mathrm{Sym}^k H_T(v)$ with eigenvalue $e^{ik\theta}$. The matrix inequality in Theorem 3.1 gives

$$x_k = \mathrm{Arg}(e^{-i\beta}\rho_T e^{ik\theta}) \in [-a, a] \qquad (0 \leq k \leq K).$$

For $0 \leq k < K$, the difference $x_{k+1} - x_k$ is congruent to θ modulo 2π and lies in $[-2a, 2a] \subset (-\pi, \pi)$. Thus $\theta \neq \pi$, $x_{k+1} - x_k = \theta$, and $x_K = x_0 + K\theta$.

For $0 \le t \le T$, geodesic reversal and $G_T = I$ give

$$\gamma_{-v}(t) = \pi G_t(v) = \gamma_v(T-t), \qquad H_T(-v) = H_T(v)^{-1}.$$

The scalar factor at $-v$ is again ρ_T. The inverse eigenvalue has principal argument $-\theta$. The preceding equality at $-v$ therefore gives $x_0 - K\theta \in [-a, a]$. Hence

$$|x_0| + K|\theta| = \max\{|x_0 + K\theta|, |x_0 - K\theta|\} \le a.$$

This proves (3.6). If all powers satisfy the hypotheses, this bound holds for every K, so $\theta = 0$. Since $H_T(v)$ is unitary, $H_T(v) = I_E$. For the tangent bundle, identity parallel transport fixes every real orthonormal frame. For the cotangent bundle, the same conclusion follows by duality. Together with $G_T = I$, this gives the stated period of the frame flow. □

The same statement holds with $E_k = E^{\otimes k}$ and the induced connections, where $E^{\otimes 0} = M \times \mathbb{C}$ has the trivial connection: the tensors $w^{\otimes k}$ satisfy the same eigenvalue identities. The period T need not be the least period.

For the trivial connection on the trivial line, the logarithmic correction below is given by [15, Theorem 1.1]. For connection Laplacians on manifolds with a common least geodesic period, the corresponding bundle construction appears in [53, proof of Theorem 5.1, p. 255].

Theorem 3.3. *Suppose that the hypotheses of Theorem 3.1 hold and that $H_T(v) = I_E$ for every $v \in S^*M$. Put $b = 2\pi/T$, and let $\alpha \in (0, b]$ be the unique number such that $e^{iT\alpha} = \rho_T$. There is a positive self-adjoint classical operator $A \in \Psi^1_{\mathrm{cl}}(M; E)$ with domain H^1, strongly commuting with P, such that*

$$A - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \mathrm{Spec}(A) \subset \alpha + b\mathbb{Z}_{\ge 0}.$$

For every sufficiently large eigenvalue μ of P, the restriction of A to $\ker(P-\mu)$ is $a_\mu I$, where a_μ is the nearest point of this lattice and $|a_\mu - \mu| = O((1+\mu)^{-1})$.

Proof. Put $\widetilde{P} = P/b$. The unitary operator $U = e^{2\pi i \widetilde{P}}$ strongly commutes with $\widetilde{P}$, and $U - \rho_T I \in \Psi^{-1}_{\mathrm{cl}}$ by (3.3). The logarithmic construction in the proof of Theorem 4.1(3), applied to $\widetilde{P}$ with $L = 1$ and $\mathcal{S} = \{\rho_T\}$, gives, after multiplication by b, a self-adjoint operator $A_0 \in \Psi^1_{\mathrm{cl}}$ on H^1, strongly commuting with P, such that

$$A_0 - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \mathrm{Spec}\, A_0 \subset \alpha + b\mathbb{Z}.$$

For an eigenvalue μ of P, put $E_\mu = \ker(P-\mu)$. The operator $(A_0 - P)(1+P)$ is bounded, so

$$\big\|(A_0 - \mu I)|_{E_\mu}\big\| \le C(1+\mu)^{-1}.$$

For all sufficiently large μ, the interval $[\mu - C/(1+\mu), \mu + C/(1+\mu)]$ is contained in $(0,\infty)$ and has length less than b. The spectrum of the self-adjoint restriction $A_0|_{E_\mu}$ therefore consists of one point $a_\mu \in \alpha + b\mathbb{Z}_{\ge 0}$, and $A_0|_{E_\mu} = a_\mu I$. Since $|a_\mu - \mu| < b/2$, this is the unique nearest lattice point.

Replace A_0 by αI on the finitely many remaining P-eigenspaces. The difference has finite rank and smooth kernel. It is self-adjoint and strongly commutes with P. The resulting operator A has domain H^1, satisfies $A \ge \alpha I > 0$, and has all the stated properties. □

Corollary 3.4. *Under the hypotheses of Theorem 3.3, there is a bounded self-adjoint classical operator $C \in \Psi^0_{\mathrm{cl}}(M; E)$, strongly commuting with D, such that*

$$\mathrm{Spec}(D + C) \subset \{(\alpha + b\,j)^2 \,:\, j \in \mathbb{Z}_{\ge 0}\}.$$

Proof. Put $B = A - P \in \Psi^{-1}_{\mathrm{cl}}$ and $C = 2PB + B^2 \in \Psi^0_{\mathrm{cl}}$. The bounded operator B is self-adjoint and strongly commutes with P. Since $B : L^2 \to H^1$, the operator PB is defined on L^2 and is bounded. It is symmetric on smooth sections, so C is bounded and self-adjoint. Each P-eigenspace is invariant under C; hence C strongly commutes with $D = P^2$. Moreover, $B : H^1 \to H^2$, and elliptic regularity gives

$$\mathrm{Dom}\, A^2 = \{u \in H^1 \,:\, Pu \in H^1\} = H^2.$$

On this domain, $D + C = A^2$. The spectral theorem gives $\mathrm{Spec}(D+C) = \{a^2 \,:\, a \in \mathrm{Spec}(A)\}$, with the multiplicities of A, since A is positive. □

3.1. **Connections with prescribed geodesic transport.** Let $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, $n \geq 2$, with its round metric. Write $x_1, \dots, x_{n+1}$ for the standard coordinates and $e_1, \dots, e_{n+1}$ for the standard basis. On the trivial complex line, put

$$\eta = x_3(x_1\, dx_2 - x_2\, dx_1), \qquad \nabla = d + i\varepsilon\eta, \qquad \varepsilon \in \mathbb{R} \setminus \{0\}.$$

The antipodal map τ satisfies $\tau^*\eta = -\eta$. Thus $\int_\gamma \eta = 0$ for every great circle γ [52, Proposition 4.4, p. 449], and the period-2π transports in all symmetric powers are the identity. The curvature $F_\nabla = i\varepsilon d\eta$ satisfies

$$(F_\nabla)_{e_3}(e_1, e_2) = 2i\varepsilon \neq 0.$$

Thus identity geodesic transport does not imply flatness. The relation between antipodally odd curvature and spectral clusters on spheres is given in [52, Theorem 4.5, p. 450]. Connections with identity transport along every closed geodesic are called transparent; see [65, Section 1].

For a second example, put

$$\eta_0 = x_1\, dx_2 - x_2\, dx_1, \qquad \nabla = d + i\varepsilon\eta_0.$$

If $\gamma(t) = u\cos t + v\sin t$, $0 \leq t \leq 2\pi$, where u, v are orthonormal in $\mathbb{R}^{n+1}$, then

$$\eta_0(\dot\gamma(t)) = u_1v_2 - u_2v_1, \qquad \int_\gamma \eta_0 = 2\pi(u_1v_2 - u_2v_1), \qquad |u_1v_2 - u_2v_1| \leq 1.$$

For $u = e_1$ and $v = e_2$, the integral is 2π. Fix an integer $K \geq 1$ and $0 < a < \pi/2$, and choose a real ε with

$$0 < |\varepsilon| < \frac{a}{2\pi K} < \frac{1}{4K}.$$

The transport in the kth power is $\exp(-ik\varepsilon \int_\gamma \eta_0)$. For $0 \leq k \leq K$, its principal argument lies in $(-a, a)$, whereas $e^{-2\pi i\varepsilon} \neq 1$ on the equator.

In either example, let ∇_k be the induced connection on the kth tensor power, with the trivial connection for $k = 0$. For a line bundle, tensor and symmetric powers agree. Choose constants $w_k > 0$, and put

$$D_k = \nabla_k^*\nabla_k + w_k I > 0, \qquad P_k = D_k^{1/2}.$$

Since $G_{2\pi} = I$, the return-symbol formula in the proof of Theorem 3.1 gives

$$e^{2\pi i P_k} \in \Psi^0_{\mathrm{cl}}, \qquad \sigma_0(e^{2\pi i P_k}) = \rho_{2\pi} H_{2\pi}^{\otimes k},$$

where $H_{2\pi}^{\otimes 0} = 1$. Fix $0 < a < \pi/2$ in the first example, and retain the chosen a in the second. Set $\beta = \operatorname{Arg} \rho_{2\pi}$, $\kappa = \cos a$, and

$$\mathcal{H}_k = \operatorname{Re}(e^{-i\beta} e^{2\pi i P_k}), \qquad h_{0,k} = \sigma_0(\mathcal{H}_k).$$

In the first example, $h_{0,k} = 1$ for every $k \geq 0$. In the second, $h_{0,k} \geq \cos(2\pi K|\varepsilon|) > \kappa$ for $0 \leq k \leq K$. In each case, choose $\delta > 0$ such that $h_{0,k} - \kappa \geq \delta$ for the indicated powers, and set

$$Q_k = \big(\mathcal{H}_k - (\kappa + \delta/2)I\big)P_k.$$

The operators $\mathcal{H}_k$ and P_k commute by spectral calculus. Thus Q_k is symmetric on smooth sections, and its principal symbol is

$$\sigma_1(Q_k) = (h_{0,k} - \kappa - \delta/2)|\xi|_g \geq (\delta/2)|\xi|_g > 0 \qquad (\xi \neq 0).$$

By the elliptic operator conventions in Section 1, its self-adjoint closure has domain H^1 and satisfies $Q_k \geq -C_k I$ for some $C_k \geq 0$. For $P_k u = \mu u$ with $\|u\| = 1$, it follows that

$$\operatorname{Re}(e^{-i\beta} e^{2\pi i\mu}) \geq \kappa + \frac{\delta}{2} - \frac{C_k}{\mu} \geq \kappa \qquad \left(\mu \geq \frac{2C_k}{\delta}\right).$$

Consequently, (3.1) holds for all sufficiently large eigenvalues in each power in the first example, and in each of the powers $0, \dots, K$ in the second.

4. Finite spectral phases and Hamiltonian return

Let M be a closed connected manifold of dimension $n \geq 2$, let $E \to M$ be a Hermitian bundle of rank $r \geq 1$, and work on E-valued half-densities. Write $\pi : T^*M \to M$ for the projection. Let $P \in \Psi^1_{\mathrm{cl}}(M;E)$ be self-adjoint and elliptic, with principal symbol $q(x,\xi)I_E$, where $q > 0$ is homogeneous of degree one. Radial rescaling identifies the unit q-cosphere $\Sigma = \{q = 1\}$ with a unit cosphere bundle; hence Σ is compact and connected. Let G_t be the Hamiltonian flow of q, and put

$$U = e^{2\pi i P}, \qquad \Phi = G_{-2\pi},$$
$$\Pi_R = \mathbf{1}_{(-\infty,R]}(P), \qquad N_P(R) = \operatorname{Tr} \Pi_R.$$

Thus Φ is the canonical transformation of U. Changing finitely many eigenvalue copies changes each matching defect by $O(1)$ (Lemma 1.2). All spectral lists and matchings retain multiplicities.

For a nonempty finite set $\mathcal{S} \subset \mathbb{S}^1$, put

$$\Lambda(\mathcal{S}) = \{x \in \mathbb{R} : e^{2\pi i x} \in \mathcal{S}\}, \qquad F(z) = \prod_{s \in \mathcal{S}} (z - s), \qquad d = |\mathcal{S}|.$$

The phases need not be roots of unity. For a positive integer L, define

$$\mathcal{T}_L(\mathcal{S}) = \{h \in \mathbb{S}^1 : \{z : z^L = h\} \subset \mathcal{S}\}. \tag{4.1}$$

Thus $h \in \mathcal{T}_L(\mathcal{S})$ precisely when its Lth roots form a regular L-gon contained in $\mathcal{S}$.

For scalar operators, concentration in shrinking neighborhoods of one arithmetic progression, uniformly in all sufficiently high intervals of a suitable fixed length, implies periodicity, even when a proportion of eigenvalue copies tending to zero is exceptional [21, Theorem 3.2, p. 55]. For a connection Laplacian on a manifold with a common least geodesic period, finite holonomy spectrum is equivalent to bounded-width clusters of Laplacian eigenvalues [53, Theorem 5.1, p. 255].

Theorem 4.1 (Finite spectral phases and Hamiltonian return). *Suppose that a multiset $\mathcal{B}$, bounded below and finite on bounded intervals, is supported on $\Lambda(\mathcal{S})$ and satisfies*

$$E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \mathcal{B}) = o(R_\nu^n)$$

for some $R_\nu \to \infty$ and $\eta_\nu \geq 0$ tending to zero.

(1) The map Φ has finite order L, with $1 \leq L \leq d$. In particular, the Hamiltonian flow of q has period $2\pi L$. The period $2\pi L$ need not be the least period of every orbit.

(2) The operator $V = U^L$ is a classical order-zero pseudodifferential operator. Its principal symbol $H \in C^\infty(\Sigma; \operatorname{End}(\pi^*E))$ is unitary, and

$$\operatorname{Spec} H(\rho) \subset \mathcal{T}_L(\mathcal{S}) \quad (\rho \in \Sigma).$$

The multiplicity of each $h \in \mathcal{T}_L(\mathcal{S})$ is constant on Σ.

(3) There is a self-adjoint $A \in \Psi^1_{\mathrm{cl}}(M;E)$ with domain H^1, strongly commuting with P, such that

$$A - P \in \Psi^{-1}_{\mathrm{cl}}(M;E), \qquad \operatorname{Spec} A \subset \Lambda(\mathcal{S}).$$

It is unique up to an operator supported on a finite-dimensional P-spectral subspace among self-adjoint operators satisfying these conditions.

Proof. For each fixed t, the scalar-principal wave parametrix makes e^{-itP} a classical order-zero Fourier integral operator with canonical relation $\operatorname{graph}(G_t)$ [21, Theorem 1.1, pp. 43–44]. Its phase is the scalar phase of q [42, Section 3, pp. 204–208]. In a bundle frame, the transport equations are linear matrix equations along the same Hamiltonian characteristics; the leading transport is determined by the subprincipal partial connection of P [48, Proposition 3.1, p. 10, arXiv version]. The construction in [53, Section 2] is the connection-Laplacian instance of this calculus.

Choose a real nonzero $\psi \in C_c^\infty((0,1))$ and put $\chi = \psi^2$. Write $\|\cdot\|_{\mathrm{HS}}$ for the Hilbert–Schmidt norm, and put $C_F = \sup_{x \in \mathbb{R}} |\frac{d}{dx} F(e^{2\pi i x})|$. A matched eigenvalue μ satisfies $|F(e^{2\pi i \mu})| \leq C_F \eta_\nu$; the same

expression is uniformly bounded on unmatched copies. The bundle Weyl law gives $O(R_\nu^n)$ copies in the cutoff. Therefore

$$\|F(U)\psi(P/R_\nu)\|_{\mathrm{HS}}^2 \le C\eta_\nu^2 R_\nu^n + o(R_\nu^n) = o(R_\nu^n). \tag{4.2}$$

If $\operatorname{supp}\psi \subset [a,b] \subset (0,1)$, each eigenvalue selected by $\psi(P/R_\nu)$ satisfies $\mu \le bR_\nu < R_\nu$. For every fixed bounded operator D, $\|DF(U)\psi(P/R_\nu)\|_{\mathrm{HS}} \le \|D\|\|F(U)\psi(P/R_\nu)\|_{\mathrm{HS}}$.

Bounded pointwise periods. Write $F(z) = \sum_{j=0}^d f_j z^j$. Suppose that some $\rho \in \Sigma$ satisfies $\Phi^j(\rho) \ne \rho$ for every $1 \le j \le d$. Choose a small conic neighborhood W of ρ whose closure contains no fixed point of any of these powers. Choose $D \in \Psi_{\mathrm{cl}}^0(M;E)$ with nonzero scalar principal symbol aI_E and all-orders microsupport in W, modulo smoothing terms.

Normality of $F(U)$, its commutation with $\chi(P/R)$, and cyclicity give the exact identity

$$\|DF(U)\psi(P/R)\|_{\mathrm{HS}}^2 = \sum_{j,k=0}^{d} f_j\overline{f_k}\,\operatorname{Tr}\big(\chi(P/R)D^*D\,U^{j-k}\big). \tag{4.3}$$

For $j = k$, the local Weyl formula (1.1) gives the positive leading coefficient

$$\frac{r}{(2\pi)^n}\int_{T^*M}\chi(q)|a|^2\,dx\,d\xi.$$

For $j \ne k$, the canonical relation has no diagonal point on the microsupport of D^*D. In a local homogeneous wave parametrix, trace restriction has no stationary point on the compact rescaled frequency annulus. Repeated integration by parts makes the trace $O(R^{-N})$ for every N. Consequently,

$$\|DF(U)\psi(P/R)\|_{\mathrm{HS}}^2 = cR^n + o(R^n), \qquad c = \frac{r}{(2\pi)^n}\sum_{j=0}^{d}|f_j|^2\int_{T^*M}\chi(q)|a|^2\,dx\,d\xi > 0.$$

This contradicts (4.2). Thus

$$\Sigma = \bigcup_{j=1}^{d}\operatorname{Fix}(\Phi^j).$$

If $N = \operatorname{lcm}(1,\dots,d)$, every point is fixed by Φ^N, so $\Phi^N = I$. Homogeneity extends this identity to the whole punctured cotangent bundle.

The global order is at most d. Let L be the order of Φ on Σ. The cyclic group $C_L = \langle\Phi\rangle$ acts properly and effectively on Σ, and Σ/C_L is connected. By the principal orbit theorem [62, Theorem 1.32, p. 15], the principal stabilizer is contained in every stabilizer, since C_L is abelian. It is therefore contained in the kernel of the action and is trivial. Hence the set of points with orbit length L is open and dense. At any such point the preceding bound gives $L \le d$.

The return polynomial identities. At $t = -2\pi L$ the canonical relation is the diagonal. Thus $V = e^{2\pi i L P} \in \Psi_{\mathrm{cl}}^0(M;E)$. Unitarity gives $H^*H = I$. Decompose the polynomial uniquely as

$$F(z) = \sum_{j=0}^{L-1} z^j F_j(z^L). \tag{4.4}$$

In the open dense full-orbit locus, choose a cutoff D as above, now avoiding the fixed sets of Φ^j, $1 \le j < L$. Since all functions of U commute, the same exact norm identity becomes

$$\|DF(U)\psi(P/R)\|_{\mathrm{HS}}^2 = \sum_{j,k=0}^{L-1}\operatorname{Tr}\big(\chi(P/R)D^*D\,U^{j-k}F_j(V)F_k(V)^*\big).$$

The off-diagonal terms are rapidly decreasing. The coefficient of R^n in the diagonal sum is

$$\frac{1}{(2\pi)^n}\int_{T^*M}\chi(q)|a|^2\sum_{j=0}^{L-1}\operatorname{tr}_E\big(F_j(H)F_j(H)^*\big)\,dx\,d\xi.$$

It is nonnegative and must vanish by (4.2). Arbitrary localization and homogeneity imply $F_j(H) = 0$ on the full-orbit locus; continuity extends this equality to all of Σ.

For a unit complex number h, the identities $F_j(h) = 0$ for all j are equivalent to F vanishing at every root of $z^L - h$. Indeed, reduction modulo $z^L - h$ gives the polynomial $\sum_{j<L} z^j F_j(h)$ of degree less than L; it vanishes at L distinct points if and only if it is zero. Applying this to each eigenvalue of the normal matrix H gives $\operatorname{Spec} H(\rho) \subset \mathcal{T}_L(\mathcal{S})$. Its spectral projectors are polynomial expressions in H: the set $\mathcal{T}_L(\mathcal{S}) \subset \{s^L : s \in \mathcal{S}\}$ is finite, and for $h \in \mathcal{T}_L(\mathcal{S})$ the projector is

$$\prod_{h' \in \mathcal{T}_L(\mathcal{S}) \setminus \{h\}} \frac{H - h' I_E}{h - h'}.$$

Their integer ranks are locally constant and hence constant on connected Σ.

Spectral projectors and the logarithmic correction. For positive scalar operators with zero subprincipal symbol and Hamiltonian period 2π, the commuting logarithmic correction is given in [15, Theorem 1.1, pp. 510–511]. For connection Laplacians with a common least geodesic period, the bundle logarithmic construction appears in [53, proof of Theorem 5.1, p. 255]. The operator $V = e^{2\pi i L P}$ is unitary, belongs to Ψ^0_{cl}, strongly commutes with P, and has finite principal-symbol spectrum. Let $\mathcal{T}$ be the finite set of eigenvalues that occur in H, and put $G(z) = \prod_{h \in \mathcal{T}} (z - h)$. The principal symbol of $G(V)$ vanishes, so $G(V) \in \Psi^{-1}_{\mathrm{cl}}$. Since $G(V)$ commutes with P, boundedness of $G(V)P$ gives

$$|G(e^{2\pi i L \mu})| \le C/\mu$$

on every sufficiently high P-eigenvector. The roots of G are simple. On the unit circle this implies that $e^{2\pi i L \mu}$ lies within C'/μ of $\mathcal{T}$.

Change V on a fixed finite-dimensional P-spectral subspace to one chosen value in $\mathcal{T}$. The resulting unitary V' still strongly commutes with P, differs from V by a smoothing operator, and has its spectrum in arbitrarily small disjoint arcs about the points of $\mathcal{T}$. For each $h \in \mathcal{T}$, let Π_h be the Riesz projection onto the corresponding arc. Then

$$\Pi_h \in \Psi^0_{\mathrm{cl}}, \qquad \Pi_h^* = \Pi_h, \qquad \Pi_h \Pi_{h'} = 0 \ (h \neq h'), \qquad \sum_h \Pi_h = I.$$

The defining contours lie in the resolvent set of V' and avoid the spectrum of H. Thus $zI - V'$ is elliptic and invertible there. Spectral invariance and continuity of inversion in the classical operator topology [6, Example 2, pp. 7–8; Theorem 5.4, pp. 15–16; proof of Theorem 7.6, p. 22, arXiv version], with the bundle reduction in the proof of Lemma 1.3, give $(zI - V')^{-1} \in \Psi^0_{\mathrm{cl}}$, continuously along each contour. Contour integration gives $\Pi_h \in \Psi^0_{\mathrm{cl}}$, with principal symbol the h-eigenprojection of H.

Define

$$Z = \sum_{h \in \mathcal{T}} h^{-1} V' \Pi_h.$$

This operator is unitary, strongly commutes with P, and has principal symbol I. The arcs may be chosen so small that $\|Z - I\| < 1/2$. Lemma 1.3 gives

$$Q = \frac{1}{2\pi i L} \log Z \in \Psi^{-1}_{\mathrm{cl}}.$$

The operator Q is self-adjoint and strongly commutes with P.

On every sufficiently high P-eigenspace, $V' = e^{2\pi i L P}$ and

$$e^{2\pi i L (P - Q)} = V Z^{-1} = \sum_{h \in \mathcal{T}} h \Pi_h.$$

Every high eigenvalue of $P - Q$ consequently has its phase among the Lth roots of some $h \in \mathcal{T}$, all of which lie in $\mathcal{S}$. Define $A = P - Q$ on these eigenspaces. On each remaining space $\ker(P - \mu) \cap \operatorname{ran} \Pi_h$, choose $a_{\mu,h} \in \mathbb{R}$ with $e^{2\pi i L a_{\mu,h}} = h$ and prescribe $A = a_{\mu,h} I$. Since $\{z : z^L = h\} \subset \mathcal{S}$, every $a_{\mu,h}$ belongs to $\Lambda(\mathcal{S})$. This finite-rank self-adjoint correction gives the exact global identity

$$e^{2\pi i L A} = \sum_{h \in \mathcal{T}} h \Pi_h. \tag{4.5}$$

The correction to P is bounded and self-adjoint, so A has domain H^1 and is self-adjoint there [49, Chapter V, Theorem 4.3, p. 287].

Finally, let A_1, A_2 be two operators with the stated properties, and let $E_\mu = \ker(P-\mu)$. Strong commutation and boundedness of $(A_i - P)P$ give

$$\left\|(A_i-\mu I)|_{E_\mu}\right\| \le C_i/\mu \qquad (i=1,2)$$

for all sufficiently large μ. Since $\Lambda(\mathcal{S})$ has positive minimum separation, the interval $[\mu - C/\mu, \mu + C/\mu]$, $C = \max(C_1, C_2)$, contains at most one point of $\Lambda(\mathcal{S})$. Both restrictions are therefore the same scalar operator. Thus $A_1 = A_2$ outside a finite-dimensional P-spectral subspace. □

Corollary 4.2. *Under Theorem 4.1, let $\mathcal{T}$ be the constant set of eigenvalues of H. For $h \in \mathcal{T}$, write $h = e^{2\pi i\theta_h}$ with $0 \le \theta_h < 1$. There are mutually orthogonal projections $\Pi_h \in \Psi^0_{\mathrm{cl}}$, $h \in \mathcal{T}$, with $\sum_h \Pi_h = I$, and strongly commuting self-adjoint classical operators K, Θ, Q such that, on H^1,*

$$LP = K + \Theta + Q, \qquad \operatorname{Spec} K \subset \mathbb{Z}, \qquad Q \in \Psi^{-1}_{\mathrm{cl}}, \qquad \Theta = \sum_h \theta_h \Pi_h \in \Psi^0_{\mathrm{cl}}.$$

Here K has domain H^1 and scalar principal symbol $Lq I_E$; Θ and Q are bounded. The principal symbols of Π_h are the spectral projections of H; their ranges are smooth subbundles of π^*E over Σ.

Proof. Take A and Π_h from the proof of Theorem 4.1, and set $\Theta = \sum_h \theta_h \Pi_h$, $K = LA - \Theta$ and $Q = L(P - A)$. The projectors, A, and P strongly commute. Equation (4.5) gives $e^{2\pi i K} = I$, hence $\operatorname{Spec} K \subset \mathbb{Z}$. □

Corollary 4.3. *Under the hypotheses of Theorem 4.1, for $\mathcal{S} = \{1\}$ the conclusion is*

$$\Phi = I, \qquad e^{2\pi i P} - I \in \Psi^{-1}_{\mathrm{cl}}, \qquad \operatorname{Spec} A \subset \mathbb{Z}.$$

For a single arbitrary phase s, the principal return is sI_E and $\operatorname{Spec} A \subset \mathbb{Z} + \theta$, where $e^{2\pi i\theta} = s$. If $q_0 \in \mathbb{Z}_{>0}$ and $\mathcal{S}$ is contained in the set of q_0th roots of unity, then $L \mid q_0$ and

$$e^{2\pi i q_0 P} - I \in \Psi^{-1}_{\mathrm{cl}}.$$

Proof. If $|\mathcal{S}| = 1$, then $L = 1$ and $H = sI_E$. Suppose every element of $\mathcal{S}$ is a q_0th root of unity. Choose $h \in \mathcal{T}_L(\mathcal{S})$, $z^L = h$, and put $\zeta = e^{2\pi i/L}$. Both z and ζz belong to $\mathcal{S}$, so $\zeta^{q_0} = 1$ and $L \mid q_0$. Each such h satisfies $h^{q_0/L} = 1$, so the principal symbol of $U^{q_0} = V^{q_0/L}$ is the identity. □

Proposition 4.4. *Let $L \in \mathbb{Z}_{>0}$, and let $\mathcal{S}$ be a nonempty union of finitely many pairwise disjoint regular L-gons in $\mathbb{S}^1$. Assign to each polygon a positive integer r_h, where h is the common Lth power of its vertices, and put $r = \sum_h r_h$. For every $n \ge 2$, there is an operator P on the trivial rank-r bundle over S^n satisfying Theorem 4.1, with canonical return of order L and*

$$\operatorname{Spec} P \subset \Lambda(\mathcal{S}), \qquad \{e^{2\pi i\mu} : \mu \in \operatorname{Spec} P\} = \mathcal{S}.$$

The multiplicity of h in its principal return is r_h.

Proof. On the unit round sphere S^n, $n \ge 2$, let

$$K_0 = \sqrt{\Delta_{S^n} + (n-1)^2/4} - (n-1)/2 + 1. \tag{4.6}$$

On spherical harmonics of degree $k \ge 0$, K_0 has eigenvalue $k+1$ [45, Lemma 1.3, p. 78]. It is a scalar classical operator with principal symbol $|\xi|$ [6, Theorem 7.9, p. 23, arXiv version]; its Hamiltonian flow has prime period 2π. For the prescribed polygons, write their Lth powers as $h = e^{2\pi i\theta_h}$ with $0 \le \theta_h < 1$. On the trivial rank-r bundle let Θ_0 be a constant Hermitian matrix with eigenvalue θ_h of the prescribed multiplicity and put

$$P = \frac{1}{L}(K_0 I_r + \Theta_0).$$

Its eigenvalues are $(k + 1 + \theta_h)/L$, $k \ge 0$, with the multiplicities supplied by the spherical harmonics and r_h. As $k+1$ runs through the residue classes modulo L, their phases run through the prescribed polygons. The time-2π canonical map has order L, and $e^{2\pi i LP} = e^{2\pi i\Theta_0}$ has precisely the specified constant return eigenvalues. The comparison multiset $\operatorname{Spec} P$ gives zero matching error and zero unmatched count; here $A = P$. □

4.1. **The limiting spectral phase measure.** Let $d\Sigma$ be the radial Liouville measure on Σ from Section 1, and write $|\Sigma| = \int_\Sigma d\Sigma$. For connection Laplacians with common least geodesic period, Kuwabara's bandwise distribution formula [53, Theorem 3.1, pp. 245–246] assumes that the holonomy spectrum omits a point of $\mathbb{S}^1$. Eigenphase distributions for quantized contact transformations are studied in [89, Theorem A(ii), p. 312, and Section 4, pp. 328–329]. In that Toeplitz setting, a contact transformation of period p admits an invariant quantization with spectrum among the pth roots of unity. The following formula concerns cumulative counts for the given operator P.

Write $\mu_j(P)$, $j \geq 1$, for the eigenvalues of P, counted with multiplicity, and let δ_z be the unit point mass at $z \in \mathbb{S}^1$.

Theorem 4.5. *Suppose that Φ has finite order L on Σ. Put $H = \sigma_0(e^{2\pi i L P})$. For all sufficiently large R, the probability measures*

$$\nu_R = \frac{1}{N_P(R)} \sum_{\mu_j(P) \leq R} \delta_{e^{2\pi i \mu_j(P)}}$$

converge weakly to the measure ν defined, for $f \in C(\mathbb{S}^1)$, by

$$\int_{\mathbb{S}^1} f \, d\nu = \frac{1}{rL|\Sigma|} \int_\Sigma \sum_{h \in \operatorname{Spec} H(\rho)} \sum_{z^L = h} f(z) \, d\Sigma(\rho). \tag{4.7}$$

The eigenvalues of H in the outer sum are counted with multiplicity.

Proof. For $f \in C(\mathbb{S}^1)$, the function

$$(\mathcal{M}_L f)(h) = \frac{1}{L} \sum_{z^L = h} f(z), \qquad h \in \mathbb{S}^1,$$

is continuous. The right side of (4.7) is $(r|\Sigma|)^{-1} \int_\Sigma \operatorname{tr}_E((\mathcal{M}_L f)(H)) \, d\Sigma$. Continuous functional calculus thus defines a probability measure, without any choice of eigenvalue labels.

Weyl's law [42, Theorem 4.4, equation (4.10), p. 212], with the diagonal fiber trace as in Section 1, gives

$$N_P(R) \sim \frac{r|\Sigma|}{n(2\pi)^n} R^n.$$

Fix $0 \leq \chi \in C_c^\infty((0,\infty))$ and put $b_\chi = n \int_0^\infty \chi(s) s^{n-1} \, ds$. Let $k \in \mathbb{Z}$. For $k = L\ell$, $\ell \in \mathbb{Z}$, the operator U^k belongs to Ψ^0_{cl} and has principal symbol H^ℓ. The local Weyl formula (1.1) gives

$$\frac{\operatorname{Tr}(U^{L\ell} \chi(P/R))}{N_P(R)} \longrightarrow \frac{b_\chi}{r|\Sigma|} \int_\Sigma \operatorname{tr}_E(H^\ell) \, d\Sigma.$$

Suppose $L \nmid k$. The fixed-set theorem [62, Proposition 1.24, p. 11] and compactness make $\mathcal{F} = \operatorname{Fix}(\Phi^k)$ a finite union of closed embedded submanifolds of Σ. A component of full dimension would be open and closed in connected Σ, hence equal to Σ, contrary to $\Phi^k \neq I$. Thus $\mathcal{F}$ has Liouville measure zero.

For $\varepsilon > 0$, choose $a_\varepsilon \in C^\infty(\Sigma; [0,1])$, equal to one near $\mathcal{F}$, with $\int_\Sigma a_\varepsilon^2 \, d\Sigma < \varepsilon$. Choose $D_\varepsilon \in \Psi^0_{\mathrm{cl}}$ with scalar principal symbol $a_\varepsilon I_E$ and with $I - D_\varepsilon$ microsupported away from $\mathcal{F}$, modulo smoothing operators. Let $T \in \Psi^0_{\mathrm{cl}}(M; E)$ be fixed. The nonstationary trace estimate gives

$$\operatorname{Tr}(U^k T (I - D_\varepsilon) \chi(P/R)) = O(R^{-N}) \quad \text{for every } N.$$

For the other term, the Hilbert–Schmidt inequality gives

$$\left| \operatorname{Tr}\big(U^k T D_\varepsilon \chi(P/R)\big) \right| \leq \|T\| \big(\operatorname{Tr} \chi(P/R)\big)^{1/2} \big(\operatorname{Tr}(D_\varepsilon^* D_\varepsilon \chi(P/R))\big)^{1/2}.$$

By (1.1),

$$\limsup_{R \to \infty} R^{-n} |\operatorname{Tr}(U^k T \chi(P/R))| \leq C_\chi \|T\| \sqrt{\varepsilon}.$$

Letting $\varepsilon \downarrow 0$ gives $\operatorname{Tr}(U^k T\chi(P/R)) = o(R^n)$.

To pass to Π_R, fix $0 < \delta < 1/2$ and choose $\chi_\delta \in C_c^\infty((0,1))$ with $0 \le \chi_\delta \le 1$ and $\chi_\delta = 1$ on $[\delta, 1-\delta]$. For every bounded operator B, $\Pi_R - \chi_\delta(P/R) \ge 0$ and

$$\begin{aligned}|\operatorname{Tr}(B(\Pi_R - \chi_\delta(P/R)))| &\le \|B\| \operatorname{Tr}(\Pi_R - \chi_\delta(P/R)) \\ &\le \|B\|\big(N_P(\delta R) + N_P(R) - N_P((1-\delta)R)\big).\end{aligned}$$

After division by $N_P(R)$, the upper limit is at most $\|B\|(\delta^n + 1 - (1-\delta)^n)$. Also $b_{\chi_\delta} \to 1$ as $\delta \downarrow 0$. Taking $R \to \infty$ first and then $\delta \downarrow 0$ gives

$$\lim_{R\to\infty} \frac{\operatorname{Tr}(U^k \Pi_R)}{N_P(R)} = \begin{cases} \dfrac{1}{r|\Sigma|}\displaystyle\int_\Sigma \operatorname{tr}_E(H^{k/L})\, d\Sigma, & L \mid k, \\ 0, & L \nmid k. \end{cases}$$

The same estimate with $B = U^k T$ gives $\operatorname{Tr}(U^k T \Pi_R) = o(R^n)$ when $L \nmid k$.

The right side of (4.7) has exactly these moments: the sum of the kth powers of the Lth roots of h is zero unless $L \mid k$, and is $Lh^{k/L}$ otherwise. Trigonometric polynomials are uniformly dense in $C(\mathbb{S}^1)$. Since all measures in question have mass one, moment convergence proves weak convergence. □

Corollary 4.6. *Under Theorem 4.1, let $\mathcal{T}$ be the constant set of eigenvalues of H, and let r_h be the multiplicity of $h \in \mathcal{T}$. Then*

$$\nu = \sum_{h\in\mathcal{T}} \frac{r_h}{rL} \sum_{z^L = h} \delta_z, \qquad r_h \in \mathbb{Z}_{>0}, \qquad \sum_h r_h = r.$$

Thus supp ν *is a disjoint union of complete regular L-gons, and every vertex in one polygon has the same mass. For a scalar operator,* supp ν *is one regular L-gon, with mass $1/L$ at each vertex.*

Proof. The multiplicities are constant on connected Σ, so substitute them into (4.7). Different values of h give disjoint root sets. If $r = 1$, only one r_h can be nonzero. □

Remark 4.7 (The principal return measure). Write X_q for the Hamiltonian vector field of q and s for the subprincipal symbol of P in a local unitary frame. The operator $X_q + is$ defines a partial connection on π^*E [48, Proposition 3.1, p. 10, arXiv version]. Denote its parallel transport along $t \mapsto G_{-t}(\rho)$ by $T_t(\rho) : E_{\pi(\rho)} \to E_{\pi(G_{-t}(\rho))}$. In a local unitary frame it satisfies

$$\frac{d}{dt} T_t(\rho) = is(G_{-t}(\rho)) T_t(\rho), \qquad T_0(\rho) = I_E.$$

The leading wave amplitude gives H as the product of the scalar Maslov factor and $T_{2\pi L}$; for the connection-Laplacian case, see [53, Section 2]. For the square root of a connection Laplacian, the bundle factor is parallel transport for the original connection [48, Proposition 3.2, p. 11, arXiv version]. Put

$$\overline{\nu}_H = \frac{1}{r|\Sigma|} \int_\Sigma \sum_{h \in \operatorname{Spec} H(\rho)} \delta_h \, d\Sigma(\rho),$$

where eigenvalues are counted with multiplicity. Then $(z \mapsto z^L)_* \nu = \overline{\nu}_H$.

Let P_1, P_2 be two such operators with finite-order canonical returns, and let $R_\nu \to \infty$. Suppose that $\eta_\nu \ge 0$, $\eta_\nu \to 0$, and $E_{\eta_\nu}(R_\nu; P_1, P_2) = o(R_\nu^n)$. Write $N_i(R) = \operatorname{Tr} \mathbf{1}_{(-\infty,R]}(P_i)$ and let $\nu_{i,R}$ be their normalized phase measures. For a trigonometric polynomial f, put $L_f = \sup_{x\in\mathbb{R}} |\frac{d}{dx} f(e^{2\pi i x})|$. The matching inequality $|N_1(R_\nu) - N_2(R_\nu)| \le E_{\eta_\nu}(R_\nu; P_1, P_2)$ gives

$$\left| \int f \, d\nu_{1,R_\nu} - \int f \, d\nu_{2,R_\nu} \right| \le L_f \eta_\nu + 2\|f\|_\infty \frac{E_{\eta_\nu}(R_\nu; P_1, P_2)}{N_1(R_\nu)} \longrightarrow 0.$$

Hence their limiting phase measures agree. If their return orders are equal to the same L, then $\overline{\nu}_{H_1} = \overline{\nu}_{H_2}$.

5. Spectral resolution and the order of correction

In the proof of Theorem C(a), Zelditch obtains equality modulo finite rank for two commuting integer-spectrum quantizers on a Zoll two-sphere whose difference has order -1 [88, Section 3, equation (3.8), p. 564].

Theorem 5.1 (Resolution and order of the correction). *Let $E \to M$ be a Hermitian bundle over a closed n-manifold, $n \geq 2$, and let $P, A \in \Psi^1_{\mathrm{cl}}(M;E)$ be strongly commuting self-adjoint elliptic operators. Suppose that their common principal symbol is qI_E, with $q > 0$, and that*

$$B = P - A \in \Psi^{-1}_{\mathrm{cl}}(M;E).$$

Let $\Lambda \subset \mathbb{R}$ satisfy $|\lambda - \lambda'| \geq \delta$ for some $\delta > 0$ and all distinct $\lambda, \lambda' \in \Lambda$, and suppose that $\operatorname{Spec} A \subset \Lambda$. *Let $\mathcal{B}$ be a multiset, bounded below and finite on bounded intervals, supported on Λ. Suppose that $R_\nu \to \infty$ and $\eta_\nu \geq 0$ satisfy*

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \mathcal{B}) = o(R_\nu^{\beta}), \qquad s > 0, \quad 0 < \beta \leq n.$$

Then, with $N = \lfloor ns/\beta \rfloor$,

$$P - A \in \Psi^{-N-1}_{\mathrm{cl}}. \tag{5.1}$$

For $N = 0$ this is the initial order assumption. For the same cutoffs and unmatched-count bound, one has $P - A \in \Psi^{-\infty}$ if $\eta_\nu = 0$ for all ν, or if $\eta_\nu = o(R_\nu^{-m})$ for every $m > 0$.

Proof. Strong commutation and compact resolvent give a common orthonormal basis (e_j) of smooth eigensections. Write

$$Pe_j = \mu_j e_j, \qquad Ae_j = a_j e_j, \qquad Be_j = b_j e_j, \qquad b_j = \mu_j - a_j.$$

The operator B is compact and self-adjoint on L^2. Since $BP \in \Psi^0_{\mathrm{cl}}$,

$$|b_j| \leq \frac{\|BP\|}{\mu_j} \qquad (\mu_j > 0).$$

Choose $\mu_0 > \max\{1, 4\|BP\|/\delta\}$. If $\mu_j > \mu_0$, then $|\mu_j - a_j| < \delta/4$. The interval $(\mu_j - \delta/4, \mu_j + \delta/4)$ contains at most one element of Λ, so A is scalar on each such P-eigenspace. For large ν, $\eta_\nu < \delta/4$. If a reference value $\lambda \in \Lambda$ is matched to a copy of $\mu_j > \mu_0$, then

$$|\lambda - a_j| \leq |\lambda - \mu_j| + |\mu_j - a_j| < \delta/2,$$

so $\lambda = a_j$. A copy with $\mu_j > \mu_0$ and $|b_j| > \eta_\nu$ is therefore unmatched.

Suppose $B \notin \Psi^{-N-1}_{\mathrm{cl}}$. The classical expansion then gives an integer $1 \leq k \leq N$ such that $B \in \Psi^{-k}_{\mathrm{cl}}$ and $b_{-k} := \sigma_{-k}(B) \not\equiv 0$. For a matrix or bounded operator T, write $|T| = (T^*T)^{1/2}$. For $t > 0$, put

$$n_B(t) = \operatorname{Tr} \mathbf{1}_{(t,\infty)}(|B|) = \#\{j : |b_j| > t\}.$$

The Birman–Solomyak singular-value asymptotic [67, Theorem 6.1, equation (6.1), p. 748], applied with order $-k$ under the half-density identification of Section 1, gives

$$n_B(t) \sim C_B t^{-n/k} \qquad (t \downarrow 0).$$

For an auxiliary Riemannian metric g, put $S^*_g M = \{(x,\xi) : |\xi|_g = 1\}$ and let $d\Sigma_g$ be its radial Liouville measure. Then

$$C_B = \frac{1}{n(2\pi)^n} \int_{S^*_g M} \operatorname{tr}_E |b_{-k}|^{n/k} \, d\Sigma_g > 0.$$

The integrand is continuous and nonnegative, and is positive on a nonempty open set.

The operator $A^k B \in \Psi^0_{\mathrm{cl}}$ is bounded, so

$$|a_j|^k |b_j| \leq \|A^k B\|.$$

Set $T_\nu = R_\nu^{\beta/n}$. Since $k\beta/n \le s$,

$$L_\eta := \limsup_{\nu\to\infty} T_\nu^k \eta_\nu < \infty.$$

Choose $D > 1 + \max\{2^k\|A^kB\|, L_\eta\}$ and put $\tau_\nu = DT_\nu^{-k}$. For large ν, $\eta_\nu < \tau_\nu$. If $|b_j| > \tau_\nu$, then $|a_j| < T_\nu/2$. Since $\beta \le n$,

$$\mu_j = a_j + b_j \le T_\nu/2 + \|B\| < R_\nu$$

for large ν. Put $m_0 = \operatorname{rank} \mathbf{1}_{(-\infty,\mu_0]}(P) < \infty$. All but at most m_0 copies counted by $n_B(\tau_\nu)$ are therefore unmatched. Hence

$$\begin{aligned} E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \mathcal{B}) &\ge n_B(\tau_\nu) - m_0 \\ &= C_B D^{-n/k} R_\nu^\beta + o(R_\nu^\beta), \end{aligned}$$

contrary to the hypothesis. This proves (5.1). If the errors vanish or are smaller than every inverse power, the conclusion holds for every $s > 0$. Thus $B \in \bigcap_{j\ge1} \Psi^{-j} = \Psi^{-\infty}$. □

Corollary 5.2. *In the setting of Theorem 4.1, assume*

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \mathcal{B}) = o(R_\nu^\beta), \qquad s > 0, \quad 0 < \beta \le n.$$

Then the operator A given by that theorem satisfies

$$P - A \in \Psi_{\mathrm{cl}}^{-\lfloor ns/\beta\rfloor - 1}.$$

Proof. The hypotheses imply $\eta_\nu \to 0$ and $E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \mathcal{B}) = o(R_\nu^n)$. Theorem 4.1 gives a self-adjoint A, strongly commuting with P, with principal symbol qI_E, $P - A \in \Psi_{\mathrm{cl}}^{-1}$, and $\operatorname{Spec} A \subset \Lambda(S)$. The finite union of integer translates $\Lambda(S)$ has a positive lower bound on the distance between distinct points. Theorem 5.1 applies. □

Proposition 5.3. *For every integer $n \ge 2$, every $s > 0$, and every $0 < \beta \le n$, there are operators satisfying Theorem 5.1 for which $P - A$ has exact order $-\lfloor ns/\beta\rfloor - 1$.*

Proof. Let K_0 be the operator (4.6) on the unit round sphere S^n; thus $\operatorname{Spec} K_0 = \{1, 2, \ldots\}$. Fix an integer $r \ge 1$ and put $A = K_0 I_r$ on the trivial rank-r bundle. Let $k = \lfloor ns/\beta\rfloor + 1$ and $0 < c < 1/4$, and put

$$P = A + cA^{-k}.$$

The inverse calculus [6, Example 2, pp. 7–8, and Theorem 5.4, pp. 15–16, arXiv version] gives $A^{-k} \in \Psi_{\mathrm{cl}}^{-k}$. The operators P and A are self-adjoint on H^1 and strongly commute. They have principal symbol $|\xi| I_r$, and

$$\sigma_{-k}(P - A) = c|\xi|^{-k} I_r \ne 0.$$

Compare P with $\operatorname{Spec} A$, supported on $\Lambda = \mathbb{Z}$. Since $0 < ca^{-k} < 1/4$, at $R_\nu = \nu + 1/2$ the truncated spectra consist of $a + ca^{-k}$ and a, respectively, for integers $1 \le a \le \nu$, each with multiplicity $\dim\ker(A - a)$. Put

$$\eta_\nu = \frac{R_\nu^{-s}}{\log R_\nu}, \qquad X_\nu = (c/\eta_\nu)^{1/k}.$$

Since $k > ns/\beta \ge s$,

$$X_\nu = c^{1/k} R_\nu^{s/k} (\log R_\nu)^{1/k} = o(R_\nu).$$

For large ν, $\eta_\nu < 1/4$, so a copy at $a + ca^{-k}$ can match only to the integer a. It matches if and only if $ca^{-k} \le \eta_\nu$, or equivalently $a \ge X_\nu$. Let $N_A(X) = \operatorname{Tr} \mathbf{1}_{(-\infty,X]}(A)$. Since $X_\nu < \nu$ for large ν, it follows that

$$\begin{aligned} E_{\eta_\nu}(R_\nu; \operatorname{Spec} P, \operatorname{Spec} A) &= 2 \sum_{\substack{a\in\mathbb{Z}_{\ge1} \\ a<X_\nu}} \dim\ker(A - a) \\ &\le 2N_A(X_\nu) = O\big(R_\nu^{ns/k} (\log R_\nu)^{n/k}\big) = o(R_\nu^\beta). \end{aligned}$$

Here $N_A(X) = O(X^n)$ by Weyl's law for K_0 [42, Theorem 4.4, equation (4.10), p. 212], and $ns/k < \beta$. The errors satisfy $\eta_\nu = o(R_\nu^{-s})$, whereas $P - A \notin \Psi^{-k-\varepsilon}$ for every $\varepsilon > 0$. □

Remark 5.4 (Strong commutation). The strong commutation hypothesis cannot be omitted, even when the spectra are equal. Let K_0 be the operator (4.6) on the unit round sphere S^n, let $q = |\xi|$, and write X_q for its Hamiltonian vector field. Let $f = x_1|_{S^n}$, also regarded as a function on T^*S^n. Put

$$S = \tfrac{1}{2}(fK_0^{-1} + K_0^{-1}f), \qquad W = e^{iS}, \qquad P_2 = WK_0W^*.$$

Then $S = S^* \in \Psi_{\mathrm{cl}}^{-1}$ has symbol f/q. The entire function

$$h(z) = \frac{e^{iz} - 1 - iz}{z^2}, \qquad h(0) = -\tfrac{1}{2},$$

satisfies $h(S) \in \Psi_{\mathrm{cl}}^0$ by the holomorphic functional calculus [6, Example 2, pp. 7–8; Theorem 5.4, pp. 15–16; proof of Theorem 7.6, p. 22, arXiv version]. Thus

$$W = I + iS + R, \qquad R = S^2h(S) \in \Psi_{\mathrm{cl}}^{-2}.$$

The operator W is unitary, and both W and W^* preserve H^1. Hence P_2 is self-adjoint on H^1, with

$$\operatorname{Spec} P_2 = \operatorname{Spec} K_0, \qquad \sigma_1(P_2) = \sigma_1(K_0) = q.$$

The scalar commutator formula [79, Section I.2, equations (2.9)–(2.11), p. 9] gives $[S, K_0] \in \Psi_{\mathrm{cl}}^{-1}$ and $[R, K_0] \in \Psi_{\mathrm{cl}}^{-2}$. Therefore

$$P_2 - K_0 = [W, K_0]W^* \equiv i[S, K_0] \pmod{\Psi_{\mathrm{cl}}^{-2}},$$
$$\sigma_{-1}(P_2 - K_0) = -\frac{X_q f}{q}.$$

On the round sphere $X_q^2 f = -f$, so $X_q f \not\equiv 0$. Thus $P_2 - K_0 \in \Psi_{\mathrm{cl}}^{-1}$ but $P_2 - K_0 \notin \Psi^{-2}$. Moreover,

$$\sigma_{-1}([K_0, P_2]) = \frac{iX_q^2 f}{q} = -\frac{if}{q} \not\equiv 0.$$

If K_0 and P_2 strongly commuted, the bounded operator $[K_0, P_2]$ would vanish on a common orthonormal basis of smooth eigensections, hence on L^2, contrary to its nonzero principal symbol.

6. Limiting distributions of correction symbols

Let M be a closed smooth manifold of dimension $n \geq 2$, and let $E \to M$ be a Hermitian vector bundle of rank $r \geq 1$. Operators act on E-valued half-densities. Let $A \geq 1$ be a self-adjoint classical elliptic operator of order one, with positive scalar principal symbol qI_E, and suppose that $\operatorname{Spec} A \subset \mathbb{N}$. Let $k > 0$, and let $B = B^* \in \Psi_{\mathrm{cl}}^{-k}(M; E)$ strongly commute with A. Since B is bounded, $P = A + B$ is self-adjoint on H^1, bounded below and has compact resolvent. Put $C = A^kB$. The complex-power theorem gives $A^k \in \Psi_{\mathrm{cl}}^k$ with principal symbol q^kI_E [6, Example 2, pp. 7–8, and Theorem 7.9, p. 23, arXiv version 1]. Since $B : L^2 \to H^k$, the operator C is defined on L^2. It belongs to Ψ_{cl}^0 and is bounded; strong commutation makes it self-adjoint. Write c for its Hermitian principal symbol restricted to

$$\Sigma_q = \{(x, \xi) : q(x, \xi) = 1\},$$

and let $d\Sigma_q$ be the radial Liouville measure of Section 1.

For a Borel subset $J \subset \mathbb{R}$, define

$$\nu_c(J) = \frac{1}{n(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E\big(\mathbf{1}_J(c(\rho))\big)\, d\Sigma_q(\rho). \tag{6.1}$$

The ordered eigenvalues $c_1(\rho) \leq \cdots \leq c_r(\rho)$ are continuous, and the integrand equals $\sum_{\ell=1}^r \mathbf{1}_J(c_\ell(\rho))$. Thus ν_c is a finite measure with compact support. Put $N_A(R) = \operatorname{Tr} \mathbf{1}_{(-\infty,R]}(A)$. Weyl's law [67, Corollary 6.9, equation (6.7), p. 749] gives

$$\nu_c(\mathbb{R}) = \frac{r}{n(2\pi)^n} \int_{\Sigma_q} d\Sigma_q = \lim_{R\to\infty} R^{-n} N_A(R).$$

The matching defect $E_\eta(R; P, A)$ compares $\operatorname{Spec} P$ and $\operatorname{Spec} A$ as in Definition 1.1. Put $R_N = N + \frac{1}{2}$, $N \in \mathbb{N}$.

6.1. **The matching reduction.** Choose a common smooth orthonormal eigenbasis (e_j) for A and B, and write (a_j, b_j) for the corresponding eigenvalues. Then $Pe_j = (a_j + b_j)e_j$ and $Ce_j = a_j^k b_j e_j$. For $R > 0$ and $\eta \geq 0$, put

$$D_\pm(R, \eta) = \#\{j \,:\, a_j \leq R,\ \pm b_j > \eta\}.$$

The bound

$$|b_j| \leq \|C\| a_j^{-k} \tag{6.2}$$

shows that, for all sufficiently large a_j, the nearest integer to $a_j + b_j$ is a_j. Hence the spectral multisets determine $D_\pm(R, \eta)$ up to a bounded additive error, uniformly in R and η.

Lemma 6.1. *Uniformly for $0 \leq \eta < 1/4$ and all sufficiently large N,*

$$E_\eta(R_N; P, A) = 2D_+(R_N, \eta) + 2D_-(R_N, \eta) + O(1).$$

Proof. Choose an integer $a_0 \geq 1$ such that $\|C\| a_0^{-k} < 1/4$, and an integer $L > a_0 + \|B\| + 1$. Let

$$\mathcal{A}_L = \{a_j \,:\, a_j \leq L\}, \qquad \mathcal{P}_L = \{a_j + b_j \,:\, a_j \leq L\}$$

be multisets. If $a_j < a_0$, then $a_j + b_j < a_0 + \|B\| < L - 1$. If $a_0 \leq a_j \leq L$, then $|b_j| < 1/4$, so $a_j + b_j < L + 1/4$. If $a_j > L$, then $a_j + b_j > L + 3/4$ and its distance to every integer other than a_j is greater than $3/4$. Thus no admissible pair joins either finite multiset to a copy with A-eigenvalue greater than L.

For $N \geq L$, both finite multisets lie below R_N, and every remaining cluster is wholly included or wholly excluded on both sides. At an included integer level, exactly the copies with $|b_j| \leq \eta$ can be matched; the two lists have the same multiplicity at that level. Consequently

$$E_\eta(R_N; P, A) = E_\eta(\mathcal{P}_L, \mathcal{A}_L) + 2\#\{j \,:\, L < a_j \leq R_N,\ |b_j| > \eta\}. \tag{6.3}$$

Both $E_\eta(\mathcal{P}_L, \mathcal{A}_L)$ and $2\#\{j \,:\, a_j \leq L,\ |b_j| > \eta\}$ belong to $[0, 2N_A(L)]$. Their difference therefore has absolute value at most $2N_A(L)$. □

6.2. **A joint trace formula.** For scalar operators, the cumulative moment formula is the weak Szegő law [33, §2, (2.1), p. 510]. For a positive self-adjoint scalar elliptic operator of order one with vanishing subprincipal symbol and 2π-periodic bicharacteristics, the cluster distributions have the expansions of [15, Theorem 3.1, pp. 518–519].

Lemma 6.2. *Let $F(s, z)$ be continuous, with compact support in $(0, \infty) \times \mathbb{R}$. Define $F(s, c(\rho))$ by the spectral calculus for Hermitian matrices. Then*

$$\lim_{R\to\infty} R^{-n} \sum_j F(a_j/R, a_j^k b_j) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \int_0^\infty s^{n-1} \operatorname{tr}_E F(s, c(\rho))\, ds\, d\Sigma_q(\rho). \tag{6.4}$$

Proof. Choose a compact interval K containing $\operatorname{Spec} C$ and all eigenvalues of c. For $\chi \in C_c^\infty((0, \infty))$ and a polynomial p, the operator $p(C)$ belongs to Ψ^0_{cl} and has principal symbol $p(c)$. Strong commutation and (1.1) give

$$\lim_{R\to\infty} R^{-n} \sum_j \chi(a_j/R) p(a_j^k b_j) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \int_0^\infty s^{n-1} \chi(s) \operatorname{tr}_E p(c(\rho))\, ds\, d\Sigma_q.$$

Choose $0 < a < b$ and $\zeta \in C_c^\infty((a, b))$, with $0 \leq \zeta \leq 1$, equal to one on the radial support of F. The Stone–Weierstrass theorem on $[a, b] \times K$, followed by multiplication by ζ, gives uniform approximation on $(0, \infty) \times K$ by finite sums $\chi(s)p(z)$, all with radial support in $[a, b]$. An approximation error ε changes the normalized sum by at most $\varepsilon R^{-n} N_A(bR) = O(\varepsilon)$; its integral has the same bound. This proves (6.4). □

6.3. **The complete critical profile.** Put $\alpha = n/k$ and $x_+ = \max(x, 0)$ for $x \in \mathbb{R}$.

Theorem 6.3 (The limiting spectral profile). *For every $t > 0$,*

$$\lim_{N\to\infty} R_N^{-n} D_\pm(R_N, tR_N^{-k}) = F_\pm(t) := \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^\alpha\}\, d\nu_c(x), \tag{6.5}$$

$$\lim_{N\to\infty} R_N^{-n} E_{tR_N^{-k}}(R_N; P, A) = F(t) := 2\int_{\mathbb{R}} \min\{1, (|x|/t)^\alpha\}\, d\nu_c(x). \tag{6.6}$$

The convergence is uniform for t in compact subintervals of $(0,\infty)$. The formulas hold at every $t > 0$, including atoms of ν_c.

Proof. Fix $t > 0$ and $0 < \varepsilon < 1$, and choose $T > \max\{\|C\|, \sup_{\Sigma_q} \|c\|\}$. For $\varepsilon R \le a_j \le R$, the inequality $\pm b_j > tR^{-k}$ is equivalent to

$$\pm(a_j^k b_j) > t(a_j/R)^k.$$

The limiting measure in Lemma 6.2 is $ns^{n-1}\, ds\, d\nu_c(z)$. The boundary of

$$\{(s, z) : \varepsilon \le s \le 1,\ |z| \le T,\ \pm z > ts^k\}$$

has zero measure: the faces $s = \varepsilon$ and $s = 1$ have zero radial measure, and $z = \pm T$ lie outside the support. For each fixed z, the equation $\pm z = ts^k$ has at most one positive solution s. Fubini's theorem gives zero measure for this remaining boundary. The joint limit therefore applies to the indicator of this set.

Weyl's law gives

$$\limsup_{R\to\infty} R^{-n}\#\{j : a_j < \varepsilon R\} \le \varepsilon^n \nu_c(\mathbb{R}).$$

Letting $\varepsilon \downarrow 0$ and integrating in s, we obtain

$$\int_0^1 s^{n-1} \mathbf{1}_{\{\pm x > ts^k\}}\, ds = \frac{1}{n} \min\{1, ((\pm x)_+/t)^\alpha\}.$$

This proves the signed limits. Lemma 6.1 gives the unsigned limit.

Dominated convergence makes $F_\pm$ and F continuous on $(0,\infty)$. They and the corresponding normalized counting functions are nonincreasing. For a partition $a = t_0 < \cdots < t_m = b$, write G_N for one of those counting functions and G for its limit. Then

$$\sup_{[a,b]} |G_N - G| \le \max_{0\le i\le m} |G_N(t_i) - G(t_i)| + \max_{0\le i<m} \big(G(t_i) - G(t_{i+1})\big).$$

Let $N \to \infty$ and then let the mesh tend to zero. □

Corollary 6.4. *The signed profiles and the total mass $\nu_c(\mathbb{R})$ determine ν_c. The right derivatives of $F_\pm$ and F, denoted by $F'_{\pm,+}$ and F'_+, exist at every $t > 0$, and*

$$\nu_c(\{x : \pm x > t\}) = F_\pm(t) + \frac{t}{\alpha} F'_{\pm,+}(t). \tag{6.7}$$

If $\nu_{|c|}$ is the pushforward of ν_c under $x \mapsto |x|$, then

$$\nu_{|c|}((t,\infty)) = \frac{1}{2}\left(F(t) + \frac{t}{\alpha} F'_+(t)\right), \qquad \nu_c(\{0\}) = \nu_c(\mathbb{R}) - \frac{1}{2}\lim_{t\downarrow 0} F(t).$$

Proof. Fix one of the two signs, and let μ be the pushforward of $\nu_c|_{\{\pm x>0\}}$ under $x \mapsto \pm x$. Put $m = \mu((0,\infty))$. If $m = 0$, the corresponding identity is immediate. Suppose $m > 0$, and let λ be the pushforward of $m^{-1}\mu$ under $y \mapsto y^{-\alpha}$. Its Williamson 2-transform is

$$\psi(v) = \int_{(0,\infty)} (1 - v/u)_+\, d\lambda(u) = 1 - m^{-1} F_\pm(v^{-1/\alpha}), \qquad v > 0.$$

The function ψ is convex, so its one-sided derivatives exist at every $v > 0$. Write ψ'_- for its left derivative. The Williamson inversion formula [60, Definition 3.2 and Proposition 3.1(ii), p. 10; proof in Appendix B, p. 34, arXiv version 1] and its left limit give

$$\lambda((0,v)) = 1 - \psi(v) + v\psi'_-(v).$$

With $v = t^{-\alpha}$,

$$F'_{\pm,+}(t) = m\alpha t^{-\alpha-1}\psi'_-(v), \qquad \mu((t,\infty)) = m\lambda((0,v)).$$

Substitution gives (6.7). Since $F = 2(F_+ + F_-)$, addition gives the formula for $\nu_{|c|}$. Monotone convergence as $t \downarrow 0$ gives the atom at zero. The two strict tails determine the restrictions of ν_c to the positive and negative half-lines; the last identity determines its remaining mass. □

The profiles depend only on the integrated eigenvalue distribution ν_c.

6.4. **Coarser shrinking scales and a noncommutative integral.**

Theorem 6.5. *Let $\eta_N \downarrow 0$ satisfy $\eta_N R_N^k \to \infty$. Then*

$$\lim_{N\to\infty} \eta_N^\alpha D_\pm(R_N, \eta_N) = \int ((\pm x)_+)^\alpha \, d\nu_c(x), \tag{6.8}$$

$$\lim_{N\to\infty} \eta_N^\alpha E_{\eta_N}(R_N; P, A) = 2C_B, \qquad C_B := \int |x|^\alpha \, d\nu_c(x). \tag{6.9}$$

If $c \not\equiv 0$, then $C_B > 0$ and $E_{\eta_N}(R_N; P, A) \sim 2C_B\eta_N^{-\alpha}$.

Proof. For $\eta > 0$, put

$$n_\pm(\eta) = \operatorname{Tr} \mathbf{1}_{(\eta,\infty)}(\pm B), \qquad \kappa_\pm = \int ((\pm x)_+)^\alpha \, d\nu_c(x).$$

The signed Birman–Solomyak law gives

$$n_\pm(\eta) = \kappa_\pm \eta^{-\alpha} + o(\eta^{-\alpha}) \qquad (\eta \downarrow 0) \tag{6.10}$$

[67, Theorem 6.1, (6.2), p. 748, and Remark 5.4, p. 744]. To identify the coefficient, let b_{-k} be the principal symbol of B, and choose a Riemannian cosphere S_g^*M with radial Liouville measure $d\Sigma_g$. The map

$$\vartheta : S_g^*M \longrightarrow \Sigma_q, \qquad \vartheta(x,\xi) = (x, \xi/q(x,\xi)),$$

satisfies

$$\vartheta^* d\Sigma_q = q^{-n} d\Sigma_g, \qquad c \circ \vartheta = q^k b_{-k}.$$

Since $k\alpha = n$, the coefficient in the cited theorem is $\kappa_\pm$. This law permits vanishing eigenvalues of b_{-k} and a zero coefficient $\kappa_\pm$.

By (6.2),

$$|b_j| > \eta_N \quad \Longrightarrow \quad a_j < \left(\|C\|/\eta_N\right)^{1/k} = o(R_N).$$

Thus $D_\pm(R_N, \eta_N) = n_\pm(\eta_N)$ for all sufficiently large N. Equation (6.10) and Lemma 6.1 give the limits, since $\kappa_+ + \kappa_- = C_B$ and $\eta_N^\alpha \to 0$. If $c \not\equiv 0$, the continuous function $\operatorname{tr}_E |c|^\alpha$ is positive on a nonempty open subset of Σ_q, so $C_B > 0$. □

Corollary 6.6. *Let $s_1(B) \geq s_2(B) \geq \cdots$ be the singular values of B, repeated according to multiplicity, and let (η_N) satisfy the hypotheses of Theorem 6.5. Then*

$$\lim_{J\to\infty} \frac{1}{\log J} \sum_{j=1}^{J} s_j(B)^\alpha = C_B = \frac{1}{2} \lim_{N\to\infty} \eta_N^\alpha E_{\eta_N}(R_N; P, A). \tag{6.11}$$

The operator $|B|^\alpha$ is measurable for Dixmier traces; every normalized Dixmier trace of it equals C_B. Let Res_W denote the Wodzicki residue, with normalization $(2\pi)^{-n}$ as in [67, §4.2, p. 741]. *If B is elliptic, then*

$$|B|^\alpha \in \Psi_{\mathrm{cl}}^{-n}, \qquad C_B = \frac{1}{n} \operatorname{Res}_W(|B|^\alpha).$$

Proof. The singular-value law [67, Theorem 6.1, (6.1), p. 748] gives

$$\lim_{j\to\infty} j\, s_j(B)^\alpha = \mathcal{C}_B.$$

In particular, $s_j(|B|^\alpha) = O(j^{-1})$, so $|B|^\alpha$ belongs to the weak trace class. Its logarithmic mean and its normalized Dixmier traces equal $\mathcal{C}_B$ by [67, Theorem 2.18, p. 733, and Proposition 5.12, pp. 746–747]. The remaining equality in (6.11) follows from Theorem 6.5.

Suppose B is elliptic. Let Π_0 be the orthogonal projection onto $\ker B$ and put $Q = B^2 + \Pi_0$. Elliptic regularity makes Π_0 smoothing and of finite rank. The elliptic parametrix and regularity theorem [79, I.4, (4.6) and (4.9), p. 15, and I.8, pp. 24–25] give an isomorphism $Q : L^2 \to H^{2k}$ and an inverse $D = Q^{-1} \in \Psi^{2k}_{\mathrm{cl}}$ with domain H^{2k}. Indeed, $Q = Q^*$ and $\langle Qu, u\rangle = \|Bu\|^2 + \|\Pi_0 u\|^2$ show that its kernel and, by elliptic regularity, its distributional cokernel are zero. The operator D is self-adjoint and satisfies $D \geq \|Q\|^{-1} I$. Its principal symbol is $(b^2_{-k})^{-1}$. The complex-power theorem [6, Example 2, pp. 7–8, and Theorem 7.9, p. 23, arXiv version 1] gives

$$|B|^\alpha = D^{-\alpha/2} - \Pi_0 \in \Psi^{-n}_{\mathrm{cl}}, \qquad \sigma_{-n}(|B|^\alpha) = |b_{-k}|^\alpha.$$

The defining residue integral, with the same change of radial coordinate as above, is

$$\operatorname{Res}_W(|B|^\alpha) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E |c|^\alpha \, d\Sigma_q = n\mathcal{C}_B.$$

□

6.5. **Stability and the limits of reconstruction.**

Proposition 6.7. *Let $\mathcal{T}$ be a fixed multiset supported on $\mathbb{Z}$, bounded below and finite on bounded intervals. For every $\eta \geq 0$ and every cutoff R_N,*

$$\left|E_\eta(R_N; P, \mathcal{T}) - E_\eta(R_N; P, A)\right| \leq E_0(R_N; A, \mathcal{T}).$$

The limit in (6.6) *is unchanged if $E_0(R_N; A, \mathcal{T}) = o(R_N^n)$. The limit in* (6.9) *is unchanged if $E_0(R_N; A, \mathcal{T}) = o(\eta_N^{-\alpha})$.*

Proof. At each integer, delete or insert the excess copies in the truncated list for A. This produces the truncated list for $\mathcal{T}$ after exactly $E_0(R_N; A, \mathcal{T})$ changes. Lemma 1.2 bounds the change of the defect at each step by one. This proves the estimate. Dividing it by R_N^n, or multiplying it by η_N^α, proves the two assertions about limits. □

For $\gamma > 0$, let $\mathcal{S}^\gamma$ denote the class of compact operators T for which $\sum_{j\geq 1} s_j(T)^\gamma < \infty$, where $s_j(T)$ are the singular values, in nonincreasing order and with multiplicity.

Proposition 6.8. *Suppose $c \not\equiv 0$. For each fixed $0 < \eta < 1/4$, the limit*

$$E_\infty(\eta) := \lim_{N\to\infty} E_\eta(R_N; P, A)$$

exists, and

$$E_\infty(\eta) \sim 2\mathcal{C}_B \eta^{-n/k} \quad (\eta \downarrow 0), \qquad \lim_{\eta\downarrow 0} \frac{\log E_\infty(\eta)}{\log(1/\eta)} = \frac{n}{k}.$$

For every $\gamma > 0$, one has $B \in \mathcal{S}^\gamma$ if and only if $\gamma > n/k$.

Proof. Fix $0 < \eta < 1/4$. Equation (6.2) gives $D_\pm(R_N, \eta) = n_\pm(\eta)$ for all sufficiently large N. In (6.3), the finite multisets are independent of N, so $E_\eta(R_N; P, A)$ is eventually constant. The uniform bound in Lemma 6.1 gives

$$E_\infty(\eta) = 2n_+(\eta) + 2n_-(\eta) + O(1), \qquad 0 < \eta < 1/4.$$

Equation (6.10) and $\mathcal{C}_B > 0$ give the asymptotic equivalence and its logarithmic consequence. Finally, the singular-value asymptotic gives $s_j(B) \asymp j^{-k/n}$. Hence $\sum_j s_j(B)^\gamma < \infty$ exactly when $k\gamma/n > 1$. □

Remark 6.9 (Lower-order perturbations). Let $B' = B'^{*} \in \Psi_{\mathrm{cl}}^{-k}$ strongly commute with A, and suppose $B' - B \in \Psi^{-k-\varepsilon}$ for some $\varepsilon > 0$. Since $A^k(B' - B) \in \Psi^{-\varepsilon}$, the principal symbol of $A^k B'$ is c. Theorem 6.3 gives the profiles $F_{\pm}$ and F for the pair $(A + B', A)$. For $t > 0$ with $t \geq \sup_{\Sigma_q} \|c\|$,

$$F(t) = 2\mathcal{C}_B t^{-\alpha}.$$

Consequently $F \equiv 0$ if and only if $c \equiv 0$.

6.6. **An exact two-component example.** On the unit round S^5, let K_0 be the operator (4.6). On the trivial rank-two bundle put

$$A = K_0 I_2, \qquad B = \begin{pmatrix} 1/8 & 0 \\ 0 & -1/16 \end{pmatrix} A^{-2}, \qquad P = A + B.$$

The levels of K_0 are $a = 1, 2, \ldots$. With the convention $\binom{b}{5} = 0$ for integers $b < 5$, equation (8.1) gives

$$Q_N := \sum_{a=1}^{N} \dim \ker(K_0 - a) = \binom{N+4}{5} + \binom{N+3}{5} = \frac{N^5}{60} + O(N^4).$$

Let δ_x denote unit point mass at x. Since $C = A^2 B$ is the displayed constant matrix,

$$\nu_c = \frac{1}{60}\big(\delta_{1/8} + \delta_{-1/16}\big).$$

Both spectral lists have $2Q_N$ copies below $R_N = N + 1/2$. For every $t > 0$, Theorem 6.3 gives

$$\lim_{N\to\infty} \frac{E_{tR_N^{-2}}(R_N; P, A)}{4Q_N} = \frac{1}{2}\left[\min\{1, (1/(8t))^{5/2}\} + \min\{1, (1/(16t))^{5/2}\}\right].$$

This function is differentiable except at $t = 1/16$ and $t = 1/8$. The signed profiles are

$$F_+(t) = \frac{1}{60}\min\{1, (1/(8t))^{5/2}\}, \qquad F_-(t) = \frac{1}{60}\min\{1, (1/(16t))^{5/2}\}.$$

7. Joint distributions of return phases and correction symbols

Retain the notation of Theorem 4.1, and put $\Sigma_q = \{q = 1\}$. Let L be the order of the time-2π canonical map and let A be the strongly commuting quantizer satisfying (4.5). Increase each eigenvalue below one by an integer on its joint eigenspace. This changes A by a finite-rank smoothing operator, preserves $e^{2\pi i A}$, and gives $A \geq 1$. Put

$$U_A = e^{2\pi i A}, \qquad V_A = U_A^L, \qquad H = \sigma_0(V_A)|_{\Sigma_q}, \qquad \mathcal{H} = \bigcup_{\rho\in\Sigma_q} \operatorname{Spec} H(\rho).$$

The set $\mathcal{H}$ is finite, and

$$V_A = \sum_{h\in\mathcal{H}} h\Pi_h, \qquad \Pi_h \in \Psi_{\mathrm{cl}}^0, \qquad \pi_h = \sigma_0(\Pi_h)|_{\Sigma_q} = \mathbf{1}_{\{h\}}(H).$$

These orthogonal projections commute with P and A. The smooth projection π_h has constant rank $r_h \geq 1$ on connected Σ_q. Every Lth root of each $h \in \mathcal{H}$ belongs to $\mathcal{S}$.

Suppose that $B = P - A \in \Psi_{\mathrm{cl}}^{-k}$, where $k > 0$, and put

$$C = A^k B \in \Psi_{\mathrm{cl}}^0, \qquad c = \sigma_0(C)|_{\Sigma_q}, \qquad \alpha = n/k.$$

The complex-power calculus of Section 6 applies to $A \geq 1$; the composition $A^k B$ is defined on L^2 because $B : L^2 \to H^k$. The order-zero operator C is bounded. Strong commutation gives $C = C^*$, $[\Pi_h, C] = 0$, and $[\pi_h, c] = 0$. For Borel sets $J \subset \mathbb{R}$, define

$$\nu_h(J) = \frac{1}{n(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E\big(\pi_h(\rho)\mathbf{1}_J(c(\rho))\big)\, d\Sigma_q(\rho). \tag{7.1}$$

The product in the integrand is an orthogonal projection. Thus ν_h is a finite positive measure, with compact support since c is continuous on Σ_q, and

$$\nu_h(\mathbb{R}) = \frac{r_h}{n(2\pi)^n} \int_{\Sigma_q} d\Sigma_q.$$

In a common orthonormal eigenbasis, write $Ae_j = a_j e_j$, $Be_j = b_j e_j$, and $\mu_j = a_j + b_j$. Then

$$|b_j| \le \|C\| a_j^{-k}.$$

Write $N_A(R) = \operatorname{Tr} \mathbf{1}_{(-\infty,R]}(A)$. The Weyl law recalled in Section 6 gives $N_A(R) = O(R^n)$. For $s \in \mathcal{S}$, let $\mathcal{A}_s$ and $\mathcal{P}_s$ be the multisets of a_j and μ_j, respectively, over the indices with $e^{2\pi i a_j} = s$. Define

$$D_{s,\pm}(R,\eta) = \#\{j \,:\, a_j \le R,\ e^{2\pi i a_j} = s,\ \pm b_j > \eta\},$$
$$E_{s,\eta}(R) = E_\eta(R; \mathcal{P}_s, \mathcal{A}_s).$$

Put

$$\delta = \inf\{|a - a'| \,:\, a, a' \in \Lambda(\mathcal{S}),\ a \ne a'\} > 0.$$

Choose $\theta \in (0,1)$ with $\delta_\theta = \operatorname{dist}(\theta, \Lambda(\mathcal{S})) > 0$ and put $R_N = N + \theta$, $N \in \mathbb{Z}_{\ge 1}$. Each cutoff has the same distance δ_θ from $\Lambda(\mathcal{S})$. For sufficiently large a_j, one has $|b_j| < \delta/4$, so a_j is the unique nearest point of $\Lambda(\mathcal{S})$ to μ_j. Moreover, $a_j \ge \mu_j - \|B\|$. Consequently the spectral multisets of A and P determine $\mathcal{A}_s$, $\mathcal{P}_s$, and the signed displacements up to finitely many copies.

Theorem 7.1. *For $h \in \mathcal{H}$, $s^L = h$, and every $t > 0$,*

$$R_N^{-n} D_{s,\pm}(R_N, t R_N^{-k}) \longrightarrow \frac{1}{L} \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^\alpha\}\, d\nu_h(x), \tag{7.2}$$

$$R_N^{-n} E_{s, t R_N^{-k}}(R_N) \longrightarrow \frac{2}{L} \int_{\mathbb{R}} \min\{1, (|x|/t)^\alpha\}\, d\nu_h(x). \tag{7.3}$$

The phase Weyl mass exists and equals

$$w_s := \lim_{R\to\infty} R^{-n} \#\{j \,:\, a_j \le R,\ e^{2\pi i a_j} = s\} = \frac{1}{L}\nu_h(\mathbb{R}). \tag{7.4}$$

The convergences in (7.2) *and* (7.3) *are uniform for t in compact subintervals of $(0,\infty)$. These limits depend on s only through $h = s^L$. Denote the limits in* (7.2) *by $F_{s,\pm}(t)$. For the given L, the functions $F_{s,\pm}$ and the masses w_s determine every measure ν_h.*

Proof. The orthogonal projector onto the s phase of A is

$$E_s = \frac{1}{L}\Pi_h \sum_{\ell=0}^{L-1} s^{-\ell} U_A^\ell. \tag{7.5}$$

On the h block, $s^{-1}U_A$ has spectrum among the Lth roots of unity. Their character sum is L at 1 and zero elsewhere. The factor Π_h annihilates the other blocks.

Let p be a polynomial with complex coefficients and let $\chi \in C_c^\infty((0,\infty))$. The operator $\Pi_h p(C)$ belongs to Ψ^0_{cl} and has principal symbol $\pi_h p(c)$ on Σ_q. The operator A has the same scalar principal symbol as P. For $1 \le \ell < L$, the map Φ^ℓ is nonidentity and has finite order; its fixed set has zero Liouville measure. The trace estimate in the proof of Theorem 4.5, applied to A with the insertion $\Pi_h p(C)$, gives, for nonnegative χ,

$$\operatorname{Tr}\big(U_A^\ell \Pi_h p(C) \chi(A/R)\big) = o(R^n).$$

Every smooth compactly supported χ is a complex linear combination of nonnegative such functions, so the estimate holds for arbitrary χ. The $\ell = 0$ term is given by (1.1). Therefore

$$R^{-n} \operatorname{Tr}(E_s p(C) \chi(A/R)) \longrightarrow \frac{1}{L(2\pi)^n} \int_{\Sigma_q} \int_0^\infty u^{n-1} \chi(u) \operatorname{tr}_E(\pi_h p(c))\, du\, d\Sigma_q.$$

The approximation argument of Lemma 6.2 now gives, for $F \in C_c((0,\infty)\times\mathbb{R})$,

$$R^{-n}\sum_{e^{2\pi i a_j}=s} F(a_j/R, a_j^k b_j) \longrightarrow \frac{n}{L}\int_{\mathbb{R}}\int_0^\infty u^{n-1}F(u,x)\,du\,d\nu_h(x).$$

On a fixed radial interval $[K^{-1},K]$, $K>1$, a uniform approximation error $\varepsilon>0$ changes the normalized sum by at most $\varepsilon R^{-n}N_A(KR) = O(\varepsilon)$. The threshold argument of Theorem 6.3 therefore applies with ν_h/L in place of ν_c: the boundary $\pm x = tu^k$ has zero measure by radial integration, and $a_j < \varepsilon R$ contributes $O(\varepsilon^n)$ after normalization. This proves (7.2). Approximating $\mathbf{1}_{[\varepsilon,1]}$ by continuous radial cutoffs in the same joint limit, and then letting $\varepsilon\downarrow 0$, gives (7.4).

Choose $a_0\ge 1$ so that $\|C\|a_0^{-k} < \min(\delta_\theta/2, \delta/4)$, and put $m_0 = N_A(a_0)$. Delete from both truncated phase lists the copies whose joint index satisfies $a_j \le a_0$. By Lemma 1.2, this changes the matching defect by at most $2m_0$. For the remaining indices, $a_j \le R_N$ if and only if $\mu_j \le R_N$, and for $a\in\Lambda(S)\backslash\{a_j\}$,

$$|\mu_j - a| \ge \delta - |b_j| > 3\delta/4.$$

For $0\le\eta<\delta/4$, admissible pairs therefore lie within one lattice level. If that level has multiplicity d_a and g_a of its displacements have absolute value at most η, its matching defect is $2(d_a-g_a)$. Deleting the low indices changes $D_{s,+}(R_N,\eta)+D_{s,-}(R_N,\eta)$ by at most m_0. Consequently

$$\left|E_{s,\eta}(R_N) - 2D_{s,+}(R_N,\eta) - 2D_{s,-}(R_N,\eta)\right| \le 4m_0, \qquad 0\le\eta<\delta/4, \tag{7.6}$$

uniformly in N. This proves (7.3). The locally uniform convergence follows from monotonicity and continuity, as in Theorem 6.3.

Corollary 6.4, applied to ν_h/L, shows that the right derivatives $F'_{s,\pm,+}(t)$ exist for $t>0$ and satisfy

$$\nu_h(\{x : \pm x > t\}) = L\left(F_{s,\pm}(t) + \frac{t}{\alpha}F'_{s,\pm,+}(t)\right).$$

Moreover,

$$\nu_h(\{0\}) = Lw_s - L\lim_{t\downarrow 0}\big(F_{s,+}(t)+F_{s,-}(t)\big).$$

The open tails and this atom determine ν_h. □

Simultaneous fiberwise unitary conjugacy of (H,c) leaves every measure ν_h unchanged. Let ν be the probability measure of Theorem 4.5. It is invariant under multiplication by every Lth root of unity, and it may be invariant under other rotations. Take $L=1$ and the two phases $1,-1$, each with multiplicity one, in Proposition 4.4. Then $\Phi = I$ and Corollary 4.6 gives

$$\nu = \tfrac{1}{2}(\delta_1+\delta_{-1}),$$

which is invariant under $z\mapsto -z$.

Part 2. Geodesic return and spherical rigidity

Under the hypotheses of Theorem 2.2 or Theorem 2.4, fixed frequency windows yield a common geodesic period and a commuting operator whose spectrum lies on the reference lattice. The discrepancy from the reference multiplicities bounds the localized wave trace differences. At times L/s, where L is the prime geodesic period on the universal cover and $s\ge 2$ is an integer, the return components of maximal dimension have Morse index zero and a common leading trace phase. The resulting restrictions on the return families and their evaluation maps, together with the contact-volume identity, yield a round universal cover. For a homogeneous reference, the harmonic multiplicities further determine the quotient up to isometry.

8. Harmonic multiplicities and localized spectral traces

For a metric on a d-dimensional manifold, the replacement $g \mapsto r^{-2}g$ multiplies scalar eigenvalues by r^2 and volume by r^{-d}. Applied to both comparison metrics, it preserves the frequencies in (2.3). Rescale to $r = 1$. For a closed Riemannian manifold Z, put

$$P_Z = (\Delta_Z + m^2)^{1/2}, \qquad n = 2m + 1 \geq 5.$$

No equality between $\dim M$ and n is assumed.

Let $\mathcal{P}_k$ be the complex homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$, and let $\mathcal{H}_k$ be their harmonic subspace. Put $\mathcal{P}_k = 0$ for $k < 0$, and set $\binom{b}{n} = 0$ for integers $b < n$. A finite free orthogonal group is a finite subgroup of $O(n+1)$ whose nonidentity elements have no fixed point on $\mathbb{S}^n(1)$.

Lemma 8.1. *For $k \geq 0$, restriction identifies $\mathcal{H}_k$ with the scalar eigenspace on $\mathbb{S}^n(1)$ of eigenvalue $k(k+n-1)$, and*

$$d_k := \dim \mathcal{H}_k = \binom{k+n}{n} - \binom{k+n-2}{n}. \tag{8.1}$$

For a finite free orthogonal group Γ, let $\mathcal{H}_k^\Gamma$ be the subspace of Γ-invariant harmonics and put $r_k = \dim \mathcal{H}_k^\Gamma$. These are the scalar multiplicities on $\mathbb{S}^n(1)/\Gamma$; the corresponding shifted frequencies are $k+m$. For $|z| < 1$,

$$F_\Gamma(z) := \sum_{k\geq 0} r_k z^k = \frac{1-z^2}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{1}{\det(I - z\gamma)}. \tag{8.2}$$

Proof. The restriction and quotient identifications are [45, Lemma 1.3 and Proposition 1.5, pp. 78–79]. The decomposition

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}$$

is [45, (1.1), p. 78]; taking dimensions gives (8.1). Since $n = 2m+1$,

$$\sqrt{k(k+n-1)+m^2} = k + m.$$

The group Γ lies in $\mathrm{SO}(2m+2)$ by [45, Lemma 1.1(ii), pp. 77–78]. Thus [45, Theorem 2.2, p. 80] gives (8.2). For lens spaces the latter formula is also [46, Theorem 3.2, p. 452]. □

For an element γ of a finite group, $\mathrm{ord}(\gamma)$ denotes its order.

Lemma 8.2. *Let $\Gamma \subset O(2m+2)$ be finite and free on the unit sphere. Then $\Gamma \subset \mathrm{SO}(2m+2)$. Every eigenvalue of $\gamma \in \Gamma$ has multiplicative order exactly $\mathrm{ord}(\gamma)$. Every involution in Γ equals $-I$. Each element other than $I, -I$ has only nonreal eigenvalues, each of complex multiplicity at most $m+1$.*

Proof. The inclusion in $\mathrm{SO}(2m+2)$ is [45, Lemma 1.1(ii), pp. 77–78]. The eigenvalues of I have order one. Let $\gamma \neq I$ and $b = \mathrm{ord}(\gamma)$. The subgroup $\langle\gamma\rangle$ acts freely on $\mathbb{S}^{2m+1}(1)$. By [45, §1, Example (ii), p. 77], γ is orthogonally conjugate to a direct sum of rotations through angles $2\pi p_j/b$, with $\gcd(p_j, b) = 1$ for every j. Its eigenvalues are therefore primitive b-th roots of unity. For $b = 2$, every rotation is $-I_2$, so $\gamma = -I$. For $b > 2$, the eigenvalues are nonreal. Each eigenvalue and its conjugate have the same complex multiplicity, which is at most $(2m+2)/2 = m+1$. □

Lemma 8.3. *Let $\Gamma_0 \subset O(n+1)$ be finite and free on $\mathbb{S}^n(1)$, put $q = |\Gamma_0|$, and set $r_k = \dim \mathcal{H}_k^{\Gamma_0}$. If q is even, then*

$$r_{2k} = \frac{2}{q} d_{2k} + O(k^m), \qquad r_{2k+1} = 0. \tag{8.3}$$

If q is odd, then

$$r_k = \frac{1}{q} d_k + O(k^m). \tag{8.4}$$

Put $a = 2$ when q is even and $a = 1$ when q is odd. Every sufficiently large level $m + ak$ has positive multiplicity. For each fixed integer $b \geq 1$ and $v \in \{0, \dots, b-1\}$,

$$\sum_{\substack{k\geq 0,\ m+ak\leq R\\ k\equiv v \pmod b}} r_{ak} = \frac{2}{qb\ n!}R^n + O(R^{n-1}) \qquad (R \to \infty).$$

Proof. If q is even, Cauchy's theorem gives an involution in Γ_0, which is $-I$ by Lemma 8.2. Its action on $\mathcal{H}_k$ is $(-1)^k I$, so $r_{2k+1} = 0$. The terms $I, -I$ in (8.2) contribute

$$\frac{1 + (-1)^k}{q} d_k.$$

For every other group element, the poles have order at most $m + 1$. Writing $[z^k]F$ for the coefficient of z^k in a power series F, one has, for $j \geq 1$ and $|\alpha| = 1$,

$$[z^k](1 - z/\alpha)^{-j} = \binom{k+j-1}{j-1}\alpha^{-k} = O(k^{j-1}) \qquad (k \to \infty).$$

Finite partial fractions therefore give a total contribution $O(k^m)$. This proves (8.3). If q is odd, the identity is the only element with real eigenvalues. Its term contributes $q^{-1}d_k$, and the remaining coefficients are $O(k^m)$. Thus (8.4) holds.

Equation (8.1) gives

$$d_k = \frac{2}{(n-1)!}k^{n-1} + O(k^{n-2}).$$

Since $m \leq n - 2$, the two parity cases give

$$r_{ak} = \frac{2a^n}{q(n-1)!}k^{n-1} + O(k^{n-2}).$$

The leading coefficient is positive. For fixed b, v and $X \to \infty$,

$$\sum_{\substack{0\leq k\leq X\\ k\equiv v \pmod b}} k^{n-1} = \frac{X^n}{bn} + O(X^{n-1}).$$

Summing the error term contributes $O(X^{n-1})$. With $X = (R - m)/a$, the main term is $2(R - m)^n/(qb\ n!) = 2R^n/(qb\ n!) + O(R^{n-1})$. This proves the last formula. □

Lemma 8.4. *Let $F(z) = \sum_{k\geq 0} a_k z^k$ be rational with all finite poles at roots of unity, and let $t \in \mathbb{R}$. Fix an integer $s \geq 1$. Suppose that $\xi = e^{it}$ is regular or a pole of order at most s, and put*

$$L_\xi(F) = \lim_{z\to\xi}(1 - z/\xi)^s F(z).$$

For $\chi \in C_c^\infty((0,1))$, as $R \to +\infty$,

$$\sum_{k\geq 0} a_k e^{it(k+m)}\chi((k+m)/R) = \frac{e^{itm}L_\xi(F)}{(s-1)!}R^s\int_0^\infty \chi(u)u^{s-1}\,du + O(R^{s-1}). \tag{8.5}$$

If F is regular at ξ, the left side is $O(R^{-A})$ for every $A > 0$. Both assertions remain valid when m is replaced by any fixed real number. All parameters other than R are fixed.

Proof. Write

$$F(z) = B(z) + \sum_\alpha \sum_{j=1}^{s_\alpha} c_{\alpha,j}(1 - z/\alpha)^{-j}, \qquad p_j(x) = \frac{1}{(j-1)!}\prod_{\ell=1}^{j-1}(x + \ell),$$

where B is a polynomial, α runs through the finite poles, s_α is the order of the pole at α, and an empty product is one. For $k \geq 0$, $[z^k](1 - z/\alpha)^{-j} = \alpha^{-k}p_j(k)$. The coefficients of B have zero weight for

large R. Put $f_{j,R}(x) = p_j(x)\chi((x+m)/R)$, extending χ by zero. Since $\operatorname{supp} f_{j,R} \subset (0,\infty)$ for large R, the contribution of $c_{\alpha,j}$ is

$$e^{itm} c_{\alpha,j} \sum_{k\in\mathbb{Z}} (\xi/\alpha)^k f_{j,R}(k).$$

Put $g_{j,R}(u) = R^{1-j} f_{j,R}(Ru)$. Thus

$$g_{j,R}(u) = \frac{\chi(u+m/R)}{(j-1)!} \prod_{\ell=1}^{j-1} (u+\ell/R).$$

These functions are supported in a common compact set and have uniformly bounded derivatives. With $\widehat{g}(\zeta) = \int_{\mathbb{R}} g(u) e^{-2\pi i u\zeta}\, du$, the estimates in the proof of [75, Chapter 5, Theorem 1.3, p. 137] give

$$|\widehat{g}_{j,R}(\zeta)| \le C_N (1+|\zeta|)^{-N} \qquad (N \ge 0),$$

uniformly for large R. Poisson summation [75, Chapter 5, Theorem 3.1, p. 154] gives

$$\sum_{k\in\mathbb{Z}} f_{j,R}(k) = R^j \sum_{\ell\in\mathbb{Z}} \widehat{g}_{j,R}(R\ell) = \int_{\mathbb{R}} f_{j,R}(x)\, dx + O(R^{-A})$$

for every $A > 0$, by taking $N > j + A + 1$.

For $\omega \in \mathbb{C}$ with $|\omega| = 1$ and $\omega \ne 1$, set $\Delta f(k) = f(k+1) - f(k)$. Repeated summation by parts [75, Chapter 2, Exercise 7(a), p. 60] gives

$$\left| \sum_{k\in\mathbb{Z}} \omega^k f_{j,R}(k) \right| \le |1-\omega|^{-N} \sum_{k\in\mathbb{Z}} |\Delta^N f_{j,R}(k)| = O(R^{j-N}) \qquad (N \ge 1).$$

Here $\|f_{j,R}^{(N)}\|_\infty = O(R^{j-1-N})$, and the finite differences have support of length $O(R)$.

Set $c_{\xi,j} = 0$ when the corresponding partial fraction is absent. The left side of (8.5) is therefore

$$e^{itm} \sum_{j=1}^{s} \frac{c_{\xi,j} R}{(j-1)!} \int_0^1 \chi(u) \prod_{\ell=1}^{j-1} (Ru + \ell - m)\, du + O(R^{-A})$$

for every $A > 0$. Since $c_{\xi,s} = L_\xi(F)$, the coefficient of R^s is the one in (8.5); every remaining term has degree at most $s-1$. If F is regular at ξ, every $c_{\xi,j}$ vanishes. □

9. Scalar return and lattice quantization

Let (M, g) be a closed connected Riemannian manifold with $\dim M \ge 2$. On scalar half-densities, put $P = (\Delta_M + c)^{1/2}$, where $c > 0$. Write $N_P(R) = \operatorname{Tr} \mathbf{1}_{(0,R]}(P)$, and let G_t be the Hamiltonian flow of $|\xi|_g$. The subprincipal symbol of P is zero [21, p. 58]. If $T > 0$ and $G_T = I$, Colin de Verdière's theorem [15, Theorem 1.1 and its proof, p. 510], applied to $TP/(2\pi)$, gives

$$e^{iTP} - \rho_T I \in \Psi_{\mathrm{cl}}^{-1}, \qquad \rho_T \in \{1, -1, i, -i\}.$$

The Maslov factor is locally constant on the connected cosphere bundle S^*M, hence constant.

Corollary 9.1. *Let $T > 0$, $\zeta \in \{1,-1,i,-i\}$, and $\kappa > 0$. Suppose that, along some $R_j \to \infty$, all but $o(N_P(R_j))$ eigenvalues $\mu \le R_j$ of P, counted with multiplicity, satisfy*

$$\operatorname{Re}(\bar{\zeta} e^{iT\mu}) \ge \kappa.$$

Then $G_T = I$ and $e^{iTP} - \zeta I \in \Psi_{\mathrm{cl}}^{-1}$. There is a constant $C > 0$ such that

$$|e^{iT\mu} - \zeta| \le C(1+\mu)^{-1} \qquad (\mu \in \operatorname{Spec} P).$$

Proof. Since $N_P(R_j) \to \infty$ and the exceptional proportion tends to zero, the hypothesis implies $\kappa \le 1$. Choose $\beta \in \mathbb{R}$ with $e^{i\beta} = \zeta$, and apply Theorem 3.1 to the trivial line bundle with its flat connection and constant potential c. It gives $G_T = I$ and $\operatorname{Re}(\bar{\zeta}\rho_T) \ge \kappa$. The fourth root of unity $\bar{\zeta}\rho_T$ has positive real part, so it equals 1. Thus $\rho_T = \zeta$. The operator $(e^{iTP} - \zeta I)(1+P)$ has order zero and is bounded on L^2. Its restriction to $\ker(P-\mu)$ gives the stated estimate. □

Corollary 9.2. *Let $m > 0$ and $a > 0$ be real numbers, and put $T = 2\pi/a$ and $\zeta = e^{iTm}$. If $e^{iTP} - \zeta I \in \Psi_{\mathrm{cl}}^{-1}$, there is a positive self-adjoint $A \in \Psi_{\mathrm{cl}}^{1}$ with domain H^1, strongly commuting with P, such that*

$$[A, P] = 0, \qquad A - P \in \Psi_{\mathrm{cl}}^{-1}, \qquad \mathrm{Spec}(A) \subset m + a\mathbb{Z}_{\geq 0}. \tag{9.1}$$

For every sufficiently large eigenvalue μ of P, there is a unique nearest point $\lambda_\mu \in m + a\mathbb{Z}$ to μ, and $A|_{\ker(P-\mu)} = \lambda_\mu I$.

Proof. The order-zero estimate in the proof of Corollary 9.1 gives $|e^{iT\mu} - \zeta| = O((1+\mu)^{-1})$. Hence

$$\mathrm{Re}(\bar{\zeta} e^{iT\mu}) = 1 - \tfrac{1}{2}|e^{iT\mu} - \zeta|^2 \geq \tfrac{1}{2}$$

for all sufficiently large μ. Theorem 3.1 applies with $\beta = Tm$ and $\kappa = 1/2$. It gives $G_T = I$. The holonomy of the trivial line with its flat connection is the identity, so the return symbol is ρ_T. Since $e^{iTP} - \zeta I$ has order -1, comparison of principal symbols gives $\rho_T = \zeta$.

Let $\alpha \in (0, a]$ be the unique number such that $m - \alpha \in a\mathbb{Z}$. Then $e^{iT\alpha} = \rho_T$. Theorem 3.3 gives a positive self-adjoint $A_0 \in \Psi_{\mathrm{cl}}^{1}$ with domain H^1, strongly commuting with P, such that

$$A_0 - P \in \Psi_{\mathrm{cl}}^{-1}, \qquad \mathrm{Spec}(A_0) \subset \alpha + a\mathbb{Z}_{\geq 0}.$$

For every sufficiently large eigenvalue μ of P, $A_0|_{\ker(P-\mu)}$ is multiplication by the nearest point of $\alpha + a\mathbb{Z}_{\geq 0}$ to μ. Since $m > 0$, one has $m = \alpha + aj_0$ for some $j_0 \in \mathbb{Z}_{\geq 0}$. The spectral projector $E = \mathbf{1}_{(-\infty,m)}(A_0)$ has finite rank and smooth kernel by the elliptic spectral theory in Section 1. Strong commutation of A_0 and P implies that E strongly commutes with P. Its range is therefore contained in a finite sum of P-eigenspaces. Put

$$A = A_0 + (m - A_0)E.$$

The correction is self-adjoint and smoothing and strongly commutes with P. Thus A is self-adjoint on H^1 and strongly commutes with P. On $\mathrm{ran}\, E$, it equals mI; on $\ker E$, it equals A_0. It follows that $A \geq mI$, $\mathrm{Spec}(A) \subset m + a\mathbb{Z}_{\geq 0}$, and $A - P \in \Psi_{\mathrm{cl}}^{-1}$. The correction vanishes on every sufficiently high P-eigenspace. Since $m + a\mathbb{Z} = \alpha + a\mathbb{Z}$, the nearest-point assertion follows from Theorem 3.3. □

Remark 9.3. The condition $\kappa > 0$ cannot be replaced by $\kappa = 0$. On the unit sphere S^{2m+1}, with $m \in \mathbb{Z}_{\geq 2}$, Lemma 8.1 gives $\mathrm{Spec}\,\sqrt{\Delta + m^2} = m + \mathbb{Z}_{\geq 0}$. For $\zeta = i(-1)^m$ and every $k \geq 0$,

$$\mathrm{Re}\big(\bar{\zeta} e^{i\pi(k+m)}\big) = 0.$$

Under the metric identification of the unit tangent and cotangent bundles, $G_\pi(x, v) = (-x, -v) \neq (x, v)$.

10. Spectral matchings and common geodesic periods

Retain the normalization $r = 1$ of Section 8. Let $Q = \mathbb{S}^n(1)/\Gamma_0$ be a round spherical space form, with $|\Gamma_0| = q$, and put $a = 2$ if q is even and $a = 1$ if q is odd. Fix $0 \leq \eta < a/4$. Suppose there are $K_j \to \infty$ such that, with $R_j = R_{K_j}$,

$$\mathcal{E}_\eta(K_j) = o(R_j^n). \tag{10.1}$$

The exponents in Theorems 2.2, 2.3, and 2.4 are strictly smaller than n.

For a positive self-adjoint operator B with compact resolvent, write

$$\mathrm{mult}_B(\omega) = \dim\ker(B - \omega I), \qquad N_B(R) = \sum_{\substack{\omega \in \mathrm{Spec}\, B \\ \omega \leq R}} \mathrm{mult}_B(\omega).$$

Put $N_M = N_{P_M}$ and $N_Q = N_{P_Q}$. Let $b_\eta(K)$ be the maximum cardinality of an η-matching at the cutoff R_K in Definition 2.1.

Proposition 10.1. *The comparison manifold has dimension n, volume V_n/q, and least positive common geodesic period $T = 2\pi/a$. Moreover, $e^{iTP_M} - e^{iTm}I \in \Psi^{-1}_{\mathrm{cl}}$. There are a positive self-adjoint $A \in \Psi^1_{\mathrm{cl}}$ with domain H^1, strongly commuting with P_M and satisfying* (9.1)*, and a constant $C_0 \geq 0$ such that*

$$\mathcal{E}_A(R_K) := \sum_{\substack{\omega \in m + a\mathbb{Z}_{\geq 0} \\ \omega \leq R_K}} |\operatorname{mult}_A(\omega) - \operatorname{mult}_{P_Q}(\omega)| \leq \mathcal{E}_\eta(K) + C_0 \tag{10.2}$$

for all sufficiently large K. For every $t \in \mathbb{R}$, every $\chi \in C_c^\infty((0,1))$, and every such K,

$$\left|\operatorname{Tr}(e^{itA}\chi(A/R_K)) - \operatorname{Tr}(e^{itP_Q}\chi(P_Q/R_K))\right| \leq \|\chi\|_\infty \mathcal{E}_A(R_K) \leq \|\chi\|_\infty \big(\mathcal{E}_\eta(K) + C_0\big). \tag{10.3}$$

Proof. By Lemma 1.2,

$$|N_M(R_K) - N_Q(R_K)| \leq \mathcal{E}_\eta(K).$$

For each integer $\ell \geq 1$, let B^ℓ be the Euclidean unit ball and put $c_\ell = (2\pi)^{-\ell}\operatorname{vol}(B^\ell)$. Set $d = \dim M \geq 1$. The scalar Weyl law [21, Corollary 2.5, equation (2.23), p. 52] gives

$$N_M(R) = c_d \operatorname{Vol}(M) R^d + o(R^d), \qquad N_Q(R) = c_n(V_n/q)R^n + o(R^n).$$

If $d < n$, divide the counting inequality along R_j by R_j^n. Its left side tends to $c_n V_n/q > 0$, whereas its right side tends to zero. If $d > n$, division by R_j^d gives the same contradiction, with limit $c_d \operatorname{Vol}(M) > 0$. Thus $d = n$. Division by R_j^n now gives

$$\operatorname{Vol}(M) = V_n/q. \tag{10.4}$$

Put $T = 2\pi/a$ and $\zeta = e^{iTm} \in \{1, -1\}$. Since every reference frequency belongs to $m + a\mathbb{Z}$, each matched pair (μ, β) satisfies

$$\operatorname{Re}(\bar{\zeta} e^{iT\mu}) = \cos(T(\mu - \beta)) \geq \cos(T\eta) > 0.$$

In a maximum matching at R_j, at most $\mathcal{E}_\eta(K_j) = o(N_M(R_j))$ copies of the P_M-spectrum below R_j are unmatched. Since $\dim M = n \geq 5$, Corollaries 9.1 and 9.2 apply and give

$$G_T = I, \qquad e^{iTP_M} - e^{iTm}I \in \Psi^{-1}_{\mathrm{cl}}, \tag{10.5}$$

and an operator A with the stated properties.

Choose a common orthonormal eigenbasis of A and P_M. By Corollary 9.2, for all sufficiently large P_M-eigenvalues μ, the restriction of A to $\ker(P_M - \mu I)$ is a scalar $\alpha \in m + a\mathbb{Z}_{\geq 0}$. Since $(A - P_M)P_M \in \Psi^0_{\mathrm{cl}}$,

$$|\alpha - \mu| \leq \frac{\|(A - P_M)P_M\|}{\mu}.$$

Choose μ_0 so that this difference is less than $a/4$ for $\mu > \mu_0$. If such a copy of μ is matched to β, then

$$|\alpha - \beta| < a/4 + \eta < a/2.$$

Both α and β belong to $m + a\mathbb{Z}$, so $\alpha = \beta$. Each R_K has distance $a/2$ from this lattice. Hence μ and α lie on the same side of every R_K.

Let $B_0 = N_M(\mu_0)$. From a maximum η-matching at R_K, discard the pairs with $\mu \leq \mu_0$; there are at most B_0 such pairs. Every remaining pair gives an equal-frequency pair for A and P_Q, with distinct labelled copies remaining distinct. If $b_A(R_K)$ is the maximum number of such pairs, then

$$b_A(R_K) \geq b_\eta(K) - B_0.$$

For all sufficiently large K, the finite-dimensional sum of the P_M-eigenspaces with $\mu \leq \mu_0$ lies below R_K for both operators. The remaining joint eigenvalues lie on the same side of R_K. Thus $N_A(R_K) = N_M(R_K)$, and

$$\begin{aligned} \mathcal{E}_A(R_K) &= N_A(R_K) + N_Q(R_K) - 2b_A(R_K) \\ &\leq N_M(R_K) + N_Q(R_K) - 2b_\eta(K) + 2B_0 \\ &= \mathcal{E}_\eta(K) + 2B_0. \end{aligned}$$

This proves (10.2) with $C_0 = 2B_0$. The spectral expansion on $m + a\mathbb{Z}_{\geq 0}$ gives (10.3) by the triangle inequality.

The subgroup $\mathscr{P} = \{t \in \mathbb{R} : G_t = I\}$ is closed. The geodesic vector field has no zero on the compact manifold S^*M. Choose a finite flow-box cover and smaller boxes whose closures lie in those boxes and still cover S^*M. For some $\epsilon > 0$, every trajectory starting in a smaller box stays in the corresponding box for $|t| < \epsilon$. Its flow coordinate changes by t. Thus

$$G_t(v) \neq v \qquad (v \in S^*M,\ 0 < |t| < \epsilon).$$

Consequently $\mathscr{P} = T_0\mathbb{Z}$ for some $T_0 > 0$. Write $T = b\,T_0$, where $b \in \mathbb{Z}_{\geq 1}$.

Suppose $b > 1$, and put $V = e^{iT_0(A-m)}$. Its canonical map is $G_{-T_0} = I$, so $V \in \Psi^0_{\mathrm{cl}}$ [21, Theorem 1.1 and p. 44]. The arithmetic spectrum gives $V^b = I$. Its scalar principal symbol v_0 satisfies $v_0^b = 1$, and is constant on the connected cosphere bundle S^*M. Thus $V - v_0 I \in \Psi^{-1}_{\mathrm{cl}}$ is compact. For $v \in \{0, \dots, b-1\}$, put

$$N_{A,v}(R) = \sum_{\substack{k \geq 0,\ m+ak \leq R \\ k \equiv v \pmod b}} \mathrm{mult}_A(m + ak),$$

and define $N_{Q,v}$ by replacing A with P_Q. Then

$$|N_{A,v}(R_K) - N_{Q,v}(R_K)| \leq \mathcal{E}_A(R_K).$$

Lemma 8.3 gives

$$N_{Q,v}(R) = \frac{2}{qb\ n!} R^n + O(R^{n-1}).$$

By (10.2) and (10.1),

$$N_{A,v}(R_j) = \frac{2}{qb\ n!} R_j^n + o(R_j^n) \longrightarrow \infty.$$

On the class $k \equiv v \pmod b$, the operator V acts by $e^{2\pi i v/b}$, whose eigenspace therefore has infinite dimension. At least one of these values differs from v_0. This contradicts the finite multiplicity of every nonzero eigenvalue of the compact operator $V - v_0 I$ [49, Chapter III, Theorem 6.26]. Hence $b = 1$ and $T_0 = T$. □

Lemma 10.2. *Assume the conclusions of Proposition 10.1. Let $s \geq 1$ be an integer and let $R_j = R_{K_j} \to \infty$. Suppose $\mathcal{E}_\eta(K_j)/R_j^s \to \delta < \infty$. Let $t \in \mathbb{R}$, $L \in \mathbb{C}$, and $\theta \in \mathbb{R}$ be fixed. If, for every nonnegative $\chi \in C_c^\infty((0,1))$,*

$$\mathrm{Tr}(e^{itA}\chi(A/R_j)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_j)) = \frac{e^{i\theta}L}{(s-1)!} R_j^s \int_0^1 \chi(u) u^{s-1}\,du + o(R_j^s),$$

then

$$\delta \geq \frac{|L|}{s!}. \tag{10.6}$$

Proof. For $0 \leq \chi \leq 1$, divide (10.3) by R_j^s and let $j \to \infty$. Since $C_0/R_j^s \to 0$,

$$\delta \geq \frac{|L|}{(s-1)!} \int_0^1 \chi(u) u^{s-1}\,du.$$

Choose $\chi_\ell \in C_c^\infty((0,1))$ with $0 \leq \chi_\ell \leq 1$ and $\chi_\ell(u) \to 1$ for every $u \in (0,1)$. For each fixed ℓ, apply the preceding inequality. Dominated convergence as $\ell \to \infty$ gives $\int_0^1 \chi_\ell(u) u^{s-1}\,du \to 1/s$. □

10.1. **Clean geodesic returns.** Let $T > 0$ and suppose $G_T = I$. The period subgroup of each unit covector is $\tau\mathbb{Z}$, with $T = k\tau$ for some integer $k \geq 1$. The uniform bound in the preceding proof gives $\tau \geq \epsilon$, so $k \leq T/\epsilon$. Every positive return time belongs to

$$\bigcup_{1 \leq k \leq \lfloor T/\epsilon \rfloor} \frac{T}{k}\mathbb{Z}_{>0},$$

a locally finite subset of $(0,\infty)$. If t is a positive return time, then $t = jT/k$ for positive integers j,k, and $G_t^k = I$. The finite group $\langle G_t\rangle$ acts smoothly and properly on S^*M. By the fixed-set theorem [62, Theorem 1.23 and Proposition 1.24], its fixed locus is closed, has smooth components, and satisfies

$$T_v \operatorname{Fix}(G_t) = \ker(dG_t(v) - I).$$

Its components are open in the compact fixed locus, hence finite in number. Thus the return is clean.

Put $p(x,\xi) = |\xi|_g$. Its Hamiltonian flow commutes with positive fiber dilations, so the fixed locus in $T^*M \setminus 0$ is a cone. The relation $G_t^k = I$ holds on $T^*M \setminus 0$ as well. The finite cyclic group $\langle G_t\rangle$ acts symplectically; each component of its fixed cone is therefore symplectic [23, Example 3.3, p. 18]. Let $\pi : T^*M \to M$ be the projection and let λ be the canonical one-form, defined for $w \in T_{(x,\xi)}T^*M$ by

$$\lambda_{(x,\xi)}(w) = \xi(d\pi_{(x,\xi)} w).$$

Put $\omega = d\lambda$ and let $Z = \sum_j \xi_j \partial_{\xi_j}$ be the Liouville vector field. On a fixed-cone component of dimension $2h$, the vector field Z is tangent and $Zp = p$; thus it is transverse to $p = 1$. Since

$$\iota_Z \omega^h = h\lambda \wedge \omega^{h-1},$$

the restriction of $\lambda \wedge (d\lambda)^{h-1}$ to that section is nonvanishing. Each component of $\operatorname{Fix}(G_t) \subset S^*M$ is therefore a contact submanifold of dimension $2h-1$.

Let B be a positive self-adjoint scalar classical operator of order one on half-densities, with principal symbol p and zero subprincipal symbol. These conditions hold for P_M and for the operator A of Proposition 10.1, since $A - P_M \in \Psi_{\mathrm{cl}}^{-1}$. Fix a positive return time t. Let d be the largest dimension of a component of $\operatorname{Fix}(G_t)$, and suppose every component of dimension d has periodic Morse index σ. For $\xi \in T_x^*M \setminus 0$ and $w \in T_x^*M \setminus 0$,

$$d_\xi p(w) = 0 \quad \Longrightarrow \quad D_\xi^2 p(w,w) = \frac{|w|_{g^{-1}}^2}{p(x,\xi)} > 0.$$

This is the fiberwise convexity condition in [21, pp. 69–70]. Hence the integer in the trace phase is the Morse index for periodic boundary conditions.

For $f \in \mathcal{S}(\mathbb{R})$, put

$$\widehat{f}(\rho) = \int_{\mathbb{R}} e^{-it\rho} f(t)\, dt, \qquad f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\rho}\widehat{f}(\rho)\, d\rho.$$

Distribution pairings are complex bilinear. For each fixed component C, let $d\nu_C$ be the positive Duistermaat–Guillemin density: divide the canonical fixed-point density of G_t on the fixed cone by $|dp|$ and restrict to $p = 1$ [21, pp. 60–61]. Put

$$\gamma_{d,\sigma} = (2\pi)^{-(d+1)/2} \exp\!\left(i\pi \left(\frac{\sigma}{2} + \frac{d-1}{4} \right) \right).$$

The clean trace formula [21, Theorem 4.5, equations (4.6)–(4.8)] gives

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = \gamma_{d,\sigma} R^{(d+1)/2} \left(\int_0^1 \chi(u) u^{(d-1)/2}\, du \right) \sum_{\substack{C \subset \operatorname{Fix}(G_t) \\ \dim C = d}} \int_C d\nu_C + o(R^{(d+1)/2}) \tag{10.7}$$

for every $\chi \in C_c^\infty((0,1))$, where the sum is over connected components.

To obtain this form of the formula, extend χ by zero and put

$$W_B(s) = \sum_{\mu \in \operatorname{Spec} B} \operatorname{mult}_B(\mu) e^{is\mu}, \qquad \varphi_{R,t}(s) = \frac{R}{2\pi}\widehat{\chi}\big(R(s-t)\big).$$

Weyl's law makes W_B a tempered distribution. Fourier inversion gives

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = \langle W_B, \varphi_{R,t}\rangle;$$

the sum converges by the rapid decay of the Fourier transform of a Schwartz function. Choose $\rho \in C_c^\infty(\mathbb{R})$ equal to one near t, with support containing no other return time. For every N,

$$(1-\rho)\varphi_{R,t} = O(R^{-N}) \quad \text{in } \mathcal{S}(\mathbb{R}).$$

Its pairing with W_B is therefore $O(R^{-N})$ for every N.

In the negative half-wave convention of [21, equations (4.7)–(4.8)], the leading coefficient for a component of dimension d is $(2\pi)^{-(d+1)/2} i^{-\sigma} e^{-i\pi(d-1)/4} \int_C d\nu_C$; the subprincipal factor is 1. Complex conjugation gives $\gamma_{d,\sigma} \int_C d\nu_C$. The model distributions are tempered, so the preceding estimate removes ρ from their pairings with an $O(R^{-N})$ error for every N. Pairing the leading homogeneous term with $\varphi_{R,t}$ gives

$$\int_0^\infty \chi(\nu/R)\nu^{(d-1)/2}\, d\nu = R^{(d+1)/2} \int_0^1 \chi(u) u^{(d-1)/2}\, du.$$

The next symbol degree is $(d-3)/2$, with contribution $O(R^{(d-1)/2})$. Every smaller fixed component has odd dimension at most $d-2$, hence has the same bound. Multiplication of a smooth local remainder by ρ gives a compactly supported smooth function. Its Fourier transform is rapidly decreasing, and $\operatorname{supp} \chi(\nu/R) \subset [cR, CR]$ for some $0 < c < C$. Its contribution is therefore $O(R^{-N})$ for every N. This proves (10.7).

If $0 \le \chi \not\equiv 0$, the displayed moment and all the density integrals are positive. Thus the leading coefficient is nonzero. For two returns with the same d and σ, the ratio of their leading coefficients for the same such χ is positive real. In particular, A and P_M have the same leading coefficient. If $\operatorname{Fix}(G_t) = \emptyset$, then W_B is smooth near t [21, Corollary 1.2]; the same localization gives

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = O(R^{-N}) \qquad \text{for every } N.$$

Put $D = A - P_M$. It is bounded and self-adjoint, belongs to Ψ_{cl}^{-1}, and strongly commutes with P_M. The joint spectral theorem gives

$$e^{-itP_M} e^{itA} = e^{itD}.$$

For fixed $t \in \mathbb{R}$, put $f_t(z) = (e^{itz} - 1)/z$ for $z \neq 0$ and $f_t(0) = it$. This function is entire. The resolvent argument in the proof of Lemma 1.3, applied to f_t on a circle $|z| = r_0 > \|D\|$, gives $f_t(D) \in \Psi_{\mathrm{cl}}^0$. Hence

$$e^{itD} - I = D f_t(D) \in \Psi_{\mathrm{cl}}^{-1}. \tag{10.8}$$

11. Universal coverings and Morse indices

A Riemannian metric is Besse if every unit-speed geodesic is periodic. The prime length of a geodesic is the least positive period of its unit tangent vector. For smooth closed connected manifolds of dimension at least two, the following results hold.

(i) By the Bott–Samelson–McCleary theorem, a simply connected Besse manifold has the integral cohomology ring of a compact rank-one symmetric space. The universal Riemannian cover of an odd-dimensional Besse manifold is homeomorphic to a sphere [5, Theorem 1.1 and the following paragraph].

(ii) On a topological sphere of dimension at least four, every Besse metric has one common prime geodesic length. On a simply connected Besse manifold, the free-loop energy is perfect for rational S^1-equivariant cohomology relative to the constant loops [70, Theorems A and D].

(iii) A Besse metric on a smooth n-manifold admitting an even-degree smooth covering by a smooth homotopy n-sphere has constant sectional curvature for $n \neq 3$ [59, Theorem 2].

Lemma 11.1. *Let M^n be a closed connected Riemannian manifold, where $n \ge 5$ is odd, and let $T > 0$ be the least positive time for which $G_T = I$. Let $p_X : X \to M$ be its universal Riemannian covering, let Γ be the deck group, and write $\widetilde{G}_t$ for the geodesic flow on S^*X. For $D \in \Gamma$, let $\widehat{D}$ denote its cotangent lift to S^*X. Then X is a compact smooth homotopy sphere, Γ is finite, and there is a unique $D \in \Gamma$ such that*

$$\widetilde{G}_T = \widehat{D}, \qquad D^2 = I, \qquad \widetilde{G}_{2T} = I. \tag{11.1}$$

All prime geodesics of X have one length L, and

$$L \in \{T, 2T\}, \qquad D = I \iff L = T, \qquad \mathrm{Vol}(X) = |\Gamma|\,\mathrm{Vol}(M). \tag{11.2}$$

Proof. The metric on M is Besse. By [5, Theorem 1.1 and the following paragraph], X is homeomorphic to $\mathbb{S}^n$, hence is a compact smooth homotopy sphere. Each fiber of p_X is a closed discrete subset of X, so it is finite. Thus Γ is finite.

The induced covering $S^*X \to S^*M$ is connected and regular, with deck group consisting of the cotangent lifts of Γ. Since $G_T = I$, the map $\widetilde{G}_T$ is a lift of the identity on S^*M, hence equals $\widehat{D}$ for a unique $D \in \Gamma$. The reversal $J(x,\xi) = (x,-\xi)$ commutes with each cotangent deck lift and conjugates $\widetilde{G}_t$ to $\widetilde{G}_{-t}$. Consequently,

$$\widehat{D} = J\widehat{D}J = J\widetilde{G}_T J = \widetilde{G}_{-T} = \widehat{D}^{-1}.$$

It follows that $D^2 = I$ and $\widetilde{G}_{2T} = I$.

The lifted metric is Besse. Since X is a topological sphere of dimension at least four, [70, Theorem A] gives a common prime length L. The global period group of G_t is $T\mathbb{Z}$. The identity $\widetilde{G}_L = I$ descends to $G_L = I$, so $L = bT$ for an integer $b \geq 1$. Since L is the prime period of every lifted unit covector and $\widetilde{G}_{2T} = I$, $2T = aL$ for an integer $a \geq 1$. Hence $ab = 2$. If $D = I$, then T is a period of every lifted unit covector, so $L = T$; conversely, $L = T$ gives $\widehat{D} = I$ and $D = I$. The volume identity follows because p_X is a $|\Gamma|$-sheeted local isometry. □

For a Riemannian manifold Y, write SY for its unit tangent bundle, identified with S^*Y by the metric. For $v \in SY$, let γ_v be the unit-speed geodesic with initial velocity v. On the free-loop space $\Lambda Y = H^1(\mathbb{R}/\mathbb{Z}, Y)$, put

$$\mathcal{E}_Y(c) = \frac{1}{2}\int_0^1 |\dot{c}(t)|^2\, dt.$$

For a closed geodesic γ, let $\mathrm{ind}_{\mathrm{per}}(\gamma)$ denote the negative index of the Hessian of $\mathcal{E}_Y$ on periodic H^1 vector fields. Here the negative index is the maximal dimension of a subspace on which the Hessian is negative definite. Let $\mathrm{ind}_D(\gamma)$ denote the index of its restriction to fields vanishing at both endpoints. The same notation applies after a constant rescaling of the parameter.

Lemma 11.2. *Let X^n, $n \geq 5$ odd, be a Besse homotopy sphere whose prime geodesics all have length $L > 0$. Their periodic Morse index is $n-1$. In particular, their endpoint-fixed Morse index is at most $n-1$.*

Proof. The critical manifold of prime geodesics in ΛX is identified with SX by

$$v \longmapsto \big(t \longmapsto \gamma_v(Lt)\big).$$

Since X and the fibers S^{n-1} are connected, SX is connected. The energy is Morse–Bott, and its index is constant on this critical manifold [70, Section 1, pp. 915–916]; denote the index by b. Every nonconstant critical loop is an iterate of a prime geodesic. For a prime geodesic γ, let γ^k denote its k-fold iterate. Bott's iteration inequality [70, Section 6, p. 952] gives

$$\mathrm{ind}_{\mathrm{per}}(\gamma^k) \geq \mathrm{ind}_{\mathrm{per}}(\gamma) = b \qquad (k \geq 1).$$

Thus b is the least index of a nonconstant critical loop. The manifold X is simply connected and has the rational cohomology of $\mathbb{S}^n$. For a Besse manifold with these properties, the least nonconstant index is $n-1$ [70, Section 4, pp. 941–942]. Hence $b = n-1$. The endpoint-fixed variation space is a subspace of the periodic variation space, so its index is at most $n-1$. □

Lemma 11.3. *Let X^n be a closed Riemannian manifold and let $L > 0$. Suppose the time-L geodesic map on SX is the identity. For every length-L unit-speed geodesic on X, the periodic and endpoint-fixed indices are equal.*

Proof. Fix such a geodesic γ, and put $V = \dot\gamma(0)^\perp \subset T_{\gamma(0)}X$. Normal Jacobi initial data identify $V \oplus V$ with the contact hyperplane at $\dot\gamma(0)$. The normal Poincaré map P_γ is the restriction of the differential of the time-L geodesic map to this hyperplane. Thus $P_\gamma = I$. In the concavity formula [70, Section 4, p. 939], the concavity form is then zero on $V \oplus V$. The difference between the normal periodic and endpoint-fixed indices is therefore

$$0 + \dim(V \oplus V) - \dim\ker(P_\gamma - I) = 0 + (2n-2) - (2n-2) = 0.$$

The normal and tangential fields split both index forms orthogonally. In arclength parametrization, the form on $f\dot\gamma$ is $\int_0^L |f'(t)|^2\,dt \geq 0$. The change of parameter $t = Ls$ multiplies this form by $L > 0$. Thus the full indices equal the normal indices. □

Lemma 11.4. *Under the hypotheses of Lemma 11.2, for any $c > 0$ the principal symbol of $e^{iL\sqrt{\Delta_X + c}}$ is* 1.

Proof. Put $P_X = \sqrt{\Delta_X + c}$. Since the time-$L$ geodesic map is the identity, Section 9 gives $e^{iLP_X} \in \Psi^0_{\mathrm{cl}}$ with constant principal symbol u_0, where $|u_0| = 1$. Fix $0 \leq \chi \in C_c^\infty((0,1))$, $\chi \not\equiv 0$. The local Weyl formula (1.1) gives, as $R \to \infty$,

$$\operatorname{Tr}(e^{iLP_X}\chi(P_X/R)) = u_0 W_X R^n + o(R^n), \qquad W_X = (2\pi)^{-n}\int_{T^*X} \chi(|\xi|_g)\,dx\,d\xi > 0.$$

The fixed locus is S^*X, of dimension $2n-1$, and all its geodesics have periodic Morse index $n-1$ by Lemma 11.2. The subprincipal symbol of P_X on half-densities is zero. Since n is odd,

$$\gamma_{2n-1,n-1} = (2\pi)^{-n} e^{i\pi(n-1)} = (2\pi)^{-n}.$$

Equation (10.7) therefore gives

$$\operatorname{Tr}(e^{iLP_X}\chi(P_X/R)) = C_X R^n + o(R^n), \qquad C_X = (2\pi)^{-n}\left(\int_0^1 \chi(u)u^{n-1}\,du\right)\int_{S^*X} dv_{S^*X} > 0.$$

Comparison of the two leading coefficients gives $u_0 W_X = C_X > 0$. Together with $|u_0| = 1$, this implies $u_0 = 1$. □

Lemma 11.5. *Let $p_X : X \to M$ be a finite regular Riemannian covering of closed connected manifolds, and let $T \in \mathbb{R}$ and $c > 0$. If both time-T geodesic maps are the identity, the principal symbols of $e^{iT\sqrt{\Delta_M + c}}$ and $e^{iT\sqrt{\Delta_X + c}}$ agree under the induced covering of cotangent bundles.*

Proof. Let Γ be the deck group and put $d = |\Gamma|$. The unitary map

$$\mathcal{U} : L^2(M) \longrightarrow L^2(X)^\Gamma, \qquad \mathcal{U}f = d^{-1/2}p_X^* f,$$

intertwines the Laplacians and their spectral functions. Let K_X and K_M be the propagator kernels with respect to Riemannian volume. For $x, y \in X$, the kernel identity is

$$K_M(T; p_X(x), p_X(y)) = \sum_{\gamma\in\Gamma} K_X(T; x, \gamma y),$$

where the equality is distributional. It follows by integrating over the sheets of an evenly covered coordinate neighbourhood. Both propagators are pseudodifferential, since their canonical maps at time T are the identity. Choose one sheet U over such a neighbourhood. For $x, y \in U$ and $\gamma \neq I$, the point $(x, \gamma y)$ lies off the diagonal in $X \times X$, so $K_X(T; x, \gamma y)$ is smooth. Hence, on $U \times U$, the pulled-back kernel K_M and K_X differ by a smooth kernel. Their principal symbols agree under the local isometry p_X. The unitary identification with half-densities preserves this equality. □

12. Integral cohomology and contact volume

Let (X^n, g), $n \geq 2$, be a closed smooth homotopy sphere with a Besse metric whose prime geodesics all have length $L > 0$. For $j \geq 1$, put $V_j = \mathrm{Vol}(\mathbb{S}^j(1))$. Identify SX with S^*X by metric duality and put $E = SX$. Let G_t be the unit-speed geodesic flow and let λ be the canonical contact form on E. The action of $\mathbb{R}/\mathbb{Z}$ given by $[s] \cdot v = G_{Ls}(v)$ is free. The quotient map is a principal circle bundle

$$p : E \longrightarrow \mathcal{G} = E/(\mathbb{R}/\mathbb{Z}).$$

The form λ/L is a connection form, and $\int_{p^{-1}(z)} \lambda = L$ for each positively oriented fiber. Write $e \in H^2(\mathcal{G}; \mathbb{Z})$ for the Euler class and $e_{\mathbb{R}}$ for its image in $H^2(\mathcal{G}; \mathbb{R})$. With the convention of [83, Section 2, p. 515], the two-form ω on $\mathcal{G}$ determined by

$$p^*\omega = d\lambda/L$$

is symplectic and satisfies $[\omega] = e_{\mathbb{R}}$. Give the closed $(2n-2)$-manifold $\mathcal{G}$ its symplectic orientation, and denote its integral fundamental class by $[\mathcal{G}]$.

Lemma 12.1 (Yang [87, Lemma 1, pp. 92–93]). *Suppose $n = 2m+1$, $m \geq 1$. The integral cohomology of E is free of rank one in degrees $0, 2m, 2m+1, 4m+1$ and zero in all other degrees. The integral cohomology of $\mathcal{G}$ is torsion-free, zero in odd degrees, of rank one in each even degree $0 \leq j \leq 4m$ other than $2m$, and of rank two in degree $2m$. Moreover,*

$$H^{2j}(\mathcal{G}; \mathbb{Z}) = \mathbb{Z}e^j \qquad (0 \leq j < m),$$

and

$$0 \longrightarrow \mathbb{Z}e^m \longrightarrow H^{2m}(\mathcal{G}; \mathbb{Z}) \xrightarrow{p^*} H^{2m}(E; \mathbb{Z}) \longrightarrow 0. \tag{12.1}$$

The class e^m is primitive, and the pairing

$$H^{2m}(\mathcal{G}; \mathbb{Z}) \times H^{2m}(\mathcal{G}; \mathbb{Z}) \longrightarrow \mathbb{Z}, \qquad (a, b) \longmapsto \langle a \smile b, [\mathcal{G}] \rangle,$$

is unimodular.

Proof. The manifold X has the integral cohomology of $\mathbb{S}^{2m+1}$, and every orbit of its geodesic flow has least period L. Yang's lemma therefore gives the stated cohomology groups and the middle Gysin sequence. Since $H^j(E; \mathbb{Z}) = 0$ for $0 < j < 2m$, the circle-bundle Gysin sequence gives $H^{2j}(\mathcal{G}; \mathbb{Z}) = \mathbb{Z}e^j$ for $0 \leq j < m$. Its middle sequence is thus (12.1). The quotient is $\mathbb{Z}$, so e^m is primitive. The integral intersection pairing of the closed oriented $4m$-manifold $\mathcal{G}$ is unimodular by [37, Proposition 3.38, p. 250], since $H^{2m}(\mathcal{G}; \mathbb{Z})$ is torsion-free. □

Lemma 12.2 (Weinstein [83, Section 2, pp. 515–516]). *The positive integer $\mathfrak{a} = \langle e^{n-1}, [\mathcal{G}] \rangle$ satisfies*

$$\mathfrak{a} = 2\frac{\mathrm{Vol}_g(X)}{V_n} \left(\frac{2\pi}{L} \right)^n. \tag{12.2}$$

Proof. In Weinstein's notation, X is a C_r-manifold with $r = L/(2\pi)$. With the Euler class and orientation fixed above, his formula is

$$L^n \langle e^{n-1}, [\mathcal{G}] \rangle = (n-1)! V_{n-1} \mathrm{Vol}_g(X).$$

The identity $(n-1)! V_{n-1} V_n = 2(2\pi)^n$ gives (12.2). □

Proposition 12.3 ([84, p. 282]). *Let X^n, $n \geq 5$, be a closed smooth homotopy sphere with a Besse metric g. Its prime geodesics have a common length $L > 0$, and*

$$\mathrm{Vol}_g(X) = V_n \left(\frac{L}{2\pi} \right)^n. \tag{12.3}$$

Proof. By Smale's theorem [73, Theorem A], X is homeomorphic to $\mathbb{S}^n$. Theorem A of [70] gives the common prime length L. For $\widehat{g} = (\pi/L)^2 g$, the prime geodesics have length π. Thus $(X, \widehat{g})$ is a C_π-manifold in the notation of [84, p. 282]. The volume formula there for a C_π-manifold homeomorphic to a compact rank-one symmetric space gives

$$\mathrm{Vol}_{\widehat{g}}(X) = \mathrm{Vol}(\mathbb{S}^n(1/2)) = 2^{-n} V_n.$$

Since $\mathrm{Vol}_{\widehat{g}}(X) = (\pi/L)^n \mathrm{Vol}_g(X)$, equation (12.3) follows. □

13. Cyclic actions and evaluation of returning geodesics

For an isometry h of X, let $\widehat{h}(x,v) = (h(x), dh_x(v))$ be its tangent lift. If every unit-speed geodesic of X has least period 2π, put $E = SX$ and let S^1 act on E by $e^{it} \cdot v = G_t(v)$. Write

$$\varpi : E \longrightarrow \mathcal{G} = E/S^1, \qquad \pi : E \longrightarrow X, \quad \pi(x,v) = x.$$

The induced map of the geodesic space is denoted by $\bar{h}$; thus $\bar{h}\varpi = \varpi\widehat{h}$. The reversal $J(x,v) = (x,-v)$ satisfies $JG_t = G_{-t}J$ and $J^*\lambda = -\lambda$. The forms λ, ω and the Euler class e are as in Section 12, with $L = 2\pi$.

For a compact Lie group K acting on a space Y and a coefficient ring R, write $Y_K = EK \times_K Y$ and $H^*_K(Y;R) = H^*(Y_K;R)$, where $EK \to BK = EK/K$ is a universal principal K-bundle. For a field $\Bbbk$, put $b_j(Y;\Bbbk) = \dim_\Bbbk H^j(Y;\Bbbk)$.

Lemma 13.1. *Let X be a closed connected Riemannian manifold and let γ be a fixed-point-free isometry. Its minimum displacement b is positive and is attained. A minimizing segment realizing that minimum has matching endpoint velocities under $d\gamma$ and extends by γ to a complete invariant geodesic. If γ has finite order s, that geodesic closes after time sb. If, in addition, every unit-speed geodesic has least period 2π, then $sb \in 2\pi\mathbb{Z}_{>0}$.*

Proof. The function $x \mapsto d(x, \gamma x)$ is continuous and strictly positive on compact X. Let $b > 0$ be its minimum and choose a unit-speed minimizing geodesic $c : [0,b] \to X$ with $c(b) = \gamma c(0)$. For $w \in T_{c(0)}X$, choose a smooth field W along c with $W(0) = w$ and $W(b) = d\gamma(w)$. For sufficiently small $\varepsilon \in \mathbb{R}$, put $c_\varepsilon(t) = \exp_{c(t)}(\varepsilon W(t))$. Since γ is an isometry, $c_\varepsilon(b) = \gamma c_\varepsilon(0)$; hence $\mathrm{length}(c_\varepsilon) \geq b$. The first variation formula gives

$$0 = \langle \dot{c}(b), d\gamma(w) \rangle - \langle \dot{c}(0), w \rangle.$$

Since $d\gamma$ is an isometry and w is arbitrary, $\dot{c}(b) = d\gamma\, \dot{c}(0)$. The curves $\gamma^j c$, $j \in \mathbb{Z}$, have matching endpoints and velocities. By uniqueness for the geodesic equation, their concatenation is a smooth complete geodesic. If γ has order s, its unit tangent vector returns after time sb. For least period 2π, this gives $sb \in 2\pi\mathbb{Z}_{>0}$. □

Lemma 13.2. *Let a cyclic group C_r of odd prime order r act smoothly on a closed manifold W. Put $\Bbbk = \mathbb{F}_r$ and $\mathbb{K} = \Bbbk(v)$, where*

$$H^*(BC_r;\Bbbk) = \Bbbk[v] \otimes \Lambda(\epsilon), \qquad |v| = 2, \quad |\epsilon| = 1,$$

and $\Lambda(\epsilon)$ is the exterior algebra on ϵ. Restriction to the fixed set induces an isomorphism

$$H^*_{C_r}(W;\Bbbk) \otimes_{\Bbbk[v]} \mathbb{K} \cong H^*(W^{C_r};\Bbbk) \otimes \Lambda(\epsilon) \otimes \mathbb{K}. \tag{13.1}$$

If the action on $H^*(W;\Bbbk)$ is trivial, then

$$\sum_j b_j(W^{C_r};\Bbbk) \leq \sum_j b_j(W;\Bbbk). \tag{13.2}$$

For every page E_s, $s \geq 2$, of the Borel spectral sequence,

$$2\sum_j b_j(W^{C_r};\Bbbk) \leq \dim_{\mathbb{K}}\big(E_s \otimes_{\Bbbk[v]} \mathbb{K}\big),$$

where E_s denotes the direct sum of all its bidegrees.

Proof. The cohomology ring of BC_r is given in [37, Example 3E.2, p. 304]. Put $Z = W^{C_r}$ and $D = \dim W$. Apply the localization theorem [43, Chapter III, Theorem III.1, p. 40] with the multiplicative system $S = \{v^j : j \geq 0\}$. A stabilizer is either trivial or C_r. In the first case v restricts to zero; in the second no element of S does. Thus the set W^S in that theorem is Z. After extending scalars from $\Bbbk[v, v^{-1}]$ to $\mathbb{K}$, restriction is an isomorphism. The action on Z is trivial, so

$$H^*_{C_r}(Z;\Bbbk) = H^*(Z;\Bbbk) \otimes \Bbbk[v] \otimes \Lambda(\epsilon),$$

which gives (13.1).

The differentials $d_s : E_s^{i,j} \to E_s^{i+s,j-s+1}$ are $\Bbbk[v]$-linear. Localization is exact, so the dimensions of the localized pages cannot increase. Only vertical degrees $0 \leq j \leq D$ occur; hence $E_{D+2} = E_\infty$. Let $F^i H^N_{C_r}(W;\Bbbk)$ be the Serre filtration and set

$$\mathcal{F}_j H^*_{C_r}(W;\Bbbk) = \bigoplus_{N \geq 0} F^{N-j} H^N_{C_r}(W;\Bbbk), \qquad -1 \leq j \leq D.$$

Here $F^i H^N = H^N$ for $i \leq 0$ and $F^i H^N = 0$ for $i > N$. Multiplication by v sends $F^i H^N$ into $F^{i+2} H^{N+2}$; thus each $\mathcal{F}_j$ is a $\Bbbk[v]$-submodule. The vertical bound gives $\mathcal{F}_{-1} = 0$, $\mathcal{F}_D = H^*_{C_r}(W;\Bbbk)$, and

$$\mathcal{F}_j / \mathcal{F}_{j-1} \cong \bigoplus_{N \geq 0} E_\infty^{N-j,j}.$$

Exactness of localization and additivity of dimension for this finite filtration give

$$\dim_{\mathbb{K}}\big(E_\infty \otimes_{\Bbbk[v]} \mathbb{K}\big) = \dim_{\mathbb{K}}\big(H^*_{C_r}(W;\Bbbk) \otimes_{\Bbbk[v]} \mathbb{K}\big) = 2 \sum_j b_j(Z;\Bbbk).$$

This proves the page bound. If the action on $H^*(W;\Bbbk)$ is trivial, then

$$E_2 = \Bbbk[v] \otimes \Lambda(\epsilon) \otimes H^*(W;\Bbbk),$$

whose localized dimension is $2\sum_j b_j(W;\Bbbk)$. The case $s = 2$ gives (13.2). □

Lemma 13.3. *Let X^{2m+1}, $m \geq 2$, be a closed homotopy sphere with a Besse metric of common prime geodesic length 2π. An isometry of odd order acts trivially on $H^*(\mathcal{G};\mathbb{Z})$. For an isometry a of odd prime order r,*

$$\sum_j b_j(\mathrm{Fix}(\bar{a});\mathbb{F}_r) \leq 2m + 2.$$

Proof. An isometry preserves the canonical contact form and commutes with the geodesic circle, so it fixes the Euler class e. Every nonmiddle cohomology group of $\mathcal{G}$ is either zero or infinite cyclic by Lemma 12.1. Its automorphism has finite odd order and is therefore the identity. In the middle degree, choose an integral basis (e^m, z) by (12.1). The first vector is fixed. The action on the quotient by $\mathbb{Z}e^m$ is an automorphism of $\mathbb{Z}$ of odd order, hence is also the identity. Its matrix is consequently

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Finite order gives $b = 0$. All integral cohomology groups are torsion-free, so the universal coefficient theorem identifies $H^*(\mathcal{G};\mathbb{F}_r)$ with $H^*(\mathcal{G};\mathbb{Z}) \otimes \mathbb{F}_r$, naturally with respect to the action. Lemma 13.2 and the Betti numbers in Lemma 12.1 give

$$\sum_j b_j(\mathrm{Fix}(\bar{a});\mathbb{F}_r) \leq \sum_j b_j(\mathcal{G};\mathbb{F}_r) = 2m + 2.$$

□

Lemma 13.4. *Let I be a symmetric bilinear form on a real vector space with $I(Y,Y) \geq 0$ for every Y. If $I(Y,Y) = 0$, then $I(Y,Z) = 0$ for every Z.*

Let N be a Riemannian manifold and let $\gamma : [0,T] \to N$, $T > 0$, be a unit-speed geodesic whose index form on fields vanishing at 0 and T is nonnegative. For $0 < t_0 < T$, a Jacobi field along $\gamma|_{[0,t_0]}$ that vanishes at both endpoints is zero. In particular, this holds when γ is minimizing.

Proof. The first assertion follows from Cauchy–Schwarz for a positive semidefinite form: $|I(Y,Z)|^2 \le I(Y,Y)I(Z,Z)$. The Dirichlet index form has index zero. The Morse index theorem [61, Theorem 15.1, p. 83] therefore excludes every conjugate instant in $(0,T)$. For a minimizing geodesic the Dirichlet index form is nonnegative by [61, Lemma 13.6, p. 76]. □

Lemma 13.5. *Let X^n, $n = 2m+1$ and $m \ge 2$, be a closed smooth homotopy sphere with a Besse metric of common prime geodesic length 2π and volume V_n. Choose an orientation class $\mu_X \in H^n(X;\mathbb{Z})$. Let $x \in H^{2m}(E;\mathbb{Z})$ restrict to a generator on each unit tangent sphere, and put $y = \pi^*\mu_X$. In the integral Borel spectral sequence of the geodesic circle,*

$$d_2 y = \pm 2ux, \tag{13.3}$$

where $u \in H^2(BS^1;\mathbb{Z})$ is a generator.

Proof. The Euler class of TX vanishes, since n is odd. The sphere-bundle Gysin homomorphism $\pi_! : H^j(E;\mathbb{Z}) \to H^{j-2m}(X;\mathbb{Z})$ satisfies $\pi_! x = \pm 1$ and is an isomorphism in degree $j = 4m+1$. By the projection formula, $\pi_!(xy) = \pm\mu_X$. Together with Lemma 12.1, this gives

$$H^*(E;\mathbb{Z}) = \mathbb{Z}\langle 1, x, y, xy\rangle, \qquad |x| = 2m, \quad |y| = 2m+1, \quad x^2 = y^2 = 0.$$

The action is free, hence $E_{S^1} \simeq \mathcal{G}$ and the edge homomorphism sends u to e, up to sign. The second page is $\mathbb{Z}[u] \otimes H^*(E;\mathbb{Z})$. Write

$$d_2 y = d\, ux, \qquad d \in \mathbb{Z}.$$

At bidegree $(2, 2m)$ this is the only possible incoming differential: every later incoming differential has negative source column. The only possible outgoing differential has target $(2m+3, 0)$, which is zero. Thus

$$E_\infty^{2,2m} = \mathbb{Z}/d\mathbb{Z}.$$

In total degree $N = 2m+2$, the only possibly nonzero terms are $(N,0)$ and $(2,2m)$. The edge image is the last filtration term [43, Chapter III, Section 1(C), p. 37]; consequently

$$H^{2m+2}(\mathcal{G};\mathbb{Z})/\mathbb{Z}e^{m+1} \simeq \mathbb{Z}/d\mathbb{Z}.$$

By Lemma 12.1, e^{m-1} generates $H^{2m-2}(\mathcal{G};\mathbb{Z})$ and $H^{2m+2}(\mathcal{G};\mathbb{Z}) \cong \mathbb{Z}$. Integral Poincaré duality gives a generator z of the latter group with $\langle e^{m-1}z, [\mathcal{G}]\rangle = 1$. Lemma 12.2 gives $\langle e^{2m}, [\mathcal{G}]\rangle = 2$, so $e^{m+1} = 2z$. Therefore $\mathbb{Z}/d\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, and $d = \pm 2$. Reduction modulo an odd prime preserves the nonzero coefficient. □

Proposition 13.6. *Let X^n, $n = 2m+1$ and $m \ge 2$, be a closed smooth homotopy sphere with a Besse metric of common prime geodesic length 2π and volume V_n. Let $\langle h\rangle$ act freely by isometries, with odd order $q \ge 5$, and put $\ell = 2\pi/q$. If*

$$F \subset \mathrm{Fix}(\widehat{h}^{-1} G_\ell)$$

is a connected component of dimension $2m+1$, then F is a mod-p cohomology sphere for every prime $p \mid q$. The evaluation map $\pi_F = \pi|_F : F \to X$ has degree nonzero modulo each such p and is surjective.

Proof. Put $A = \widehat{h}^{-1} G_\ell$. The map A has order dividing q and preserves λ. Its homogeneous extension to $T^*X \setminus 0$ is symplectic, has finite order, preserves $|\xi|$, and commutes with radial dilations. By Section 10.1, F is a closed contact submanifold. The geodesic circle commutes with A. Since it is connected, it preserves F. Put

$$B_+ = \varpi(F), \qquad B_- = \varpi(J(F)).$$

The map $F/S^1 \to \mathcal{G}$ is an injective immersion of a compact manifold, hence an embedding. The Reeb field is tangent to F; it spans the kernel of $d\lambda|_{TF}$. Thus $\omega|_{B_+}$ is nondegenerate. The identities for J give the same conclusions for $J(F)$ and B_-. Consequently $B_\pm$ are connected compact symplectic submanifolds of dimension $2m$, and $F = \varpi^{-1}(B_+)$, $J(F) = \varpi^{-1}(B_-)$. If $B_+ \cap B_- \ne \emptyset$, a unit tangent

vector v on a common orbit would satisfy both $\widehat{h}v = G_\ell v$ and $\widehat{h}v = G_{-\ell}v$. This would give $G_{2\ell}v = v$, contrary to $0 < 2\ell < 2\pi$.

Fix a prime $p \mid q$ and put $h_p = h^{q/p}$. For $z \in \mathrm{Fix}(\bar{h}_p)$ there is a unique $\theta(z) \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$\widehat{h}_p v = G_{\theta(z)} v \qquad (v \in \varpi^{-1}(z)).$$

It is independent of v, since $\widehat{h}_p$ commutes with the geodesic flow. Local trivializations of ϖ show that θ is continuous. Since $h_p^p = I$, its values lie in $(2\pi/p)\mathbb{Z}/(2\pi\mathbb{Z})$; hence it is constant on each connected component. On $B_\pm$ its values are $\pm 2\pi/p$. As p is odd, B_+ and B_- lie in distinct components C_+ and C_- of $\mathrm{Fix}(\bar{h}_p)$. These components are closed symplectic submanifolds [23, Example 3.3, p. 18]; write $\dim C_\pm = 2d_\pm$. Since $B_\pm \subset C_\pm$, one has $d_\pm \geq m$. The powers of $[\omega|_{C_\pm}]$ through exponent $d_\pm$ are nonzero in real cohomology. Therefore

$$\sum_j b_j(C_\pm; \mathbb{Q}) = \sum_j b_j(C_\pm; \mathbb{R}) \geq d_\pm + 1.$$

The universal coefficient theorem and Lemma 13.3 give

$$2m + 2 \geq \sum_j b_j(\mathrm{Fix}(\bar{h}_p); \mathbb{F}_p) \geq \sum_j b_j(C_+; \mathbb{Q}) + \sum_j b_j(C_-; \mathbb{Q}) \geq 2m + 2.$$

Equality forces $d_+ = d_- = m$ and excludes every other fixed component. Each $B_\pm$ is an embedded submanifold of the same dimension as $C_\pm$, and is both open and closed in it. Thus $C_\pm = B_\pm$. It follows that

$$\mathrm{Fix}(\widehat{h}_p^{-1} G_{2\pi/p}) = F.$$

Indeed, above B_+ the shift is $2\pi/p$ and the full inverse image is F; above B_- the time difference is $4\pi/p$, which is not a return time for odd p.

Put $\Bbbk = \mathbb{F}_p$. Reduce the classes u, x, y of Lemma 13.5 modulo p, retaining their notation, and choose a generator $v \in H^2(B\langle h_p\rangle; \Bbbk)$. Regard u, v as classes on $B(S^1 \times \langle h_p\rangle)$. Consider the Borel spectral sequence of the product action on E. Both factors act trivially on $H^*(E; \Bbbk)$: the circle is connected, and an odd-order automorphism of each cyclic integral cohomology group is the identity. Thus

$$d_2 y = (au + bv)x, \qquad a, b \in \Bbbk.$$

Restriction to the circle factor and Lemma 13.5 give $a = \pm 2$. The map $\pi : E \to X$ is $\langle h_p\rangle$-equivariant and pulls the top class of X back to y. In the $\langle h_p\rangle$-Borel spectral sequence of X, the d_2 of this top class is zero because $H^{2m}(X; \Bbbk) = 0$. After restriction to the finite factor, naturality therefore gives $bvx = 0$, so $b = 0$.

Let Δ_p be the cyclic subgroup generated by $(e^{2\pi i/p}, h_p^{-1})$. Its action on E has fixed set F. The restriction $w = u|_{B\Delta_p}$ generates $H^2(B\Delta_p; \Bbbk)$. Choose a generator $\epsilon \in H^1(B\Delta_p; \Bbbk)$; then $H^*(B\Delta_p; \Bbbk) = \Bbbk[w] \otimes \Lambda(\epsilon)$. Restriction of the product action gives

$$d_2 y = \pm 2wx, \qquad d_2 x = 0.$$

After extending scalars to $\Bbbk(w)$, the differential sends y to $\pm 2wx$ and ϵy to $\mp 2w\epsilon x$. Since $2w$ is invertible and $x^2 = 0$, the localized third page has basis $1, xy, \epsilon, \epsilon xy$, and hence dimension four. Lemma 13.2 gives

$$2 \sum_j b_j(F; \Bbbk) \leq 4.$$

Give F the orientation defined by $\lambda|_F$. Since F is closed and connected, its cohomology is nonzero in degrees zero and $n = 2m + 1$. All intermediate cohomology therefore vanishes.

Put $C_p = \langle h_p\rangle$ and choose an orientation of X. On F one has $\widehat{h}_p = G_{2\pi/p}$, so F is C_p-invariant. The action is free on F and X, and π_F is equivariant. For $Z = F, X$, the odd-order action preserves the integral orientation class. Its Borel fibration is therefore $\Bbbk$-orientable [47, Proposition 1.3, p. 281].

Let $\eta_Z \in H^{n+1}(BC_p;\Bbbk)$ be its Euler class and let $D \in \Bbbk$ be the reduction of $\deg(\pi_F)$ modulo p. The equivariant degree formula [47, Theorem 1.9, p. 282] gives

$$D\eta_F = \eta_X.$$

By [47, Theorem 2.3, pp. 283–284], η_X is the transgression of the chosen top class. The Borel spectral sequence of X has only rows zero and n, and $X_{C_p} \simeq X/C_p$ has zero cohomology in degree $n+1$. Thus the only possible differential into $H^{n+1}(BC_p;\Bbbk) \cong \Bbbk$ is nonzero. Hence $\eta_X \neq 0$ and $D \neq 0$. This holds for every prime $p \mid q$, so the integral degree is nonzero. If π_F omitted a point, its map on top cohomology would factor through the zero top cohomology of the punctured manifold X. Its degree would then be zero. Thus π_F is surjective. □

14. Contact volume and Clifford translations

Let $m \geq 1$ and let ω be a symplectic form on $\mathbb{R}^{2m}$. Write ω also for its matrix in the standard basis, and put

$$\mathrm{Sp}(\omega) = \{U \in \mathrm{GL}(2m,\mathbb{R}) \,:\, U^t\omega U = \omega\}, \qquad \mathfrak{sp}(\omega) = \{C \in M_{2m}(\mathbb{R}) \,:\, C^t\omega + \omega C = 0\}.$$

A linear map J with $J^2 = -I$ is ω-compatible if ωJ is symmetric positive definite. Set

$$\mathfrak{sp}^+(\omega) = \{C \in \mathfrak{sp}(\omega) \,:\, \omega C \text{ is symmetric positive definite}\}.$$

A smooth path $U : [0,T] \to \mathrm{Sp}(\omega)$, with $T > 0$, is positive if $U' = CU$ with $C(t) \in \mathfrak{sp}^+(\omega)$. Its Lorentz–Finsler length is

$$\mathsf{L}(U) = \int_0^T (\det C(t))^{1/(2m)}\, dt \tag{14.1}$$

[2, Proposition B.1]. The positively elliptic region consists of maps symplectically conjugate to direct sums of compatible planar rotations with angles in $(0,\pi)$. It is contractible [2, Proposition ii.2]. Its distinguished lift to the universal covering group is represented by the corresponding rotation paths starting at the identity. If a positive path from the identity ends in this lift, with rotation angles $\theta_1, \dots, \theta_m$, then

$$\mathsf{L}(U) \leq \frac{1}{m}\sum_{j=1}^{m} \theta_j \tag{14.2}$$

by [2, Theorem J.1(i)]. This bound applies to every positive path in the same fixed-endpoint homotopy class. Hedicke [39, Corollary 1.2, arXiv version] proves the sharper bound $\mathsf{L}(U) \leq (\theta_1 \cdots \theta_m)^{1/m}$ in this lift.

Write μ_{RS} and μ_{CZ} for the Robbin–Salamon and Conley–Zehnder indices, normalized so that a positive planar rotation through an angle in $(0,2\pi)$ has Conley–Zehnder index one. A crossing of U is a time t with $\ker(U(t)-I) \neq 0$. For a positive path $U : [0,T] \to \mathrm{Sp}(\omega)$ with $U' = CU$, the crossing form on $\ker(U(t)-I)$ is

$$v \longmapsto \omega(v, C(t)v) > 0 \qquad (v \neq 0).$$

Every crossing is regular and isolated, including one-sided isolation at an endpoint. Since the crossing set is closed, it is finite on a compact interval. Its Robbin–Salamon index is

$$\mu_{\mathrm{RS}}(U) = \frac{1}{2}\dim\ker(U(0)-I) + \sum_{0<t<T} \dim\ker(U(t)-I) + \frac{1}{2}\dim\ker(U(T)-I). \tag{14.3}$$

For a path from the identity with $\ker(U(T)-I) = 0$, one has $\mu_{\mathrm{RS}}(U) = \mu_{\mathrm{CZ}}(U)$. Inserting a based loop of winding $b \in \pi_1(\mathrm{Sp}(\omega)) \cong \mathbb{Z}$ changes this index by $2b$; a positive full rotation in one symplectic plane has winding one. These formulas and the direct-sum property are [35, Propositions 2.6, 4.6 and 4.7].

Proposition 14.1 (A contact-volume inequality). *Let (X^{2m+1}, g) be closed, with $m \geq 1$. Suppose a smooth unit field V generates a free circle action Φ_t of period 2π by geodesics, and suppose $\alpha = g(V, \cdot)$ is contact. Give X the contact orientation. Let $G_t : SX \to SX$ be the geodesic flow, and write ind_D for the Morse index with fixed endpoints. Let $q \geq 5$ be odd, put $\ell = 2\pi/q$, and write $\gamma_x(t) = \Phi_t(x)$. Assume:*

(i) Φ_ℓ is an isometry;
(ii) the full geodesic flow satisfies $G_{2\pi} = I$;
(iii) the endpoint-fixed Morse index satisfies $\mathrm{ind}_D(\gamma_x|_{[0,2\pi]}) \leq 2m$ for every $x \in X$;
(iv) for every $x \in X$, no point $\gamma_x(t)$ with $0 < t \leq 2\ell$ is conjugate to x along γ_x.

Then

$$\mathrm{Vol}_g(X) \geq \int_X \frac{\alpha \wedge (d\alpha)^m}{2^m m!}. \tag{14.4}$$

If equality holds, $\mathcal{L}_V g = 0$.

Proof. Since $|V| = 1$ and $\nabla_V V = 0$, one has

$$\alpha(V) = 1, \qquad d\alpha(V, Y) = \langle \nabla_V V, Y\rangle - \langle \nabla_Y V, V\rangle = 0.$$

Cartan's formula gives $\mathcal{L}_V \alpha = 0$, hence $\Phi_t^* \alpha = \alpha$.

Fix $x \in X$ and a basis $E_1(0), \dots, E_{2m}(0)$ of $\ker \alpha_x$. Along γ_x, set

$$E_i(t) = d\Phi_t E_i(0), \qquad H_{ij}(t) = g(E_i(t), E_j(t)), \qquad \Omega_{ij} = d\alpha(E_i(t), E_j(t)).$$

Let $E(t) : \mathbb{R}^{2m} \to V^{\perp}_{\gamma_x(t)}$ be the frame map with columns $E_i(t)$, and let D_t denote covariant differentiation along γ_x. Define $B(t)$ by $D_t E = EB$. The fields E_i are Jacobi fields obtained by varying the geodesic orbits. They form a normal frame, since $d\Phi_t$ is invertible and $\Phi_t^* \alpha = \alpha$. Thus H is positive definite and Ω is a constant invertible skew matrix. Torsion-freeness gives $D_t E_i = \nabla_{E_i} V$, whence

$$H' = B^t H + HB, \qquad \Omega = B^t H - HB, \qquad 2HB = H' - \Omega. \tag{14.5}$$

For a normal Jacobi field $J = Ey$, the Jacobi equations for the columns of E give $y'' + 2By' = 0$. Therefore

$$(Hy')' = \Omega y', \qquad Hy' - \Omega y = c$$

for a constant vector c. Setting $u = y + \Omega^{-1} c$ gives

$$u' = C(t)u, \qquad C = H^{-1}\Omega. \tag{14.6}$$

Let $U' = CU$, $U(0) = I$. If $y(0) = 0$, then $c = H(0)y'(0)$ and

$$y(t) = (U(t) - I)\Omega^{-1} H(0) y'(0). \tag{14.7}$$

A tangential Jacobi field vanishing at both endpoints is zero. Since $\Omega^{-1}H(0)$ is invertible, the multiplicity of a conjugate time $t > 0$ is therefore $\dim \ker(U(t) - I)$.

Put $\omega = -\Omega$. Then

$$C^t \Omega + \Omega C = 0, \qquad \omega C = -\Omega H^{-1} \Omega > 0, \tag{14.8}$$

since $-v^t \Omega H^{-1} \Omega v = (\Omega v)^t H^{-1} (\Omega v)$. Thus $U(t) \in \mathrm{Sp}(\omega)$ and U is positive. The flow law and hypothesis (i) give

$$E_i(t + \ell) = d\Phi_\ell E_i(t), \qquad H(t + \ell) = H(t), \qquad C(t + \ell) = C(t).$$

Uniqueness for the matrix differential equation gives, with $S = U(\ell)$,

$$U(t + \ell) = U(t)S. \tag{14.9}$$

By hypothesis (ii), every Jacobi field obtained by varying the initial unit velocity at x vanishes at time 2π. Equation (14.7) therefore gives

$$U(2\pi) = I, \qquad S^q = I. \tag{14.10}$$

Average an inner product over $\langle S \rangle$ to obtain an S-invariant inner product h. The polar construction of [12, Proposition 12.3 and Remark 1, pp. 68–69, author's version] gives an ω-compatible complex structure commuting with S, since S preserves both h and ω. The unitary spectral theorem then gives a symplectically orthogonal decomposition into planes on which

$$S = \exp\bigl((2\pi r_j/q)J_j\bigr), \qquad r_j \in \{1, \dots, q-1\}, \quad 1 \le j \le m,$$

with J_j compatible. Here $r_j = 0$ is excluded by (iv) at $t = \ell$. Write $R(t) = \bigoplus_j \exp(t r_j J_j)$.

By (iv) and (14.7), the path $U|_{[0,\ell]}$ has no crossing after zero, so its Conley–Zehnder index is m. The same is true of $R|_{[0,\ell]}$, whose rotation angles lie in $(0, 2\pi)$. The based loop $t \mapsto U(t)R(t)^{-1}$ has winding zero by the loop formula. It is null-homotopic, so the two paths have the same lift.

Equation (14.9) identifies the lift of $U|_{[0,2\ell]}$ with the square of the lift of $U|_{[0,\ell]}$, represented by $R|_{[0,2\ell]}$. Since q is odd, its endpoint is nondegenerate. Its Conley–Zehnder index is

$$m + 2\sum_{j=1}^{m} \left\lfloor \frac{2r_j}{q} \right\rfloor.$$

By (iv), this index is m. Every floor is nonnegative, hence zero. Thus

$$0 < \theta_j := 2\pi r_j/q < \pi. \tag{14.11}$$

The path $R|_{[0,\ell]}$ lies in the positively elliptic region for every positive time. It therefore represents the distinguished lift required in (14.2).

Likewise, $U|_{[0,2\pi]}$ has the same lift as $R|_{[0,2\pi]}$, whose Robbin–Salamon index is $2\sum_j r_j$. The actual path has endpoint crossing spaces of dimension $2m$. By (14.7) and the endpoint-fixed Morse index theorem [61, Theorem 15.1, p. 83], its interior crossing dimensions sum to $\operatorname{ind}_D(\gamma_x|_{[0,2\pi]})$. Terminal conjugacy does not contribute to this negative index. Consequently, (iii) gives

$$2\sum_j r_j = 2m + \operatorname{ind}_D(\gamma_x|_{[0,2\pi]}) \le 4m. \tag{14.12}$$

Applying (14.2) in the distinguished lift yields

$$\int_0^{\ell} (\det C(t))^{1/(2m)}\, dt \le \frac{1}{m}\sum_{j=1}^{m} \theta_j \le 2\ell. \tag{14.13}$$

Let dv_g be the Riemannian volume density and write

$$d\mu_\alpha = \frac{\alpha \wedge (d\alpha)^m}{2^m m!}, \qquad dv_g = f\, d\mu_\alpha.$$

Denote the Pfaffian of Ω by $\operatorname{Pf}\Omega$, so that $(\operatorname{Pf}\Omega)^2 = \det\Omega$. In the frame $(V, E_1, \dots, E_{2m})$, the Riemannian density is $\sqrt{\det H}$ and the contact density is $|\operatorname{Pf}\Omega|/2^m$. Hence

$$f = \frac{2^m\sqrt{\det H}}{|\operatorname{Pf}\Omega|}, \qquad (\det C)^{1/(2m)} = \left(\frac{\det\Omega}{\det H}\right)^{1/(2m)} = 2f^{-1/m}. \tag{14.14}$$

Put $a(x) = \ell^{-1}\int_0^{\ell} f(\Phi_t x)\, dt$. Equation (14.13) and Jensen's inequality give

$$a(x)^{-1/m} \le \frac{1}{\ell}\int_0^{\ell} f(\Phi_t x)^{-1/m}\, dt \le 1. \tag{14.15}$$

Thus $a(x) \ge 1$ for every x. Since μ_α is Φ_t-invariant, Fubini's theorem gives

$$\int_X f\, d\mu_\alpha = \int_X \frac{1}{\ell}\int_0^{\ell} f(\Phi_t x)\, dt\, d\mu_\alpha(x) \ge \int_X d\mu_\alpha.$$

This proves (14.4).

Suppose equality holds. Since $a-1$ is continuous and nonnegative, the equality $\int_X (a-1)\,d\mu_\alpha = 0$ gives $a \equiv 1$. Both inequalities in (14.15) are therefore equalities. Strict convexity of $s^{-1/m}$ makes $t \mapsto f(\Phi_t x)$ constant on $[0,\ell]$, with value one. Hence

$$f \equiv 1, \qquad (\det C)^{1/(2m)} \equiv 2. \tag{14.16}$$

The path $U|_{[0,\ell]}$ has length 2ℓ, so both bounds in (14.13) are equalities. By (14.2), it maximizes length among positive paths with the same endpoint lift. For $Y \in C_c^\infty((0,\ell), \mathfrak{sp}(\omega))$, the paths $U_s(t) = \exp(sY(t))U(t)$ have the same endpoints and fixed-endpoint homotopy class as U. Compactness of $[0,\ell]$ and openness of $\mathfrak{sp}^+(\omega)$ imply that these paths are positive for sufficiently small $|s|$. Thus U is stationary for the length functional. Proposition i.3 of [2] gives

$$U(t) = \exp(\tau(t)A), \qquad A \in \mathfrak{sp}^+(\omega), \quad \tau(0) = 0, \quad \tau' > 0.$$

Consequently $C(t) = \tau'(t)A$, and (14.16) gives

$$2 = \tau'(t)(\det A)^{1/(2m)}.$$

Hence τ' and C are constant. The identity $H = \Omega C^{-1}$ then gives $H' = 0$ on $[0,\ell]$, and periodicity gives $H' = 0$ for every t.

It follows that Φ_t preserves g on $\ker\alpha$. It also preserves V and α, with $|V| = 1$ and $\ker\alpha = V^\perp$. Hence $\Phi_t^* g = g$, and $\mathcal{L}_V g = 0$. □

14.1. **Clifford translations on Besse homotopy spheres.**

Theorem 14.2 (Clifford translations on Besse homotopy spheres). *Let (X,g) be a smooth homotopy sphere of odd dimension $n = 2m+1 \geq 5$. Suppose that every unit-speed geodesic has prime period 2π. If h_0 is a nonidentity isometry of constant displacement and of finite odd order $q \geq 5$, then (X,g) is isometric to the unit round sphere. There is an isometry $X \to \mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ under which*

$$h_0(z) = e^{2\pi i k/q} z \qquad \text{for some } k \in \mathbb{Z} \text{ with } \gcd(k,q) = 1.$$

Proof. Write $E = SX$ and let G_t be its geodesic flow. The constant displacement c_0 of the nonidentity isometry h_0 is positive: if it were zero, h_0 would fix every point. Thus h_0 is fixed-point-free. Every minimizing segment from x to $h_0 x$ realizes the global minimum displacement. Lemma 13.1 shows that it extends to an oriented geodesic on which h_0 is the time-c_0 shift. Such a geodesic passes through every $x \in X$. Since $h_0^q = I$, its unit tangent vector returns after qc_0, whence

$$qc_0 = 2\pi k \qquad (k \in \mathbb{Z}_{>0}).$$

If $d = \gcd(k,q) > 1$, then on every one of these geodesics the isometry $h_0^{q/d}$ acts by time $2\pi k/d$ and fixes every point of X. This contradicts the exact order of h_0. Thus $\gcd(k,q) = 1$. Choose $a \in \mathbb{Z}$ with $ak \equiv 1 \pmod q$, and put

$$h = h_0^a, \qquad \ell = \frac{2\pi}{q}.$$

The isometries h and h_0 generate the same cyclic group and have the same fixed-point set. Hence h has no fixed point. On each h_0-invariant geodesic constructed above, h acts by time ℓ modulo 2π, so $d(x,hx) \leq \ell$ for every x. If $b > 0$ is the minimum displacement of h, Lemma 13.1 and $h^q = I$ give $qb \in 2\pi\mathbb{Z}_{>0}$. Therefore

$$d(x,hx) = \ell \qquad (x \in X). \tag{14.17}$$

Let $p : E \to \mathcal{G} = E/S^1$ be the free geodesic circle bundle, and let $J(x,v) = (x,-v)$. Define

$$\mathcal{F} = \mathrm{Fix}(\widehat{h}^{-1} G_\ell).$$

The map $\widehat{h}^{-1} G_\ell$ has finite order dividing q, and its homogeneous extension to $T^*X \setminus 0$ is symplectic. The fixed-cone argument of Section 10.1 makes $\mathcal{F}$ a finite union of closed smooth contact submanifolds.

The connected geodesic circle preserves each component. Equation (14.17) and Lemma 13.1 imply that evaluation $\mathcal{F} \to X$ is surjective. If all its components had dimension less than n, Sard's theorem would make the image of each a set of measure zero in X. Their finite union cannot be all of X. Thus some component $F_0 \subset \mathcal{F}$ has $\dim F_0 \geq n$. Set $B_0 = p(F_0)$. Since F_0 is circle-invariant, $F_0 = p^{-1}(B_0)$ and $B_0 = F_0/S^1$ is embedded in $\mathcal{G}$.

Every isometry b induces a map $\bar{b}$ on $\mathcal{G}$. Write J also for the induced reversal on $\mathcal{G}$. Fix an odd prime $r \mid q$, and put $h_r = h^{q/r}$. On the oriented geodesics in B_0 and $J(B_0)$, the time shifts of h_r are respectively $2\pi/r$ and $-2\pi/r$. On a connected component of $\mathrm{Fix}(\bar{h}_r : \mathcal{G} \to \mathcal{G})$, the shift is a continuous function with values in $\{2\pi j/r : 0 \leq j < r\}$; it is therefore constant. Thus B_0 and $J(B_0)$ lie in distinct fixed components C_+ and C_-. Since $\bar{h}_r$ is a finite-order symplectomorphism, these components are closed symplectic submanifolds. Write $\dim C_\pm = 2d_\pm \geq 2m$. The powers of their symplectic classes give $\sum_j b_j(C_\pm; \mathbb{R}) \geq d_\pm + 1$. For a compact manifold, rational and real Betti numbers coincide. Lemma 13.3 and the universal coefficient theorem give

$$2m + 2 = \sum_j b_j(\mathcal{G}; \mathbb{F}_r) \geq \sum_j b_j(\mathrm{Fix}(\bar{h}_r); \mathbb{F}_r) \geq (d_+ + 1) + (d_- + 1) \geq 2m + 2.$$

All inequalities are equalities. Hence $\dim F_0 = n$, the only fixed components are $C_\pm$, and they have dimension $2m$. The compact embedded submanifolds $B_0, J(B_0)$ have that dimension, so they are open and closed in the respective components. Consequently

$$\mathrm{Fix}(\bar{h}_r) = B_0 \sqcup J(B_0).$$

The prime divisor r of q was arbitrary.

Every point of $\mathcal{F}$ projects into this union. Above B_0, $\widehat{h} = G_\ell$, whereas above $J(B_0)$, $\widehat{h} = G_{-\ell}$. The second possibility is incompatible with $\widehat{h} = G_\ell$, since $G_{2\ell}$ has no fixed unit tangent vector. Therefore $\mathcal{F} = F_0$. Write

$$F = \mathcal{F}, \qquad B_+ = F/S^1, \qquad B_- = J(F)/S^1.$$

In particular, F is connected, has dimension n, and evaluates onto X. Every nonidentity $b \in \langle h \rangle$ generates a subgroup containing a subgroup of prime order. Since both B_+ and B_- are fixed by all of $\langle h \rangle$, it follows that

$$\mathrm{Fix}(\bar{b}) = B_+ \sqcup B_- \qquad (b \in \langle h \rangle \setminus \{I\}). \tag{14.18}$$

The action of $\langle h \rangle$ on X is free. Indeed, if a nontrivial power fixed x, a prime-order subgroup $\langle h_r \rangle$ would also fix x. The orthogonal transformation $d(h_r)_x$ has odd order on the odd-dimensional real vector space T_xX and hence has eigenvalue 1. A corresponding unit vector defines an oriented geodesic fixed pointwise by h_r. Its shift is zero, contrary to (14.18), where the shifts are $\pm 2\pi/r$.

For $1 \leq j < q$, surjectivity of $F \to X$ gives

$$d(x, h^j x) \leq \ell \min\{j, q - j\}.$$

Put $\delta_j = \min_x d(x, h^j x) > 0$. Let $v \in SX$ be the initial unit vector of a segment attaining this minimum. Lemma 13.1 gives equality of unit tangent vectors

$$G_{\delta_j} v = \widehat{h}^j v.$$

Equation (14.18) places $p(v)$ in B_+ or B_-. On these two sets, respectively, $\widehat{h}^j v = G_{j\ell} v$ or $G_{-j\ell} v$. The prime period is 2π; hence

$$\delta_j \equiv j\ell \pmod{2\pi} \quad \text{or} \quad \delta_j \equiv -j\ell \pmod{2\pi}.$$

Since $\delta_j > 0$, either congruence implies $\delta_j \geq \ell \min\{j, q - j\}$. Together with the upper bound this gives

$$d(x, h^j x) = \ell \min\{j, q - j\} \qquad (x \in X, 1 \leq j < q). \tag{14.19}$$

The evaluation map $\pi_F = \pi|_F : F \to X$ is a diffeomorphism. Let $v, w \in F$ have the same base point x. Their first length-ℓ segments c_v, c_w satisfy

$$\operatorname{length}\big(c_v * (hc_w)\big) = 2\ell = d(x, h^2x),$$

where the last equality requires $q \geq 5$. Varying the junction with velocity $z \in T_{hx}X$, while fixing the two outer endpoints, gives

$$0 = \langle dh(v) - dh(w), z\rangle.$$

Hence $dh(v) = dh(w)$ and $v = w$. If $\xi \in T_vF$ is vertical, differentiation of $G_\ell v = \widehat{h}v$ gives a normal Jacobi field with

$$Y(0) = Y(\ell) = 0.$$

The segment of length 2ℓ minimizes distance, and $0 < \ell < 2\ell$. Lemma 13.4 gives $Y = 0$, so its initial data, and therefore ξ, vanish. Thus $d\pi_F : T_vF \to T_xX$ is injective. Its domain and codomain both have dimension n, so it is an isomorphism. The inverse function theorem and bijectivity give a smooth global inverse.

Write $\pi_F^{-1}(x) = (x, V(x))$ and let Φ_t be the flow of the smooth unit field V. Geodesic-flow invariance of F gives

$$G_t(x, V(x)) = (\Phi_t(x), V(\Phi_t(x))), \qquad \nabla_V V = 0, \qquad \Phi_{2\pi} = I, \qquad \Phi_\ell = h.$$

If $\Phi_s(x) = x$, the displayed identity gives $G_s(x, V(x)) = (x, V(x))$, so $s \in 2\pi\mathbb{Z}$. Thus Φ_t defines a free circle action. Let λ be the canonical contact form on SX. Then

$$\alpha = g(V, \cdot) = (\pi_F^{-1})^*(\lambda|_F)$$

is contact, and V is its Reeb field.

Set $T = 2\ell$, $v = (x, V(x)) \in F$, and $\gamma(t) = \Phi_t(x)$ for $0 \leq t \leq T$. By (14.19), γ minimizes distance. Lemma 13.4 excludes conjugate times in $(0, T)$. Put

$$\mathscr{V}_h = \{Z \in C^\infty(\gamma^*TX) : Z(T) = d(h^2)_x Z(0)\}.$$

Every smooth curve $c : [0, T] \to X$ with $c(T) = h^2c(0)$ satisfies

$$E(c) := \frac{1}{2}\int_0^T |\dot{c}|^2 dt \geq \frac{\operatorname{length}(c)^2}{2T} \geq \frac{T}{2} = E(\gamma).$$

Let D_t be covariant differentiation along γ, and let R be the Riemannian curvature tensor. Define

$$I(Y, Z) = \int_0^T \Big(\langle D_tY, D_tZ\rangle - \langle R(Y, \dot{\gamma})\dot{\gamma}, Z\rangle\Big)\, dt.$$

For $Z \in \mathscr{V}_h$, the variation $c_s(t) = \exp_{\gamma(t)}(sZ(t))$ satisfies $c_s(T) = h^2c_s(0)$, since h^2 is an isometry. Its endpoint curves are geodesics in the parameter s. The second variation formula [66, Chapter 6, Theorem 21, p. 158] therefore gives

$$I(Z, Z) = \frac{d^2}{ds^2}E(c_s)\bigg|_{s=0} \geq 0.$$

If a Jacobi field Y satisfies $Y(0) = Y(T) = 0$, integration by parts gives $I(Y, Y) = 0$. Lemma 13.4 then gives $I(Y, Z) = 0$ for every $Z \in \mathscr{V}_h$. A second integration by parts gives

$$0 = \langle dh^{-2}D_tY(T) - D_tY(0), Z(0)\rangle.$$

Every vector of T_xX occurs as $Z(0)$ for some $Z \in \mathscr{V}_h$. Consequently

$$D_tY(T) = dh^2D_tY(0).$$

The tangential part of Y is $f\dot{\gamma}$ with $f'' = 0$ and $f(0) = f(T) = 0$; hence it vanishes. The normal initial data of Y define a vertical vector $\xi \in T_vSX$ fixed by $d(\widehat{h}^{-2}G_T)_v$. Put $A = \widehat{h}^{-2}G_T$. The commuting

relations and $qT = 4\pi$ give $A^q = I$. If $w \in SX$ satisfies $A(w) = w$, then $p(w)$ is fixed by $\overline{h^2}$, so (14.18) gives $p(w) \in B_+ \sqcup B_-$. Above B_+ one has $A = I$; above B_- one has $A = G_{4\ell}$. The latter has no fixed unit tangent vector, since $q \nmid 4$. Thus $\mathrm{Fix}(A) = F$. Section 10.1 gives

$$\xi \in \ker(dA_v - I) = T_vF.$$

The vertical kernel of $d\pi_F$ is zero. Hence $\xi = 0$, and uniqueness for the Jacobi equation gives $Y = 0$.

Put $B = X/S^1$ for the action Φ_t and let $p_B : X \to B$ be the quotient map. Since $i_V d\alpha = 0$ and $\Phi_t^* d\alpha = d\alpha$, there is a two-form ω_B with $p_B^*\omega_B = d\alpha$. It is symplectic because α is contact. Orient B by ω_B^m and choose the Euler class $e_B \in H^2(B;\mathbb{Z})$ so that $(e_B)_{\mathbb{R}} = [\omega_B/(2\pi)]$. The homotopy-sphere cohomology gives $H^j(X;\mathbb{Z}) = 0$ for $0 < j < 2m+1$. The integral Gysin sequence [37, Section 4.D, p. 438] gives

$$H^1(B;\mathbb{Z}) = 0, \qquad \smile e_B : H^{j-2}(B;\mathbb{Z}) \xrightarrow{\cong} H^j(B;\mathbb{Z}) \quad (2 \le j \le 2m).$$

Since $H^0(B;\mathbb{Z}) = \mathbb{Z}$ and $\dim B = 2m$, it follows that

$$H^*(B;\mathbb{Z}) = \mathbb{Z}[e_B]/(e_B^{m+1}).$$

In particular e_B^m generates $H^{2m}(B;\mathbb{Z})$, so its pairing with $[B]$ is 1 or -1. Its representative $(\omega_B/(2\pi))^m$ is positive, so $\langle e_B^m, [B]\rangle = 1$. Finally $\int_{p_B^{-1}(b)} \alpha = 2\pi$, and integration over the fibers gives

$$\int_X \frac{\alpha \wedge (d\alpha)^m}{2^m m!} = \frac{(2\pi)^{m+1}}{2^m m!} = V_n. \tag{14.20}$$

Lemma 11.2 bounds the endpoint-fixed index on $[0, 2\pi]$ by $2m$. Moreover, $\Phi_\ell = h$ is an isometry, $G_{2\pi} = I$, and no conjugate point occurs in $(0, 2\ell]$. Proposition 14.1 and (14.20) give $\mathrm{Vol}_g(X) \ge V_n$. Proposition 12.3 gives equality. Hence every Φ_t is an isometry.

Its time-π map is a fixed-point-free isometric involution. The quotient is Besse and has a degree-two smooth covering by the homotopy sphere X. It has constant sectional curvature by Lin–Schmidt [59, Theorem 2]; so does X. Since X is compact and simply connected, it is a round sphere. Its prime period fixes its radius as one. On the unit sphere, the unit Killing field V has the form $V(x) = Ax$ for an orthogonal complex structure A [8, Theorem 15, with $l = 1$]. In complex coordinates for A, one has $\Phi_t(z) = e^{it}z$. Thus $h(z) = e^{2\pi i/q}z$ and $h_0 = h^k$, as required. □

Corollary 14.3. *Let X be a smooth homotopy sphere of odd dimension $n \ge 5$, with a Besse metric. If it admits a constant-displacement isometry of finite odd order at least five, then its sectional curvature is constant.*

Proof. By [73, Theorem A], X is homeomorphic to $\mathbb{S}^n$. Radeschi–Wilking [70, Theorem A] therefore give a common prime geodesic length L. The metric $(2\pi/L)^2 g$ satisfies Theorem 14.2. □

Proposition 14.4. *Let X be a smooth homotopy sphere of dimension at least five with a Besse metric. If it admits a constant-displacement isometry of infinite order, then its sectional curvature is constant.*

Proof. Put $n = \dim X$. By [73, Theorem A], X is homeomorphic to $\mathbb{S}^n$. Radeschi–Wilking [70, Theorem A] give one common prime geodesic length L. Replace g by $(2\pi/L)^2 g$. Let h have constant displacement $c > 0$. For each x, Lemma 13.1 extends a unit-speed minimizing segment from x to hx to a complete geodesic γ_x satisfying $h\gamma_x(t) = \gamma_x(t + c)$, and hence $h^j x = \gamma_x(jc)$ for every $j \in \mathbb{Z}$. If $c/(2\pi)$ were rational, a positive power of h would fix every point. Hence $c/(2\pi)$ is irrational.

By Deng–Xu [19, Theorem 3.4, arXiv version], there is $\varepsilon > 0$ such that every constant-displacement isometry of X with displacement less than ε is generated by a constant-length Killing field. Let $\mathrm{inj}(X) > 0$ be the global injectivity radius. Irrationality gives $j \in \mathbb{Z}$ and

$$0 < \delta < \min\{\mathrm{inj}(X), \varepsilon, 2\pi\}, \qquad jc \equiv \delta \pmod{2\pi}.$$

For every x, one has $h^j x = \gamma_x(\delta)$; hence $d(x, h^j x) = \delta$. Thus h^j is generated by a nonzero constant-length Killing field. Let V be its unit normalization and Φ_t its flow. The orbits of V are geodesics

[19, Corollary 3.2, arXiv version]. Their prime period is 2π, so Φ_t defines a free circle action by isometries, as in the proof of Theorem 14.2. Its time-π map is a fixed-point-free isometric involution. The Besse quotient $X/\langle\Phi_\pi\rangle$ has a degree-two smooth covering by X. Lin–Schmidt [59, Theorem 2] gives constant sectional curvature on the quotient and hence on X. □

Remark 14.5. Let $m \geq 1$, let $q \geq 5$ be odd, and put $\zeta = e^{2\pi i/q}$. On the unit sphere in $\mathbb{C}^{m+1}$, the isometry

$$h = \operatorname{diag}(\zeta, \dots, \zeta, \zeta^2)$$

generates a free cyclic action: if h^j has eigenvalue one, then $q \mid j$ or $q \mid 2j$, and either condition gives $q \mid j$. For unit coordinate vectors e_1, e_{m+1},

$$d(e_1, he_1) = \frac{2\pi}{q}, \qquad d(e_{m+1}, he_{m+1}) = \frac{4\pi}{q}.$$

Thus the displacement of h is not constant.

15. Reciprocal times and orders of deck transformations

Write SY for the unit tangent bundle of a Riemannian manifold Y and G_t for its geodesic flow. For an isometry h of Y, put $\widehat{h}(x, v) = (h(x), dh_x v)$.

Lemma 15.1. *Let X be a closed connected Riemannian manifold of positive dimension, and suppose every unit-speed geodesic has the same prime period $L > 0$. Let a finite group Γ act freely by isometries and put $M = X/\Gamma$. For every integer $s \geq 2$,*

$$\operatorname{Fix}(G_{L/s} : SM \to SM) \neq \emptyset \quad \Longleftrightarrow \quad \Gamma \text{ contains an element of order } s.$$

Every constant-speed closed geodesic of length L/s on M minimizes energy in its free homotopy class, with the parameter interval fixed. In particular, its periodic Morse index is zero. If $\mathcal{O}(\Gamma) = \{\operatorname{ord}(h) : h \in \Gamma\}$ and $\mathscr{L}(M)$ is the set of lengths of nonconstant closed geodesics, including their iterates, then

$$\mathscr{L}(M) = \bigcup_{s \in \mathcal{O}(\Gamma)} \frac{L}{s}\mathbb{Z}_{>0}. \tag{15.1}$$

Proof. Let $h \in \Gamma$ have order $s \geq 2$ and put $b = \min_x d(x, hx) > 0$. Lemma 13.1 gives a complete unit-speed geodesic γ with $h\gamma(t) = \gamma(t + b)$. Since $h^s = I$, its unit tangent vector returns after sb, so $sb = kL$ for some $k \in \mathbb{Z}_{>0}$. If $d = \gcd(k, s) > 1$, the nonidentity element $h^{s/d}$ shifts γ by $(k/d)L$ and fixes a point of X, contrary to freeness. Thus k is a unit modulo s. Choose $a \in \mathbb{Z}$ with $ak \equiv 1 \pmod{s}$. Then h^a acts on γ by time L/s modulo L, so its projection to M returns after L/s.

Conversely, lift a returning tangent state to $v \in SX$ and write $G_{L/s}v = \widehat{h}v$ with $h \in \Gamma$. Isometries commute with geodesic flow. Hence $\widehat{h}^s v = G_L v = v$, and freeness gives $h^s = I$. If h had order $r < s$, then $G_{rL/s}v = v$, contrary to the prime period L. Thus $\operatorname{ord}(h) = s$.

The displacement argument also gives, for every element h of order $s \geq 2$,

$$s \min_x d(x, hx) \in L\mathbb{Z}_{>0}, \qquad \min_x d(x, hx) \geq L/s. \tag{15.2}$$

Let c be a quotient geodesic of length L/s, with deck monodromy h. The preceding paragraph gives $\operatorname{ord}(h) = s$. Parametrize c with constant speed on $[0, T]$. Every piecewise smooth loop $u : [0, T] \to M$ freely homotopic to c has a lift from some $y \in X$ to $h'y$, where h' is conjugate to h in Γ. Along a lifted free homotopy the endpoint deck transformation is locally constant; changing the initial lift conjugates that transformation. Hence $\operatorname{length}(u) \geq d(y, h'y) \geq L/s$ by (15.2). Cauchy–Schwarz gives

$$E(u) := \frac{1}{2}\int_0^T |\dot{u}|^2\, dt \geq \frac{\operatorname{length}(u)^2}{2T} \geq \frac{(L/s)^2}{2T} = E(c).$$

Every periodic variation of c is a free homotopy, so its second variation is nonnegative.

Finally, a return at time $t > 0$ lifts to $G_t v = \widehat{h} v$ for some $h \in \Gamma$. If $s = \operatorname{ord}(h)$, then $G_{st} v = v$, and hence $t \in (L/s)\mathbb{Z}_{>0}$. This proves one inclusion in (15.1). For every element order $s \geq 2$, the first paragraph produces a quotient geodesic of length L/s; its iterates give all positive multiples. A length-L geodesic on X supplies the case $s = 1$. □

Lemma 15.2. *Under the hypotheses of Lemma 15.1, let Δ_M be the scalar Laplacian. Choose $c \in \mathbb{R}$ such that $P = \sqrt{\Delta_M + c}$ is positive. Suppose A is self-adjoint and $A - P \in \Psi_{\mathrm{cl}}^{-1}(M)$. Fix an integer $s \geq 2$ and put $t = L/s$. If $\operatorname{Fix}(G_t : SM \to SM)$ is nonempty and its maximum component dimension is d_s, then, for every nonzero $0 \leq \chi \in C_c^\infty((0,1))$,*

$$\operatorname{Tr}(e^{itA}\chi(A/R)) = C_{s,\chi} R^{(d_s+1)/2} + o(R^{(d_s+1)/2}), \qquad C_{s,\chi} \neq 0.$$

Here $d_s \geq 1$, so this trace is not $o(R)$. If the fixed set is empty, then, for every $\chi \in C_c^\infty((0,1))$, the trace is $O(R^{-N})$ for every N.

Proof. Since $G_L = I$ on SM, the time-t map has finite order. By Section 10.1, its fixed set is a finite union of closed clean contact submanifolds. Each nonempty component has positive odd dimension. By Lemma 15.1, every corresponding periodic Morse index is zero.

The difference $A - P$ is bounded, so A has domain $H^1(M)$. It is elliptic and bounded below. For each nonpositive eigenvalue λ of A, let Π_λ be its spectral projector, and put

$$A_+ = A + \sum_{\lambda \leq 0} (1 - \lambda)\Pi_\lambda.$$

The sum is finite, its terms are smoothing, and $A_+ > 0$. For sufficiently large R,

$$e^{itA_+}\chi(A_+/R) = e^{itA}\chi(A/R).$$

On scalar half-densities, P has principal symbol $|\xi|_g$ and zero subprincipal symbol [21, p. 58]. Since $A_+ - P \in \Psi_{\mathrm{cl}}^{-1}(M)$, the same holds for A_+. Apply (10.7) with $B = A_+$ and $\sigma = 0$. With the density and phase defined in Section 10.1, this gives

$$C_{s,\chi} = \gamma_{d_s,0} \left(\int_0^1 \chi(u) u^{(d_s-1)/2}\, du \right) \sum_{\substack{C \subset \operatorname{Fix}(G_t) \\ \dim C = d_s}} \int_C dv_C.$$

The factor $\gamma_{d_s,0}$ is nonzero, and the moment and every density integral are positive. Thus $C_{s,\chi} \neq 0$. Components of smaller dimension contribute lower powers. If the fixed set is empty, the estimate for an empty fixed set in Section 10.1, applied to A_+, gives rapid decay. □

Theorem 15.3. *Let $M_i = X_i/\Gamma_i$, $i = 1, 2$, satisfy Lemma 15.1 with the same upstairs prime period L, and let A_i satisfy Lemma 15.2. Suppose that, along $R_j \to \infty$, for every fixed time t and every $\chi \in C_c^\infty((0,1))$,*

$$\operatorname{Tr}(e^{itA_1}\chi(A_1/R_j)) - \operatorname{Tr}(e^{itA_2}\chi(A_2/R_j)) = o(R_j).$$

Then Γ_1 and Γ_2 have the same sets of element orders.

Proof. Suppose precisely one group contains an element of order $s \geq 2$. By Lemma 15.1, precisely one quotient returns at $t = L/s$. Choose a nonzero $0 \leq \chi \in C_c^\infty((0,1))$. Lemma 15.2 makes the trace difference $CR_j^\alpha + o(R_j^\alpha)$, where $C \neq 0$ and $\alpha \geq 1$. Dividing by R_j^α contradicts the hypothesis. Both groups contain the identity. □

For $X = \mathbb{S}^n(1)$, $n \geq 2$, formula (15.1) is the positive-length form of Prüfer's formula [68, Corollary 1, p. 135]. For odd-dimensional isospectral round spherical space forms, equality of the element-order sets is due to Ikeda [45, Corollary 2.8, p. 84].

Lemma 15.4. *Let a finite group Γ of odd order act freely and smoothly on a closed homotopy sphere X. Every Sylow subgroup of Γ is cyclic, and its exponent $\operatorname{lcm}\{\operatorname{ord}(h) : h \in \Gamma\}$ equals $|\Gamma|$.*

Proof. If $\dim X$ is even, the covering identity $2 = \chi(X) = |\Gamma|\chi(X/\Gamma)$ gives $|\Gamma| = 1$. Suppose $\dim X$ is odd. A finite CW structure on the smooth quotient X/Γ lifts to a finite free Γ-CW structure on X. Since X is an integral homology sphere, [11, Proposition VII.10.2, p. 180] gives periodic cohomology for Γ. By [11, Theorem VI.9.5, p. 157], its Sylow subgroups are cyclic or generalized quaternion. Their odd orders exclude the latter case.

A generator of each Sylow subgroup realizes the full prime-power factor of $|\Gamma|$. Hence $|\Gamma|$ divides the exponent. Lagrange's theorem gives the reverse divisibility. □

15.1. **The covering radius.**

Lemma 15.5. *Let $n = 2m + 1$, where $m \geq 2$ is even. Let $M = \mathbb{S}^n(2)/\Gamma$ and $Q = \mathbb{S}^n(1)/\Gamma_0$ be round spherical space forms with*

$$|\Gamma| = d = 2^{2m+1}q, \qquad |\Gamma_0| = q \text{ odd.}$$

Put

$$P = \sqrt{\Delta_M + m^2}, \qquad P_0 = \sqrt{\Delta_M + m^2/4}, \qquad P_Q = \sqrt{\Delta_Q + m^2}.$$

Only even harmonic degrees $2l$ occur on M. Their P-frequencies satisfy

$$\sqrt{l(l+m) + m^2} = l + \frac{m}{2} + O(l^{-1}), \tag{15.3}$$

and their P_0-frequencies are exactly $l + m/2$. For every $\chi \in C_c^\infty((0,1))$,

$$\operatorname{Tr} \chi(P_0/R) - \operatorname{Tr} \chi(P_Q/R) = \frac{m-1}{8q(2m-2)!} R^{2m-1} \int_0^1 \chi(u) u^{2m-2}\, du + O(R^{2m-2}). \tag{15.4}$$

Proof. Cauchy's theorem and Lemma 8.2 give $-I \in \Gamma$. It acts by $(-1)^k$ on degree-k harmonics, so the odd degrees have no invariants. At degree $2l$, the radius-two Laplace eigenvalue is $l(l+m)$. If $b = l + m/2$, then

$$\sqrt{l(l+m) + m^2} = \sqrt{b^2 + 3m^2/4} = b + \frac{3m^2}{8b} + O(b^{-3}), \qquad \sqrt{l(l+m) + m^2/4} = b.$$

The harmonic dimensions in (8.1) satisfy $d_k = D_m(k+m)$, where

$$D_m(u) = \frac{2}{(2m)!} u^2 \prod_{j=1}^{m-1} (u^2 - j^2). \tag{15.5}$$

Put $c_m = 2/(2m)!$ and $S_m = \sum_{j=1}^{m-1} j^2 = m(m-1)(2m-1)/6$. Then

$$D_m(b) = c_m\big(b^{2m} - S_m b^{2m-2} + O(b^{2m-4})\big).$$

Since m is even, the P_0- and P_Q-frequencies lie on the integer lattice. Apart from finitely many initial values of b, the identity contribution for Q at frequency b is $D_m(b)/q$, whereas the identity and antipodal contributions for M sum to $2D_m(2b)/d$. Hence

$$\frac{2}{d} D_m(2b) - \frac{1}{q} D_m(b) = \frac{3c_m S_m}{4q} b^{2m-2} + O(b^{2m-4}) = \frac{m-1}{8q(2m-2)!} b^{2m-2} + O(b^{2m-4}).$$

By partial fractions in (8.2), the coefficient of every remaining group term is a finite sum of expressions $p(k)\alpha^{-k}$, where p is a polynomial and α is a root of unity. On M, restriction to $k = 2l$ gives the phase $(\alpha^{-2})^l$. This phase is constant only if $\alpha^2 = 1$. By Lemma 8.2, such poles occur only for I and $-I$. On Q, a constant phase requires $\alpha = 1$, which occurs only for the identity. Lemma 8.4 therefore makes every remaining annular sum $O(R^{-N})$ for each N. The finitely many initial terms vanish for large R. Finally,

$$\sum_{b \in \mathbb{Z}} b^{2m-2} \chi(b/R) = R^{2m-1} \int_0^1 \chi(u) u^{2m-2}\, du + O(R^{2m-2}).$$

Together with the displayed polynomial difference, this proves (15.4). □

Theorem 15.6. *Let $m \geq 2$ be an integer, put $n = 2m+1$, and let $Q = \mathbb{S}^n(1)/\Gamma_0$ be a round spherical space form with $q = |\Gamma_0|$ odd. Let M be a closed connected Riemannian manifold. For the scalar frequencies $\sqrt{\lambda + m^2}$ of M and Q, let $E_\eta(R)$ be the matching defect of Definition 2.1. Put $R_K = m + K + \frac{1}{2}$ for $K \in \mathbb{Z}_{\geq 0}$. If, for some fixed $0 \leq \eta < 1/4$ and a sequence $K_j \to \infty$,*

$$E_\eta(R_{K_j}) = o(R_{K_j}),$$

then M has dimension n. Its universal Riemannian cover X is a smooth homotopy sphere on which all prime geodesics have length 2π, and

$$|\pi_1(M)| = q, \qquad \mathrm{Vol}(M) = V_n/q, \qquad \mathrm{Vol}(X) = V_n.$$

The groups $\pi_1(M)$ and Γ_0 have the same sets of element orders.

Proof. Put $R_j = R_{K_j}$ and $P_M = \sqrt{\Delta_M + m^2}$. A zero-dimensional M has finite spectrum, so the counting inequality $|N_M(R_j) - N_Q(R_j)| \leq E_\eta(R_j)$ and Weyl's law for Q exclude this case. Proposition 10.1 gives dimension n, volume V_n/q, least common geodesic period 2π, and scalar return symbol 1. Its quantizer A, constructed in Corollary 9.2, is positive and self-adjoint, strongly commutes with P_M, and satisfies

$$A - P_M \in \Psi^{-1}_{\mathrm{cl}}, \qquad \mathrm{Spec}(A) \subset m + \mathbb{Z}_{\geq 0}.$$

The trace comparison (10.3) is $o(R_j)$ for every fixed time and every smooth annular cutoff. Lemma 11.1 gives a finite universal covering by a smooth homotopy sphere X. Let Γ be its deck group, put $d = |\Gamma|$, and denote its common prime length by L. Then $L \in \{2\pi, 4\pi\}$ and $\mathrm{Vol}(X) = dV_n/q$.

Suppose d is even. The Besse manifold M has an even-degree smooth covering by the homotopy n-sphere X, with $n \neq 3$. Lin–Schmidt rigidity [59, Theorem 2] gives constant sectional curvature. Its compact simply connected Riemannian cover X is therefore a round sphere. Its prime length fixes its radius $R_0 \in \{1, 2\}$, and volume gives $d = qR_0^n$. Since d is even and q is odd, one has $R_0 = 2$ and $d = 2^n q$.

The deck group contains $-I$, so only degrees $2l$ occur and their P_M-frequencies satisfy

$$\mu_l = \sqrt{l(l+m) + m^2} = l + \frac{m}{2} + O(l^{-1}).$$

If m is odd, then $e^{2\pi i\mu} \to -1$ as $\mu \to \infty$ in $\mathrm{Spec}(P_M)$. Hence $e^{2\pi i P_M} + I$ is compact. But Proposition 10.1 gives $e^{2\pi i P_M} - I \in \Psi^{-1}_{\mathrm{cl}}$, which is also compact. Their difference is $2I$, a contradiction.

If m is even, $l + m/2$ is the nearest integer to μ_l for large l. The nearest-lattice property in Corollary 9.2 gives

$$A = P_0 := \sqrt{\Delta_M + m^2/4}$$

outside a finite sum of eigenspaces. These exceptional eigenspaces make no contribution to an annular trace for large R. For a nonnegative nonzero cutoff, Lemma 15.5 therefore gives a nonzero time-zero trace difference of order R_j^{2m-1}. Since $m \geq 2$, this contradicts the $o(R_j)$ trace comparison. Thus d is odd. The deck involution in Lemma 11.1 is then the identity, so $L = 2\pi$.

Theorem 15.3 applies to $M = X/\Gamma$ with A and to $Q = \mathbb{S}^n(1)/\Gamma_0$ with P_Q, with common upstairs prime length 2π. Their sets of element orders coincide. Lemma 15.4 identifies the order of each odd-order group with its exponent. Hence $d = q$, and $\mathrm{Vol}(X) = dV_n/q = V_n$. □

An isometry γ of a round sphere is a Clifford translation if $x \mapsto d(x, \gamma x)$ is constant.

Theorem 15.7 (Rigidity from a Clifford subgroup). *Under the hypotheses of Theorem 15.6, assume that Γ_0 contains a Clifford translation of odd order $r \geq 5$. Then M is a round unit spherical space form of degree q, and its complete scalar spectrum equals that of Q.*

Proof. Put $t = 2\pi/r$. Let $h_0 \in \Gamma_0$ be a Clifford translation of order r, and let c be its constant displacement. On the unit sphere, $h_0 + h_0^{-1} = 2\cos(c)I$. Lemma 8.2 gives $c = 2\pi a/r$ for some $a \in \mathbb{Z}$ with $\gcd(a, r) = 1$. Choose $b \in \mathbb{Z}$ with $ab \equiv 1 \pmod r$. Then h_0^b has eigenvalues e^{it}

and e^{-it}, each with complex multiplicity $m+1$. For $g \in \Gamma_0$, let $g_{\mathbb{C}}$ be its complexification and put $\rho_g = \dim_{\mathbb{C}} \ker(g_{\mathbb{C}} - e^{it} I)$. If $\rho_g > 0$, Lemma 8.2 gives $\operatorname{ord}(g) = r$. Its free homotopy class contains no geodesic of length $k\pi$, $k \geq 1$: a lift would give $g^2 x = x$, whereas $g^2 \neq I$. By Prüfer's description [69, Theorem 1, p. 291, and its proof, p. 295], the lifted initial points of the time-t returns with monodromy g form the great sphere

$$\mathbb{S}^n(1) \cap \ker\bigl(g^2 - 2\cos(t) g + I\bigr),$$

of dimension $2\rho_g - 1$; there are no such returns when $\rho_g = 0$. Since $\sin t \neq 0$, the initial velocity is uniquely determined by

$$v = \frac{gx - \cos(t) x}{\sin(t)}.$$

Thus the lifted returning states form a graph over this sphere. Freeness makes the graphs for distinct monodromies disjoint, and passage to the finite quotient preserves their dimensions. Since $\rho_g \leq m+1$, with equality for the chosen Clifford element, the largest return component on Q has dimension n. Lemma 15.2 gives a nonzero leading trace term of order R^{m+1}.

By Theorem 15.6, the comparison cover X has prime length 2π, volume V_n, and a deck element of order r. Its quotient therefore returns at t. If the largest return-component dimension on M differed from n, Lemma 15.2 would give a nonzero unmatched leading power in the $o(R_j)$ trace comparison. Hence this largest dimension is n.

Let $C \subset \operatorname{Fix}(G_t : SM \to SM)$ be a component of dimension n, and let F be a connected component of its inverse image in SX. The covering $SX \to SM$ is a local diffeomorphism, so $\dim F = n$. For each $v \in F$, there is a unique deck transformation h_v with $G_t v = \widehat{h_v} v$. The map $v \mapsto h_v$ is locally constant and therefore constant on F; write it as h. The set F is a connected component of $\operatorname{Fix}(\widehat{h^{-1}} G_t)$. Since $rt = 2\pi$, freeness gives $h^r = I$. An order smaller than r would give an upstairs return before 2π. Thus h has exact order r.

The subgroup $\langle h \rangle$ acts freely on X. Proposition 13.6, applied to this subgroup, makes evaluation $F \to X$ onto. It gives $d(x, hx) \leq t$ for every x, while (15.2) gives the reverse inequality. Thus h has constant displacement $2\pi/r$. Theorem 14.2 makes X round of radius one.

Let a_k and b_k be the degree-k harmonic multiplicities on M and Q. Their shifted frequencies are $k+m$, so the tolerance $\eta < 1/4$ permits only equal-frequency pairs. Consequently

$$\sum_{k=0}^{K_j} |a_k - b_k| = E_\eta(R_{K_j}) = o(K_j).$$

Put $J_j = \{k \in \mathbb{Z} : 0 \leq k \leq K_j,\ a_k = b_k\}$. Since unequal integer multiplicities differ by at least one, for $v = 0, \dots, q-1$ one has

$$\#\bigl(J_j \cap (v + q\mathbb{Z})\bigr) \geq \left\lfloor \frac{K_j}{q} \right\rfloor - E_\eta(R_{K_j}) \longrightarrow \infty.$$

Put $G = \mathrm{SO}(2m+2)$, choose a maximal torus and a system Φ^+ of positive roots, and let ω_1 be the highest weight of the standard complex representation. The harmonic representation $\mathcal{H}_k$ is irreducible of highest weight $k\omega_1$ [57, §2.2 and proof of Lemma 3.6, arXiv version 4]. Both deck groups lie in G by Lemma 8.2, and both have order q. Moreover, $a_k = \dim \mathcal{H}_k^{\Gamma}$, $b_k = \dim \mathcal{H}_k^{\Gamma_0}$, and $|\Phi^+| = m(m+1)$. For sufficiently large j, the set J_j has at least $|\Phi^+| + 1$ elements in each residue class modulo q. Thus [56, Proposition 3.1, arXiv version 2], applied to the string with base weight zero and direction ω_1, gives $a_k = b_k$ for every $k \geq 0$. The complete scalar spectra agree. □

Corollary 15.8. *Let $m \geq 2$ and $q \geq 3$ be integers, with q odd, and let $a \in \mathbb{Z}$ satisfy $\gcd(a, q) = 1$. Theorem 15.7 applies to $L(q; 1, \dots, 1, a)$ whenever a divisor $r \mid q$, $r \geq 5$, satisfies $a \equiv \pm 1 \pmod{r}$. In particular, it applies to $L(25; 1, \dots, 1, 6)$ in every odd dimension at least five. It also applies to each member of the pair $L(49; 1, 6, 15)$, $L(49; 1, 6, 20)$ of Lauret–Miatello–Rossetti* [57, §8, arXiv version 4].

Proof. In the two-weight case, the generator's q/rth power has order r and rotation angles $\pm 2\pi/r$. Hence it is a Clifford translation. For the two order-49 triples, reduction modulo 7 gives $(1,-1,1)$ and $(1,-1,-1)$, respectively. Their seventh powers are Clifford translations of order 7. □

For a lens space $L(q;1,\dots,1,a)$ of dimension $n = 2m+1 \geq 5$, with q an odd prime and $a \not\equiv 0, \pm 1 \pmod q$, consider the return time $2\pi/q$. Write g for the deck generator, and let a^{-1} denote the inverse of a modulo q. The powers $g^{\pm 1}$ give return graphs over the unit sphere in $\mathbb{C}^m$, whereas $g^{\pm a^{-1}}$ give return graphs over the unit sphere in $\mathbb{C}$. These four powers are distinct, and no other power has eigenvalue $e^{\pm 2\pi i/q}$. The four graphs are disjoint and invariant under the cyclic deck group. Their quotients are the return components, of dimensions $2m-1 = n-2$ and 1. Thus the largest return component has dimension $n-2$. Theorem 15.6 still determines the covering degree and volume.

16. Even-order spherical quotients

Retain the normalization $r = 1$ of Section 8. Assume (E) and (2.6), and put

$$\delta = \liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{E}}(n,q).$$

Choose nonnegative integers $K_j \to \infty$ and put $R_j = R_{K_j}$, so that

$$\frac{\mathcal{E}_\eta(K_j)}{R_j^{m+1}} \longrightarrow \delta.$$

Since $m+1 < n$, one has $\mathcal{E}_\eta(K_j) = O(R_j^{m+1}) = o(R_j^n)$. Proposition 10.1 gives $\dim M = n$, $\mathrm{Vol}(M) = V_n/q$, and least positive common geodesic period $T = \pi$. Let A denote the operator of that proposition. Let $p_X : X \to M$ be the universal Riemannian covering, let Γ be its deck group, and put $d = |\Gamma|$. By Lemma 11.1, X is a compact smooth homotopy sphere whose prime geodesics have a common length L, and

$$L \in \{\pi, 2\pi\}, \qquad \mathrm{Vol}(X) = dV_n/q. \tag{16.1}$$

Lemma 16.1. *If $L = 2\pi$, then X is the round unit sphere and $d = q$. If $L = \pi$, then d is odd.*

Proof. Suppose $L = 2\pi$. Lemma 11.1 gives a nonidentity deck involution D. The Riemannian quotient $Y = X/\langle D\rangle$ is smooth and Besse, and $X \to Y$ is a smooth covering of degree two. Since X is a smooth homotopy n-sphere and $n \geq 5$, Lin–Schmidt's theorem [59, Theorem 2] gives constant sectional curvature on Y, and hence on X. Since X is compact and simply connected, it is a round sphere. Its prime length 2π gives radius one. Equation (16.1) then gives $d = q$.

Suppose $L = \pi$ and d is even. The same theorem, applied to $p_X : X \to M$, gives constant sectional curvature. Thus X is a round sphere of radius $1/2$. After an isometry $X \cong \mathbb{S}^n(1/2)$, the deck group is a finite subgroup of $\mathrm{O}(n+1)$ acting freely on the sphere. By Cauchy's theorem and Lemma 8.2, it contains $-I$. The cotangent lift of $-I$ is the time-$\pi/2$ geodesic map on X, so $G_{\pi/2} = I$ on S^*M. This contradicts the least positive common period π. Hence d is odd. □

Lemma 16.2. *If m is odd, the alternative $L = \pi$ is impossible.*

Proof. Put $P_X = \sqrt{\Delta_X + m^2}$, and suppose $L = \pi$ and m is odd. By (10.5),

$$\sigma_0(e^{i\pi P_M}) = e^{i\pi m} = -1.$$

The time-π geodesic maps on M and X are identities. Lemma 11.5, with $c = m^2$, therefore gives $\sigma_0(e^{i\pi P_X}) = -1$. Since π is the common prime length on X, Lemma 11.4 gives $\sigma_0(e^{i\pi P_X}) = 1$, a contradiction. □

Lemma 16.3. *Under the hypotheses of this section, if $4 \mid q$, then M has a unit-speed closed geodesic of length $\pi/2$.*

Proof. Let S be a Sylow 2-subgroup of Γ_0. Its order is at least four. If it had no element of order four, every nonidentity element would be an involution: an element of order 2^j, $j \geq 2$, has a power of order four. Thus S would contain at least three involutions, contrary to Lemma 8.2. Let $b_4 > 0$ be the number of order-four elements in Γ_0. They occur in distinct inverse pairs, so $b_4 \geq 2$.

The order-four terms in (8.2) sum to [45, proof of Proposition 2.9, p. 84]

$$\frac{b_4}{q}\frac{1-z^2}{(1+z^2)^{m+1}}.$$

No other element has eigenvalue i or $-i$, by Lemma 8.2. The remaining terms are therefore holomorphic at i. Set $s = m+1$ in the notation of Lemma 8.4. Since $1+z^2 = (1-z/i)(1+z/i)$,

$$L_i(F_{\Gamma_0}) = \lim_{z\to i}(1-z/i)^{m+1}F_{\Gamma_0}(z) = \frac{b_4}{q2^m} > 0. \tag{16.2}$$

Suppose $\mathrm{Fix}(G_{\pi/2}) = \emptyset$ on S^*M. For every $\chi \in C_c^\infty((0,1))$ and every $N > 0$, Section 10.1 gives, as $R \to \infty$,

$$\mathrm{Tr}(e^{i\pi A/2}\chi(A/R)) = O(R^{-N}).$$

By Lemma 8.4 at $t = \pi/2$, with $s = m+1$, and (16.2),

$$\begin{aligned}&\mathrm{Tr}(e^{i\pi A/2}\chi(A/R)) - \mathrm{Tr}(e^{i\pi P_Q/2}\chi(P_Q/R))\\ &\quad= -\frac{e^{i\pi m/2}}{m!}\frac{b_4}{q2^m}R^{m+1}\int_0^1 \chi(u)u^m\,du + O(R^m).\end{aligned}$$

Lemma 10.2, with $s = m+1$, $t = \pi/2$, $\theta = \pi m/2$, and coefficient $-b_4/(q2^m)$, gives

$$\delta \geq \frac{b_4}{q2^m(m+1)!} \geq \frac{1}{q2^{m-1}(m+1)!} = c_{\mathrm{E}}(n,q),$$

contrary to (2.6). Hence $G_{\pi/2}$ has a fixed unit covector; its unit-speed geodesic is closed after time $\pi/2$. □

Proof of the covering conclusion in Theorem 2.2. Suppose $L = \pi$. By Lemma 16.1, d is odd, whereas Proposition 12.3 gives

$$\frac{dV_n}{q} = 2^{-n}V_n.$$

Thus $q = 2^n d$, and in particular $4 \mid q$. Lemma 16.3 gives a closed geodesic on M of length $\pi/2$. Lemma 15.1, with $L = \pi$ and $s = 2$, then gives an element of order two in Γ, contrary to the oddness of d. Therefore $L = 2\pi$, and Lemma 16.1 gives $X \cong \mathbb{S}^n(1)$ and $d = q$. Under this isometry, Γ acts freely and orthogonally on $\mathbb{S}^n(1)$. The scaling in Section 8 gives $M \cong \mathbb{S}^n(r)/\Gamma$, with $|\Gamma| = q$ and sectional curvature r^{-2}. □

17. Homogeneity and harmonic multiplicities

Under the hypotheses of Theorem 2.2, Section 16 gives, after rescaling to $r = 1$,

$$M \cong \mathbb{S}^{2m+1}(1)/\Gamma, \qquad Q \cong \mathbb{S}^{2m+1}(1)/\Gamma_0, \qquad |\Gamma| = |\Gamma_0| = q,$$

where both groups are finite and act freely and orthogonally. By Lemma 8.3, their shifted spectra lie in $m + 2\mathbb{Z}_{\geq 0}$. Since $\eta < 1/2$, an admissible matching pairs only equal frequencies. Hence, for every $K \geq 0$,

$$\mathcal{E}_\eta(K) = \sum_{\substack{\omega \in m+2\mathbb{Z}_{\geq 0}\\ \omega \leq R_K}} |\,\mathrm{mult}_{P_M}(\omega) - \mathrm{mult}_{P_Q}(\omega)|. \tag{17.1}$$

Definition 17.1. For a finite subgroup $\Gamma \subset O(2m+2)$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\xi \neq \pm 1$, let $c_\Gamma(\xi)$ be the number of elements of Γ whose complex eigenvalues are $\xi, \bar{\xi}$, each with multiplicity $m+1$.

Lemma 17.2 (Ikeda). *Let $\Gamma \subset O(2m+2)$ be finite, act freely on the unit sphere, and have order q. For $|\xi| = 1$, $\xi \neq \pm 1$, put*

$$L_\xi(F_\Gamma) = \lim_{z \to \xi} (1 - z/\xi)^{m+1} F_\Gamma(z).$$

Then

$$c_\Gamma(\xi) = q(1-\xi^2)^m L_\xi(F_\Gamma). \tag{17.2}$$

The exponent in the limit is fixed at $m+1$; the limit is zero at a regular point or a pole of smaller order. The elements counted by $c_\Gamma(\xi)$ are precisely the Clifford translations with eigenvalue pair $\{\xi, \bar\xi\}$.

Proof. Put $s = m+1$. By Lemma 8.2, $\Gamma \subset \mathrm{SO}(2m+2)$. Ikeda's formula [45, Lemma 5.2, p. 90], with his dimension parameter equal to s, is

$$\lim_{z \to \xi} (z-\xi)^s F_\Gamma(z) = \frac{c_\Gamma(\xi)}{q} \frac{1-\xi^2}{(\xi - \bar\xi)^s}.$$

Since $1 - z/\xi = -\xi^{-1}(z-\xi)$ and $\xi - \bar\xi = -\xi^{-1}(1-\xi^2)$, this gives (17.2).

For an orthogonal map γ on the unit sphere, $d(x, \gamma x) = \arccos\langle x, \gamma x\rangle$. Thus constant displacement is equivalent to $\gamma + \gamma^t = 2cI$ for some $c \in [-1,1]$. For $c = \pm 1$ one has $\gamma = \pm I$. For $-1 < c < 1$, the spectral theorem for orthogonal matrices gives the two eigenvalues $c \pm i\sqrt{1-c^2}$ with equal multiplicities $m+1$. Conversely, an orthogonal map with eigenvalues $\xi, \bar\xi$ satisfies $\gamma + \gamma^t = 2\,\mathrm{Re}(\xi) I$. □

Lemma 17.3 (Wolf). *If $\mathbb{S}^{2m+1}(1)/\Gamma$ is homogeneous, every element of its deck group Γ is a Clifford translation.*

Proof. By Lemma 8.2, $\Gamma \subset \mathrm{SO}(2m+2)$. Wolf's classification, in the form of [45, Theorem 5.1, (1) ⇒ (2), p. 90], gives the eigenvalue condition of Lemma 17.2 for each element other than $I, -I$. These two maps have constant displacements $0, \pi$. □

Completion of Theorem 2.2. Assume Q is homogeneous. Choose $K_j \to \infty$ and put $R_j = R_{K_j}$, so that

$$\frac{\mathcal{E}_\eta(K_j)}{R_j^{m+1}} \longrightarrow \delta := \liminf_{K \to \infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{E}}(n,q).$$

Let $\mathcal{R}$ be the finite set of nonreal eigenvalues of elements of $\Gamma \cup \Gamma_0$. Suppose that $c_\Gamma(\xi) \neq c_{\Gamma_0}(\xi)$ for some $\xi \in \mathcal{R}$, and put $\Delta_\xi = L_\xi(F_\Gamma) - L_\xi(F_{\Gamma_0})$. Inversion is a fixed-point-free involution on each set counted at ξ: an element and its inverse have the same eigenvalue pair, whereas an element equal to its inverse has only real eigenvalues. Both counts are therefore even. By Lemma 17.2 and $|1-\xi^2| \le 2$,

$$|\Delta_\xi| = \frac{|c_\Gamma(\xi) - c_{\Gamma_0}(\xi)|}{q|1-\xi^2|^m} \ge \frac{1}{q 2^{m-1}}.$$

Fix $t \in \mathbb{R}$ with $e^{it} = \xi$ and a nonnegative $\chi \in C_c^\infty((0,1))$. Lemma 8.2 bounds the pole orders at ξ by $m+1$. Lemma 8.4 therefore gives

$$\begin{aligned} &\mathrm{Tr}(e^{itP_M}\chi(P_M/R_j)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_j)) \\ &\quad = \frac{e^{itm}\Delta_\xi}{m!} R_j^{m+1} \int_0^1 \chi(u) u^m \, du + O(R_j^m). \end{aligned}$$

For each fixed χ and t, equation (17.1) gives

$$\left|\mathrm{Tr}(e^{itP_M}\chi(P_M/R_j)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_j))\right| \le \|\chi\|_\infty \mathcal{E}_\eta(K_j).$$

The proof of Lemma 10.2, with P_M in place of A and this bound, gives

$$\delta \ge \frac{|\Delta_\xi|}{(m+1)!} \ge \frac{1}{q 2^{m-1}(m+1)!} = c_{\mathrm{E}}(n,q),$$

a contradiction. Hence $c_\Gamma(\xi) = c_{\Gamma_0}(\xi)$ for every $\xi \in \mathcal{R}$.

Both groups contain $I, -I$, by Lemma 8.2 and the evenness of q. For $H \in \{\Gamma, \Gamma_0\}$, let $\mathcal{C}_H$ be the set of Clifford translations in $H \setminus \{I, -I\}$. Each element of $\mathcal{C}_H$ is counted once at each of its two conjugate eigenvalues. Thus, as in [45, proof of Theorem III, p. 91],

$$2 + |\mathcal{C}_\Gamma| = 2 + \frac{1}{2}\sum_{\xi\in\mathcal{R}} c_\Gamma(\xi) = 2 + \frac{1}{2}\sum_{\xi\in\mathcal{R}} c_{\Gamma_0}(\xi) = q,$$

where the last equality follows from Lemma 17.3. Since $\mathcal{C}_\Gamma \subset \Gamma \setminus \{I, -I\}$, every element of Γ is a Clifford translation. For each unordered pair $\{\xi, \bar{\xi}\}$, the determinant $(1 - z\xi)^{m+1}(1 - z\bar{\xi})^{m+1}$ occurs $c_\Gamma(\xi) = c_{\Gamma_0}(\xi)$ times in each group. The terms from $I, -I$ also agree. Equation (8.2) gives $F_\Gamma = F_{\Gamma_0}$, hence equality of the complete scalar spectra by Lemma 8.1. Ikeda's theorem [45, Theorem III, p. 76], also proved in [85, Theorem 8.2, p. 334], applies to these two spherical space forms, one homogeneous, and gives $M \cong Q$. The same isometry holds after restoring the common radius r. □

Corollary 17.4. *Let Q be any reference in (E), and let $v_2(q)$ be the exponent of 2 in q. Suppose $0 \le \eta < 1/2$ and*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^n} = 0.$$

If $n \equiv 3 \pmod 4$, or if $n \equiv 1 \pmod 4$ and $v_2(q) \neq n$, then $M \cong \mathbb{S}^n(r)/\Gamma$, where $\Gamma \subset O(n+1)$ is finite, acts freely on $\mathbb{S}^n(r)$, and has order q.

Proof. Normalize $r = 1$. Choose $K_j \to \infty$ such that $\mathcal{E}_\eta(K_j) = o(R_{K_j}^n)$. Proposition 10.1 gives

$$\dim M = n, \qquad \mathrm{Vol}(M) = V_n/q, \qquad \sigma_0(e^{i\pi P_M}) = e^{i\pi m},$$

and the least positive common geodesic period is π. Let $X \to M$ be the universal Riemannian covering, let Γ be its deck group, and put $d = |\Gamma|$. By Lemma 11.1, X is a compact smooth homotopy sphere with a common prime geodesic length L, and

$$L \in \{\pi, 2\pi\}, \qquad \mathrm{Vol}(X) = dV_n/q.$$

The proofs of Lemmas 16.1 and 16.2 apply to these covering, period, volume, and return-symbol hypotheses. Thus $L = 2\pi$ gives a round unit cover of degree q, whereas $L = \pi$ forces d odd and m even. In the latter case, Proposition 12.3 gives

$$dV_n/q = 2^{-n}V_n, \qquad q = 2^n d, \qquad v_2(q) = n.$$

Since $n = 2m + 1$, this alternative has $n \equiv 1 \pmod 4$. The stated hypotheses therefore imply $L = 2\pi$. Rescaling gives $M \cong \mathbb{S}^n(r)/\Gamma$ with $|\Gamma| = q$. □

18. Scalar quotients of odd order

Let $Q = Q_{q,n}(r)$ be as in (O). Put

$$\xi = e^{2\pi i/q}, \qquad \gamma_0(z) = \xi z \quad (z \in \mathbb{C}^{m+1}),$$

and write $\Gamma_0 = \langle \gamma_0 \rangle$ and $F_Q = F_{\Gamma_0}$. The elements counted by $c_{\Gamma_0}(\xi)$ are γ_0 and γ_0^{-1}. Thus $c_{\Gamma_0}(\xi) = 2$, and Lemma 17.2 gives

$$L_\xi(F_Q) = \frac{2}{q}(1 - \xi^2)^{-m}. \tag{18.1}$$

Proof of Theorem 2.4. Rescale both metrics to $r = 1$ as in Section 8. Put $s = m + 1$, and choose $K_j \to \infty$ such that, with $R_j = R_{K_j}$,

$$\frac{\mathcal{E}_\eta(K_j)}{R_j^s} \longrightarrow \delta := \liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^s} < c_{\mathrm{O}}(n, q).$$

Since $s < n$, one has $\mathcal{E}_\eta(K_j) = o(R_j^n)$. Proposition 10.1 gives $\dim M = n$, $\mathrm{Vol}(M) = V_n/q$, and least positive common geodesic period 2π. It also gives

$$e^{2\pi i P_M} - I \in \Psi_{\mathrm{cl}}^{-1}, \qquad A - P_M \in \Psi_{\mathrm{cl}}^{-1}, \qquad \mathrm{Spec}(A) \subset m + \mathbb{Z}_{\geq 0},$$

where A is positive and self-adjoint, strongly commutes with P_M, and satisfies (10.3). Let $X \to M$ be the universal Riemannian covering, let Γ be its deck group, and put $d = |\Gamma|$. By Lemma 11.1, X is a compact smooth homotopy sphere, d is finite, and the common prime geodesic length L satisfies

$$L \in \{2\pi, 4\pi\}, \qquad \mathrm{Vol}(X) = dV_n/q.$$

Write d_X for the Riemannian distance on X.

Suppose d is even. The Besse manifold M admits the even-degree smooth covering $X \to M$, with $n \geq 5$. Lin–Schmidt [59, Theorem 2] gives constant sectional curvature. Its compact simply connected Riemannian cover X is therefore round, of radius $L/(2\pi) \in \{1, 2\}$. The volume identity gives $d = q(L/(2\pi))^n$. Since q is odd, the radius is two and $d = 2^n q$. The compactness argument in the proof of Theorem 15.6 applies to this radius-two quotient and the displayed time-2π return; it excludes odd m. For even m, (15.3) and the nearest-lattice property in Corollary 9.2 give

$$A = P_0 := \sqrt{\Delta_M + m^2/4}$$

outside finitely many eigenspaces. For each fixed $\chi \in C_c^\infty((0,1))$, those eigenspaces contribute zero to the annular traces for all sufficiently large R. Lemma 15.5 therefore gives

$$\mathrm{Tr}\,\chi(A/R) - \mathrm{Tr}\,\chi(P_Q/R) = \frac{m-1}{8q(2m-2)!} R^{2m-1} \int_0^1 \chi(u) u^{2m-2}\, du + O(R^{2m-2}).$$

Apply Lemma 10.2 at time zero, with exponent $2m-1$ and coefficient $(m-1)/(8q)$. It gives

$$\frac{m-1}{8q(2m-1)!} \leq \lim_{j\to\infty} \frac{\mathcal{E}_\eta(K_j)}{R_j^{2m-1}} = \begin{cases} \delta, & m = 2, \\ 0, & m \geq 4. \end{cases}$$

For $m = 2$, the left side is $1/(48q) > \delta$; for even $m \geq 4$, it is positive. Both alternatives are impossible. Thus d is odd. The time-2π deck involution in Lemma 11.1 is the identity, so $L = 2\pi$. Proposition 12.3 gives $\mathrm{Vol}(X) = V_n$, and hence $d = q$.

Put $\ell = 2\pi/q$. For every $h \in \Gamma \setminus \{I\}$, $\mathrm{ord}(h)$ divides q, so (15.2) gives

$$\min_{x\in X} d_X(x, hx) \geq \frac{2\pi}{\mathrm{ord}(h)} \geq \ell.$$

For every fixed $\chi \in C_c^\infty((0,1))$, (18.1) and Lemma 8.4 give

$$\mathrm{Tr}(e^{i\ell P_Q}\chi(P_Q/R)) = \frac{e^{i\ell m} L_\xi(F_Q)}{m!} R^s \int_0^1 \chi(u) u^m\, du + O(R^{s-1}). \tag{18.2}$$

If the time-ℓ return set on SM is nonempty, let d_* be the largest dimension of its connected components. Lemma 15.2 gives, for every fixed nonzero $0 \leq \chi \in C_c^\infty((0,1))$,

$$\mathrm{Tr}(e^{i\ell A}\chi(A/R)) = C_\chi R^{(d_*+1)/2} + o(R^{(d_*+1)/2}), \qquad C_\chi \neq 0.$$

The bound (10.3) is $O(R_j^s)$. Together with (18.2), it excludes $d_* > n$, since $(n+1)/2 = s$. If the return set is empty, the trace for A is $O(R^{-N})$ for every N by Lemma 15.2.

If the return set is empty or $d_* < n$, the trace for A is $o(R^s)$. Equation (18.2) and Lemma 10.2, with coefficient $-L_\xi(F_Q)$ and phase ℓm, imply

$$\delta \geq \frac{|L_\xi(F_Q)|}{(m+1)!} = \frac{2}{q|1 - e^{4\pi i/q}|^m (m+1)!}.$$

For $m = 2$, this is at least $1/(12q) > 1/(48q)$. For $m \geq 3$, it equals $c_{\mathrm{O}}(n, q)$. Both cases contradict the hypothesis. Thus the return set has a component C of dimension n.

Let F be a connected component of the inverse image of C under $SX \to SM$. The lifting and exact-order arguments in the proof of Theorem 15.7, applied at time $\ell = 2\pi/q$, give $\dim F = n$ and a deck transformation h of order q such that F is a connected component of $\mathrm{Fix}(\widehat{h}^{-1} G_\ell)$. Since $|\Gamma| = q$, one has $\Gamma = \langle h \rangle$. Proposition 13.6 applies to X and $\langle h \rangle$: evaluation $F \to X$, $(x, v) \mapsto x$, is surjective. For every $x \in X$, a vector $(x, v) \in F$ determines a unit-speed geodesic segment of length ℓ from x to hx. Thus $d_X(x, hx) \le \ell$ for every $x \in X$. The opposite inequality was proved above, so h has constant displacement ℓ. Theorem 14.2 identifies X with the unit round sphere and h with scalar multiplication by a primitive qth root of unity. This isometry conjugates $\Gamma = \langle h \rangle$ to Γ_0. Hence $M \cong Q_{q,n}(1)$; rescaling gives $M \cong Q_{q,n}(r)$. □

Proof of Theorem 2.3. The implication from (2.7) to (2.9) in Section 2 reduces this to Theorem 2.4. □

Part 3. **Perturbations and sharpness**

For a homogeneous reference in (E) or a reference in (O), isospectrality after symmetric scalar classical perturbations of order less than one implies isometry. At the form bound $|b[u]| \le C\langle (1+\Delta)^{1/2} u, u \rangle$, isospectral form perturbations exist between closed connected manifolds of positive dimension precisely when their dimensions and volumes agree (Theorem 20.1). For the same references, classical perturbations of order one retain rigidity under the constant orbit-average hypothesis of Theorem 20.8. Without this hypothesis, first-order differential perturbations can produce isospectral operators on nonhomeomorphic manifolds. Round lens-space pairs attain an unmatched count of order $R^{(n+1)/2}$, so this exponent cannot be increased for odd scalar references. The shrinking-resolution hypotheses of Theorem 23.2 give rigidity for prime two-weight lens spaces in dimension five.

19. Spectral separation and perturbations

Retain the notation of Section 2. Let M be a smooth closed connected Riemannian manifold of positive dimension.

Proof of Corollary 2.5. For $x > 0$,

$$0 \le \sqrt{x^2 + m^2} - x = \frac{m^2}{\sqrt{x^2 + m^2} + x} \longrightarrow 0 \qquad (x \to \infty).$$

Since both Laplace eigenvalue sequences tend to infinity,

$$\limsup_{j \to \infty} |\mu_j - \beta_j| = \limsup_{j \to \infty} r \left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right| < a/4.$$

Choose η strictly between this upper limit and $a/4$. For some $J \ge 1$, one has $|\mu_j - \beta_j| \le \eta$ whenever $j \ge J$. Every β_j belongs to $m + a\mathbb{Z}$, and $R_K = m + aK + a/2$. Hence $|\beta_j - R_K| \ge a/2$, so

$$\mu_j \le R_K \quad \Longleftrightarrow \quad \beta_j \le R_K \qquad (j \ge J).$$

Matching these equal indices leaves at most J copies in each list unmatched. Thus $\mathcal{E}_\eta(K) \le 2J$ for every K. Theorems 2.2 and 2.3 imply that M has a round cover of radius r; if Q is homogeneous, $M \cong Q$.

In case (E), write $M \cong \mathbb{S}^{2m+1}(r)/\Gamma$, where $|\Gamma| = |\Gamma_0| = q$. By Lemma 8.3, both shifted spectra belong to $m + 2\mathbb{Z}_{\ge 0}$. Since $\eta < 1/2$, one has $\mu_j = \beta_j$ for $j \ge J$. For every degree $k \ge 0$ with $k + m > \max\{\mu_{J-1}, \beta_{J-1}\}$, all copies of this frequency occur at indices at least J. Hence the harmonic multiplicities agree for all sufficiently large k.

Put $G = \mathrm{SO}(2m + 2)$. Both deck groups lie in G by Lemma 8.2. Choose a maximal torus and positive roots of G, and let ω_1 be the highest weight of its standard complex representation. The complex harmonic representation of degree k is irreducible with highest weight $k\omega_1$ [57, §2.2 and proof of Lemma 3.6, arXiv version 4]. Lauret–Miatello's theorem [56, Corollary 3.2, arXiv version 2], with base weight 0 and direction ω_1, gives equality of the invariant multiplicities in every degree. By Lemma 8.1, the complete scalar spectra agree. □

Proposition 19.1. *Let Q be homogeneous in (E), or let Q belong to (O). For $Z \in \{M, Q\}$, let b_Z be a symmetric sesquilinear form on $H^1(Z)$, and put $b_Z[u] = b_Z(u,u)$. Suppose that, for constants $C_Z \geq 0$ and $0 \leq \alpha < 1/2$,*

$$|b_Z[u]| \leq C_Z \langle (1+\Delta_Z)^\alpha u, u\rangle, \qquad u \in H^1(Z). \tag{19.1}$$

The form sum $\Delta_Z + b_Z$ defines a semibounded self-adjoint operator H_Z with compact resolvent. If $\operatorname{Spec}(H_M) = \operatorname{Spec}(H_Q)$, including multiplicities, then M and Q are isometric. This applies, in particular, to self-adjoint classical pseudodifferential perturbations of order $\rho < 1$.

Proof. Fix one of the manifolds and omit its subscript. Put $\mathfrak{q}_\Delta[u] = \int |du|^2 \, dv_g$ on H^1. For every $\epsilon > 0$ there is $A_\epsilon \geq 0$ such that

$$C(1+t)^\alpha \leq \epsilon t + A_\epsilon \qquad (t \geq 0).$$

Spectral calculus gives

$$|b[u]| \leq \epsilon \mathfrak{q}_\Delta[u] + A_\epsilon \|u\|_2^2.$$

Thus b has relative form bound zero. Since Δ has compact resolvent, Kato's form perturbation and representation theorems [49, Chapter VI, Theorems 3.4 and 2.6] give a closed form $\mathfrak{h} = \mathfrak{q}_\Delta + b$ on H^1 and a semibounded self-adjoint operator H with compact resolvent.

Let $(e_j)_{j\geq 0}$ be an orthonormal Laplace eigenbasis, with $\Delta e_j = \lambda_j e_j$ and $\lambda_j \leq \lambda_{j+1}$, and put

$$V_j = \operatorname{span}(e_0, \dots, e_j), \qquad V_{-1} = \{0\}.$$

Let $(\nu_j)_{j\geq 0}$ be the eigenvalues of H in nondecreasing order, repeated according to multiplicity. For $u \in V_j$ with $\|u\|_2 = 1$,

$$\mathfrak{h}[u] \leq \lambda_j + C(1+\lambda_j)^\alpha.$$

The function $f(t) = t - C(1+t)^\alpha$, $t \geq 0$, has $f'(t) \to 1$ as $t \to \infty$. For all sufficiently large j, it is therefore increasing on $[\lambda_j, \infty)$. For such j and $u \in H^1 \cap V_{j-1}^\perp$ with $\|u\|_2 = 1$,

$$\mathfrak{h}[u] \geq \sum_{i \geq j} f(\lambda_i) |\langle u, e_i\rangle|^2 \geq f(\lambda_j).$$

The min–max theorem on the form domain [80, Theorem 4.10 and the following paragraph, p. 119] gives, for all sufficiently large j,

$$\lambda_j - C(1+\lambda_j)^\alpha \leq \nu_j \leq \lambda_j + C(1+\lambda_j)^\alpha.$$

If the two perturbed spectra agree, let ν_j denote their common ordered list, including every negative eigenvalue. The preceding bounds and $\alpha < 1$ give $\lambda_j(M) \sim \nu_j \sim \lambda_j(Q)$. Consequently

$$\left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right| = \frac{|\lambda_j(M) - \lambda_j(Q)|}{\sqrt{\lambda_j(M)} + \sqrt{\lambda_j(Q)}} = O(\nu_j^{\alpha - 1/2}) \longrightarrow 0.$$

Corollary 2.5 gives the isometry.

Let B be a symmetric classical pseudodifferential operator of order $0 \leq \rho < 1$. The operator $1 + \Delta$ is strictly positive and elliptic of order two. By [6, Theorem 7.9, arXiv version 1], $(1+\Delta)^{-\rho/4}$ has order $-\rho/2$. The composition and boundedness theorems [79, Sections I.2, I.3, and I.8] give

$$T = (1+\Delta)^{-\rho/4} B (1+\Delta)^{-\rho/4} \in \Psi^0_{\mathrm{cl}}, \qquad T : L^2 \longrightarrow L^2 \text{ bounded}.$$

For $u \in C^\infty$,

$$|\langle Bu, u\rangle| = |\langle T(1+\Delta)^{\rho/4} u, (1+\Delta)^{\rho/4} u\rangle| \leq \|T\| \langle (1+\Delta)^{\rho/2} u, u\rangle.$$

The form extends continuously to $H^{\rho/2}$, hence to H^1. Thus (19.1) holds with $\alpha = \rho/2 < 1/2$. If $\rho < 0$, then B is bounded on L^2, and $\alpha = 0$ suffices. □

Proof of Corollary 2.6. For $Z \in \{M, Q\}$, the form $b_Z[u] = \langle B_Z u, u\rangle$ satisfies

$$|b_Z[u]| \le \|B_Z\| \|u\|_2^2.$$

Its sum with the Laplace form is represented on $H^2(Z)$ by $\Delta_Z + B_Z$. This operator is self-adjoint by [49, Chapter V, Theorem 4.3], and hence is the operator associated with that form. Proposition 19.1 applies with $\alpha = 0$. □

Remark 19.2. Let Q be as in Proposition 19.1. Suppose that (19.1) holds with $\alpha = 1/2$ and that the two form sums have the same spectrum, including multiplicities. The preceding form construction and eigenvalue bounds remain valid. They give $\lambda_j(M)/\lambda_j(Q) \to 1$ and

$$\begin{aligned}\delta_r(M, Q) &\le r \limsup_{j\to\infty} \frac{C_M\sqrt{1+\lambda_j(M)} + C_Q\sqrt{1+\lambda_j(Q)}}{\sqrt{\lambda_j(M)} + \sqrt{\lambda_j(Q)}} \\ &= \frac{r}{2}(C_M + C_Q).\end{aligned}$$

Thus Corollary 2.5 gives isometry if $r(C_M + C_Q) < 2\varepsilon_Q$. The critical form class is determined in Theorem 20.1. For classical operators of order one, Theorem 20.8 gives rigidity under the constant reference orbit-average hypothesis; Theorem 21.1 gives two-sided differential counterexamples without that hypothesis.

Corollary 19.3. *For a homogeneous reference in (E) or a reference in (O), exact scalar isospectrality implies isometry. The same is true if the scalar spectral multisets agree outside a bounded interval.*

Proof. Choose $C \ge 0$ such that the scalar spectral multisets agree outside $[0, C]$. Matching equal eigenvalues above C gives

$$\mathcal{E}_0(K) \le \#\{j : \lambda_j(M) \le C\} + \#\{j : \lambda_j(Q) \le C\} \qquad (K \ge 0).$$

The right-hand side is finite and independent of K. Hence $\mathcal{E}_0(K)/R_K^{m+1} \to 0$. Theorem 2.2 or 2.4 gives the isometry. □

Corollary 19.4. *Let Q be homogeneous in (E), or let Q belong to (O), and fix $0 \le \eta < a/4$. If $M \not\cong Q$, then*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} \ge \begin{cases} c_{\mathrm{E}}(n, q), & \text{in case (E),} \\ c_{\mathrm{O}}(n, q), & \text{in case (O).} \end{cases}$$

Proof. If the lower bound failed, Theorem 2.2 or Theorem 2.4 would give $M \cong Q$. □

20. Perturbations of order one

All manifolds in this section are smooth, closed and connected. Eigenvalues are numbered from zero in nondecreasing order and repeated according to multiplicity. Every equality of spectra includes multiplicities. Forms are Hermitian sesquilinear forms; we write $b[u] = b(u, u)$. The Dirichlet form of the scalar Laplacian is $q_Z[u] = \|\Delta_Z^{1/2} u\|^2_{L^2(Z)}$, with domain $H^1(Z)$. The notation $\Delta_Z + b$ denotes the self-adjoint operator associated with $q_Z + b$.

Theorem 20.1. *Let M and N be Riemannian manifolds of positive dimension. The following conditions are equivalent.*

(i) $\dim M = \dim N$ *and* $\operatorname{Vol} M = \operatorname{Vol} N$.

(ii) *There are a form b on $H^1(M)$ and a constant $C \ge 0$ such that*

$$|b[u]| \le C\langle (1 + \Delta_M)^{1/2} u, u\rangle, \qquad \operatorname{Spec}(\Delta_M + b) = \operatorname{Spec} \Delta_N.$$

(iii) *For $Z = M, N$, there are a form b_Z on $H^1(Z)$ and a constant $C_Z \ge 0$ such that*

$$|b_Z[u]| \le C_Z\langle (1 + \Delta_Z)^{1/2} u, u\rangle, \qquad \operatorname{Spec}(\Delta_M + b_M) = \operatorname{Spec}(\Delta_N + b_N).$$

In (ii)*, b can be chosen diagonal in a Laplace eigenbasis. The corresponding diagonal operator B extends continuously from $H^s(M)$ to $H^{s-1}(M)$ for every $s \in \mathbb{R}$.*

Proof. Let Z have dimension d, let $\lambda_j(Z)$ be the eigenvalues of Δ_Z, and put $N_Z(R) = \#\{j : \sqrt{\lambda_j(Z)} \le R\}$. If $B^d \subset \mathbb{R}^d$ is the Euclidean unit ball, then

$$N_Z(R) = c_d \operatorname{Vol}(Z) R^d + O(R^{d-1}), \qquad c_d = (2\pi)^{-d} \operatorname{Vol}(B^d). \tag{20.1}$$

Indeed, integrate [42, Theorem 5.1, equation (5.3), p. 215] for $1 + \Delta_Z$ at the spectral parameter $1 + R^2$. Put $a = c_d \operatorname{Vol}(Z)$. There is $A > 0$ such that, for all sufficiently large x,

$$N_Z(x - A) < a x^d < N_Z(x + A).$$

For all sufficiently large j, take $a x^d = j + 1$. The finitely many remaining indices can be included in the constant, giving

$$\sqrt{\lambda_j(Z)} = ((j+1)/a)^{1/d} + O(1). \tag{20.2}$$

Thus (i) implies $D := \sup_j |\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(N)}| < \infty$. Put $\lambda_j = \lambda_j(M)$ and $\nu_j = \lambda_j(N)$. Then

$$|\nu_j - \lambda_j| \le D(2\sqrt{\lambda_j} + D) \le (2D + D^2)\sqrt{1 + \lambda_j}. \tag{20.3}$$

Choose an orthonormal Laplace eigenbasis (e_j) on M. For $u, v \in H^{1/2}(M)$, put $u_j = \langle u, e_j \rangle$, $v_j = \langle v, e_j \rangle$, and define

$$b(u, v) = \sum_{j \ge 0} (\nu_j - \lambda_j) u_j \overline{v_j}.$$

The series converges absolutely on $H^{1/2}(M) \times H^{1/2}(M)$, by Cauchy–Schwarz and (20.3). It satisfies the bound in (ii). For $u \in H^1(M)$,

$$q_M[u] + b[u] = \sum_{j \ge 0} \nu_j |u_j|^2.$$

Since $1 + \nu_j \asymp 1 + \lambda_j$, this is a closed nonnegative form with domain $H^1(M)$. Its operator has eigenvectors e_j and eigenvalues ν_j. If $B e_j = (\nu_j - \lambda_j) e_j$, then

$$\sum_j (1 + \lambda_j)^{s-1} |(Bu)_j|^2 \le (2D + D^2)^2 \sum_j (1 + \lambda_j)^s |u_j|^2.$$

This proves the stated Sobolev bounds and (ii).

Condition (ii) implies (iii) with $b_N = 0$. The form construction and eigenvalue estimates in the proof of Proposition 19.1 apply with $\alpha = 1/2$ on every closed manifold. They give a semibounded self-adjoint form sum with compact resolvent, whose ordered eigenvalues satisfy

$$|\eta_j - \lambda_j| \le C\sqrt{1 + \lambda_j} \qquad (j \text{ sufficiently large}).$$

In particular, $\eta_j / \lambda_j \to 1$. If (iii) holds, let η_j be the common ordered list, including its negative eigenvalues. For $Z = M, N$, (20.2) gives

$$\eta_j \sim \lambda_j(Z) \sim \left(\frac{j+1}{c_{\dim Z} \operatorname{Vol} Z} \right)^{2/\dim Z}.$$

Equality of the powers of $j + 1$ gives $\dim M = \dim N$; equality of the leading coefficients then gives $\operatorname{Vol} M = \operatorname{Vol} N$. □

Example 20.2. Let $n \ge 2$, $a = 2^{-1/n}$, $M = \mathbb{S}^n(a)$ and $N = \mathbb{RP}^n(1)$. Then

$$\operatorname{Vol} M = \operatorname{Vol} N = \tfrac{1}{2} \operatorname{Vol} \mathbb{S}^n(1), \qquad \pi_1(M) = 0, \quad \pi_1(N) = \mathbb{Z}/2\mathbb{Z}.$$

The diagonal construction in Theorem 20.1 therefore gives nonisometric manifolds satisfying (ii).

20.1. **A constant translation of scalar frequencies.** For $c > 0$, write $P_{Z,c} = (\Delta_Z + c)^{1/2}$.

Lemma 20.3. *Let Γ denote Euler's gamma function. If $\dim Z = n \geq 2$, then, as $t \downarrow 0$,*

$$\operatorname{Tr} e^{-tP_{Z,c}} = \kappa_n \operatorname{Vol}(Z) t^{-n} + O(t^{2-n}), \qquad \kappa_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}. \tag{20.4}$$

Proof. The scalar heat expansion gives

$$\operatorname{Tr} e^{-s(\Delta_Z + c)} = (4\pi s)^{-n/2} \operatorname{Vol}(Z) + O(s^{1-n/2}) \qquad (0 < s \leq 1).$$

The heat parametrix has only integral powers of s on a closed manifold; see [28, Lemmas 1.7.4 and 4.8.4]. Subordination gives

$$e^{-t\sqrt{x}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty s^{-3/2} e^{-t^2/(4s)} e^{-sx}\, ds \qquad (x > 0).$$

All integrands in the spectral sum are nonnegative, so the same identity holds for the traces. The substitution $v = t^2/(4s)$ gives

$$\frac{t}{2\sqrt{\pi}} (4\pi)^{-n/2} \int_0^\infty s^{-(n+3)/2} e^{-t^2/(4s)}\, ds = \kappa_n t^{-n}.$$

The integral of the error on $0 < s \leq 1$ is bounded by

$$Ct \int_0^1 s^{-(n+1)/2} e^{-t^2/(4s)}\, ds \leq C' t^{2-n} \int_0^\infty v^{(n-3)/2} e^{-v}\, dv.$$

This integral is finite for $n \geq 2$. On $s \geq 1$, both the actual trace and the subtracted leading expression have integrable $s^{-3/2}$ multiples, and their contribution is $O(t)$. □

Lemma 20.4. *Let M, N have the same dimension $n \geq 2$, and let $d \in \mathbb{R}$. Let μ_j, ν_j be the eigenvalues of $P_{M,c}, P_{N,c}$ respectively. If*

$$\nu_j = \mu_j + d + O(\mu_j^{-1}),$$

then $\operatorname{Vol} M = \operatorname{Vol} N$ and $d = 0$.

Proof. Put $N(R) = \#\{j : \mu_j \leq R\}$. By (20.1), $N(R) = O(R^n)$. For $0 < t \leq 1$, Stieltjes integration by parts gives

$$\sum_j \mu_j^{-1} e^{-t\mu_j} \leq C \int_{\sqrt{c}}^\infty R^n (R^{-2} + tR^{-1}) e^{-tR}\, dR = O(t^{1-n}).$$

For all sufficiently large j and $0 < t \leq 1$, the mean value theorem gives

$$|e^{-t\nu_j} - e^{-t(\mu_j + d)}| \leq C' t \mu_j^{-1} e^{-t\mu_j}.$$

Consequently,

$$\sum_j e^{-t\nu_j} = e^{-td} \sum_j e^{-t\mu_j} + O(t^{2-n}).$$

The finitely many exceptional indices contribute $O(t)$ after subtraction. Substitution of (20.4) yields

$$\kappa_n \operatorname{Vol}(N) t^{-n} = \kappa_n \operatorname{Vol}(M) t^{-n} - d\kappa_n \operatorname{Vol}(M) t^{1-n} + O(t^{2-n}).$$

Multiplication first by t^n and then, after subtracting the leading term, by t^{n-1} gives the two conclusions. □

Lemma 20.5. *Let $P_0 = P_{M,c}$, let $d \in \mathbb{R}$, and let $R \in \Psi_{\mathrm{cl}}^{-1}(M)$ be self-adjoint. If μ_j are the eigenvalues of P_0, then the ordered eigenvalues of $P_0 + d + R$ satisfy*

$$\eta_j = \mu_j + d + O(\mu_j^{-1}).$$

Proof. The operator $P_0^{1/2} R P_0^{1/2}$ has order zero and is bounded. Thus, for some $C > 0$,

$$P_0 + d - CP_0^{-1} \le P_0 + d + R \le P_0 + d + CP_0^{-1}$$

as forms. The functions $x + d \pm C/x$ are increasing for all sufficiently large x. Their values on the bounded initial part of the spectrum lie below every sufficiently large value. Consequently, for all large j, the jth ordered eigenvalues of the two bounding operators are exactly $\mu_j + d \pm C/\mu_j$. The variational inequalities give the result. □

20.2. **Scalar return and conjugation.** We identify functions with half-densities by the unitary map $u \mapsto u\,(d\,\mathrm{vol}_g)^{1/2}$. All symbols below refer to this identification; $\mathrm{sub}(A)$ denotes the scalar subprincipal symbol of A. The Hamiltonian flow of $p(x,\xi) = |\xi|_g$ is denoted by G_t. The Poisson bracket is normalized by $H_p f = \{p, f\}$. Thus, if $A \in \Psi^1_{\mathrm{cl}}(M)$ has principal symbol a and $F \in \Psi^0_{\mathrm{cl}}(M)$ has principal symbol f, then $\sigma_0(i[F,A]) = \{f, a\}$.

Lemma 20.6. *Let $A \in \Psi^2_{\mathrm{cl}}(M)$ be a strictly positive self-adjoint scalar operator with positive principal symbol a_2. Then $A^{1/2} \in \Psi^1_{\mathrm{cl}}(M)$, and*

$$\sigma_1(A^{1/2}) = \sqrt{a_2}, \qquad \mathrm{sub}(A^{1/2}) = \frac{\mathrm{sub}(A)}{2\sqrt{a_2}}.$$

In particular, if $c > 0$, $B \in \Psi^1_{\mathrm{cl}}(M)$ is symmetric, and $\Delta_g + B + c > 0$, then

$$(\Delta_g + B + c)^{1/2} - (\Delta_g + c)^{1/2} \in \Psi^0_{\mathrm{cl}}(M),$$

with principal symbol $\sigma_1(B)/(2|\xi|_g)$.

Proof. The complex-powers theorem [6, Example 2 and Theorem 7.9, arXiv version 1], applied to the classical calculus on the closed manifold M, gives $P = A^{1/2} \in \Psi^1_{\mathrm{cl}}$ and $\sigma_1(P) = \sqrt{a_2}$. The subprincipal product formula [21, equation (1.4), p. 41] gives

$$\mathrm{sub}(A) = \mathrm{sub}(P^2) = 2\sqrt{a_2}\,\mathrm{sub}(P).$$

For $P_0 = (\Delta_g + c)^{1/2}$ on half-densities, $\mathrm{sub}(P_0) = 0$; see [21, p. 61]. Hence $\mathrm{sub}(\Delta_g + c) = 0$ and $\mathrm{sub}(\Delta_g + B + c) = \sigma_1(B)$. The two square roots have the same principal symbol $|\xi|_g$; the principal symbol of their difference is therefore $\sigma_1(B)/(2|\xi|_g)$. □

Lemma 20.7. *Let $\dim M \ge 2$, fix $c, T > 0$, and put $P_0 = (\Delta_g + c)^{1/2}$. Let $P = P_0 + B_0$ be self-adjoint, where $B_0 \in \Psi^0_{\mathrm{cl}}(M)$ has principal symbol b. Put*

$$\overline{b} = T^{-1} \int_0^T b \circ G_t \, dt.$$

(i) *If $e^{iTP} - \zeta I$ is compact for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, then $G_T = I$ and $\overline{b}$ is a real constant.*
(ii) *If $G_T = I$ and $\overline{b} = d \in \mathbb{R}$ is constant, there is a unitary $U \in \Psi^0_{\mathrm{cl}}(M)$ such that*

$$UPU^* = P_0 + d + R, \qquad R \in \Psi^{-1}_{\mathrm{cl}}(M).$$

Proof. For (i), choose $a \in \mathbb{R}$ such that $P + aI > 0$. This operator has principal symbol p and subprincipal symbol $b + a$. Choose $\beta \in \mathbb{R}$ with $e^{iT\beta} = e^{iTa}\zeta$. If θ_j are the eigenvalues of $P + aI$, the compactness of $e^{iT(P+aI)} - e^{iT\beta} I$ gives

$$\mathrm{dist}\left(\theta_j, \beta + \frac{2\pi}{T}\mathbb{Z}\right) \longrightarrow 0.$$

For any fixed $\varepsilon, K > 0$, every eigenvalue in $[\mu - K, \mu + K]$ therefore lies within ε of this progression once μ is sufficiently large. Thus condition (3.8) of [21, Theorem 3.2, p. 55] holds. That theorem gives $G_T = I$ and constancy of the orbit average of $b + a$. Hence $\overline{b}$ is a real constant.

Whenever $G_T = I$, the wave transport equation [21, proof of Theorem 3.1, equation (3.3), p. 53] gives $e^{iTP} \in \Psi^0_{\mathrm{cl}}(M)$ and

$$\sigma_0(e^{iTP})(\rho) = \rho_T \exp\left(i \int_0^T b(G_t\rho)\, dt\right), \qquad \rho \in S^*M. \tag{20.5}$$

Here $\rho_T \in \{1, -1, i, -i\}$ is the Maslov factor. It is locally constant, hence constant on the connected space S^*M. For a semibounded P, apply the formula to $P + aI > 0$ and cancel the factor e^{iTa}.

For (ii), put

$$f = \frac{1}{T} \int_0^T t\,(b - d) \circ G_t\, dt.$$

Integration by parts gives

$$H_p f = \frac{1}{T}\,[t(b - d) \circ G_t]_0^T - \frac{1}{T} \int_0^T (b - d) \circ G_t\, dt = b - d.$$

Choose $F = F^* \in \Psi^0_{\mathrm{cl}}(M)$ with principal symbol f. The holomorphic functional calculus [6, Example 2, Theorem 5.4, and proof of Theorem 7.6, arXiv version 1] gives a smooth family

$$U_s = e^{isF} \in \Psi^0_{\mathrm{cl}}(M), \qquad \sigma_0(U_s) = e^{isf} \qquad (0 \le s \le 1).$$

Each U_s is unitary and preserves $H^1(M)$. Put $U = U_1$. The scalar commutator formula gives

$$\sigma_0(i[F, P]) = \{f, p\} = -(b - d), \qquad [F, [F, P]] \in \Psi^{-1}_{\mathrm{cl}}(M).$$

On $H^1(M)$, twice integrating the derivative of $U_s P U_s^*$ gives

$$UPU^* = P + i[F, P] - \int_0^1 (1 - s) U_s [F, [F, P]] U_s^*\, ds.$$

The remainder integral has order -1, so $UPU^* = P_0 + d + R$ with $R \in \Psi^{-1}_{\mathrm{cl}}(M)$. Both sides have domain $H^1(M)$, and the bounded operator R is self-adjoint. □

20.3. **The spherical reference.** Let $n = 2m + 1 \ge 5$. Let Q be either a homogeneous unit spherical space form of even group order, or the scalar lens space

$$Q = \mathbb{S}^{2m+1}(1)/\langle e^{2\pi i/q} I\rangle, \qquad q \ge 5 \text{ odd}.$$

Put $T = \pi$ in the first case and $T = 2\pi$ in the second. In both cases $G_T^Q = I$: in the even-order case, the deck group contains $-I$ by Lemma 8.2. These references satisfy the ordered scalar isolation statement in Corollary 2.5: if

$$\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \longrightarrow 0,$$

then M is isometric to Q.

Theorem 20.8 (Perturbations of order one). *Let $B_M \in \Psi^1_{\mathrm{cl}}(M)$ and $B_Q \in \Psi^1_{\mathrm{cl}}(Q)$ be symmetric scalar operators. Let b_Q be the degree-zero symbol*

$$b_Q(x, \xi) = \frac{\sigma_1(B_Q)(x, \xi)}{2|\xi|_Q}.$$

Suppose that

$$\frac{1}{T} \int_0^T b_Q(G_t^Q \rho)\, dt \tag{20.6}$$

is independent of $\rho \in S^*Q$. Then

$$\mathrm{Spec}(\Delta_M + B_M) = \mathrm{Spec}(\Delta_Q + B_Q) \quad \Longrightarrow \quad M \cong Q.$$

In particular, the conclusion holds for arbitrary symmetric $B_M \in \Psi^1_{\mathrm{cl}}(M)$ when $B_Q = 0$.

Proof. The common spectrum is infinite, so $\dim M > 0$. For $Z = M, Q$, the operator $(1+\Delta_Z)^{-1/4} B_Z (1+\Delta_Z)^{-1/4}$ has order zero and is bounded. Hence B_Z satisfies the form bound in Theorem 20.1. The form sums are semibounded and self-adjoint, with compact resolvent. Elliptic regularity [28, Lemma 1.3.5] identifies their operator domains with $H^2(Z)$ and their restrictions there with $\Delta_Z + B_Z$. Their equal spectra and Theorem 20.1 give $\dim M = \dim Q = n$ and $\operatorname{Vol} M = \operatorname{Vol} Q$. Choose $c > 0$ so large that both $\Delta_Z + B_Z + c$ are strictly positive. For $Z = M, Q$, put

$$P_Z = (\Delta_Z + B_Z + c)^{1/2}, \qquad P_Z^0 = (\Delta_Z + c)^{1/2}.$$

Lemma 20.6 gives $P_Z - P_Z^0 \in \Psi^0_{\mathrm{cl}}$ with principal symbol $b_Z = \sigma_1(B_Z)/(2|\xi|_Z)$. Let d_Q denote (20.6). Lemma 20.7(ii) gives

$$U_Q P_Q U_Q^* = P_Q^0 + d_Q + R_Q, \qquad R_Q \in \Psi^{-1}_{\mathrm{cl}}.$$

The reference scalar frequencies satisfy

$$\sqrt{\lambda_j(Q) + m^2} \in \begin{cases} m + 2\mathbb{Z}_{\geq 0}, & T = \pi, \\ m + \mathbb{Z}_{\geq 0}, & T = 2\pi. \end{cases}$$

Write μ_j^Z for the eigenvalues of P_Z^0 and η_j for the common eigenvalues of P_M, P_Q. Lemma 20.5 gives

$$\eta_j = \mu_j^Q + d_Q + O((\mu_j^Q)^{-1}).$$

Moreover,

$$\mu_j^Q - \sqrt{\lambda_j(Q) + m^2} = \frac{c - m^2}{\mu_j^Q + \sqrt{\lambda_j(Q) + m^2}} = O((\mu_j^Q)^{-1}).$$

The displayed lattice therefore implies $e^{iT\eta_j} \to e^{iT(m+d_Q)}$. In an eigenbasis of either P_Z, the diagonal entries of $e^{iTP_Z} - e^{iT(m+d_Q)} I$ tend to zero. These two differences are compact. Lemma 20.7 gives a constant d_M and a unitary U_M such that

$$U_M P_M U_M^* = P_M^0 + d_M + R_M, \qquad R_M \in \Psi^{-1}_{\mathrm{cl}}.$$

The common leading Weyl term gives $\mu_j^M \sim \mu_j^Q$. Lemma 20.5 and equality of the spectra give

$$\mu_j^Q = \mu_j^M + d_M - d_Q + O((\mu_j^M)^{-1}).$$

Lemma 20.4 gives $d_M = d_Q$. Hence $\mu_j^M - \mu_j^Q \to 0$. Since

$$\sqrt{\lambda + c} - \sqrt{\lambda} = \frac{c}{\sqrt{\lambda + c} + \sqrt{\lambda}} \longrightarrow 0,$$

the unshifted scalar frequencies also have difference tending to zero. Ordered scalar isolation gives $M \cong Q$. □

Remark 20.9. For every $n \geq 2$, the diagonal operator B in Example 20.2 cannot belong to $\Psi^1_{\mathrm{cl}}(M)$. If it did, the equality of its perturbed spectrum with that of $\mathbb{RP}^n(1)$ would give

$$e^{i\pi(\Delta_M + B + (n-1)^2/4)^{1/2}} = e^{i\pi(n-1)/2} I.$$

Lemma 20.7(i) would force $G_\pi^M = I$. On $\mathbb{S}^n(a)$ the prime geodesic period is $2\pi a$, whereas $0 < \pi/(2\pi a) = 2^{1/n-1} < 1$. This is impossible.

21. First-order differential perturbations on different manifolds

All operators act on complex-valued functions, and the Laplacian is nonnegative. For a Killing field V, put $D_V u = -iV(u)$ on smooth functions.

Theorem 21.1. *Let $m, p \geq 1$ be integers. Put*

$$Q_p = L(p; 1, \dots, 1) = \mathbb{S}^{2m+1}(1)/\langle e^{2\pi i/p} I \rangle, \qquad M_p = \left(\prod_{j=1}^{m} \mathbb{S}^2 \left(\frac{1}{2\sqrt{j}} \right) \right) \times \mathbb{S}^1 \left(\frac{1}{p} \right).$$

Let K be the vector field on Q_p induced by

$$(z_0, \dots, z_m) \longmapsto (e^{imt} z_0, e^{i(m-2)t} z_1, \dots, e^{-imt} z_m) \quad \text{on } \mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}.$$

For $1 \leq j \leq m$, let V_j be the rotation field of angular speed one about the third coordinate axis of the jth sphere factor of M_p. Set

$$B_Q = 2D_K, \qquad B_M = 4 \sum_{j=1}^{m} j D_{V_j}.$$

These are symmetric scalar differential operators of order one. The operators $\Delta_{Q_p} + B_Q$ and $\Delta_{M_p} + B_M$ are nonnegative, self-adjoint on H^2, and have the same spectrum, with multiplicities. More precisely, their common spectrum is

$$\coprod_{\substack{a \in \mathbb{Z} \\ n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}}} \left\{ p^2 a^2 + 4 \sum_{j=1}^{m} j n_j \right\}, \tag{21.1}$$

with one copy for each displayed tuple. Thus, for every $t > 0$,

$$\operatorname{Tr} e^{-t(\Delta_{Q_p} + B_Q)} = \operatorname{Tr} e^{-t(\Delta_{M_p} + B_M)} = \frac{\displaystyle\sum_{a \in \mathbb{Z}} e^{-t p^2 a^2}}{\displaystyle\prod_{j=1}^{m} (1 - e^{-4jt})}. \tag{21.2}$$

Their fundamental groups are

$$\pi_1(Q_p) = \mathbb{Z}/p\mathbb{Z}, \qquad \pi_1(M_p) = \mathbb{Z}.$$

In particular, M_p and Q_p are not isometric. Both have volume $2\pi^{m+1}/(p\, m!)$.

Proof. The flow defining K commutes with scalar multiplication and is isometric; hence it descends to Q_p. We also write K for its lift to $\mathbb{S}^{2m+1}$, and put $B_{\mathbb{S}^{2m+1}} = 2D_K$. Each V_j is Killing. For a Killing field V, one has $\operatorname{div} V = 0$, so D_V is symmetric on smooth functions. For $Z \in \{Q_p, M_p, \mathbb{S}^{2m+1}\}$ and every $\varepsilon > 0$,

$$\|B_Z u\|_{L^2} \leq C_Z \|u\|_{H^1} \leq \varepsilon \|\Delta_Z u\|_{L^2} + C_{Z,\varepsilon} \|u\|_{L^2}, \qquad u \in H^2(Z).$$

Thus B_Z has relative operator bound zero. By [49, Chapter V, Theorem 4.3, p. 287], the operator $\Delta_Z + B_Z$ is self-adjoint on $H^2(Z)$. Its graph norm is equivalent to the H^2 norm, so $C^\infty(Z)$ is a core. The isometric flows commute with the corresponding Laplacians. Their infinitesimal generators are commuting Hermitian operators on each Laplace eigenspace. They therefore admit complete orthogonal systems of simultaneous eigenfunctions.

Harmonic polynomials.

For integers $d \geq 3$ and $k \geq 0$, let $\mathcal{P}_k(\mathbb{R}^d)$ be the complex homogeneous polynomials of degree k, and let $\mathcal{H}_k(\mathbb{R}^d)$ be their harmonic subspace. Put $\mathcal{P}_k = 0$ for $k < 0$. The Fischer decomposition

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}$$

is $O(d)$-equivariant [18, Theorems 1.3 and 1.7]. The restrictions of the spaces $\mathcal{H}_k$ give the complete orthogonal decomposition into scalar eigenspaces on $\mathbb{S}^{d-1}(1)$, with eigenvalues $k(k + d - 2)$ [45, Lemma 1.3 and Proposition 1.5, pp. 78–79]. We identify harmonic polynomials with their restrictions to the sphere.

Under the Hopf action $z \mapsto e^{it}z$, a function has weight $c \in \mathbb{Z}$ if $f(e^{it}z) = e^{ict}f(z)$. For integers $r, s \geq 0$, let $\mathcal{P}_{r,s}$ be the polynomials on $\mathbb{C}^{m+1}$ that are homogeneous of degree r in z and degree s in $\overline{z}$, and let $\mathcal{H}_{r,s}$ be their harmonic subspace. Put $\mathcal{P}_{r,s} = 0$ when either index is negative. Within $\mathcal{P}_{r+s}$, the Hopf weight-$(r-s)$ subspace is $\mathcal{P}_{r,s}$. Restricting the Fischer decomposition to this weight gives

$$\mathcal{P}_{r,s} = \mathcal{H}_{r,s} \oplus |z|^2 \mathcal{P}_{r-1,s-1}. \tag{21.3}$$

The summands are invariant under the coordinate torus. The restrictions of $\mathcal{H}_{r,s}$ have Laplace eigenvalue $(r+s)(r+s+2m)$ and Hopf weight $r-s$. The scalar deck transformation acts on $\mathcal{H}_{r,s}$ by $e^{2\pi i(r-s)/p}$. Hence the spaces with $r - s \in p\mathbb{Z}$ give the complete orthogonal decomposition on Q_p; cf. [45, Lemma 1.4 and equation (1.3), p. 79].

The product.

In spherical coordinates (θ, ϕ) on $\mathbb{S}^2(1)$, put $D = -i\partial_\phi$. The standard orthogonal basis $Y_{k,a}(\theta, \phi)$, $-k \leq a \leq k$, of $\mathcal{H}_k$ satisfies

$$\Delta Y_{k,a} = k(k+1)Y_{k,a}, \qquad DY_{k,a} = aY_{k,a};$$

see [18, Theorem 4.5 and equation (6.6)]. Thus

$$\operatorname{Spec}(\Delta_{\mathbb{S}^2(1)} + D) = \coprod_{k\geq 0} \{k(k+1) + a \,:\, -k \leq a \leq k\} = \{0, 1, 2, \ldots\},$$

with multiplicity one: the kth interval consists of the integers from k^2 through $(k+1)^2 - 1$.

Identify each sphere factor with the unit sphere by radial dilation, and let $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ be the angular coordinate on the circle factor. Write $\Delta^{(j)}_{\mathbb{S}^2(1)}$ for the unit-sphere Laplacian acting on the jth factor. Then

$$\Delta_{M_p} + B_M = 4 \sum_{j=1}^{m} j\left(\Delta^{(j)}_{\mathbb{S}^2(1)} + D_{V_j}\right) + p^2(-\partial_\vartheta^2).$$

The product eigenfunctions therefore have eigenvalues

$$p^2 r^2 + 4 \sum_{j=1}^{m} j n_j, \qquad r \in \mathbb{Z}, \quad n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0},$$

with one copy for each displayed tuple. Fix $q \in (0, 1)$. Then

$$\operatorname{Tr} q^{(\Delta_{M_p} + B_M)/4} = \frac{\sum_{r\in\mathbb{Z}} q^{p^2 r^2/4}}{\prod_{j=1}^{m}(1 - q^j)}. \tag{21.4}$$

The joint character on the sphere.

For formal variables u, v, write $[u^r v^s]F$ for the coefficient of $u^r v^s$ in a power series $F(u, v)$, and similarly for $[u^r]$. Put $w_j = m - 2j$ for $0 \leq j \leq m$. The coordinates $z_j, \overline{z}_j$ have D_K-weights $w_j, -w_j$. The multiset of the numbers $w_j/2$ is $\{-m/2, -m/2 + 1, \ldots, m/2\}$, as is the multiset of their negatives. Thus

$$\operatorname{Tr}_{\mathcal{P}_{r,s}} q^{D_K/2} = q^{-m(r+s)/2}[u^r v^s] \prod_{j=0}^{m} (1 - q^j u)^{-1}(1 - q^j v)^{-1}.$$

Put

$$F_m(u, v) = \frac{1 - q^m uv}{\prod_{j=0}^{m}(1 - q^j u)(1 - q^j v)}.$$

Since $|z|^2$ has bidegree $(1, 1)$ and D_K-weight zero, equation (21.3) gives

$$\operatorname{Tr}_{\mathcal{H}_{r,s}} q^{D_K/2} = q^{-m(r+s)/2}[u^r v^s]F_m(u, v). \tag{21.5}$$

On total degree $k \geq 0$, every D_K-weight has absolute value at most mk, so every eigenvalue of $\Delta + 2D_K$ is at least k^2. Moreover,

$$\dim \mathcal{H}_k(\mathbb{R}^{2m+2}) \leq \binom{k+2m+1}{2m+1}, \qquad \sum_{k\geq 0} \binom{k+2m+1}{2m+1} q^{k^2/4} < \infty.$$

Thus the operator is nonnegative and its q-trace converges. For $c \in \mathbb{Z}$, let Tr_c denote the trace on the closed Hopf weight-c subspace of $L^2(\mathbb{S}^{2m+1})$. On $\mathcal{H}_{r,s}$, put $k = r + s$; the operator $\Delta/4$ has eigenvalue $k(k+2m)/4$. Hence

$$\mathrm{Tr}_c\, q^{(\Delta+2D_K)/4} = \sum_{\substack{r,s\geq 0 \\ r-s=c}} q^{(r+s)^2/4}[u^r v^s] F_m(u,v). \tag{21.6}$$

For multi-indices $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{m+1}$ with $|\alpha| = r$ and $|\beta| = s$, the monomial $z^\alpha \overline{z}^\beta$ has Hopf weight $c = r - s$ and D_K-weight $d = \sum_{j=0}^m w_j(\alpha_j - \beta_j)$. Then

$$\frac{k(k+2m) + 2d - c^2}{4} = rs + \sum_{j=0}^m (m-j)\alpha_j + \sum_{j=0}^m j\beta_j \in \mathbb{Z}_{\geq 0}.$$

Every weight of the harmonic subspace is a polynomial weight. Thus each Hopf weight-c trace, after division by $q^{c^2/4}$, is a power series with nonnegative integral exponents.

A power-series identity.

Put

$$(a;q)_j = \prod_{h=0}^{j-1}(1 - aq^h), \qquad (a;q)_\infty = \prod_{h=0}^{\infty}(1 - aq^h),$$

with $(a;q)_0 = 1$. The q-binomial theorem [13, p. 4, following equation (3.5)] gives

$$A_j := [u^j] \prod_{h=0}^m (1 - q^h u)^{-1} = \frac{(q^{m+1};q)_j}{(q;q)_j} \quad (j \geq 0); \qquad A_{-1} = 0.$$

Write

$$Z_{m,c} = \sum_{\substack{r,s\geq 0 \\ r-s=c}} q^{(r+s)^2/4}[u^r v^s] F_m(u,v).$$

Since $F_m(u,v) = F_m(v,u)$, one has $Z_{m,-c} = Z_{m,c}$. Suppose $c \geq 0$, and put

$$C_{m,c} = \frac{(q;q)_{m+c-1}}{(q;q)_m (q;q)_c}.$$

For $j \geq 0$, substitution of the formula for A_j gives

$$\begin{aligned}[u^{j+c}v^j]F_m &= A_{j+c}A_j - q^m A_{j+c-1}A_{j-1} \\ &= C_{m,c}\,\frac{(q^{m+c};q)_j (q^m;q)_j}{(q^{c+1};q)_j (q;q)_j}(1 - q^{m+c+2j}).\end{aligned}$$

Apply the Rogers–Fine identity [13, Corollary 1, equation (2.6), p. 3] with $a = q^{m+c}$ and $b = q^{c+1}$. Its hypotheses hold since $0 < q < 1$ and $b \neq q^{-h}$ for every $h \geq 0$. As $b^j q^{j^2-j} = q^{j^2+cj}$, it gives

$$Z_{m,c} = q^{c^2/4} C_{m,c} \frac{(q^{m+c};q)_\infty}{(q^{c+1};q)_\infty} = \frac{q^{c^2/4}}{\prod_{j=1}^m (1 - q^j)}. \tag{21.7}$$

By symmetry, the last equality holds for every $c \in \mathbb{Z}$. Equations (21.6) and (21.7) show that the spectrum of $\Delta + 2D_K$ on the Hopf weight-c subspace is

$$\coprod_{n_1,\ldots,n_m \in \mathbb{Z}_{\geq 0}} \left\{ c^2 + 4\sum_{j=1}^m j n_j \right\}.$$

The equality of multiplicities follows by comparing the coefficients of the convergent power series after division by $q^{c^2/4}$. Retaining $c = pa$ for $a \in \mathbb{Z}$ proves (21.1) for Q_p, and hence equality of all eigenvalues and multiplicities in the theorem.

Heat trace and geometry.

Every bounded interval contains finitely many tuples in (21.1). Summing their exponentials gives (21.2).

All the sphere factors of M_p are simply connected, while its circle factor has fundamental group $\mathbb{Z}$. The universal cover of Q_p is $\mathbb{S}^{2m+1}$, with deck group $\mathbb{Z}/p\mathbb{Z}$. These fundamental groups differ.

Finally,

$$\operatorname{Vol}(Q_p) = \frac{2\pi^{m+1}}{p\,m!}, \qquad \operatorname{Vol}(M_p) = \left(\prod_{j=1}^{m} \frac{\pi}{j}\right)\frac{2\pi}{p} = \frac{2\pi^{m+1}}{p\,m!}.$$

□

Remark 21.2. For $m \geq 2$, the theorem applies in every odd dimension at least five. The case $p = 2$ gives a homogeneous even-order spherical reference; every odd $p \geq 5$ gives precisely a scalar odd-order lens reference. Theorem 21.1 therefore excludes a two-sided extension of Theorem 20.8 without its orbit-average hypothesis. If C denotes complex conjugation, then $CB_ZC = -B_Z$ on $C^\infty(Z)$ for $Z \in \{Q_p, M_p\}$. Since $B_Z \neq 0$, these perturbations do not commute with C.

Write g for the metric on Q_p and ∇ for its Levi–Civita connection. By Lemma 20.6, the principal symbol of $\sqrt{\Delta_{Q_p} + B_Q + c} - \sqrt{\Delta_{Q_p} + c}$, for a real $c > 0$, is

$$b_Q(x,\xi) = \frac{\xi(K(x))}{|\xi|}, \qquad (x,\xi) \in T^*Q_p \setminus 0.$$

For a unit-speed geodesic γ,

$$\frac{d}{dt}\langle K(\gamma(t)), \dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma} K, \dot\gamma\rangle = \tfrac{1}{2}(\mathcal{L}_K g)(\dot\gamma,\dot\gamma) = 0.$$

Thus b_Q is constant along each geodesic. The unit-speed geodesics $\gamma_\pm(t) = [(e^{\pm it}, 0, \dots, 0)]$ satisfy

$$\langle K(\gamma_\pm(t)), \dot\gamma_\pm(t)\rangle = \pm m.$$

Hence the orbit average of b_Q is not constant on S^*Q_p.

For $m = p = 1$, the isometry $J(z_0, z_1) = (z_0, \overline{z}_1)$ sends K to the unit Hopf field $H(z) = iz$. Let α be the metric dual of H, and let d denote exterior differentiation. The magnetic Laplacian is

$$\Delta_\alpha = (d + i\alpha)^*(d + i\alpha) = \Delta_{\mathbb{S}^3} + 2D_H + 1$$

by [90, equation (1.2), p. 1297]. Its spectrum is given by [90, Appendix A.4, equation (A.5), p. 1339] with Berger parameter $\varepsilon = 1$ and magnetic parameter $t = 1$. Thus the sphere operator in this case is unitarily equivalent to $\Delta_\alpha - 1$. An earlier computation, with magnetic potential $-\alpha$ and mass $1/2$, appears in [91, §VI.A, equations (93)–(99), pp. 21–22]. Since $C\Delta_\alpha C = \Delta_{-\alpha}$, it gives the same spectrum.

22. Sharpness of the multiplicity exponent

Let $s \geq 3$ be an integer, and let $Y = \mathbb{S}^{2s-1}(1)/\Gamma$ be a unit round spherical space form. For $k \in \mathbb{Z}_{\geq 0}$, let $r_k(Y)$ be the scalar Laplace multiplicity of $k(k + 2s - 2)$, and put

$$F_Y(z) = \sum_{k\geq 0} r_k(Y) z^k.$$

Thus $F_Y = F_\Gamma$ in (8.2). We put $\binom{\ell}{r} = 0$ for integers $0 \leq \ell < r$.

Lemma 22.1. *Let $q \ge 1$ and $d \ge 2$ be integers, and let $F(z) = \sum_{k\ge 0} c_k z^k$ be a rational function analytic at zero, with real Taylor coefficients. Suppose all its poles are qth roots of unity, their orders are at most d, and at least one has order d. Then there are real numbers $b_0, \dots, b_{q-1}$, not all zero, such that, as $k \to \infty$,*

$$c_k = b_v k^{d-1} + O(k^{d-2}) \qquad (k \equiv v \pmod q).$$

Consequently, as $H \to +\infty$,

$$\sum_{0\le k\le H} |c_k| = \frac{1}{qd}\left(\sum_{v=0}^{q-1} |b_v|\right) H^d + O(H^{d-1}).$$

Proof. Subtracting the polynomial part of F changes only finitely many coefficients. By [74, Proposition 4.4.1, p. 546, author's version], the remaining proper rational function has coefficients of the form

$$c_k = \sum_{\xi^q=1} Q_\xi(k)\xi^{-k}, \qquad \deg Q_\xi \le d-1,$$

for all sufficiently large k, and at least one Q_ξ has degree $d-1$. Put

$$a_\xi = [X^{d-1}]Q_\xi(X), \qquad b_v = \sum_{\xi^q=1} a_\xi \xi^{-v}.$$

Then $c_k = b_v k^{d-1} + O(k^{d-2})$ for $k \equiv v \pmod q$. The coefficient b_v is real, being the limit of c_k/k^{d-1} on that residue class. Fourier inversion gives

$$a_\xi = \frac{1}{q}\sum_{v=0}^{q-1} b_v \xi^v,$$

so the b_v are not all zero. Moreover,

$$|c_k| = |b_v| k^{d-1} + O(k^{d-2}) \qquad (k \equiv v \pmod q),$$

and

$$\sum_{\substack{0\le k\le H\\ k\equiv v \pmod q}} k^{d-1} = \frac{H^d}{qd} + O(H^{d-1}).$$

Summing over v proves the second formula; the accumulated error is $O(H^{d-1})$. □

22.1. **Even-order lens spaces.** Let $Y = \mathbb{S}^{2s-1}(1)/\Gamma$ be a cyclic spherical space form with $|\Gamma| \ge 3$. Let h be the number of distinct conjugate eigenvalue pairs of a generator, and let ρ_ν, $1 \le \nu \le h$, be the common complex multiplicity of the two eigenvalues in the νth pair. Then

$$r_2(Y) = \sum_{\nu=1}^{h} \rho_\nu^2 - 1 \tag{22.1}$$

by [69, Corollary 4, p. 308].

Proof of Theorem 2.7. Put $n = 2s-1$ and $\zeta = e^{\pi i/4}$. The two generating matrices are

$$\operatorname{diag}(\zeta, \dots, \zeta), \qquad \operatorname{diag}(\zeta, \dots, \zeta, \zeta^3).$$

Every weight is a unit modulo eight, so no nonidentity power has eigenvalue one. Thus both actions are free. Their fourth powers are $-I$; hence, by Lemma 8.1, only even harmonic degrees occur, and both shifted spectra lie in $s - 1 + 2\mathbb{Z}_{\ge 0}$. Since $\eta < 1/2$, a matching pairs only equal frequencies.

The conjugate-eigenvalue multiplicity lists are (s) and $(s-1, 1)$. Formula (22.1) gives $s^2 - 1$ and $(s-1)^2$, respectively. Their difference is $2(s-1) > 0$, so the quotients are not isometric. The scalar action commutes with the transitive group $U(s)$, which descends to Q. Thus Q is homogeneous.

For the group powers with exponents $0, 2, 4, 6$, the determinant polynomials agree in the two representations. Put

$$a(z) = 1 - \sqrt{2}z + z^2, \qquad b(z) = 1 + \sqrt{2}z + z^2, \qquad a(z)b(z) = 1 + z^4.$$

The remaining powers form the conjugate pairs $\{1, 7\}$ and $\{3, 5\}$. Subtracting their terms in (8.2) gives

$$F_Q - F_{Q'} = \frac{1 - z^2}{4}\left(a^{-s} + b^{-s} - \frac{1}{a^{s-1}b} - \frac{1}{b^{s-1}a}\right). \tag{22.2}$$

At a root of a, the factor b and the numerator $1 - z^2$ are nonzero. The pole from a^{-s} has order s, while every other term has order at most $s - 1$. The same holds at roots of b. These are precisely the primitive eighth roots.

Let $c_k = r_k(Q) - r_k(Q')$, and let $b_0, \dots, b_7$ be the coefficients supplied by Lemma 22.1 for $F_Q - F_{Q'}$. Then

$$\sum_{0 \le k \le H} |c_k| = C_s H^s + O(H^{s-1}), \qquad C_s = \frac{1}{8s}\sum_{v=0}^{7} |b_v| > 0.$$

At $R_K = 2K + s$, the possible harmonic degrees are even and at most $2K$. At each such degree the largest matching has the smaller of the two multiplicities. Hence

$$\mathcal{E}_\eta(K) = \sum_{k=0}^{2K} |c_k|.$$

Since $2K = R_K - s$,

$$(2K)^s = R_K^s + O(R_K^{s-1}).$$

This gives (2.13).

For $s = 4$,

$$a^4 + b^4 - ab(a^2 + b^2) = (a - b)^2(a^2 + ab + b^2) = 8z^2(3 + 8z^2 + 3z^4).$$

Substitution in (22.2) gives

$$F_Q - F_{Q'} = \frac{6z^2 + 10z^4 - 10z^6 - 6z^8}{(1 + z^4)^4}. \tag{22.3}$$

Since $(1 + z^4)^{-4} = \sum_{j \ge 0}(-1)^j \binom{j+3}{3} z^{4j}$, odd coefficients vanish. For $j \ge 1$,

$$c_{4j} = (-1)^{j-1}\left(10\binom{j+2}{3} + 6\binom{j+1}{3}\right),$$

and for $j \ge 0$,

$$c_{4j+2} = (-1)^j\left(6\binom{j+3}{3} + 10\binom{j+2}{3}\right).$$

Consequently

$$|c_k| = \begin{cases} k(k+4)(2k+7)/48, & k \equiv 0 \pmod 4, \\ (k+2)(k+6)(2k+5)/48, & k \equiv 2 \pmod 4, \end{cases}$$

with $c_0 = 0$. Thus $|c_k| = k^3/24 + O(k^2)$ on even indices, and

$$\sum_{\substack{0 \le k \le H \\ k \text{ even}}} \frac{k^3}{24} = \frac{1}{24}\,8 \sum_{0 \le j \le H/2} j^3 = \frac{H^4}{192} + O(H^3).$$

This proves $C_4 = 1/192$. □

22.2. **Sharpness for odd-order scalar lens spaces.**

Proposition 22.2. *Let $q \geq 5$ be an odd integer, let $s \geq 3$ be an integer, and put $n = 2s-1$. The unit round lens spaces*

$$Q = L(q; 1, \dots, 1), \qquad Q' = L(q; 1, \dots, 1, 2)$$

are not isometric. In Definition 2.1, take $M = Q'$, $r = 1$, and $m = s-1$, so that $R_K = K + s - 1/2$. For every $0 \leq \eta < 1/4$, their unmatched multiplicity satisfies

$$\mathcal{E}_\eta(K) = C_{q,s} R_K^s + O(R_K^{s-1}), \qquad C_{q,s} > 0. \tag{22.4}$$

For $q = 5$ and $s = 3$, one has $C_{5,3} = 4/75$.

Proof. Both actions are free because 1 and 2 are units modulo q. Since $2 \not\equiv \pm 1 \pmod{q}$, their conjugate-eigenvalue multiplicity lists are (s) and $(s-1, 1)$. Equation (22.1) gives $r_2(Q) - r_2(Q') = 2(s-1) > 0$. Hence the quotients are not isometric.

Put $\xi = e^{2\pi i/q}$, and write

$$D(z) = F_Q(z) - F_{Q'}(z) = \sum_{k \geq 0} c_k z^k.$$

The identity terms in (8.2) cancel. For a nonidentity group element, neither a weight-one nor a weight-two eigenvalue equals 1, and the two eigenvalues in each conjugate pair are distinct because q is odd. Thus every pole of D is a nonidentity qth root of unity and has order at most s.

By (18.1), with $m = s-1$,

$$\lim_{z \to \xi} (1 - z/\xi)^s F_Q(z) = \frac{2}{q} (1 - \xi^2)^{1-s} \neq 0.$$

For a term of the mixed action to have an order-s pole there, its weight-one and weight-two blocks would both have to contain ξ^{-1} as an eigenvalue. Writing the corresponding group element as a power with exponent j gives $\xi^{\pm j} = \xi^{-1} = \xi^{\pm 2j}$. Since ξ is primitive, j is a unit modulo q; hence this equality implies $2 \equiv \pm 1 \pmod{q}$. Neither congruence holds for odd $q \geq 5$. The mixed sum therefore has pole order at most $s-1$ at ξ. Consequently

$$\lim_{z \to \xi} (1 - z/\xi)^s D(z) = \frac{2}{q} (1 - \xi^2)^{1-s} \neq 0.$$

Let $b_0, \dots, b_{q-1}$ be the coefficients supplied by Lemma 22.1 for D. Then

$$\sum_{0 \leq k \leq H} |c_k| = C_{q,s} H^s + O(H^{s-1}), \qquad C_{q,s} = \frac{1}{qs} \sum_{v=0}^{q-1} |b_v| > 0.$$

Both shifted spectra have frequency $k + s - 1$ in harmonic degree k. Since $\eta < 1/4$, a matching pairs only equal frequencies. The cutoff $R_K = K + s - 1/2$ includes exactly the degrees $k \leq K$. Matching the smaller multiplicity at each frequency gives

$$\mathcal{E}_\eta(K) = \sum_{k=0}^{K} |c_k|.$$

Since $K = R_K + O(1)$, the preceding asymptotic formula proves (22.4).

For $q = 5$ and $s = 3$, put

$$a(z) = 1 - \frac{\sqrt{5}-1}{2} z + z^2, \qquad b(z) = 1 + \frac{\sqrt{5}+1}{2} z + z^2.$$

The two nonidentity conjugate pairs give

$$\begin{aligned} D(z) &= \frac{2(1-z^2)}{5} \left(\frac{1}{a^3} + \frac{1}{b^3} - \frac{1}{a^2 b} - \frac{1}{a b^2} \right) \\ &= \frac{2(1-z^2)(a+b)(a-b)^2}{5 a^3 b^3}. \end{aligned}$$

Since

$$ab = 1 + z + z^2 + z^3 + z^4, \qquad a + b = 2 + z + 2z^2, \qquad (a - b)^2 = 5z^2,$$

one obtains the exact identity

$$D(z) = \frac{4z^2 - 10z^3 + 6z^4 + 6z^7 - 10z^8 + 4z^9}{(1 - z^5)^3}. \tag{22.5}$$

The expansion $(1 - z^5)^{-3} = \sum_{j \geq 0} \binom{j+2}{2} z^{5j}$ gives $c_{5j} = c_{5j+1} = 0$ and

$$\begin{aligned} c_{5j+2} &= 4\binom{j+2}{2} + 6\binom{j+1}{2}, \\ c_{5j+3} &= -10\binom{j+2}{2} - 10\binom{j+1}{2}, \\ c_{5j+4} &= 6\binom{j+2}{2} + 4\binom{j+1}{2} \qquad (j \geq 0). \end{aligned}$$

The coefficients of k^2 on the five residue classes are therefore

$$(b_0, b_1, b_2, b_3, b_4) = \left(0, 0, \frac{1}{5}, -\frac{2}{5}, \frac{1}{5}\right),$$

and hence $C_{5,3} = (4/5)/15 = 4/75$. □

For every real $\alpha > s = (n+1)/2$, the preceding pair satisfies

$$\frac{\mathcal{E}_\eta(K)}{R_K^\alpha} \longrightarrow 0, \qquad Q' \not\cong Q.$$

Thus the exponent $(n+1)/2$ in Theorem 2.4 cannot be increased. The same pair satisfies $\mathcal{E}_\eta(K) = O(R_K^{(n+1)/2})$, so this estimate with unrestricted constant cannot replace that theorem's positive-threshold hypothesis.

23. Five-dimensional lens spaces at shrinking resolution

Lemma 23.1. *Let Y be a closed Riemannian 5-manifold and put $P_Y = (\Delta_Y + 4)^{1/2}$. Define $a_j(Y)$ by*

$$\operatorname{Tr}(e^{-t\Delta_Y}) \sim (4\pi t)^{-5/2} \sum_{j \geq 0} a_j(Y) t^j \qquad (t \downarrow 0).$$

For $\chi \in C_c^\infty((0,\infty))$, as $R \to \infty$,

$$\begin{aligned} \operatorname{Tr} \chi(P_Y/R) = {} & \frac{a_0(Y)}{12\pi^3} R^5 \int_0^\infty \chi(u) u^4 \, du \\ & + \frac{a_1(Y) - 4a_0(Y)}{8\pi^3} R^3 \int_0^\infty \chi(u) u^2 \, du \\ & + \frac{a_2(Y) - 4a_1(Y) + 8a_0(Y)}{16\pi^3} R \int_0^\infty \chi(u) \, du + O_{Y,\chi}(R^{-1}). \end{aligned} \tag{23.1}$$

Let Rm *and* Ric *denote the Riemann curvature tensor and the Ricci tensor of Y, and let* Scal *denote its scalar curvature. Norms are induced by the metric, and dV_Y denotes the Riemannian volume density. Then*

$$\begin{aligned} a_0(Y) &= \operatorname{Vol}(Y), \qquad a_1(Y) = \frac{1}{6} \int_Y \operatorname{Scal} \, dV_Y, \\ 360\, a_2(Y) &= \int_Y \left(2|\operatorname{Rm}|^2 - 2|\operatorname{Ric}|^2 + 5 \operatorname{Scal}^2\right) dV_Y. \end{aligned} \tag{23.2}$$

Proof. Put $L = \Delta_Y + 4$ and $h = R^{-1}$. The integrated semiclassical functional calculus gives, for $f \in C_c^\infty(\mathbb{R})$,

$$\operatorname{Tr} f(h^2 L) \sim \sum_{j\geq 0} h^{-5+2j} C_j(f)$$

[16, Section 5.4 and Remark 2, p. 2445]. The Laplace symbol has even parity; the shift by 4 preserves the even powers of h, since $f(h^2 L) = f_h(h^2 \Delta_Y)$ with $f_h(s) = f(s + 4h^2)$. The distributions C_j are supported in $[0, \infty)$, and scaling gives

$$C_j(f(\tau\,\cdot)) = \tau^{-5/2+j} C_j(f) \qquad (\tau > 0).$$

For $j = 0, 1, 2$, it follows that

$$C_j(f) = \gamma_j \int_0^\infty f(s) s^{3/2-j}\, ds.$$

Indeed these powers are locally integrable at zero, and no nonzero distribution supported at zero has the stated homogeneity. Define $\widetilde{a}_j$ by

$$\operatorname{Tr}(e^{-tL}) \sim (4\pi t)^{-5/2} \sum_{j\geq 0} \widetilde{a}_j t^j \qquad (t \downarrow 0).$$

In the heat specialization of the same calculus [16, Remark 2, p. 2445], take a Schwartz function g equal to e^{-s} for $s \geq 0$. Since $L \geq 4$, one has $g(h^2 L) = e^{-h^2 L}$. Thus, with Γ denoting Euler's gamma function,

$$\gamma_j \Gamma(5/2 - j) = (4\pi)^{-5/2} \widetilde{a}_j, \qquad j = 0, 1, 2.$$

The identity $e^{-tL} = e^{-4t} e^{-t\Delta_Y}$ gives

$$\widetilde{a}_0 = a_0(Y), \qquad \widetilde{a}_1 = a_1(Y) - 4a_0(Y), \qquad \widetilde{a}_2 = a_2(Y) - 4a_1(Y) + 8a_0(Y).$$

Take $f(s) = \chi(\sqrt{s})$ for $s > 0$, extended by zero. This function belongs to $C_c^\infty(\mathbb{R})$. Substitution $s = u^2$ gives (23.1); the next power is $h = R^{-1}$. Formulas (23.2) follow from [28, Theorem 4.8.18(d), p. 330]: the integral of the scalar Laplacian term is zero because Y is closed. □

Theorem 23.2 (Prime two-weight lens spaces in dimension five). *Let $p \geq 5$ be prime and let $a \in \mathbb{Z}$ satisfy $a \not\equiv 0, 1, -1 \pmod{p}$. Let $Q = L(p; 1, 1, a)$ with sectional curvature one, and let M be a smooth closed connected Riemannian manifold. Put*

$$P_M = (\Delta_M + 4)^{1/2}, \qquad P_Q = (\Delta_Q + 4)^{1/2}.$$

If $R_\nu \to \infty$ and $\eta_\nu \geq 0$ satisfy

$$\eta_\nu = O(R_\nu^{-3/5}), \qquad E_{\eta_\nu}(R_\nu; P_M, P_Q) = o(R_\nu), \tag{23.3}$$

then M is isometric to Q.

Proof. Write $E_\nu = E_{\eta_\nu}(R_\nu; P_M, P_Q)$. The inequality $|N_{P_M}(R_\nu) - N_{P_Q}(R_\nu)| \leq E_\nu$ and Weyl's law give

$$\dim M = 5, \qquad \operatorname{Vol}(M) = V_5/p. \tag{23.4}$$

Let g and dV_M denote the metric and volume density of M. Under the identification of functions with half-densities, P_M is scalar, self-adjoint and elliptic, with principal symbol $|\xi|_g$. By Lemma 8.1, the eigenvalue of P_Q in harmonic degree k is $k + 2$. Since $\eta_\nu \to 0$ and $E_\nu = o(R_\nu^5)$, Theorem 4.1, with $\mathcal{S} = \{1\}$, gives a self-adjoint classical quantizer A strongly commuting with P_M, such that

$$\operatorname{Spec}(A) \subset \mathbb{Z}, \qquad P_M - A \in \Psi_{\mathrm{cl}}^{-1}.$$

Theorem 5.1, with $n = 5$, $s = 3/5$ and $\beta = 1$, has $\lfloor ns/\beta \rfloor = 3$. Hence

$$P_M - A \in \Psi_{\mathrm{cl}}^{-4}. \tag{23.5}$$

Put $B = P_M - A$. Choose a joint orthonormal eigenbasis (u_j) and write $P_M u_j = \mu_j u_j$, $A u_j = k_j u_j$, where $k_j \in \mathbb{Z}$. The operators B and BA^4 are bounded on L^2, so

$$|\mu_j - k_j| \le \|B\|, \qquad |\mu_j - k_j| \le \|BA^4\| k_j^{-4} \quad (k_j \ge 1).$$

Fix $0 \le \chi \in C_c^\infty((0,1))$, $\chi \not\equiv 0$, and choose $0 < \alpha < \beta < 1$ with $\operatorname{supp}\chi \subset [\alpha, \beta]$. For $t \in \mathbb{R}$, put $f_{t,R}(x) = e^{itx}\chi(x/R)$. Take a maximum η_ν-matching of the spectra of P_M and P_Q below R_ν. If a matched pair (μ_j, λ) has $f_{t,R_\nu}(k_j) \ne 0$ or $f_{t,R_\nu}(\lambda) \ne 0$, then $k_j \asymp R_\nu$ and

$$|\lambda - k_j| \le \eta_\nu + |\mu_j - k_j| < 1$$

for large ν. Both λ and k_j are integers, so $\lambda = k_j$. Moreover, $f_{t,R_\nu}(k_j) \ne 0$ implies $\mu_j \le \beta R_\nu + O(R_\nu^{-4}) < R_\nu$. Thus only unmatched eigenvalues contribute to the difference, and

$$\left|\operatorname{Tr} f_{t,R_\nu}(A) - \operatorname{Tr} f_{t,R_\nu}(P_Q)\right| \le \|\chi\|_\infty E_\nu = o(R_\nu). \tag{23.6}$$

Let $J_R = \{j : \chi(\mu_j/R) \ne 0 \text{ or } \chi(k_j/R) \ne 0\}$. For $j \in J_R$, both eigenvalues lie in $[\alpha R - \|B\|, \beta R + \|B\|]$. Hence $\#J_R = O(R^5)$ by Weyl's law, and $|\mu_j - k_j| = O(R^{-4})$ uniformly on J_R. Since

$$\|f'_{t,R}\|_\infty \le |t|\,\|\chi\|_\infty + R^{-1}\|\chi'\|_\infty,$$

one obtains

$$\begin{aligned} |\operatorname{Tr} f_{t,R}(P_M) - \operatorname{Tr} f_{t,R}(A)| &\le \sum_{j \in J_R} |f_{t,R}(\mu_j) - f_{t,R}(k_j)| \\ &\le C_\chi(1 + |t|R). \end{aligned}$$

Put $\ell = 2\pi/p$. Taking $t = 0, \ell$ gives

$$\operatorname{Tr}\chi(P_M/R_\nu) - \operatorname{Tr}\chi(P_Q/R_\nu) = o(R_\nu), \tag{23.7}$$

$$\operatorname{Tr}\big(e^{i\ell P_M}\chi(P_M/R_\nu)\big) - \operatorname{Tr}\big(e^{i\ell P_Q}\chi(P_Q/R_\nu)\big) = o(R_\nu^2). \tag{23.8}$$

The three moments in (23.1) are positive. Subtract that expansion for M and Q and apply (23.7). Successive division by R_ν^5, R_ν^3 and R_ν gives

$$a_j(M) = a_j(Q), \qquad j = 0, 1, 2. \tag{23.9}$$

Let W, Ric and Scal be the Weyl tensor, Ricci tensor and scalar curvature of M, and put $\mathrm{Ric}_0 = \mathrm{Ric} - (\mathrm{Scal}/5)g$. Formula (23.2) and Tanno's curvature identity [78, equation (2.9), p. 393] give, in dimension five,

$$360\,a_2(M) = \int_M \left(2|W|^2 + \frac{2}{3}|\,\mathrm{Ric}_0\,|^2 + \frac{24}{5}\,\mathrm{Scal}^2\right) dV_M.$$

This identity involves only curvature norms and volume densities; it therefore holds without an orientability hypothesis. Since Q has curvature one, (23.9) gives

$$\int_M \mathrm{Scal}\; dV_M = 20\,\mathrm{Vol}(M), \qquad 360\,a_2(M) = 1920\,\mathrm{Vol}(M).$$

It follows that

$$0 = \int_M \left(2|W|^2 + \frac{2}{3}|\,\mathrm{Ric}_0\,|^2 + \frac{24}{5}(\mathrm{Scal} - 20)^2\right) dV_M.$$

The integrand is continuous and nonnegative. Thus $W = 0$, $\mathrm{Ric}_0 = 0$ and $\mathrm{Scal} = 20$ everywhere, so M has sectional curvature one.

The complete simply connected cover of M is the unit sphere. Hence $M = \mathbb{S}^5/\Gamma$ for a finite free orthogonal group Γ [45, Section 1, p. 77]. Equation (23.4) gives $|\Gamma| = p$. Thus Γ is cyclic and lies in $\mathrm{SO}(6)$. A generator has three nonzero rotation weights modulo p [45, Section 1, Example (ii), p. 77]. Their multiplicities modulo sign form one of the partitions (3), (2, 1) and (1, 1, 1).

Set $\zeta = e^{2\pi \mathrm{i}/p}$. For $X = M, Q$, write F_X for its harmonic generating function (8.2), and let r_X be the largest weight multiplicity modulo sign. Every nonidentity group element has order p and the same weight multiplicities as a generator. By [45, Proposition 2.7, p. 83], F_X has a pole of order r_X at ζ. In particular, $r_Q = 2$. Put

$$c_X = \frac{\zeta^2}{(r_X - 1)!} \lim_{z \to \zeta} (1 - z/\zeta)^{r_X} F_X(z) \neq 0.$$

Lemma 8.4, with shift 2, gives

$$\operatorname{Tr}\bigl(e^{i\ell P_X} \chi(P_X/R)\bigr) = c_X R^{r_X} \int_0^1 \chi(u) u^{r_X - 1}\, du + O(R^{r_X - 1}).$$

If $r_M \neq 2$, division of (23.8) by $R_\nu^{\max(r_M,2)}$ contradicts this formula, since its leading coefficient is nonzero. Thus $r_M = 2$, and division by R_ν^2 gives $c_M = c_Q$.

Changing the generator and conjugating individual complex coordinates gives $M = L(p; 1, 1, b)$ with $2 \le b \le (p-1)/2$. Replacing a by its representative modulo sign in the same interval does not change the isometry class of Q. In (8.2), the double poles at ζ come only from the generator and its inverse. Direct evaluation gives

$$\lim_{z \to \zeta} (1 - z/\zeta)^2 F_M(z) = \frac{2}{p(1 - \zeta^2)(1 - \zeta^{b+1})(1 - \zeta^{1-b})}.$$

For $c \in \{a, b\}$, put

$$t_c = \frac{2\zeta^2}{p(1 - \zeta^2)(1 - \zeta^{c+1})(1 - \zeta^{1-c})}. \tag{23.10}$$

Then $c_M = t_b$ and $c_Q = t_a$. All denominator factors are nonzero, and

$$|t_b| = \frac{2}{p\,[2\sin(2\pi/p)]\,4[\sin^2(\pi b/p) - \sin^2(\pi/p)]}.$$

The function $b \mapsto \sin^2(\pi b/p)$ is strictly increasing on $[2, (p-1)/2]$. Hence $t_b = t_a$ implies $b = a$, and M is isometric to Q. □

Question 23.3. Let $p \ge 5$ be prime and let $a \in \mathbb{Z}$ satisfy $a \not\equiv 0, 1, -1 \pmod p$. Put $Q = L(p; 1, 1, a)$ with sectional curvature one, and let M be a smooth closed connected Riemannian manifold. For a fixed $0 < \eta < 1/4$, does

$$E_\eta\Bigl(R_\nu; \sqrt{\Delta_M + 4}, \sqrt{\Delta_Q + 4}\Bigr) = o(R_\nu) \qquad (R_\nu \longrightarrow \infty)$$

imply that M and Q are isometric?

Part 4. Closed geodesics and cyclic actions

On a finite free isometric quotient of a closed connected Zoll manifold of dimension at least two, the scalar spectrum determines the closed-geodesic lengths and the maximal dimension of a returning component at each length. For cyclic deck groups, the cyclotomic parts of the normalized cluster character determine the eventual cluster ranks of intermediate covers and rational holonomy bundles (Section 26). Under the prime two-weight comparison hypotheses of Section 31, trace coefficients and Morse indices determine the marked phases, signed normal weights and homotopy types of the return families. For $p \ge 11$, the large and small families have two disjoint embedded images with linking number of absolute value one; these images are the minimum and maximum sets of a suitable deck transformation’s displacement.

24. Periods and the wave trace

For a closed connected Riemannian manifold Y, let SY and S^*Y denote its unit tangent and cotangent bundles, identified by the metric, and let G_t^Y be its geodesic flow. Put

$$\operatorname{Fix} G_t^Y = \{v \in SY \,:\, G_t^Y v = v\}, \qquad \mathscr{L}(Y) = \{t > 0 \,:\, \operatorname{Fix} G_t^Y \neq \emptyset\}.$$

Thus $\mathscr{L}(Y)$ includes all iterates of closed geodesics. A Zoll manifold has a common prime geodesic period; simplicity of the geodesics is not assumed. Let Δ_Y be the nonnegative scalar Laplacian and write

$$W_Y(t) = \operatorname{Tr} e^{-it\sqrt{\Delta_Y}}.$$

For a Schwartz function u on $\mathbb{R}$, the Fourier convention is $\widehat{u}(\lambda) = \int_{\mathbb{R}} e^{it\lambda} u(t)\, dt$, extended to tempered distributions by duality.

Theorem 24.1 (The Poisson relation). *Let X be a closed connected Zoll manifold of dimension $n \geq 2$ and common prime period L. Let a finite group Γ act freely by isometries on X, and put $M = X/\Gamma$. Then*

$$\operatorname{sing\,supp} W_M = \{0\} \cup \mathscr{L}(M) \cup (-\mathscr{L}(M)).$$

For $\tau \in \mathscr{L}(M)$, put

$$D(\tau) = \max\{\dim C \,:\, C \text{ is a connected component of } \operatorname{Fix} G_\tau^M\}.$$

Let $\rho \in C_c^\infty(\mathbb{R})$ equal one near τ, with support containing no other element of $\{0\} \cup \mathscr{L}(M) \cup (-\mathscr{L}(M))$. Then, as $\lambda \to +\infty$,

$$e^{-i\tau\lambda}\widehat{\rho W_M}(\lambda) = c_\tau \lambda^{(D(\tau)-1)/2} + O\big(\lambda^{(D(\tau)-3)/2}\big), \qquad c_\tau \neq 0.$$

Thus the highest local frequency order is $(D(\tau)-1)/2$. The scalar spectrum determines $\mathscr{L}(M)$ and $D(\tau)$ for every such τ.

For a Zoll manifold itself, the Poisson equality follows from the clean trace formula at a full return; see [77, Section 1.3]. Prüfer obtained the equality for spherical space forms from the shifted cosine trace $\operatorname{Tr}\cos(t\sqrt{\Delta + (n-1)^2/4})$ [68, Theorems 2–3 and Corollary 4, pp. 142, 150, 152]. In odd dimension $n = 2m+1$, its highest coefficient is nonzero at every return time. His proof first treats $2\pi/s$ by positivity and then transfers the conclusion by automorphisms of the sth cyclotomic field. Theorem 24.1 extends this argument to finite free quotients of arbitrary Zoll manifolds. The rational function is the cluster character of Colin de Verdière's periodic quantizer [15, Theorems 1.1 and 1.4, pp. 510–513]; the minimizing property is Lemma 15.1.

Write $\mathcal{O}(\Gamma) = \{\operatorname{ord}(h) \,:\, h \in \Gamma\}$, and let Φ_s denote the sth cyclotomic polynomial.

Lemma 24.2. *Every returning time in Theorem 24.1 is clean. Each connected component of its fixed set has positive odd dimension. After the normalization $L = 2\pi$, all returning times lie in $(2\pi/|\Gamma|)\mathbb{Z}$.*

Proof. Equation (15.1) and Lagrange's theorem place every return time in $(L/|\Gamma|)\mathbb{Z}$. Since $G_L^M = I$, every returning time map has finite order. Section 10.1 then gives a finite union of clean contact components. Their dimensions are positive and odd. The stated grid follows on setting $L = 2\pi$. □

Lemma 24.3. *Normalize $L = 2\pi$ and put $q = |\Gamma|$ and $P = \sqrt{\Delta_M}$. There are a real number β and a self-adjoint $Q_{-1} \in \Psi_{\mathrm{cl}}^{-1}(M)$, strongly commuting with P, such that the self-adjoint operator*

$$A = P + Q_{-1} - \beta \qquad \text{satisfies} \qquad \operatorname{Spec} A \subset \mathbb{Z}_{\geq 0}.$$

If $a_k = \dim\ker(A - k)$, then a_k is eventually a quasipolynomial with quasiperiod q and degree at most $n-1$. In particular,

$$F_A(z) := \sum_{k \geq 0} a_k z^k \in \mathbb{Q}(z), \qquad (1 - z^q)^n F_A(z) \in \mathbb{Z}[z]. \tag{24.1}$$

Changing finitely many eigenvalue labels within $\mathbb{Z}_{\geq 0}$ changes F_A by an integer polynomial.

For odd-dimensional spherical space forms, an exact quasipolynomial multiplicity formula is given in [69, Lemma 7, pp. 302–303].

Proof. Let g be the metric on M, and let Π_0 be the orthogonal projection onto $\ker \Delta_M$. The positive operator $P + \Pi_0$ has principal symbol $|\xi|_g$ and zero subprincipal symbol on scalar half-densities. Its Hamiltonian flow has period 2π, and S^*M is connected since $n \geq 2$. Apply the spectral construction in [15, Theorem 1.1 and its proof, pp. 510–511] to $P + \Pi_0$ in an orthonormal eigenbasis of P. The resulting correction is diagonal in this basis with real eigenvalues. It is therefore self-adjoint and strongly commutes with P. The Maslov shift is a single real number β. Absorb Π_0 into the correction. Moving any negative integer labels to zero adds a finite-rank smoothing operator commuting with P.

By [15, Theorem 1.4, pp. 512–513], for all sufficiently large k there is an exact expression

$$a_k = \sum_{\zeta} \zeta^k p_\zeta(k), \qquad \deg p_\zeta \leq n - 1,$$

where the sum is finite and each ζ is a root of unity associated with a geodesic return time. Lemma 24.2 gives $\zeta^q = 1$. Hence, on each residue class modulo q, a_k agrees eventually with a polynomial of degree at most $n - 1$. Extend these polynomials to all nonnegative indices. By [74, Proposition 4.4.1, p. 546, author's version], the generating function of the extended sequence has denominator dividing $(1 - z^q)^n$. The extension changes F_A by a polynomial, so $(1 - z^q)^n F_A \in \mathbb{C}[z]$. Its Taylor coefficients are integers; hence $(1 - z^q)^n F_A \in \mathbb{Z}[z]$, as required. □

Proof of Theorem 24.1. Rescale the metric by $(2\pi/L)^2$, so that $L = 2\pi$. Take A from Lemma 24.3 and put

$$B = A + \beta = \sqrt{\Delta_M} + Q_{-1}.$$

The operator $B - \sqrt{\Delta_M + 1}$ has order -1 and strongly commutes with $\sqrt{\Delta_M + 1}$. Thus Lemma 15.2 applies to B. Moreover, B and $\sqrt{\Delta_M}$ have the same principal and subprincipal symbols, so their highest clean coefficients agree [21, Theorem 4.5, equations (4.7)–(4.8)]. Changing finitely many nonpositive eigenvalues adds a smooth function to the wave trace and leaves the annular traces below unchanged for large R.

Fix $0 \leq \chi \in C_c^\infty((0,1))$, $\chi \neq 0$. For $s \in \mathcal{O}(\Gamma)$ with $s \geq 2$, put $t_s = 2\pi/s$ and $D_s = D(t_s)$. Lemma 15.2 gives

$$\operatorname{Tr}(e^{it_s B}\chi(B/R)) = C_{s,\chi} R^{(D_s+1)/2} + o(R^{(D_s+1)/2}), \qquad C_{s,\chi} \neq 0. \tag{24.2}$$

The nonvanishing follows from the zero periodic Morse indices in Lemma 15.1.

By Lemma 8.4, with $m = \beta$, a pole of order $r \geq 1$ of F_A at $\xi = e^{it}$ gives

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = \frac{e^{it\beta}}{(r-1)!}\left(\lim_{z\to\xi}(1 - z/\xi)^r F_A(z)\right) R^r \int_0^1 \chi(u) u^{r-1}\, du + O(R^{r-1}).$$

The displayed coefficient is nonzero. At a regular point ξ, the same lemma gives $O(R^{-N})$ for every $N > 0$. Comparison with (24.2) shows that F_A has a pole of order $(D_s + 1)/2$ at $e^{2\pi i/s}$.

Write $F_A = U/V$ in lowest terms over $\mathbb{Q}$. The irreducibility of Φ_s over $\mathbb{Q}$ implies that F_A has the same pole order at every primitive sth root. Moreover, if $(a, s) = 1$, then

$$\operatorname{Fix} G^M_{2\pi a/s} = \operatorname{Fix} G^M_{2\pi/s}.$$

Indeed, for integers u, v with $ua + vs = 1$, $G^M_{2\pi/s} = (G^M_{2\pi a/s})^u (G^M_{2\pi})^v$; the reverse inclusion follows by taking the ath power. The annular trace at $2\pi a/s$ therefore has a nonzero coefficient of order $R^{(D_s+1)/2}$. If the clean frequency coefficient of order $(D_s - 1)/2$ vanished, the localization formula in Section 10.1 would give $o(R^{(D_s+1)/2})$, a contradiction. Complex conjugation gives the corresponding nonzero coefficient for $\operatorname{Tr} e^{-itB}$. Its highest coefficient agrees with that of W_M.

At $t = 0$ the positive leading Weyl coefficient gives a pole of order n at 1. Since $\operatorname{Spec} A \subset \mathbb{Z}$, for $j \in \mathbb{Z}$,

$$e^{i(t+2\pi j)B} = e^{2\pi i j\beta} e^{itB}.$$

Thus the frequency coefficient of order $n-1$ is nonzero at every $2\pi j$, and nonvanishing at every other return time is invariant under translation by $2\pi\mathbb{Z}$. Every positive geometric length has the form $2\pi k/s$ with $s \in \mathcal{O}(\Gamma)$ and $k \geq 1$, by (15.1). Reducing k/s gives a denominator d dividing s. A power of an element of order s has order d, so this length is among the times just treated. Negative times follow by complex conjugation. The opposite inclusion is [21, Corollary 1.2]. The length set is discrete by Lemma 24.2.

The clean expansion [21, Theorem 4.5], with the Fourier convention above, gives the stated localized formula. Components of smaller dimension contribute only to lower powers; their dimensions are odd by Lemma 24.2. The spectrum determines W_M, its singular support and its highest local frequency order. The latter equals $(D(\tau)-1)/2$. Scaling back gives both assertions for the original metric. □

Corollary 24.4. *In the normalization $L = 2\pi$, put $q = |\Gamma|$, $D_1 = 2n-1$ and $D_s = D(2\pi/s)$ for $s \in \mathcal{O}(\Gamma)$, $s > 1$. The reduced denominator of F_A, up to a nonzero rational scalar, is*

$$\prod_{s \in \mathcal{O}(\Gamma)} \Phi_s(z)^{(D_s+1)/2}. \tag{24.3}$$

Proof. The proof of Theorem 24.1 gives these orders at all primitive roots whose orders lie in $\mathcal{O}(\Gamma)$. At every other qth root there is no geometric return, by (15.1) and closure of $\mathcal{O}(\Gamma)$ under divisors. The Poisson inclusion makes the trace of $B = A + \beta$ smooth there. Its annular trace then decays faster than every power, whereas Lemma 8.4 gives a nonzero leading term at a pole. Thus there is no pole there. Equation (24.1) excludes poles away from the qth roots of unity. □

Corollary 24.5. *The conclusion of Theorem 24.1 holds on every closed connected Besse manifold of odd dimension $n \geq 5$.*

Proof. Wadsley's period theorem, in the formulation of [5, Theorem 4.6], gives a nonzero global period. The global periods form a closed subgroup of $\mathbb{R}$. Since the geodesic vector field never vanishes, a flow box excludes small nonzero global periods. Hence this subgroup is $T\mathbb{Z}$ for some $T > 0$. Lemma 11.1 gives a compact Zoll homotopy-sphere universal cover with finite deck group. Theorem 24.1 applies. □

Proposition 24.6. *Under the hypotheses of Theorem 24.1, let T be the least positive global period of G_t^M. Then $T = L$ or $L/2$. If $|\Gamma|$ is odd, then $T = L$, and the scalar spectrum determines L and $\mathcal{O}(\Gamma)$.*

Proof. The flow-box argument in the proof of Corollary 24.5 applies to G_t^M. Since $G_L^M = I$, the global period group is $T\mathbb{Z}$ with $0 < T \leq L$. For $h \in \Gamma$, write $\widehat{h}(x,v) = (h(x), dh_x v)$. The map G_T^X lifts the identity of SM. Since SX is connected, $G_T^X = \widehat{h}$ for a single $h \in \Gamma$. Velocity reversal $R(x,v) = (x,-v)$ commutes with $\widehat{h}$ and satisfies $RG_T^X R = G_{-T}^X$. Hence

$$G_{-T}^X = \widehat{h} = G_T^X, \qquad G_{2T}^X = I, \qquad h^2 = 1.$$

Every upstairs geodesic has prime period L, so $2T \in L\mathbb{Z}_{>0}$. Together with $T \leq L$, this gives $T = L$ or $L/2$. If $|\Gamma|$ is odd, then $h = 1$; hence T is an upstairs period and $T = L$.

A fixed component has dimension $2n-1$ if and only if it is open in SM. Such a component is also closed, and SM is connected; hence it is all of SM. By Theorem 24.1, T is therefore the least positive time with a nonzero wave coefficient of frequency order $n-1$. The Weyl law determines n, so this describes T spectrally. In the odd-order case it gives L, and Lemma 15.1 gives

$$\mathcal{O}(\Gamma) = \{s \in \mathbb{Z}_{>0} \,:\, L/s \in \mathscr{L}(M)\}.$$ □

25. Arithmetic of eigenvalue multiplicities

Let M be a closed connected Riemannian manifold of dimension $n \geq 2$, and let $\Delta = \Delta_M$ be its nonnegative scalar Laplacian. Write

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad m_j = \dim \ker(\Delta - \lambda_j),$$

for its distinct eigenvalues and their multiplicities. If the geodesic flow is periodic, let T be its least positive global period and put $P = (T/2\pi)\sqrt{\Delta}$. By periodic quantization [15, Theorem 1.1, pp. 510–511], there are constants $\sigma \in \mathbb{R}$, $C > 0$ and $k_0 \in \mathbb{Z}_{>0}$ such that every sufficiently large eigenvalue of P belongs to one of the disjoint intervals

$$I_k = [k + \sigma - C/k, k + \sigma + C/k], \qquad k \geq k_0.$$

The kth T-cluster consists of the eigenvalues whose normalized frequencies belong to I_k. Write R_k for their number without multiplicity, and d_k for the sum of their multiplicities. If R_k is eventually constant, denote its value by R_{cl}.

A sequence $(u_j)_{j\geq 0}$ is eventually quasipolynomial with quasiperiod $d \in \mathbb{Z}_{>0}$ if there are real polynomials $p_0, \dots, p_{d-1}$ such that $u_j = p_r(j)$ for every sufficiently large $j \equiv r \pmod{d}$, for each $0 \leq r < d$. The quasiperiod need not be minimal.

For metrics on real projective space whose multiplicities agree with those of the round metric, Zelditch obtained eventual constancy of R_k [88, Section 6, p. 583].

Lemma 25.1. *Let M be a closed connected Besse manifold of dimension $n \geq 2$ and suppose*

$$m_j = cj^{n-1} + O(j^{n-2}), \qquad c > 0. \tag{25.1}$$

Then each sufficiently high T-cluster contains the same positive integer R_{cl} of distinct eigenvalues. Its total rank d_k satisfies

$$d_k = cR_{\mathrm{cl}}^n k^{n-1} + O(k^{n-2}). \tag{25.2}$$

If m_j is eventually quasipolynomial with quasiperiod d, then d_k is eventually quasipolynomial with quasiperiod

$$q_d = \frac{d}{\gcd(R_{\mathrm{cl}}, d)}. \tag{25.3}$$

Proof. Wadsley's theorem [5, Theorem 4.6] gives a global period. The period-subgroup argument in the proof of Corollary 24.5 gives the least positive period T. Put $P = (T/2\pi)\sqrt{\Delta}$. Its principal symbol is $(T/2\pi)|\xi|_g$, where g is the metric on M, and its subprincipal symbol on scalar half-densities is zero. Its Hamiltonian flow has period 2π, and S^*M is connected. The construction in the first paragraph of the proof of Lemma 24.3, based on [15, Theorem 1.1, pp. 510–511], therefore gives

$$P = A + \sigma + Q_{-1}, \qquad \operatorname{Spec} A \subset \mathbb{Z}_{\geq 0},$$

with the same σ as in the intervals I_k, where $A \in \Psi^1_{\mathrm{cl}}(M)$ is self-adjoint and $Q_{-1} \in \Psi^{-1}_{\mathrm{cl}}(M)$ is self-adjoint and strongly commutes with P and A. The operator $Q_{-1}(A+1)$ has order zero and is bounded on $L^2(M)$. Its restriction to $\ker(A-k)$ therefore gives $\|Q_{-1}|_{\ker(A-k)}\| = O(k^{-1})$. After increasing C and k_0, the intervals I_k are disjoint, and the normalized frequencies in the kth cluster are precisely the spectrum of P on $\ker(A-k)$, with multiplicities. Thus $d_k = \dim\ker(A-k)$ for all large k.

Since T is least, the circle action on S^*M is effective. By [15, Proposition 1.3, pp. 511–512], every nonempty fixed component of a nonidentity circle element has odd dimension. A component of dimension $2n-1$ would be open and closed in the connected manifold S^*M, hence would make that element act identically. Thus these dimensions are at most $2n-3$. The corresponding terms in [15, Theorem 1.4, pp. 512–513] have degree at most $n-2$. The identity term has positive leading coefficient, so

$$d_k = ak^{n-1} + O(k^{n-2}), \qquad a > 0, \tag{25.4}$$

with one leading coefficient for all residue classes.

Let J_k be the last distinct-eigenvalue index in cluster k. All high clusters are nonempty by (25.4). Let ω_n be the Euclidean unit-ball volume. The Weyl law [42, Theorem 1.1, p. 194], at the normalized frequency $k + \sigma + 1/2$, gives

$$\sum_{j=0}^{J_k} m_j = wk^n + O(k^{n-1}), \qquad w = \frac{\omega_n \operatorname{Vol}(M)}{T^n}.$$

On the other hand, (25.1) gives

$$\sum_{j=0}^{J} m_j = \frac{c}{n} J^n + O(J^{n-1}).$$

Comparison gives $J_k/k \to \gamma = (nw/c)^{1/n}$ and then $J_k^n - \gamma^n k^n = O(k^{n-1})$. Factoring this difference yields $J_k = \gamma k + O(1)$. Thus $R_k = J_k - J_{k-1}$ is bounded, and, uniformly over the indices in cluster k,

$$m_j = c\gamma^{n-1} k^{n-1} + O(k^{n-2}).$$

Summing over that cluster and using (25.4) gives

$$R_k = \frac{a}{c\gamma^{n-1}} + O(k^{-1}).$$

This convergent sequence of positive integers is eventually constant, equal to R_{cl}. Summing (25.4) and comparing with the Weyl law gives $a = nw$. Hence $R_{\mathrm{cl}} = \gamma$ and $a = cR_{\mathrm{cl}}^n$.

Under the quasipolynomial hypothesis, (25.1) forces every residue polynomial to have degree $n-1$ and leading coefficient c. For some integer b and all large k,

$$J_{k-1} = R_{\mathrm{cl}} k + b, \qquad d_k = \sum_{i=1}^{R_{\mathrm{cl}}} m_{R_{\mathrm{cl}} k + b + i}. \tag{25.5}$$

Since $d \mid R_{\mathrm{cl}} q_d$, each index on the right has a fixed residue modulo d when k varies in a residue class modulo q_d. The sum is therefore polynomial on each such class. □

For a prime ℓ and an integer $b > 0$, let $v_\ell(b)$ be the exponent of ℓ in b.

Theorem 25.2. *Let M be closed and connected of odd dimension $n \geq 5$, and assume* (25.1)*. Its universal Riemannian cover X is a compact Zoll homotopy sphere. Let L be its prime period, $\Gamma = \pi_1(M)$, and T the least global period on M. Then $\kappa = L/T \in \{1, 2\}$ and*

$$(n-1)!\, c\, |\Gamma| R_{\mathrm{cl}}^n = 2\kappa^n. \tag{25.6}$$

Moreover, $\kappa = 2$ if and only if $|\Gamma|$ is even. In this case M has constant positive curvature and $R_{\mathrm{cl}} = 1$.

The number $(n-1)!c$ is rational. Write it as a_0/b_0, where $a_0, b_0 \in \mathbb{Z}_{>0}$ are coprime. If

$$v_\ell(2b_0) < n \qquad \text{for every prime } \ell, \tag{25.7}$$

then $R_{\mathrm{cl}} = 1$.

Proof. By [88, Theorem A, p. 548], the asymptotic (25.1) makes the geodesic flow periodic. Lemma 11.1 gives the compact Zoll homotopy-sphere cover X and $\kappa \in \{1, 2\}$. In the notation of that lemma, $D^2 = I$ and $D = I$ if and only if $\kappa = 1$. If $|\Gamma|$ is odd, then $D = I$, so $\kappa = 1$. If $|\Gamma|$ is even, $X \to M$ is an even-degree smooth covering by a smooth homotopy sphere. Since M is Besse and $n \neq 3$, [59, Theorem 2] gives constant curvature. Its compact simply connected cover is a round sphere. Cauchy's theorem and Lemma 8.2 give $-I \in \Gamma$. Thus $L/2$ is a global period downstairs; since $\kappa \in \{1, 2\}$, one has $\kappa = 2$.

Proposition 12.3 gives

$$\mathrm{Vol}(X) = \mathrm{Vol}(S^n) \left(\frac{L}{2\pi}\right)^n, \qquad \mathrm{Vol}(M) = \frac{\mathrm{Vol}(X)}{|\Gamma|}.$$

Lemma 25.1 gives $cR_{\mathrm{cl}}^n = n\omega_n \mathrm{Vol}(M)/T^n$. The identity

$$\frac{n\omega_n \mathrm{Vol}(S^n)}{(2\pi)^n} = \frac{2}{(n-1)!}$$

therefore gives (25.6) and the rationality of $(n-1)!c$.

Suppose $\kappa = 2$ and scale the universal round sphere to radius one. Then $L = 2\pi$, $T = \pi$, and $-I \in \Gamma$. Lemma 8.3, specifically (8.3), gives zero multiplicity in odd harmonic degrees and, in even degrees,

$$\dim \mathcal{H}_l^\Gamma = \frac{2}{|\Gamma|} \dim \mathcal{H}_l + O(l^{n-2}). \tag{25.8}$$

Its leading coefficient is positive, so every sufficiently high even degree $l = 2k$ occurs. The normalized frequency is

$$\frac{1}{2}\sqrt{2k(2k+n-1)} = k + \frac{n-1}{4} + O(k^{-1}).$$

For large k, consecutive normalized frequencies are separated by more than $1/2$, and each cluster has width less than $1/2$. Every sufficiently high cluster is nonempty by (25.4), so $R_{\mathrm{cl}} = 1$.

If $\kappa = 1$, equation (25.6) becomes $a_0|\Gamma|R_{\mathrm{cl}}^n = 2b_0$. A prime divisor ℓ of R_{cl} would give $v_\ell(2b_0) \geq n$, contrary to (25.7). □

For spherical space forms, the coefficient set and its maximal value below follow from Prüfer's multiplicity formula and eventual spectral support [69, Lemma 7 and Corollary 1, pp. 302–305].

For $r \in \mathbb{Z}_{>0}$, write C_r for the cyclic group of order r.

Theorem 25.3 (Leading multiplicity coefficients). *Fix an odd integer $n \geq 5$. The possible positive coefficients c in* (25.1)*, among all closed connected Riemannian n-manifolds, are exactly*

$$\left\{ \frac{2^n}{(n-1)!N} \,:\, N \in \mathbb{Z}_{>0} \right\}. \tag{25.9}$$

Every value is realized by the round scalar lens space S^n/C_{2N}, where S^n is the unit sphere in $\mathbb{C}^{(n+1)/2}$ and the generator acts by $z \mapsto e^{\pi i/N} z$.

Proof. For a manifold satisfying (25.1), take $\Gamma, \kappa, R_{\mathrm{cl}}$ from Theorem 25.2. For $\kappa = 1$, equation (25.6) gives

$$(n-1)!c = \frac{2}{|\Gamma|R_{\mathrm{cl}}^n} = \frac{2^n}{2^{n-1}|\Gamma|R_{\mathrm{cl}}^n}.$$

For $\kappa = 2$, it gives $R_{\mathrm{cl}} = 1$ and $(n-1)!c = 2^n/(|\Gamma|/2)$. Both have the form in (25.9).

The stated scalar action of C_{2N} is free and contains $-I$. Equations (25.8) and (8.1) give

$$\dim \mathcal{H}_{2k}^{C_{2N}} = \frac{1}{N} \dim \mathcal{H}_{2k} + O(k^{n-2}) = \frac{2^n}{N(n-1)!} k^{n-1} + O(k^{n-2}).$$

All high even degrees occur and odd degrees do not. The index of the corresponding distinct eigenvalue differs from k by a fixed integer, which does not change this asymptotic. □

Corollary 25.4. *Under the hypotheses of Theorem 25.2, write $c = 2^n/((n-1)!N)$ as in* (25.9)*. If $v_2(N) \not\equiv n-1 \pmod{n}$, then M has constant positive curvature, $R_{\mathrm{cl}} = 1$, and $|\pi_1(M)| = 2N$. In particular this holds if $2^{n-1} \nmid N$, or if $c > 2/(n-1)!$.*

Proof. If $|\Gamma|$ is odd, then $\kappa = 1$ and $N = 2^{n-1}|\Gamma|R_{\mathrm{cl}}^n$. Hence $v_2(N) = n - 1 + nv_2(R_{\mathrm{cl}})$, contrary to the hypothesis. Theorem 25.2 gives constant curvature, $\kappa = 2$, and $R_{\mathrm{cl}} = 1$, whence $|\Gamma| = 2N$. If $2^{n-1} \nmid N$, then $v_2(N) < n-1$, so the stated congruence cannot hold. If $c > 2/(n-1)!$, the alternative $\kappa = 1$ is excluded by (25.6). □

Corollary 25.5. *For each odd dimension $n \geq 5$, the largest possible coefficient in* (25.1) *is $2^n/(n-1)!$. It occurs precisely on round real projective space, up to scale.*

Proof. The largest value in (25.9) has $N = 1$. Corollary 25.4 gives a round universal cover and a deck group of order two. By Lemma 8.2, this group is $\{I, -I\}$. The converse is the case $N = 1$ of Theorem 25.3. □

For a finite group Γ, let $\exp \Gamma$ be the least common multiple of the orders of its elements.

Proposition 25.6. *Under the hypotheses of Theorem 25.2, suppose m_j is eventually quasipolynomial with quasiperiod d. Retain L, T, Γ, κ from that theorem and put $q_d = d/\gcd(R_{\mathrm{cl}}, d)$. Then*

$$\exp \pi_1(M) \mid \kappa q_d. \tag{25.10}$$

Furthermore,

$$v_d := (n-1)!\, d^{n-1} c \in \mathbb{Z}_{>0}, \qquad v_d |\Gamma| R_{\mathrm{cl}}^n = 2\kappa^n d^{n-1}. \tag{25.11}$$

Proof. Replace the metric g by $(2\pi/T)^2 g$, so that $T = 2\pi$. The multiplicities, R_{cl}, and κ are unchanged. Take A, σ, Q_{-1} from the proof of Lemma 25.1, and put $d_k = \dim\ker(A - k)$ also for the finitely many remaining indices. The sequence (d_k) is eventually quasipolynomial with quasiperiod q_d and degree $n-1$. Changing finitely many terms adds a polynomial to its generating function. By [74, Proposition 4.4.1, p. 546, author's version], its generating function has denominator dividing $(1-z^{q_d})^n$. Since $d_k \in \mathbb{Z}$, this gives

$$F_A(z) = \sum_{k \geq 0} d_k z^k \in \mathbb{Q}(z), \qquad (1 - z^{q_d})^n F_A(z) \in \mathbb{Z}[z].$$

Put $B = A + \sigma = \sqrt{\Delta} - Q_{-1}$, so that $B - (\Delta+1)^{1/2} \in \Psi_{\mathrm{cl}}^{-1}(M)$. Let $s \geq 2$ be the order of an element of Γ, and put $t = L/s$. Fix a nonzero $0 \leq \chi \in C_c^\infty((0,1))$. Lemmas 15.1 and 15.2, with positive reference operator $(\Delta+1)^{1/2}$, give

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = C_{s,\chi} R^{\alpha_s} + o(R^{\alpha_s}), \qquad C_{s,\chi} \neq 0, \quad \alpha_s \geq 1.$$

On the other hand,

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = \sum_{k \geq 0} d_k e^{it(k+\sigma)} \chi((k+\sigma)/R).$$

If F_A were regular at e^{it}, Lemma 8.4, with shift σ, would make this sum $O(R^{-N})$ for every $N > 0$, a contradiction. Thus e^{it} is a pole of F_A. Every finite pole is a q_dth root of unity, so

$$e^{iq_d t} = 1, \qquad \frac{q_d L}{2\pi s} = \frac{\kappa q_d}{s} \in \mathbb{Z}.$$

Taking the least common multiple of the element orders gives (25.10).

For every sufficiently large j, the finite-difference identity [74, (1.97) and Proposition 1.9.2, pp. 85, 87, author's version] gives

$$\sum_{a=0}^{n-1} (-1)^{n-1-a} \binom{n-1}{a} m_{j+ad} = (n-1)!\, d^{n-1} c.$$

The left side is an integer, and the right side is positive. Multiplying (25.6) by d^{n-1} proves the second identity in (25.11). □

Corollary 25.7. *If the quasiperiod d in Proposition 25.6 is odd and squarefree, then $R_{\mathrm{cl}} = 1$.*

Proof. Write $(n-1)!\,c = a_0/b_0$ with coprime positive integers a_0, b_0. Equation (25.11) gives $b_0 \mid d^{n-1}$. Thus b_0 is odd, $v_2(2b_0) = 1 < n$, and $v_\ell(2b_0) \leq n-1$ for every odd prime ℓ. Apply Theorem 25.2. □

Corollary 25.8. *Let M be closed and connected of odd dimension $n \geq 5$. Suppose $m_j = p(j)$ for every sufficiently large j, where p is a polynomial of degree $n-1$ and positive leading coefficient. Then every geodesic has the same prime period and $R_{\mathrm{cl}} = 1$. Exactly one of the following holds:*

(1) M is a Zoll smooth homotopy sphere;

(2) M is round real projective space, up to scale.

Proof. Take X, Γ, κ from Theorem 25.2 and put $d = 1$. Corollary 25.7 gives $R_{\mathrm{cl}} = 1$, and (25.10) gives $\exp\Gamma \mid \kappa$. If $\kappa = 1$, then $\Gamma = 1$, so $M = X$ is a Zoll smooth homotopy sphere. If $\kappa = 2$, Theorem 25.2 makes M round and Γ nontrivial. Every nonidentity element of Γ is an involution, hence equals $-I$ by Lemma 8.2. Thus $\Gamma = C_2$. Both alternatives have a common prime geodesic period, and they are disjoint because their fundamental groups differ. □

Corollary 25.9. *Suppose* (25.1) *holds in odd dimension $n \geq 5$ and $\pi_1(M) = C_p$ for an odd prime p. If m_j is eventually quasipolynomial with quasiperiod p, then*

$$R_{\mathrm{cl}} = 1, \qquad c = \frac{2}{p(n-1)!}.$$

The same conclusion $R_{\mathrm{cl}} = 1$ follows from (25.1) *alone when its coefficient equals $2/(p(n-1)!)$.*

Proof. With Γ and κ as in Theorem 25.2, $|\Gamma| = p$ is odd, so $\kappa = 1$. In the quasipolynomial case, Corollary 25.7, with $d = p$, gives $R_{\mathrm{cl}} = 1$; equation (25.6) then determines c. If $c = 2/(p(n-1)!)$, the same identity gives $R_{\mathrm{cl}}^n = 1$ directly. □

26. Spectral clusters on finite coverings and flat bundles

Let X be a closed connected Zoll manifold of dimension $n \geq 2$ and common prime period L, and let Γ be a finite free isometry group of order q. Put $M = X/\Gamma$ and $\mathcal{O}(\Gamma) = \{\mathrm{ord}(g) : g \in \Gamma\}$. For $H \leq \Gamma$, write $Y = X/H$, let Δ_Y be its nonnegative scalar Laplacian, and put $P_Y = (L/2\pi)\sqrt{\Delta_Y}$.

All representations and function spaces are complex. Write $R_g f = f \circ g^{-1}$. For a finite-dimensional unitary representation $\rho_V : \Gamma \to U(V)$, let V^* be its complex dual and put

$$\chi_V(g) = \operatorname{Tr} \rho_V(g), \qquad \rho_V^*(g)\ell = \ell \circ \rho_V(g)^{-1} \quad (g \in \Gamma,\ \ell \in V^*).$$

Define the Hermitian vector bundle

$$\mathcal{E}_V = (X \times V^*)/\Gamma, \qquad g \cdot (x, \ell) = (gx, \rho_V^*(g)\ell).$$

Its smooth sections are the smooth maps $s : X \to V^*$ satisfying $s(gx) = \rho_V^*(g)s(x)$. The product connection descends to a flat unitary connection. Denote its nonnegative connection Laplacian by $\Delta_{\mathcal{E}_V}$, and put $P_V = (L/2\pi)\sqrt{\Delta_{\mathcal{E}_V}}$. The action of Γ on $L^2(X) \otimes V^*$ is $R_g \otimes \rho_V^*(g)$.

Let $\beta \in [0, 1)$ be the scalar quantizer shift on X, modulo $\mathbb{Z}$. For every sufficiently large integer k, put

$$b_Y(k) = \#\{\nu \in \operatorname{Spec} P_Y : |\nu - (k + \beta)| < 1/2\}, \tag{26.1}$$

counting multiplicities. Assigning the finitely many remaining frequencies to nonnegative integer labels defines $F_Y(z) = \sum_{k \geq 0} b_Y(k) z^k$ modulo $\mathbb{Q}[z]$. For $d \in \mathbb{Z}_{\geq 1}$ and a rational function F whose finite poles are roots of unity, $\mathrm{PP}_d F$ denotes the sum of its proper partial-fraction terms at the primitive dth roots. If $F \in \mathbb{Q}(z)$, then $\mathrm{PP}_d F \in \mathbb{Q}(z)$, since the primitive dth roots form a Galois-invariant set. This operation vanishes on polynomials.

Lemma 26.1. *There is a self-adjoint quantizer*

$$A_X = P_X + Q_X - \beta, \qquad Q_X \in \Psi_{\mathrm{cl}}^{-1}(X), \qquad \operatorname{Spec} A_X \subset \mathbb{Z}_{\geq 0},$$

which commutes with P_X and every isometry of X. It descends, with the same shift β, to every X/H and every $\mathcal{E}_V$. For all sufficiently large k, uniformly in H and V, the k-eigenspaces of the descended operators are the spectral subspaces of P_Y and P_V for the interval $(k + \beta - 1/2, k + \beta + 1/2)$.

Proof. On scalar half-densities, P_X has zero subprincipal symbol. Its Hamiltonian flow has period 2π, and S^*X is connected. Move the zero eigenvalue of P_X to a positive number below its first positive eigenvalue. This smoothing perturbation leaves the eigenspaces unchanged. Theorem 1.1 of [15] then gives a self-adjoint quantizer commuting with P_X. Absorb the perturbation into Q_X and subtract an integer from the shift to obtain $\beta \in [0, 1)$. The resulting spectrum is integral.

The operator $Q_X P_X$ has order zero and is bounded. Hence there is $C > 0$ such that, on every P_X-eigenspace of eigenvalue $\nu > 0$,

$$\|Q_X u\| \leq C\nu^{-1}\|u\|.$$

Every eigenvalue of A_X on that space is an integer within C/ν of $\nu - \beta$. For large ν, this interval contains at most one integer. Since A_X is self-adjoint, its restriction is the scalar nearest integer to

$\nu - \beta$. Replace its restriction on each remaining P_X-eigenspace by a nonnegative scalar integer. These eigenspaces are finite-dimensional and smooth, so the change is finite-rank and smoothing. Every isometry preserves the P_X-eigenspaces and hence commutes with the resulting A_X.

Restrict A_X to H-invariant functions, or restrict $A_X \otimes 1$ to $(L^2(X) \otimes V^*)^\Gamma$. The corresponding Laplacian is the restriction of Δ_X or $\Delta_X \otimes 1$, so functional calculus gives the corresponding restriction of P_X. Choose a covering chart whose distinct translates are disjoint. In its descended correction kernel, the identity sheet contributes Q_X or $Q_X \otimes 1$; the other sheets contribute smooth kernels. The correction therefore has order -1 on the quotient or flat bundle. The restrictions remain self-adjoint and commute with the corresponding P. On the unchanged eigenspaces, tensoring by 1_{V^*} preserves the bound C/ν, and restriction cannot increase it. Choose $\nu_0 > 2C$ above the changed frequencies, and choose K above all A_X-eigenvalues on the P_X spectral subspace for $[0, \nu_0]$. For $k > \max\{K, \nu_0 + 1\}$, the estimate $|\nu - (k + \beta)| < 1/2$ identifies the k-eigenspaces with the stated spectral subspaces. These choices are independent of H and V. □

The scalar quantizer on X/H supplied by [15, Theorem 1.1] has the same shift modulo $\mathbb{Z}$. Indeed, if its shift is $\beta_H \in [0, 1)$, comparison along any sequence of P_Y-eigenvalues $\nu \to \infty$ gives $\operatorname{dist}(\beta_H - \beta, \mathbb{Z}) = O(\nu^{-1})$. Thus $\beta_H = \beta$. Its high labels coincide with the descended labels. Every return on X/H lies in $(L/|H|)\mathbb{Z}$, by lifting the returning tangent vector and applying Lagrange's theorem. The multiplicity formula [15, Theorem 1.4] consequently gives, as in Lemma 24.3,

$$(1 - z^{|H|})^n F_{X/H}(z) \in \mathbb{Z}[z].$$

In particular these functions are rational, with finite poles only at $|H|$th roots of unity.

Put $E_k = \ker(A_X - k)$ and $\chi_k(g) = \operatorname{Tr}(R_g | E_k)$. Choose the finitely many initial labels by restriction from A_X. The kth cluster on $\mathcal{E}_V$ is then $(E_k \otimes V^*)^\Gamma$. The character inner-product formula [24, Theorem 4.5.1, p. 66] gives

$$b_{X/H}(k) = \frac{1}{|H|} \sum_{g \in H} \chi_k(g), \tag{26.2}$$

$$b_V(k) = \frac{1}{q} \sum_{g \in \Gamma} \overline{\chi_V(g)} \chi_k(g). \tag{26.3}$$

Here $b_V(k) = \dim \operatorname{Hom}_\Gamma(V, E_k)$ and $F_V(z) = \sum_{k \geq 0} b_V(k) z^k$. For irreducible V, $b_V(k)$ is its multiplicity in E_k. The conjugate in (26.3) is the character of V^*.

Lemma 26.2. *For $d \in \mathcal{O}(\Gamma)$, put*

$$r_d(k) = \sum_{\operatorname{ord}(g) = d} \chi_k(g), \qquad R_d(z) = \sum_{k \geq 0} r_d(k) z^k.$$

Then $r_d(k) \in \mathbb{Z}$, $R_d \in \mathbb{Q}(z)$, and every finite pole of R_d is a primitive dth root of unity.

Proof. Let $C \leq \Gamma$ be cyclic of order d, and let C_e be its subgroup of order $e \mid d$. Let μ denote the arithmetic Möbius function. Since $\sum_{g \in C_e} \chi_k(g) = e \dim E_k^{C_e}$, Möbius inversion on the divisors of d [74, Proposition 3.7.1, p. 303, and Example 3.8.4, pp. 306–307] gives

$$\sum_{\substack{g \in C \\ \operatorname{ord}(g) = d}} \chi_k(g) = \sum_{e \mid d} \mu(d/e) e \dim E_k^{C_e}.$$

Each element of order d generates exactly one cyclic subgroup of order d. Summing over these subgroups proves integrality and expresses R_d as an integer linear combination of F_{X/C_e}. Thus $R_d \in \mathbb{Q}(z)$ and its finite poles are dth roots.

Write $\Phi_t = G^X_{Lt/(2\pi)}$, and let $\widehat{g}$ denote the tangent lift of g to SX. The coefficients of $T_g(z) = \sum_{k \geq 0} \chi_k(g) z^k$ have polynomial growth, since $|\chi_k(g)| \leq \dim E_k$. Its Abel boundary, parametrized by $z = e^{-it}$, is the distribution $\operatorname{Tr}(R_g e^{-itA_X})$. The finite-rank positive adjustment in the proof of

Lemma 15.2 commutes with R_g and changes this trace by a smooth finite sum. The propagator theorem and the trace wavefront inclusion [21, Theorem 1.1, p. 43, and equations (1.20)–(1.23), p. 44] therefore apply. Composition with the canonical graph of R_g shows that the twisted trace can be singular at t only if

$$\Phi_t v = \widehat{g}^{-1} v \qquad \text{for some } v \in SX.$$

If g has order d, this equality implies $dt = 2\pi a$ for some $a \in \mathbb{Z}$. If $r = \gcd(a, d) > 1$, then

$$\widehat{g}^{\,d/r} v = \Phi_{-2\pi a/r} v = v,$$

contradicting freeness. Thus, for $d > 1$, $t/(2\pi) = a/d$ in lowest terms; for $d = 1$, $t \in 2\pi\mathbb{Z}$. The twisted trace is smooth at every other time. The Abel boundary of R_d is the sum of these traces over the elements of order d. It is smooth at every nonprimitive dth root. A pole on the unit circle has a singular Abel boundary, by the calculation in Lemma 8.4. Such poles are therefore absent. □

Theorem 26.3. *Let $H \le \Gamma$ be a union of complete element-order classes, and write*

$$H = \{g \in \Gamma \,:\, \operatorname{ord}(g) \in \mathcal{D}_H\}.$$

Then

$$F_{X/H} \equiv \frac{q}{|H|} \sum_{d \in \mathcal{D}_H} \mathrm{PP}_d \, F_M \pmod{\mathbb{Q}[z]}. \tag{26.4}$$

In particular $F_X \equiv q\,\mathrm{PP}_1 F_M$. For $\Gamma = C_q$ and its subgroup H_h of order $h \mid q$,

$$F_{X/H_h} \equiv \frac{q}{h} \sum_{d \mid h} \mathrm{PP}_d \, F_M \pmod{\mathbb{Q}[z]}. \tag{26.5}$$

Proof. Equation (26.2) gives $F_M = q^{-1} \sum_{d \in \mathcal{O}(\Gamma)} R_d$. The finite pole sets in Lemma 26.2 are pairwise disjoint, so

$$R_d \equiv q\,\mathrm{PP}_d \, F_M \pmod{\mathbb{Q}[z]}.$$

Since H contains every element of each order in $\mathcal{D}_H$, averaging over H gives (26.4). For $H = \{1\}$, one has $\mathcal{D}_H = \{1\}$. In C_q, the subgroup of order h consists exactly of the elements whose orders divide h, giving (26.5). □

Theorem 26.4. *Let V be a finite-dimensional unitary representation of Γ whose character is constant on each complete element-order class: $\chi_V(g) = v_d$ when $\operatorname{ord}(g) = d$. Then $v_d \in \mathbb{Z}$, and the cluster character of the flat-bundle Laplacian on $\mathcal{E}_V$ satisfies*

$$F_V \equiv \sum_{d \in \mathcal{O}(\Gamma)} v_d \, \mathrm{PP}_d \, F_M \pmod{\mathbb{Q}[z]}. \tag{26.6}$$

For $\Gamma = C_q = \langle g_0 \rangle$ and $e \mid q$, put

$$W_e = \bigoplus_{\substack{a \bmod e \\ (a,e)=1}} \chi_{a/e}, \qquad \chi_{a/e}(g_0) = e^{2\pi i a/e}.$$

Then

$$F_{W_e} \equiv \sum_{d \mid q} c_e(q/d)\,\mathrm{PP}_d \, F_M \pmod{\mathbb{Q}[z]}, \qquad c_e(j) = \sum_{\substack{a \bmod e \\ (a,e)=1}} e^{2\pi i a j/e}. \tag{26.7}$$

Proof. Let N be the exponent of Γ, and put $\zeta_N = e^{2\pi i/N}$. The eigenvalues of $\rho_V(g)$ are Nth roots of unity, so $\chi_V(g)$ is an algebraic integer in $\mathbb{Q}(\zeta_N)$. For $(a, N) = 1$, the Galois automorphism $\zeta_N \mapsto \zeta_N^a$ sends $\chi_V(g)$ to $\chi_V(g^a)$. The elements g and g^a have the same order, so every v_d is fixed by the Galois group. Hence $v_d \in \mathbb{Q}$, and a rational algebraic integer is an integer [24, Propositions 5.2.4–5.2.5, pp. 94–95]. In particular $\overline{v_d} = v_d$. Grouping (26.3) by element order gives $F_V = q^{-1} \sum_d v_d R_d$, and the preceding partial-fraction identity proves (26.6).

For an element g_0^j of order d, $\gcd(j,q) = q/d$. Since $e \mid q$, $\gcd(j,e) = \gcd(\gcd(j,q),e) = \gcd(q/d,e)$. The Ramanujan-sum identity [71, Section 2, equations (2.3) and (2.7)]

$$c_e(j) = \sum_{r \mid \gcd(e,j)} r\mu(e/r)$$

therefore gives $\chi_{W_e}(g_0^j) = c_e(q/d)$. This proves (26.7). Changing the generator g_0 permutes the primitive characters modulo e, so the isomorphism class of W_e is unchanged. □

Corollary 26.5. *Within the class of free cyclic Zoll quotients of odd order, the scalar spectrum determines L, q, and β, and therefore the eventual cluster ranks of every corresponding intermediate cover and every representation W_e in Theorem 26.4.*

Proof. Proposition 24.6 determines L. Since $\mathcal{O}(C_q)$ is the set of divisors of q, equation (15.1) gives $\mathscr{L}(M) = (L/q)\mathbb{Z}_{>0}$. Theorem 24.1 determines its least positive element $\ell_{\min}$ and hence $q = L/\ell_{\min}$. For any sequence of scaled scalar frequencies $\nu_j \to \infty$, the common quantizer gives

$$e^{2\pi i \nu_j} \longrightarrow e^{2\pi i \beta}.$$

This determines $\beta \in [0,1)$ and the eventual coefficients of F_M. Apply Theorems 26.3 and 26.4. □

For round lens spaces, Lauret [55, Theorem 4.3 and its proof, p. 15] proves that scalar isospectrality passes to corresponding intermediate covers. His proof separates the generating function by primitive orders of finite poles. For $\Gamma = C_q$ and $e \mid q$, the characters trivial on $H_{q/e}$ are precisely the summands of W_f with $f \mid e$. Frobenius reciprocity [24, Theorem 5.10.1, p. 107] gives, with the common labels chosen above,

$$F_{X/H_{q/e}} = \sum_{f \mid e} F_{W_f}.$$

Möbius inversion therefore gives equality of the sequences for W_e whenever the corresponding cover sequences agree for all divisors of q. In the round case, this is a consequence of Lauret's theorem. Equations (26.4) and (26.6) hold for eventual cluster ranks on free Zoll quotients, for the subgroups and characters specified above. The polynomial ambiguity leaves finitely many initial ranks undetermined; the eigenvalue locations inside each cluster are not determined by these formulas.

27. Exact deficits in normalized prime quotients

Let $m \geq 2$ be an integer, put $n = 2m+1$, and let $p \geq 5$ be prime. Choose $a \in \mathbb{Z}$ with $a \not\equiv 0, 1, -1 \pmod{p}$, and let $Q = L(p; 1^m, a)$ have the unit round metric. Suppose

$$M = X/C_p, \tag{27.1}$$

where C_p acts freely by isometries on a smooth homotopy n-sphere X with common prime period 2π and $\mathrm{Vol}(X) = \mathrm{Vol}(S^n)$. For $s \in \mathbb{R}$, write $\Psi^s(M)$ for the pseudodifferential operators of order s and type $(1,0)$. Suppose also that

$$P_M = \sqrt{\Delta_M + m^2} = A + B, \quad \operatorname{Spec} A \subset \mathbb{Z}, \quad B \in \Psi^{-1}(M), \tag{27.2}$$

where A is self-adjoint and strongly commutes with P_M. Let $P_Q = \sqrt{\Delta_Q + m^2}$. For $k \in \mathbb{Z}$, put

$$a_k = \dim\ker(A - k), \qquad q_k = \dim\ker(P_Q - k), \qquad d_k = a_k - q_k.$$

All frequency lists are counted with multiplicity. Put $E_\eta(R) = E_\eta(R; \operatorname{Spec} P_M, \operatorname{Spec} P_Q)$, as in Definition 1.1, and set $R_K = m + K + 1/2$ for $K \in \mathbb{Z}_{\geq 0}$. For a polynomial f, define $\Delta_p f(t) = f(t+p) - f(t)$.

Theorem 27.1. *Under (27.1)–(27.2), d_k is eventually quasipolynomial with quasiperiod dividing p and degree at most $n-2$. For every fixed $0 < \eta < 1$, the sequence $E_\eta(R_K)$ is eventually quasipolynomial in K, with quasiperiod dividing p. For $0 \leq r < p$, let $D_r \in \mathbb{Q}[t]$ satisfy $d_k = D_r(k)$ for all sufficiently large $k \equiv r \pmod{p}$. Precisely one of the following holds:*

(1) Every D_r is zero, and $E_\eta(R_K)$ is eventually constant.
(2) Some D_r is nonzero. Put $D = \max_{D_r \neq 0} \deg D_r$. The constants $c_r = \Delta_p^D D_r$ are integers, and

$$E_\eta(R_K) = CR_K^{D+1} + O(R_K^D), \qquad C = \frac{\sum_{r=0}^{p-1} |c_r|}{p^{D+1}(D+1)!} > 0. \tag{27.3}$$

In this case $\liminf_K E_\eta(R_K)/R_K \geq 1/p$.

Proof. Apply Lemma 24.3 to M. Since

$$P_M - \sqrt{\Delta_M} = m^2(P_M + \sqrt{\Delta_M})^{-1} \in \Psi_{\mathrm{cl}}^{-1}(M),$$

its quantizer can be written

$$\widetilde{A} = P_M + \widetilde{Q} - \beta, \qquad \widetilde{Q} \in \Psi_{\mathrm{cl}}^{-1}(M),$$

where $\widetilde{Q}$ strongly commutes with P_M. The multiplicities of $\widetilde{A}$ are eventually quasipolynomial with quasiperiod p and degree at most $n-1$. On a P_M-eigenspace of eigenvalue ν, boundedness of BP_M and $\widetilde{Q}P_M$ gives corrections of norm $O(\nu^{-1})$. The nearest-integer argument in the proof of Lemma 26.1 makes both A and $\widetilde{A}$ scalar on every sufficiently high such eigenspace. Their integer eigenvalues u_ν, v_ν satisfy

$$u_\nu - v_\nu = \beta + O(\nu^{-1}).$$

Letting $\nu \to \infty$ gives $\beta \in \mathbb{Z}$, and then $u_\nu = v_\nu + \beta$ for all sufficiently large ν. Thus $S = A - (\widetilde{A} + \beta)$ is finite-rank and smoothing, and $B = -\widetilde{Q} - S \in \Psi_{\mathrm{cl}}^{-1}(M)$. Moreover, $a_k = \dim\ker(\widetilde{A} - (k-\beta))$ for all large k. For Q, Lemma 8.1 makes P_Q an integer quantizer; the same comparison applies with $B = 0$. Hence a_k, q_k, and d_k are eventually quasipolynomial with quasiperiod p and degree at most $n-1$. Since $d_k \in \mathbb{Z}$, interpolation at n sufficiently large indices in each residue class gives $D_r \in \mathbb{Q}[t]$.

Since p is odd, Proposition 24.6 gives the least global period 2π for both M and Q. The fixed-component argument preceding (25.4) in the proof of Lemma 25.1 gives one leading coefficient on all residue classes for each multiplicity sequence. Let ω_n be the Euclidean unit-ball volume, and put $c_Z = n\omega_n \operatorname{Vol}(Z)/(2\pi)^n$ for $Z \in \{M, Q\}$. The Weyl law for $\Delta_Z + m^2$ [42, Theorem 1.1, p. 194] then gives

$$a_k = c_M k^{n-1} + O(k^{n-2}), \qquad q_k = c_Q k^{n-1} + O(k^{n-2}).$$

Since $\operatorname{Vol}(M) = \operatorname{Vol}(X)/p = \operatorname{Vol}(S^n)/p = \operatorname{Vol}(Q)$, one has $c_M = c_Q$. Thus $d_k = O(k^{n-2})$, and each D_r has degree at most $n-2$.

The bounded operator $B = P_M - A$ is self-adjoint and commutes with A. Since $BA \in \Psi^0(M)$ is bounded, there is $C_0 > 0$ such that

$$\operatorname{Spec}(P_M|_{\ker(A-k)}) \subset [k - C_0/k, k + C_0/k]$$

for all sufficiently large positive k. Choose $k_0 \in \mathbb{Z}_{>0}$ so that this inclusion holds and $C_0/k < \min(\eta, 1-\eta, 1/4)$ for every $k \geq k_0$. Every frequency in this cluster lies within η of k and at distance at least $1 - C_0/k > \eta$ from every other integer. Thus its largest matching with the reference spectrum has $\min(a_k, q_k)$ pairs.

Every frequency with A-eigenvalue below k_0 is less than $k_0 + \|B\|$, so only finitely many such copies occur. Choose an integer

$$k_1 > \max\{m, k_0 + \|B\| + \eta + 1\}.$$

No matching pair joins a P_M-frequency whose A-eigenvalue is below k_1 to a reference integer at least k_1, or conversely. For A-eigenvalues at least k_0, this follows from the preceding separation; for the remaining frequencies, it follows from the choice of k_1. For $k \geq k_1$, all frequencies in the kth cluster are below R_K when $k \leq m + K$, and above R_K when $k > m + K$. Let $e_{\eta,0} \in \mathbb{Z}_{\geq 0}$ be the matching

deficit between the finite multiset of P_M-frequencies with A-eigenvalue below k_1 and the reference frequencies below k_1. Each later cluster has deficit $a_k + q_k - 2\min(a_k, q_k) = |d_k|$. Therefore

$$E_\eta(R_K) = e_{\eta,0} + \sum_{k=k_1}^{m+K} |d_k| \tag{27.4}$$

for all sufficiently large K.

Each nonzero D_r has a constant sign for all sufficiently large positive arguments. Hence $|d_k|$ is eventually quasipolynomial with quasiperiod p. By [74, Proposition 4.4.1, p. 546, author's version], its generating function differs by a polynomial from a proper rational function whose finite poles are pth roots of unity. Multiplication by $(1-z)^{-1}$ gives the generating function of the partial sums and introduces no other finite poles. After removal of its polynomial part, the same proposition gives eventual quasipolynomiality of these partial sums. Translation of the index by m and subtraction of the finitely many initial terms give the assertion for (27.4).

If every D_r vanishes, the sum is eventually constant. Otherwise let γ_r be the coefficient of degree D in D_r. The finite-difference formula [74, (1.97) and Proposition 1.9.2(b), pp. 85 and 87, author's version], applied to the integer-valued tail in each residue class, gives

$$c_r = D!\,p^D \gamma_r \in \mathbb{Z}.$$

Suppose first that $D \geq 1$, and let $G(z)$ be the proper rational part of $\sum_{k\geq 0} d_k z^k$. By [74, Proposition 4.4.1], its poles are pth roots of unity, with maximal order exactly $D+1$. Its coefficients agree with d_k for all sufficiently large k. Lemma 22.1, with $q = p$ and $d = D+1$, therefore gives

$$\sum_{k=k_1}^{m+K} |d_k| = \frac{\sum_r |\gamma_r|}{p(D+1)}(m+K)^{D+1} + O(K^D).$$

Since $m + K = R_K - 1/2$ and $\gamma_r = c_r/(D!\,p^D)$, this proves (27.3). If $D = 0$, counting the indices in each residue class gives

$$E_\eta(R_K) = \frac{\sum_r |c_r|}{p} R_K + O(1),$$

which is the same formula. At least one c_r is a nonzero integer, so $C > 0$. For $D \geq 1$, the quotient by R_K tends to infinity. For $D = 0$, it tends to $\sum_r |c_r|/p \geq 1/p$. □

Corollary 27.2. *Within the class* (27.1)–(27.2)*, the following are equivalent:*

(1) for one $0 < \eta < 1$ *and a sequence* $K_\nu \to \infty$, $E_\eta(R_{K_\nu}) = o(R_{K_\nu})$;

(2) $a_k = q_k$ *for all sufficiently large* k;

(3) for every fixed $\epsilon > 0$, $E_\epsilon(R_K) = O_\epsilon(1)$.

The first condition can equivalently be replaced by $\liminf_K E_\eta(R_K)/R_K < 1/p$ *for one* $0 < \eta < 1$.

Proof. Theorem 27.1 gives the second condition from the first and from the displayed liminf bound. Under the second, fix $\epsilon > 0$ and take k sufficiently large that $C_0/k < \min(\epsilon, 1/4)$. The clusters have equal ranks and lie wholly on the same side of each cutoff R_K. Pairing within each cluster leaves only finitely many copies unmatched, which proves the third. Taking $\epsilon = 1/2$ gives the first. Under the second, the displayed liminf is zero. □

For each sufficiently large k, let $b_{k,j}$, $1 \leq j \leq a_k$, be the eigenvalues of B on $\ker(A-k)$, counted with multiplicity. Then $|b_{k,j}| \leq C_0/k$, uniformly in j. The corresponding scalar Laplace eigenvalues $\lambda_{k,j}$ satisfy

$$\lambda_{k,j} = k^2 - m^2 + 2kb_{k,j} + b_{k,j}^2, \qquad \Delta_M - (A^2 - m^2) = 2AB + B^2 \in \Psi^0_{\mathrm{cl}}(M).$$

Changing finitely many integer eigenvalues in a common eigenbasis of A and P_M leaves $a_k = q_k$ for all large k unchanged and alters B by a finite-rank smoothing operator.

For finite subgroups of a compact connected semisimple Lie group, Lauret–Miatello prove rationality of multiplicity generating functions and finite determination along each fixed string of irreducible representations [56, Propositions 2.2 and 3.1, pp. 4 and 6–7, arXiv version 2]. If q is the least common multiple of the orders of two finite subgroups, their Corollary 3.2, pp. 7–8, gives equality at every label of the string when the set of unequal labels has upper density less than $1/q$. For two compact orientable real hyperbolic manifolds of the same dimension, Kelmer proves scalar isospectrality when the cumulative absolute multiplicity discrepancy up to spectral parameter T is $o(T)$ [50, equation (0.1) and Theorem 1(1), pp. 3–4, arXiv version 2].

28. Equivariant spectral comparison near the round metric

Let $m \geq 2$ be an integer, put $n = 2m + 1$, and let g_0 be the standard round metric of curvature one on S^n. A normalized Zoll metric is a smooth Riemannian metric whose unit-speed geodesics have prime period 2π. Operators in this section act on complex half-densities. For a diffeomorphism h, put $\mathcal{R}_h = (h^{-1})^*$ on half-densities. Put

$$P_g = (\Delta_g + m^2)^{1/2}, \qquad P_0 = (\Delta_{g_0} + m^2)^{1/2}.$$

Their principal symbols are

$$p_g(x,\xi) = \sqrt{g_x^{-1}(\xi,\xi)}, \qquad p_0 = p_{g_0}.$$

The spectrum of P_0 is $m + \mathbb{Z}_{\geq 0}$, with the harmonic multiplicities of Lemma 8.1. The canonical and Fourier integral comparison for Zoll metrics on S^2 appears in [88, Section 3, equations (3.6)–(3.8)].

Theorem 28.1 (Equivariant Zoll spectra). *There is a C^3 neighborhood $\mathcal{U}_n$ of g_0 with the following property. Let $g \in \mathcal{U}_n$ be a normalized Zoll metric, and let $H \subset O(n+1)$ be a finite group preserving g. There is a unitary H-equivariant Fourier integral operator U of order zero such that*

$$U^* P_g U = P_0 + K, \qquad K = K^* \in \Psi^{-1}_{\mathrm{cl}}(S^n). \tag{28.1}$$

There are $C_g > 0$ and an integer k_g such that, for every integer $k \geq k_g$, the spectral subspace of P_g in $(k - 1/4, k + 1/4)$ is isomorphic to $\ker(P_0 - k)$ as an H-module, and its eigenvalues belong to $[k - C_g/k, k + C_g/k]$. These intervals contain every sufficiently large eigenvalue of P_g.

If H acts freely, write $M_g = (S^n, g)/H$ and $M_0 = (S^n, g_0)/H$. For fixed $\eta > 0$ and $N \in \mathbb{Z}_{\geq 0}$,

$$E_\eta\big(m + N + \tfrac{1}{2}; (\Delta_{M_g} + m^2)^{1/2}, (\Delta_{M_0} + m^2)^{1/2}\big) = O_{g,\eta}(1) \qquad (N \to \infty). \tag{28.2}$$

The neighborhood $\mathcal{U}_n$ is independent of H. For fixed g, the operator U and the constants C_g, k_g may also be chosen independently of H.

28.1. **The geodesic circle action.** Identify the round unit tangent bundle with

$$Y = \{Z = [\, x \; v \,] \in \mathbb{R}^{(n+1)\times 2} \,:\, Z^{\mathsf{T}} Z = I_2\}.$$

Put

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad D = \mathrm{diag}(1,-1), \quad \rho(Z) = ZD,$$

and write $\widehat{h}(Z) = hZ$ for $h \in O(n+1)$. Let G_g^t be the geodesic flow on the g-unit tangent bundle. Identify the two unit tangent bundles by

$$v_g(x,v) = \left(x, \frac{v}{\sqrt{g_x(v,v)}}\right), \qquad \Gamma_g^t = v_g^{-1} G_g^t v_g.$$

Thus $\Gamma_0^t(Z) = ZR_t$.

Lemma 28.2. *For a normalized Zoll metric g sufficiently close to g_0 in C^2, the map*

$$B_g(Z) = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_g^t(Z) R_{-t}\, dt, \qquad R_g(Z) = B_g(Z)(B_g(Z)^{\mathsf{T}} B_g(Z))^{-1/2} \tag{28.3}$$

is a diffeomorphism of Y, and

$$R_g \Gamma_g^t = \Gamma_0^t R_g, \qquad R_g \rho = \rho R_g, \qquad R_g \widehat{h} = \widehat{h} R_g \quad (h \in O(n+1),\ h^* g = g).$$

Both R_g and R_g^{-1} tend to the identity in C^1 as $g \to g_0$ in C^2.

Proof. On the compact unit tangent bundle, the coefficients of the geodesic vector field depend on the metric and its first derivatives. The first variational equation depends on its second derivatives. The integral equations for the flow and its differential, followed by Gronwall's inequality on $[0, 2\pi]$, give

$$\sup_{0\le t\le 2\pi} \|\Gamma_g^t - \Gamma_0^t\|_{C^1} \le C_n \|g - g_0\|_{C^2}$$

in a fixed neighborhood. Hence $B_g \to \mathrm{Id}$ in C^1. The polar retraction $r(Z) = Z(Z^{\mathsf{T}}Z)^{-1/2}$ is smooth on the full-rank matrices and satisfies

$$r(AZQ) = A r(Z) Q \qquad (A \in O(n+1),\ Q \in O(2)).$$

It follows that R_g is C^1-close to the identity.

Since Y is compact and has no boundary, the diffeomorphisms form an open subset of $C^1(Y, Y)$ [41, Chapter 2, Theorem 1.6, p. 38]. Thus R_g is a diffeomorphism for g sufficiently close to g_0. Its inverse tends to the identity in C^1, by $D(R_g^{-1}) = (DR_g \circ R_g^{-1})^{-1}$.

Periodicity gives

$$B_g(\Gamma_g^s Z) = \frac{1}{2\pi}\int_s^{2\pi+s} \Gamma_g^u(Z) R_{-u}\, du\, R_s = B_g(Z) R_s.$$

Reversibility gives $\Gamma_g^t \rho = \rho \Gamma_g^{-t}$. Since $DR_{-t} = R_t D$, changing t to $-t$ in the circle integral gives $B_g(\rho Z) = B_g(Z) D$. An orthogonal isometry common to g and g_0 commutes with ν_g and Γ_g^t; hence $B_g(\widehat{h} Z) = h B_g(Z)$. The polar retraction proves all three identities. None of the estimates depends on the common isometry. □

28.2. **Strict contact and homogeneous Hamiltonian conjugacy.** Let θ be the tautological form on T^*S^n, and take $\iota_{X_a} d\theta = -da$. Define

$$\iota_g(x, v) = \left(x, \frac{g_x(v, \cdot)}{\sqrt{g_x(v, v)}}\right), \qquad \alpha_g = \iota_g^* \theta.$$

The Reeb flow of α_g is Γ_g: on $\{p_g = 1\}$, Euler's identity gives $\theta(X_{p_g}) = 1$, while dp_g vanishes on the tangent space of this hypersurface.

Lemma 28.3. *After restricting $\mathcal{U}_n$, there is a homogeneous Hamiltonian isotopy κ_t on $T^*S^n \setminus 0$ such that*

$$\kappa_0 = I, \qquad p_g \circ \kappa_1 = p_0, \qquad \kappa_t^* \theta = \theta.$$

It commutes with all common orthogonal isometries and with $\jmath(x, \xi) = (x, -\xi)$. Its generators are real, homogeneous of degree one, invariant under the common isometries, and odd under $\jmath$.

Proof. Put $\beta = (R_g^{-1})^* \alpha_g$ and $\eta = \beta - \alpha_0$. Let $T_0 = \partial_t \Gamma_0^t|_{t=0}$ be the round Reeb field. The two forms have Reeb field T_0. Thus

$$\eta(T_0) = 0, \qquad \iota_{T_0} d\eta = 0, \qquad \mathcal{L}_{T_0} \eta = 0.$$

They are invariant under the common isometries and change sign under ρ. For $0 \le s \le 1$ the forms $\alpha_s = \alpha_0 + s\eta$ are contact in a sufficiently small neighborhood and satisfy

$$\alpha_s(T_0) = 1, \qquad \iota_{T_0} d\alpha_s = 0.$$

The Moser isotopy in the proof of [1, Proposition 3.2], with equal periods, is the flow B_s of the uniquely determined vector field

$$\alpha_s(Y_s) = 0, \qquad \iota_{Y_s} d\alpha_s = -\eta.$$

It satisfies $B_s^* \alpha_s = \alpha_0$. Uniqueness makes Y_s, and therefore B_s, commute with the round flow, the common isometries, and ρ; for the last symmetry both $d\alpha_s$ and η change sign. The map $\psi = R_g^{-1} B_1$ satisfies $\psi^* \alpha_g = \alpha_0$. Define

$$\kappa(r\iota_0(z)) = r\iota_g(\psi z), \qquad r > 0.$$

The tautological forms in these coordinates are $r\alpha_0$ and $r\alpha_g$, respectively. Thus $\kappa^* \theta = \theta$ and $p_g \circ \kappa = p_0$. The second variational equation contains derivatives of the geodesic field of order at most two, hence metric derivatives of order at most three. The C^3 metric bound therefore gives C^2 closeness of R_g. Substituting $d\beta = (R_g^{-1})^* d\alpha_g$ in the defining linear system for Y_s gives C^1 closeness of Y_s to zero, uniformly in s. Hence κ is C^1-close to the identity on compact cotangent annuli.

The Lagrangian neighborhood theorem identifies nearby symplectic maps with closed one-forms [7, Theorem 4.19 and Corollaries 4.20–4.21, p. 49, author's version]. For nearby $x, y \in S^n$, the round metric gives a unique midpoint q and a small $u \in T_q S^n$ such that

$$x = \exp_q^{g_0}(-u), \qquad y = \exp_q^{g_0}(u).$$

This is an $O(n+1)$-equivariant coordinate map near the diagonal. Write $\kappa(x, \xi) = (y, v)$. The graph of κ, regarded in $T^*(S^n \times S^n)$ with covectors $(-\xi, v)$, is Lagrangian. Pull its tautological form back to these midpoint coordinates and write it as $A_i\, dq^i + B_i\, du^i$. Set $\zeta_i = B_i/2$. Subtraction of $d(2\zeta_i u^i)$ gives

$$A_i\, dq^i - 2u^i\, d\zeta_i.$$

This is the canonical form on $T^*(T^* S^n)$ with base (q, ζ) and fiber $(A, -2u)$. Both the original tautological form and the differential of the pairing $2\zeta_i u^i$ are intrinsic, so this symplectic map is independent of coordinates. The diagonal corresponds to the zero section. The transformed graph of κ is the graph of a small closed one-form σ on $T^* S^n \setminus 0$. Projection to (q, ζ) is a local diffeomorphism by the inverse function theorem. All norms and cospheres in this construction are for g_0. On the unit cosphere, the angular part of this projection is a C^1-small perturbation of the identity and hence a diffeomorphism. Its radial part is multiplication by the positive function $|\zeta|/|\xi|$. Homogeneity therefore makes the full projection a diffeomorphism of the punctured cotangent bundle.

For fiber dilation δ_r and its generator E, homogeneity implies $\delta_r^* \sigma = r\sigma$. Hence

$$\sigma = d(\sigma(E))$$

by Cartan's formula. Replace σ by $t\sigma$ and invert the midpoint construction. Both projections of each inverse image, to the input covector (x, ξ) and to the output covector (y, v), have angular part C^1-close to the identity and positive radial factor, uniformly for $0 \le t \le 1$. Hence both are global diffeomorphisms. The resulting Lagrangians are therefore graphs of homogeneous symplectic diffeomorphisms κ_t. Every such map satisfies

$$\kappa_t^* \theta = \kappa_t^* (\iota_E d\theta) = \iota_E \kappa_t^* d\theta = \theta.$$

All maps are equivariant under the common orthogonal group. Simultaneous reversal of ξ, v leaves q, u unchanged and negates A, B, so $j^* \sigma = -\sigma$; this remains true for $t\sigma$. Thus $\kappa_t J = J \kappa_t$. Finally, for $W_t = \dot{\kappa}_t \kappa_t^{-1}$, set $G_t = \theta(W_t)$. Differentiation of $\kappa_t^* \theta = \theta$ gives $\iota_{W_t} d\theta = -dG_t$. Homogeneity gives degree one; the stated invariance and oddness follow from the symmetries of W_t and θ. □

28.3. **Quantization through the subprincipal term.**

Proposition 28.4. *Let $H \subset O(n+1)$ be a compact subgroup preserving g. Suppose a homogeneous Hamiltonian isotopy κ_t, with $\kappa_0 = I$, is generated by real degree-one functions G_t, commutes with H, and satisfies $p_g \circ \kappa_1 = p_0$. Then* (28.1) *holds.*

Proof. Since G_t has degree one, $G_t = \theta(\dot\kappa_t\kappa_t^{-1})$ by Euler's identity. The cotangent lifts of H preserve θ, so G_t is H-invariant. Choose a smooth family of formally self-adjoint operators $A_t \in \Psi^1_{\mathrm{cl}}$ on smooth half-densities, with principal symbol G_t and subprincipal symbol zero. In half-density coordinates, take real Weyl quantization after multiplying G_t by a smooth frequency cutoff that vanishes near zero and equals one for large frequencies. A real partition $\sum_j \chi_j^2 = 1$ and the sum $\sum_j \chi_j A_{t,j}\chi_j$ preserve both symbols: the two Poisson terms involving χ_j cancel. Averaging $\mathcal{R}_h A_t \mathcal{R}_h^{-1}$ over H with its Haar probability measure makes the operators H-equivariant without changing those symbols or formal self-adjointness. Conjugation by an orthogonal map is continuous in each classical symbol seminorm. Compactness of $O(n+1)$ therefore bounds these seminorms before and after averaging, independently of the subgroup.

Solve $\partial_t U_t = -iA_tU_t$, $U_0 = I$. The first-order evolution and Sobolev energy estimates of [79, Section I.5, equations (5.1)–(5.8)] apply on the compact time interval. The homogeneous flow is complete there: $|X_{G_t}p_0| \le Cp_0$ gives $e^{-Ct}p_0 \le p_0\circ\kappa_t \le e^{Ct}p_0$, and the angular variables lie in the compact cosphere. For smooth half-densities u, v, differentiation gives $\partial_t\langle U_tu, U_tv\rangle = 0$. By density U_t is an isometry on L^2. The backward equation supplies its inverse, so U_t is unitary. Uniqueness makes it commute with every $\mathcal{R}_h$.

The scalar Cauchy propagator construction [27, Section 2.1, equation (6)], with $\lambda = -G_t$, gives a Fourier integral operator of order zero associated with κ_t. Since A_t is classical, the homogeneous transport equations give classical amplitudes. Near the characteristic relation $\tau + G_t = 0$, where $\xi \neq 0$, the spacetime symbol satisfies $\partial_\tau(\tau + G_t) = 1$. Thus the initial hypersurface is noncharacteristic for the transport equations [22, Theorems 5.3.1–5.3.2]. The local propagators compose over $[0,1]$; the Sobolev estimates and the Duhamel formula identify their composition with U_t modulo a smoothing operator.

Let $B \in \Psi^1_{\mathrm{cl}}(S^n)$ have principal symbol b and subprincipal symbol s. The scalar Weyl product gives

$$\sigma_1(i[A_t, B]) = \{G_t, b\}, \qquad \mathrm{sub}(i[A_t, B]) = \{G_t, s\}.$$

The next odd Weyl-product term has degree $1 + 1 - 3 = -1$. Egorov's theorem [79, Theorem I.6.1 and equation (6.16), pp. 20–21] gives $U_tBU_t^* \in \Psi^1_{\mathrm{cl}}$. Put $b_t = \sigma_1(U_tBU_t^*)$ and $s_t = \mathrm{sub}(U_tBU_t^*)$. Since $\partial_t(U_tBU_t^*) = -i[A_t, U_tBU_t^*]$ on smooth half-densities,

$$\partial_t b_t + \{G_t, b_t\} = 0, \qquad \partial_t s_t + \{G_t, s_t\} = 0.$$

Their initial values are b and s; hence $b_t = b\circ\kappa_t^{-1}$ and $s_t = s\circ\kappa_t^{-1}$. Equivalently the inverse conjugation has symbols $b\circ\kappa_t$ and $s\circ\kappa_t$. Lemma 20.6 gives $\mathrm{sub}(P_g) = \mathrm{sub}(P_0) = 0$ on half-densities. Hence $U_1^*P_gU_1 - P_0$ has order -1. The propagators preserve every Sobolev space, so the two order-one operators have the same domain H^1. Their bounded difference is self-adjoint. Take $U = U_1$. □

28.4. **High eigenspaces and quotient multiplicities.**

Proof of Theorem 28.1. The common orthogonal isometry group

$$\boldsymbol{\mathcal{I}}_g = \{h \in O(n+1) \,:\, h^*g = g\}$$

is closed in $O(n+1)$ and hence compact. Apply Lemma 28.3 and Proposition 28.4 with $H = \boldsymbol{\mathcal{I}}_g$. The resulting operators U, K commute with every subgroup of $\boldsymbol{\mathcal{I}}_g$. The neighborhood and these operators are independent of the chosen finite subgroup H. Put $B_t = P_0 + tK$ and $C = \|P_0^{1/2}KP_0^{1/2}\|$. Then

$$P_0 - CP_0^{-1} \le B_t \le P_0 + CP_0^{-1} \qquad (0 \le t \le 1).$$

Lemma 20.5, with the displayed common form bound, gives uniformly in t the ordered estimate

$$|\lambda_j(B_t) - \lambda_j(P_0)| \le C/\lambda_j(P_0)$$

for all large j, without an index shift. The finitely many remaining eigenvalues are bounded uniformly in t, since $\|tK\| \le \|K\|$. Thus there is an integer k_g such that the high spectrum lies in disjoint intervals

$[k - C/k, k + C/k] \subset (k - 1/8, k + 1/8)$ for $k \geq k_g$. The resolvent identity makes the finite-rank orthogonal projections $\Pi_{t,k} = \mathbf{1}_{(k-1/4,k+1/4)}(B_t)$ norm-continuous in t.

Each $\Pi_{t,k}$ commutes with H. Lemma 1.4 identifies its representation with that of $\Pi_{0,k} = \mathbf{1}_{\{k\}}(P_0)$. The unitary U transfers this identification to P_g.

For a free action, let $\pi : S^n \to S^n/H$ be the quotient map. The unitary map $|H|^{-1/2}\pi^*$ identifies quotient eigenspaces with invariant subspaces upstairs. Choose an integer $L \geq k_g$ such that $C/L \leq \eta$. Equal invariant multiplicities match every quotient frequency in a cluster with label $k \geq L$ to its round integer center. The cutoffs $m + N + 1/2$ include the same labels on both sides. The total unmatched multiplicity is at most

$$\operatorname{rank} \mathbf{1}_{(-\infty,L+1/4]}(P_g) + \operatorname{rank} \mathbf{1}_{(-\infty,L+1/4]}(P_0).$$

This finite bound is independent of N and H, and proves (28.2). □

Remark 28.5. The functions $p_t = p_0 \circ \kappa_t^{-1}$ associated with Lemma 28.3 are positive and homogeneous of degree one. They need not be Riemannian dual norms, and their Hamiltonian generators G_t need not be positive. Theorem 28.1 holds for each normalized Zoll metric $g \in \mathcal{U}_n$. For the invariant family of Proposition 28.6, the cluster bounds are uniform on $[0, 1]$; the metrics need not belong to $\mathcal{U}_n$.

28.5. **Invariant families of Zoll metrics.** Let p and a be as in Section 27.

Proposition 28.6. *Fix a free smooth C_p-action on S^{2m+1}, $m \geq 2$, which is linear with weights $(1, \dots, 1, a)$ for the unit round metric. Suppose g_s, $s \in [0, 1]$, is a smooth family of invariant Riemannian metrics with g_0 the unit round metric, such that every prime geodesic of every g_s has length 2π. Put $M_s = (S^{2m+1}, g_s)/C_p$ and $P_s = \sqrt{\Delta_{g_s} + m^2}$, and put $Q_s = \sqrt{\Delta_{M_s} + m^2}$. For $k \in \mathbb{Z}$, the kth cluster is the spectrum in $(k - 1/4, k + 1/4)$. There are an integer K_0 and $C > 0$, independent of s, such that for every integer $k \geq K_0$:*

(1) every frequency of P_s in the kth cluster differs from k by at most C/k;

(2) the C_p-module carried by that cluster is isomorphic to $\ker(P_0 - k)$;

(3) the quotient cluster multiplicity on M_s equals that on M_0.

These clusters contain every frequency above $K_0 - 1/2$. Consequently, for each fixed $\eta > 0$, the quotient frequencies admit matchings at the cutoffs $R_K = m + K + \frac{1}{2}$, $K \in \mathbb{Z}_{\geq 0}$, satisfying

$$E_\eta(R_K; Q_s, Q_0) = O_\eta(1) \qquad (K \to \infty),$$

uniformly in $s \in [0, 1]$.

Proof. The half-density Hilbert space and the operators $\mathcal{R}_h$, $h \in C_p$, are independent of s. The complex-power construction in Lemma 20.6, applied with the parameter s, makes $s \mapsto P_s$ smooth in Ψ^1_{cl}. Its scalar subprincipal symbol vanishes, and its Hamiltonian flow has period 2π. Formula (20.5), with $B_0 = 0$, gives $\sigma_0(e^{2\pi i P_s}) = \rho_s$, where ρ_s is the Maslov factor. Under radial identification with $S^*_{g_0}S^n$, this fourth root of unity is locally constant on the connected space $[0, 1] \times S^*_{g_0}S^n$. At $s = 0$ the frequencies are the integers $k + m$, so $\rho_s = \rho_0 = 1$. Put $p_s = p_{g_s}$ and $U_s(t) = e^{itP_s}$. The local Fourier integral construction [22, Theorems 5.3.1–5.3.2] applies near $\tau - p_s = 0$, since $\partial_\tau(\tau - p_s) = 1$. The transport equations and their asymptotic sums may be solved smoothly in s on a finite cover of $[0, 1] \times [0, 2\pi] \times S^*_{g_0}S^n$ by nondegenerate phase charts. The resulting parametrices $V_s(t)$ satisfy

$$F_s(t) = (\partial_t - iP_s)V_s(t) \in \Psi^{-\infty}, \qquad F_s^0 = V_s(0) - I \in \Psi^{-\infty},$$

with kernels smooth jointly in the parameters and the two spatial variables. Duhamel's formula gives

$$U_s(t) - V_s(t) = -U_s(t)F_s^0 - \int_0^t U_s(t - u)F_s(u)\, du.$$

The Sobolev estimates [79, Section I.5, equations (5.1)–(5.8)] are uniform in s, t. Differentiate the evolution equation on smooth half-densities. For every $j, \ell \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{R}$, the derivatives extend to bounded operators $\partial_s^j \partial_t^\ell U_s(t) : H^{r+j+\ell} \to H^r$, uniformly in s, t. Every parameter derivative of the

right-hand side of Duhamel's formula therefore maps H^{-r} to H^{r} for every $r \geq 0$, with norm bounded uniformly in s, t. Hence $U_s(t) - V_s(t)$ is smooth in s, t with values in $\Psi^{-\infty}$. At $t = 2\pi$ the canonical relation is the diagonal and the principal symbol is $\rho_s = 1$. Therefore

$$e^{2\pi i P_s} = I + R_s, \qquad R_s \in \Psi^{-1}_{\mathrm{cl}},$$

and $s \mapsto R_s$ is smooth in this class.

The order-zero family $R_s P_s$ has uniformly bounded L^2 operator norm, say at most C_0. For an eigenvector with frequency μ,

$$|e^{2\pi i \mu} - 1| \leq C_0/\mu.$$

If $d = \operatorname{dist}(\mu, \mathbb{Z}) \leq 1/2$, then $|e^{2\pi i\mu} - 1| = 2\sin(\pi d) \geq 4d$. Hence $d \leq C_0/(4\mu)$. Choose K_0 large enough that all frequencies above $K_0 - 1/2$ lie in disjoint intervals $(k - 1/4, k + 1/4)$ and satisfy $|\mu - k| \leq C/k$. Increase K_0 so that $C/K_0 < 1/8$.

For each such k, let $\Pi_{s,k}$ be the Riesz projector onto $(k - 1/4, k + 1/4)$. Its contour is uniformly separated from the spectrum. The family P_s is smooth as a map $H^1 \to L^2$; elliptic estimates and the resolvent identity therefore make $\Pi_{s,k}$ norm-continuous in s. It commutes with the fixed C_p-action. Lemma 1.4 proves equality of the cluster representations and of their invariant multiplicities.

Match all copies in each quotient cluster with the copies of the round frequency k. If $k \geq \max\{K_0, C/\eta\}$, all match errors are at most η. The whole-cluster cutoffs include the same labels on both sides. Compactness of the metric family gives uniform upper and lower bounds for the metric tensors and their volume densities. The Rayleigh quotients and the min–max principle therefore give

$$\lambda_j(\Delta_{g_s}) \geq c\lambda_j(\Delta_{g_0}) \qquad (s \in [0,1],\ j \geq 0)$$

with $c > 0$ independent of s, j. The number of frequencies below any fixed bound is consequently bounded uniformly in s. The quotient multiplicity is at most the corresponding multiplicity upstairs. Leaving the remaining low frequencies unmatched gives $E_\eta(R_K; Q_s, Q_0) = O_\eta(1)$ uniformly in s. □

29. Integral topology of order-three returns

29.1. **The geodesic circle.** Let X^{2m+1} be a closed smooth homotopy sphere with Riemannian metric g, where $m \geq 2$ is an integer. Let d be its distance. Suppose that every unit-speed geodesic has prime period 2π. Write $E = SX$, let G_t denote geodesic flow, and put

$$p : E \longrightarrow \mathcal{G} = E/S^1, \qquad \pi : E \longrightarrow X.$$

The action of $\mathbb{R}/(2\pi\mathbb{Z})$ on E is free. The canonical contact form is

$$\lambda_{(x,v)}(\xi) = g_x(v, d\pi\,\xi).$$

There is a symplectic form ω on $\mathcal{G}$ with $p^*\omega = d\lambda$. Give $\mathcal{G}$ its symplectic orientation. Let $e \in H^2(\mathcal{G}; \mathbb{Z})$ be the Euler class with the positive convention of Section 12; thus $e_{\mathbb{R}} = [\omega/(2\pi)]$. Proposition 12.3 gives $\operatorname{Vol}(X) = \operatorname{Vol}(S^{2m+1})$.

Lemma 29.1. *There are integral classes x, y such that*

$$H^*(E; \mathbb{Z}) = \mathbb{Z}\langle 1, x, y, xy\rangle, \qquad |x| = 2m, \quad |y| = 2m+1, \quad x^2 = y^2 = 0,$$

where $y = \pi^* u_X$ for an oriented generator u_X of $H^{2m+1}(X; \mathbb{Z})$. The integral cohomology of $\mathcal{G}$ is torsion-free. Its odd groups vanish. For $0 \leq j \leq 2m$, the group $H^{2j}(\mathcal{G}; \mathbb{Z})$ has rank one, except when $j = m$, when it has rank two. The class e^m is primitive, and

$$\langle e^{2m}, [\mathcal{G}]\rangle = 2. \tag{29.1}$$

Every odd-order isometry of X acts trivially on $H^*(\mathcal{G}; \mathbb{Z})$. In the Borel spectral sequence of the geodesic circle action,

$$d_2 y = \pm 2ux, \qquad u \in H^2(BS^1; \mathbb{Z}) \text{ a generator.} \tag{29.2}$$

Proof. The sphere-bundle computation in Lemma 13.5 gives the ring $H^*(E;\mathbb{Z})$ and the stated class y. Lemma 12.1 gives the cohomology of $\mathcal{G}$ and the primitivity of e^m. Proposition 12.3 and (12.2) give (29.1). The odd-order action is determined in Lemma 13.3, and the integral transgression is Lemma 13.5. □

Lemma 29.2. *Let q be an odd prime and let h generate a free isometric action of C_q on X. Denote its tangent lift by $\widehat{h}$ and its induced map on $\mathcal{G}$ by $\bar{h}$. For $c \in \mathbb{F}_q^*$, set*

$$F_c = \mathrm{Fix}(\widehat{h}^{-1} G_{2\pi c/q}), \qquad B_c = F_c/S^1.$$

The return map is independent of the integer representative of c, since $G_{2\pi} = I$. Every nonempty F_c is a connected closed contact manifold and a mod-q cohomology sphere. If $\dim F_c = 2d_c - 1$, then

$$H^*(B_c;\mathbb{F}_q) = \mathbb{F}_q[e_c]/(e_c^{d_c}),$$

where e_c is the restriction of e reduced modulo q. Moreover,

$$\mathrm{Fix}(\bar{h} : \mathcal{G} \to \mathcal{G}) = \coprod_{c \neq 0} B_c.$$

Reversal interchanges F_c and F_{-c}, and

$$\sum_{\{c,-c\}\,:\,F_c \neq \emptyset} d_c = m + 1. \tag{29.3}$$

Proof. The lift $\widehat{h}$ preserves λ and commutes with G_t. Put $\Bbbk = \mathbb{F}_q$ and retain x, y for their reductions modulo q. The action of $S^1 \times C_q$ on $H^*(E;\Bbbk)$ is trivial. Let u also denote the pullback of the reduction modulo q of the class in (29.2). The product-action transgression in the proof of Proposition 13.6 gives

$$d_2 y = \pm 2ux, \qquad d_2 x = 0.$$

Let $\Delta_c \subset S^1 \times C_q$ be generated by $(e^{2\pi i c/q}, h^{-1})$. Its action is generated by $K_c = \widehat{h}^{-1} G_{2\pi c/q}$. Choose $w \in H^2(B\Delta_c;\Bbbk)$ so that $u|_{B\Delta_c} = cw$, and choose a generator $\epsilon \in H^1(B\Delta_c;\Bbbk)$. Then $H^*(B\Delta_c;\Bbbk) = \Bbbk[w] \otimes \Lambda(\epsilon)$, and restriction gives $d_2 y = \pm 2cwx$. Over $\Bbbk(w)$ this coefficient is invertible, x is a boundary, and xy is a cycle because $x^2 = 0$. The third page has basis

$$1, \quad xy, \quad \epsilon, \quad \epsilon xy$$

where $|\epsilon| = 1$ and $\epsilon^2 = 0$. Lemma 13.2 therefore gives $2\sum_j b_j(F_c;\mathbb{F}_q) \leq 4$.

For a K_c-invariant auxiliary metric, the fixed set is a closed submanifold with tangent space $\ker(dK_c - I)$ [51, Chapter II, Theorem 5.1, p. 59]. The homogeneous cotangent extension of K_c is symplectic, has finite order, preserves $|\xi|_g$, and commutes with fiber dilations. The fixed-cone argument of Section 10.1 makes every nonempty component of F_c a closed oriented contact manifold of positive odd dimension. Each contributes its degree-zero and top cohomology to the preceding bound. Hence F_c is connected and has no intermediate mod-q cohomology. The Gysin sequence of its free geodesic circle action gives the ring $H^*(B_c;\mathbb{F}_q) = \mathbb{F}_q[e_c]/(e_c^{d_c})$, including $d_c = 1$ when B_c is a point.

On a circle fiber above $\mathrm{Fix}(\bar{h})$, the map $\widehat{h}$ is a rotation. Its angle depends continuously on the base point in a local bundle trivialization. Since $\widehat{h}^{\,q} = I$, that angle belongs to $\{2\pi c/q : c \in \mathbb{F}_q\}$ and is locally constant. The value $c = 0$ would give a fixed point of h on X. Thus $\mathrm{Fix}(\bar{h}) = \coprod_{c\neq 0} B_c$. The identities $JG_t = G_{-t}J$ and $J\widehat{h} = \widehat{h}J$, for $J(x,v) = (x,-v)$, give $J(F_c) = F_{-c}$ and, when $F_c \neq \emptyset$, $d_c = d_{-c}$.

Give $\mathcal{G}$ a $\bar{h}$-invariant Riemannian metric by averaging. The Lefschetz formula for an isometry [51, Chapter II, §5, p. 63] gives $L(\bar{h}) = \chi(\mathrm{Fix}(\bar{h}))$. Lemma 29.1 and the cohomology of B_c therefore give

$$2m + 2 = \sum_{\substack{c\neq 0 \\ F_c \neq \emptyset}} \chi(B_c) = \sum_{\substack{c\neq 0 \\ F_c \neq \emptyset}} d_c.$$

Division into reversal pairs proves (29.3). □

29.2. **The short return families.** Let h be a fixed-point-free isometry of exact order three. Write $\widehat{h}$ and $\bar{h}$ for its induced maps on E and $\mathcal{G}$, respectively, and put

$$\ell = \frac{2\pi}{3}, \qquad F_+ = \operatorname{Fix}(\widehat{h}^{-1} G_\ell), \qquad F_- = \operatorname{Fix}(\widehat{h}^{-1} G_{-\ell}), \qquad B_\pm = F_\pm / S^1.$$

Since h has prime order, its action on X is free. Put $e_\pm = e|_{B_\pm} \in H^2(B_\pm; \mathbb{Z})$, and denote their reductions modulo three by $\bar{e}_\pm$.

Theorem 29.3. *The sets F_+ and F_- are connected closed contact manifolds of dimension $2m+1$. They are interchanged by $(x, v) \mapsto (x, -v)$, and*

$$H^*(F_\pm; \mathbb{F}_3) \cong H^*(S^{2m+1}; \mathbb{F}_3), \qquad H^*(B_\pm; \mathbb{F}_3) \cong \mathbb{F}_3[\bar{e}_\pm]/(\bar{e}_\pm^{m+1}).$$

Moreover,

$$\operatorname{Fix}(\bar{h} : \mathcal{G} \to \mathcal{G}) = B_+ \sqcup B_-. \tag{29.4}$$

Each evaluation map $F_\pm \to X$ has integral degree prime to three. In particular it is onto, and

$$d(x, hx) = d(x, h^2 x) = \frac{2\pi}{3} \qquad (x \in X). \tag{29.5}$$

Proof. For $q = 3$ the only possible nonzero phases are 1 and -1. Lemma 29.2 gives precisely one reversal pair, and (29.3) gives $d_1 = m + 1$. This proves the statements about dimensions, cohomology, and the fixed set.

The deck action on F_+ is free, since evaluation maps it equivariantly to the free action on X. On F_+ it equals the contact flow G_ℓ; on X its degree has odd order in $\{1, -1\}$. Both actions therefore preserve orientation. For $Z = F_+, X$, let $\eta_Z \in H^{2m+2}(BC_3; \mathbb{F}_3)$ be the Euler class of its Borel fibration with the chosen orientation. These fibrations are $\mathbb{F}_3$-oriented [47, Proposition 1.3, p. 281]. Put $D = \deg(\pi|_{F_+}) \bmod 3$. The equivariant degree formula [47, Theorem 1.9, p. 282] gives

$$D\eta_{F_+} = \eta_X.$$

The class η_X is the transgression of the oriented top class [47, Theorem 2.3, pp. 283–284]. The nonvanishing argument at the end of the proof of Proposition 13.6 applies to this free C_3-action: the Borel sequence of X has only rows zero and $2m+1$, and $H^{2m+2}(X/C_3; \mathbb{F}_3) = 0$. Thus $\eta_X \neq 0$, and $D \neq 0$. The integral degree is prime to three. Evaluation is onto, since $H_{2m+1}(X \setminus \{x\}; \mathbb{F}_3) = 0$ for every $x \in X$ [37, Proposition 3.29]. Reversal commutes with evaluation and identifies F_+ with F_-. Their evaluation degrees agree up to sign, so the assertion holds for F_- as well.

For every x, choose $v \in F_+$ based at x. Its length-ℓ geodesic ends at hx, so $d(x, hx) \leq \ell$. Equation (15.2), applied with $s = 3$ and $L = 2\pi$, gives $d(x, hx) \geq \ell$. Hence $d(x, hx) = \ell$ for every x. Finally, $d(x, h^2 x) = d(hx, h^3 x) = d(x, hx)$. □

Corollary 29.4. *For $M = X/\langle h \rangle$, every prime geodesic has length $2\pi/3$ or 2π. The prime geodesics of length $2\pi/3$ are exactly the noncontractible prime geodesics. One passes through every point of M.*

Proof. Extend a unit-speed lift to a geodesic on X, of prime period 2π. The stabilizer of its oriented geodesic in C_3 is either trivial or all of C_3. In the first case its projection has prime period 2π and trivial monodromy. In the second, Theorem 29.3 places the oriented geodesic in B_+ or B_-; its prime projected period is $2\pi/3$ and its monodromy is h or h^2. Since X is simply connected, a quotient loop is contractible exactly when its monodromy is trivial. Surjectivity of $F_+ \to X$ gives a noncontractible prime geodesic through every point of the quotient. □

Corollary 29.5. *For odd-dimensional Besse homotopy spheres of dimension at least five, the following conditions are equivalent:*

(1) every sphere admitting a nonidentity Clifford translation of order three has constant sectional curvature;

(2) every sphere admitting a free isometric C_3 action has constant sectional curvature.

Proof. Smale's theorem [73, Theorem A] makes each such homotopy sphere a topological sphere. By [70, Theorem A], its prime geodesics have one common length. Rescale that length to 2π. A nonidentity constant-displacement map has positive displacement and no fixed point. Conversely, Theorem 29.3 makes every generator of a free isometric C_3 action a Clifford translation. □

29.3. **Twisted indices and integral homotopy type.** For an isometry f of a closed Riemannian manifold X, put

$$\Lambda_f X = \{c \in H^1([0,1], X) \,:\, c(1) = f\,c(0)\}, \qquad \mathcal{E}(c) = \frac{1}{2}\int_0^1 |\dot c|^2\, dt, \qquad \mathrm{ev}_0(c) = c(0).$$

A path is critical for $\mathcal{E}$ if and only if it is a smooth geodesic satisfying $\dot c(1) = df\,\dot c(0)$ [30, Theorem 1.6(2)].

Theorem 29.6. *Under the hypotheses of Theorem 29.3, both F_+ and F_- are homotopy $(2m+1)$-spheres. Each evaluation $F_\pm \to X$ has degree ± 1 and is a homotopy equivalence. With the symplectic orientation and the positive Euler-class convention,*

$$H^*(B_\pm; \mathbb{Z}) = \mathbb{Z}[e_\pm]/(e_\pm^{m+1}), \qquad \langle e_\pm^m, [B_\pm]\rangle = 1, \qquad B_\pm \simeq \mathbb{CP}^m.$$

Moreover,

$$X/\langle h\rangle \simeq L(3; 1, \dots, 1).$$

The critical manifolds of the energy on the h-twisted path space, in increasing order of length, have dimension $2m+1$ and indices $0, 2m, 4m, 6m, \dots$.

Lemma 29.7. *The energy on $\Lambda_f X$ satisfies the Palais–Smale condition. Suppose its critical set at a value c is a compact submanifold whose Hessian kernel is its tangent bundle. Then c is isolated among the critical values. Up to homotopy, crossing c attaches the negative disk bundle along its sphere bundle. Suppose the full minimum set C is such a submanifold, every higher critical level has the same property, and every negative bundle above C has rank at least $r \geq 1$. Then*

$$\pi_j(\Lambda_f X, C) = 0 \qquad (j < r).$$

If X is a homotopy n-sphere, $n \geq 5$, and f has degree one, then

$$\pi_j(\Lambda_f X) = 0 \quad (1 \leq j \leq n-2), \qquad (\mathrm{ev}_0)_* \,:\, H_n(\Lambda_f X; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z}).$$

Proof. Take $N = \mathrm{graph}(f) \subset X \times X$. This is a closed submanifold with compact projections, and X is complete. Grove's theorem therefore gives the Palais–Smale condition [30, Theorem 2.4]. The graph of f is totally geodesic. At a critical path, the Hessian, represented for the H^1 inner product $\int_0^1 (\langle D_t Y, D_t Z\rangle + \langle Y, Z\rangle)\, dt$, is identity minus a compact self-adjoint operator: its lower-order terms factor through the compact inclusion $H^1 \hookrightarrow L^2$. It has finite negative index and is invertible on the orthogonal complement of its kernel. The Morse–Bott attachment theorem for closed sublevels applies [4, Appendix B, Theorems B.1–B.3, pp. 1625–1627]. An accumulation of critical values in a bounded interval would, by Palais–Smale, yield a critical point with distinct critical values in every neighborhood, contrary to its Morse–Bott normal form. Thus only finitely many attachments occur in each bounded interval. A CW decomposition of each compact critical manifold gives relative cells of dimension at least the negative rank. Cellular approximation gives the asserted relative connectivity on each bounded sublevel. Every map from a compact pair into $(\Lambda_f X, C)$ has bounded energy, so the same conclusion holds for the full path space.

The inclusion into the continuous twisted path space is a homotopy equivalence [30, Theorem 1.3]. Evaluation is therefore a fibration up to homotopy with fiber ΩX. Its homotopy sequence gives $\pi_j(\Lambda_f X) = 0$ for $1 \leq j \leq n-2$. The James reduced-product model $J(S^{n-1})$ for ΩS^n has cells only in dimensions divisible by $n-1$ [37, Proposition 3.22, p. 224; Theorem 4J.1, p. 471]. Since $n \geq 5$, it gives

$H_n(\Omega X;\mathbb{Z}) = 0$. Since f has degree one, it is smoothly homotopic to id_X. Such a homotopy gives a section of evaluation. In its integral homology spectral sequence,

$$E^2_{a,b} = H_a(X; H_b(\Omega X;\mathbb{Z})),$$

the only nonzero term of total degree n is $E^2_{n,0} = \mathbb{Z}$. The section makes this term permanent. The edge homomorphism is $(\mathrm{ev}_0)_*$, which is consequently an isomorphism. □

Lemma 29.8. *Let γ be a unit-speed geodesic with $(\gamma(0), \dot\gamma(0)) \in F_+$. Write D_t for covariant differentiation and take the curvature convention $D_t^2 J + R(J,\dot\gamma)\dot\gamma = 0$ for Jacobi fields. On the complexified normal bundle, let $g_{\mathbb{C}}$ be the complex bilinear extension of g and put $\langle Y, Z\rangle = g_{\mathbb{C}}(Y, \overline{Z})$. For $|z| = 1$, let $j(z)$ be the complex negative index of*

$$I_z(J) = \int_0^\ell \left(|D_t J|^2 - \langle R(J,\dot\gamma)\dot\gamma, J\rangle\right) dt$$

on the complex normal H^1 fields satisfying $J(\ell) = z\, dh\, J(0)$. Then

$$j(1) = 0, \qquad j(e^{i\theta}) = \begin{cases} m, & 0 < \theta \le 2\pi/3 \text{ or } 4\pi/3 \le \theta < 2\pi, \\ 2m, & 2\pi/3 < \theta < 4\pi/3. \end{cases}$$

For an integer $r \ge 1$, let i_r be the real Morse index of the arc $\gamma|_{[0,r\ell]}$ with endpoint twist h^r. One has

$$i_{3k+1} = 4km, \qquad i_{3k+2} = (4k+2)m, \qquad i_{3k} = (4k-2)m, \tag{29.6}$$

where $k \ge 0$ in the first two formulas and $k \ge 1$ in the last.

Proof. Uniqueness for the geodesic equation gives $\gamma(t+\ell) = h\gamma(t)$ for every $t \in \mathbb{R}$. Theorem 29.3 gives $d(x, hx) = \ell$ for every x. Thus the short arc minimizes the twisted energy and $j(1) = 0$. The endpoint terms in its Hessian cancel, since h is an isometry and $\dot\gamma(\ell) = dh\,\dot\gamma(0)$.

For normal Jacobi fields, the form

$$\Omega(Y, Z) = g_{\mathbb{C}}(D_t Y, Z) - g_{\mathbb{C}}(Y, D_t Z)$$

is independent of t. Its restriction to real initial data is symplectic. Write P_ℓ for Jacobi transport on normal initial data and set

$$T = (dh^{-1} \oplus dh^{-1}) P_\ell.$$

This map is symplectic on a real vector space of dimension $4m$. The identity $G_{3\ell} = I$ and commutation with $\widehat{h}$ give $T^3 = I$. Its fixed space is the contact tangent space of F_+, of dimension $2m$. Hence T is semisimple, and its complex eigenspaces at $\zeta = e^{2\pi i/3}$ and ζ^2 each have dimension m.

Extend a field on the r-fold arc by $J(t + r\ell) = dh^r J(t)$. Translation gives the unitary operator

$$(S_r J)(t) = dh^{-1} J(t+\ell), \qquad S_r^r = I.$$

It preserves the index form. Restriction to $[0, \ell]$ identifies its z-eigenspace with the domain of I_z, and the restricted form is rI_z. The character decomposition [31, Lemma 2.3 and (2.4)] gives

$$i_r = \sum_{z^r = 1} j(z).$$

Here complexification preserves the negative index, and the tangential form $\int_0^{r\ell} |f'|^2\, dt$ is nonnegative. In particular, Lemma 11.2 gives

$$2m = j(1) + j(\zeta) + j(\zeta^2).$$

The kernel of I_z consists of Jacobi fields satisfying also $D_t J(\ell) = z\, dh\, D_t J(0)$; it is therefore the z-eigenspace of T. At a phase θ_0, the gauge $J(t) = e^{i(\theta-\theta_0)t/\ell} Y(t)$ fixes the form domain, and its derivative on the kernel is

$$Q_{\theta_0}(Y, Z) = \frac{i}{\ell} \int_0^\ell \left(\langle Y, D_t Z\rangle - \langle D_t Y, Z\rangle\right) dt = -i\Omega(Y, \overline{Z}).$$

If E_z is an eigenspace of T, symplectic invariance gives $\Omega(E_z, E_w) = 0$ for $zw \neq 1$. Semisimplicity makes the pairing of E_z with $E_{z^{-1}}$ nondegenerate. Thus Q_{θ_0} is nondegenerate at each of the three eigenphases.

Put $\delta = \theta - \theta_0$, $z_0 = e^{i\theta_0}$ and $R_\gamma Y = R(Y, \dot\gamma)\dot\gamma$. Under this gauge the Jacobi operator is

$$A_\delta = -(D_t + i\delta/\ell)^2 - R_\gamma$$

on the fixed H^2 domain

$$Y(\ell) = z_0\, dh\, Y(0), \qquad D_t Y(\ell) = z_0\, dh\, D_t Y(0).$$

It is a holomorphic family of type (A), self-adjoint for real δ, with compact resolvent. Its eigenvalues near zero admit analytic parametrizations [49, Chapter VII, Theorem 3.9, pp. 392–393]. On the finite-dimensional spectral subspace near zero, the derivatives of the eigenvalues through zero are the eigenvalues of Q_{θ_0} relative to the L^2 inner product [49, Chapter II, Theorem 5.4, pp. 111–112]. The index is constant off the three eigenphases. At $z = 1$, the real fixed symplectic space has dimension $2m$, so Q_0 has inertia (m, m). Since $j(1) = 0$, it follows that $j(e^{i\theta}) = m$ for $0 < \theta < 2\pi/3$.

Complex conjugation gives $j(\zeta) = j(\zeta^2)$, so both equal m. At $\theta = 2\pi/3$, the index immediately before and at the crossing is m, whereas the kernel has dimension m. Every crossing eigenvalue is therefore positive on the left and negative on the right. The index is $2m$ on $(2\pi/3, 4\pi/3)$. Conjugation gives its values on the remaining arc and at $4\pi/3$.

Finally,

$$i_r = m\big(r - 1 + \#\{a \in \mathbb{Z} \,:\, r/3 < a < 2r/3\}\big).$$

The number in braces is k, $k+1$, or $k-1$ when $r = 3k+1$, $3k+2$, or $3k$, respectively. This gives (29.6). □

Proof of Theorem 29.6. Put $n = 2m + 1$. A critical path in $\Lambda_h X$ extends by h to a closed geodesic. If its length is L, its unit tangent state returns after $3L$; hence $L = r\ell$ for an integer $r \geq 1$. The integer r is not divisible by three, since h has no fixed point. For $r = 3k+1$ the initial unit state belongs to F_+; for $r = 3k+2$ it belongs to F_-. Conversely, each such state v determines the critical path $c_v(t) = \pi G_{r\ell t} v$, $0 \leq t \leq 1$. Uniqueness for the geodesic and Jacobi equations makes $v \mapsto c_v$ an injective immersion. Compactness makes it an embedding, so each critical manifold is compact and has dimension n. If $r \equiv 2 \pmod 3$, time reversal gives an r-fold arc in F_+ with endpoint twist $h^{-1} = h^r$ and preserves the index form. Formula (29.6) therefore gives the successive indices

$$0,\ n-1,\ 2(n-1),\ 3(n-1), \dots.$$

For $r \equiv \pm 1 \pmod 3$, the relative return is $\widehat{h}^{-1} G_{r\ell} = \widehat{h}^{-1} G_{\pm\ell}$. Its fixed normal Jacobi data are identified with $T_v F_\pm \cap \ker \lambda$, of dimension $2m$. These data form the normal Hessian kernel. The tangential Jacobi fields are $(A + Bt)\dot\gamma$; the twisted value condition gives $B = 0$. Thus the full kernel has dimension n. The differential of the initial-state parametrization identifies the tangent space of the critical manifold with this kernel: the fixed set of the finite-order return has tangent space equal to its fixed subspace. All critical manifolds are consequently Morse–Bott. Every $c \in \Lambda_h X$ satisfies

$$\mathcal{E}(c) \geq \tfrac{1}{2}\,\mathrm{length}(c)^2 \geq \tfrac{1}{2} d(c(0), hc(0))^2 = \tfrac{1}{2}\ell^2.$$

The family F_+ attains equality; the classification of critical paths makes it the full minimum set. The map h has degree one, since its order is odd. Lemma 29.7 gives

$$\pi_j(\Lambda_h X, F_+) = 0 \quad (j < n-1), \qquad \pi_j(F_+) = 0 \quad (1 \leq j \leq n-3).$$

For the second vanishing statement, apply the relative homotopy sequence and the vanishing for $\Lambda_h X$ in that lemma.

As $n \geq 5$, F_+ is simply connected and $H_j(F_+; \mathbb{Z}) = 0$ for $1 \leq j \leq n-3$ by Hurewicz. The universal coefficient theorem gives $H^1(F_+; \mathbb{Z}) = H^2(F_+; \mathbb{Z}) = 0$. Integral Poincaré duality then gives

$H_{n-1}(F_+;\mathbb{Z}) = H_{n-2}(F_+;\mathbb{Z}) = 0$. Thus F_+ is a simply connected integral homology sphere. Hurewicz and the homological Whitehead theorem give $F_+ \simeq S^n$ [37, Corollary 4.33]. Reversal gives $F_- \simeq S^n$.

The first attachment above F_+ has rank $n-1$ over F_-. Its negative bundle is orientable because F_- is simply connected. The Thom isomorphism gives relative homology in degrees $n, n+1$ equal to $H_1(F_-;\mathbb{Z}), H_2(F_-;\mathbb{Z})$, respectively; both vanish. Every later negative rank is at least $2(n-1) > n+1$. The exact sequences of the sublevel pairs therefore give

$$H_n(F_+;\mathbb{Z}) \xrightarrow{\cong} H_n(\Lambda_h X;\mathbb{Z}).$$

Compose with the evaluation isomorphism of Lemma 29.7. Evaluation $F_+ \to X$ has degree ± 1 and is a homotopy equivalence. Reversal gives the same conclusion for F_-.

The integral Gysin sequence of $F_\pm \to B_\pm$ gives

$$H^*(B_\pm;\mathbb{Z}) = \mathbb{Z}[e_\pm]/(e_\pm^{m+1}).$$

In particular $e_\pm^m$ generates the top cohomology. Its symplectic representative has positive integral, so $\langle e_\pm^m, [B_\pm]\rangle = 1$. The homotopy sequence of the circle bundle gives $\pi_1(B_\pm) = 0$. Since $\dim B_\pm = 2m$, cellular approximation gives a classifying map for $e_\pm$ with values in $\mathbb{CP}^m$. It induces an integral cohomology isomorphism; the homological Whitehead theorem gives $B_\pm \simeq \mathbb{CP}^m$.

The quotient $X/\langle h\rangle$ is a finite CW complex with fundamental group C_3 and universal cover homotopy equivalent to S^n. Fix an identification of its fundamental group with C_3 and a homotopy equivalence of its universal cover with S^n. With these choices, Wall's classification [81, Proposition 1, p. 117] assigns a generator of $H^{n+1}(C_3;\mathbb{Z}) \cong \mathbb{Z}/3$, which determines the homotopy type with the chosen identifications. Postcomposition of the covering equivalence with a degree--1 self-map of S^n negates this generator. Thus, after forgetting the choices, the two generators give the same homotopy type, represented by $L(3;1,\dots,1)$. □

Question 29.9. Let X^{2m+1}, $m \geq 2$, be a normalized Zoll homotopy sphere with a free isometric action of order three. Must its sectional curvature be one?

30. Prime cyclic actions and their geodesic return families

Let $m \geq 2$ be an integer, put $n = 2m+1$, and let $p \geq 5$ be prime. Set $\ell = 2\pi/p$. Let X be a closed smooth homotopy n-sphere with Riemannian metric g and distance d. Suppose that

$$\text{every unit-speed geodesic has prime period } 2\pi, \qquad \operatorname{Vol}(X) = V_n. \tag{30.1}$$

The volume equality also follows from Proposition 12.3. Let h generate a free isometric action of C_p on X, let $\widehat{h}$ be its tangent lift, and put

$$E = SX, \qquad \mathcal{G} = E/S^1, \qquad M = X/C_p.$$

The circle acts by the geodesic flow G_t, with $t \in \mathbb{R}/(2\pi\mathbb{Z})$. Write $\pi : E \to X$ for evaluation and $p_E : E \to \mathcal{G}$ for the circle projection. Let λ be the canonical contact form, and let ω be the symplectic form on $\mathcal{G}$ with $p_E^*\omega = d\lambda$. The positive Euler class $e \in H^2(\mathcal{G};\mathbb{Z})$ has real image $e_{\mathbb{R}} = [\omega/(2\pi)]$, as in Section 12. For $b \in \mathbb{F}_p$, interpret $b\ell$ modulo 2π and put

$$F_b = \{v \in E : \widehat{h}v = G_{b\ell}v\}, \qquad B_b = F_b/S^1.$$

Freeness gives $F_0 = \emptyset$. Reversal $J(x,v) = (x,-v)$ sends F_b to F_{-b}. Empty families are allowed.

30.1. Cohomology and the partition of return dimensions.

Lemma 30.1. *The integral cohomology of E has additive basis $1, x, y, xy$, where $|x| = 2m$, $|y| = 2m+1$, $x^2 = y^2 = 0$, and $y = \pi^* u_X$ for an oriented generator $u_X \in H^{2m+1}(X;\mathbb{Z})$. The geodesic circle action has Borel differential*

$$d_2 y = \pm 2ux, \qquad u \in H^2(BS^1;\mathbb{Z}) \text{ a generator.}$$

Every nonempty F_b, $b \neq 0$, is a connected closed contact manifold which is a mod-p cohomology sphere. If $\dim F_b = 2d_b - 1$, *then*

$$H^*(B_b; \mathbb{F}_p) \cong \mathbb{F}_p[e_b]/(e_b^{d_b}), \qquad |e_b| = 2,$$

where e_b is the restriction of e reduced modulo p.

Proof. The integral cohomology and circle differential are those of Lemma 29.1. Apply Lemma 29.2 with $q = p$ to the return families F_b. □

Theorem 30.2. *Under* (30.1)*, the nonempty F_b occur in pairs F_b, F_{-b} of the same dimension. Choosing one representative from each pair, their integers $d_b = (\dim F_b + 1)/2$ satisfy*

$$\sum_{\{b,-b\}\colon F_b \neq \emptyset} d_b = m + 1.$$

In particular, a free order-p isometry determines a partition of $m+1$ with at most $(p-1)/2$ parts, together with nonzero phase labels modulo sign.

Proof. This is (29.3). There are $(p-1)/2$ nonzero phase pairs, and every nonempty pair contributes a positive integer. □

30.2. **Two return dimensions.**

Theorem 30.3. *Assume* (30.1)*, and suppose the largest component of* $\mathrm{Fix}(G_\ell : S(X/C_p) \to S(X/C_p))$ *has dimension $n-2$. After replacing h by a generator of the same group, there is $b \in \mathbb{F}_p^* \setminus \{1, -1\}$ such that the only nonempty phase families are*

$$F_1, F_{-1}, F_b, F_{-b},$$

where

$$\dim F_1 = \dim F_{-1} = n - 2, \qquad \dim F_b = \dim F_{-b} = 1.$$

The first two are mod-p cohomology $(n-2)$-spheres, the last two are circles. Their quotients have respectively the mod-p cohomology of $\mathbb{CP}^{m-1}$ and of a point.

Proof. Let $\rho : E \to SM$ be the tangent covering. For $a \in \mathbb{F}_p^*$,

$$\mathrm{Fix}(\widehat{h}^{-a} G_\ell) = F_{a^{-1}}, \qquad \rho^{-1}\,\mathrm{Fix}(G_\ell : SM \to SM) = \coprod_{c \neq 0} F_c.$$

Indeed, $G_\ell v = \widehat{h}^{\,a} v$ is equivalent to $G_{a^{-1}\ell} v = \widehat{h} v$, since $G_{2\pi} = \widehat{h}^{\,p} = I$. The case $a = 0$ is excluded by the prime period 2π. The deck group preserves each F_c. The nonempty F_c are connected; their quotients are precisely the connected components of the fixed set in SM, with the same dimensions. The largest part of the partition in Theorem 30.2 is therefore m. Its parts are positive and sum to $m+1$, so they are m and 1. If $\dim F_c = n-2$, replacing h by $h^{c^{-1}}$ multiplies every phase by c^{-1}. The phases of the $(n-2)$-dimensional families become $1, -1$, and the remaining phases are $b, -b$ with $b \neq \pm 1$. Lemma 30.1 gives the cohomology statements. A connected closed one-dimensional manifold is a circle, and its free circle quotient is a point. □

Theorem 30.4. *In the setting of Theorem 30.3, put $|c|_p = \min\{r, p-r\}$ for $r \in \{0, \dots, p-1\}$ representing c. Then for every $j \in \{1, \dots, p-1\}$,*

$$\min_{x \in X} d(x, h^j x) = \ell \min\{|j|_p, |jb|_p\}.$$

Proof. Put $a_j = \min\{|j|_p, |jb|_p\}$. For $c \in \{1, b\}$, an arc on a geodesic in F_c or F_{-c} joins a point x to $h^j x$ with length $\ell |jc|_p$. Thus the minimum displacement L_j satisfies $L_j \leq \ell a_j$. The isometry h^j is fixed-point-free. Lemma 13.1 gives $L_j > 0$ and extends every minimizing segment attaining L_j to an h^j-invariant oriented geodesic. Its initial unit state v satisfies $\widehat{h}^{\,j} v = G_{L_j} v$. Since j is a unit modulo p, this geodesic is also h-invariant. Thus $v \in F_c$ for some $c \in \{1, -1, b, -b\}$, and

$$L_j \equiv jc\ell \pmod{2\pi}.$$

Every positive representative of this class is at least $\ell |jc|_p \geq \ell a_j$. Hence $L_j = \ell a_j$. □

Corollary 30.5. *In the setting of Theorem 30.3, let $N_0 = \pi(F_1) = \pi(F_{-1})$ and $C_0 = \pi(F_b) = \pi(F_{-b})$, where $\pi : SX \to X$ is evaluation. For every $j \in \mathbb{F}_p^*$, the two numbers $|j|_p$ and $|jb|_p$ are distinct, and*

$$\operatorname*{argmin}_{x \in X} d(x, h^j x) = \begin{cases} N_0, & |j|_p < |jb|_p, \\ C_0, & |jb|_p < |j|_p. \end{cases}$$

In particular, no nonidentity element of the deck group is a Clifford translation. These identities hold for every $p \geq 5$.

Proof. Equality of the two least absolute residues would give $jb \equiv \pm j \pmod{p}$, contrary to $b \not\equiv \pm 1$. Put $a_j = \min\{|j|_p, |jb|_p\}$ and let A_j be the indicated set N_0 or C_0. Every $x \in A_j$ lies on an invariant geodesic whose shorter arc from x to $h^j x$ has length ℓa_j. By Theorem 30.4, this is the global minimum. Conversely, the phase calculation in that theorem's proof places the initial state of any globally minimizing segment in some F_c and gives its length at least $\ell|jc|_p$. Since ℓa_j is the minimum and the two residues are distinct, c belongs to the pair defining A_j. Its initial point therefore lies in A_j.

Each A_j is the image of a smooth map from a compact manifold of dimension at most $n-2$. Every point of the domain is critical, so Sard's theorem [41, Chapter 3, Theorem 1.3] gives that A_j has zero n-dimensional measure. Thus $A_j \neq X$. A constant-displacement isometry has all of X as its minimum set, so no h^j, $j \neq 0$, is a Clifford translation. □

30.3. **Embedded minimum sets.**

Lemma 30.6. *Let $b \in \mathbb{F}_p^* \setminus \{1, -1\}$. If $p \geq 11$, there is $k \in \{2, 3\}$ such that*

$$k \leq (p-1)/2, \qquad |kb|_p \geq k.$$

One can take $k = 2$ unless $b = \pm 2^{-1}$; in that case one can take $k = 3$. For $p = 7$, $k = 2$ satisfies these inequalities when $b = \pm 2$, whereas neither $k = 2$ nor $k = 3$ does when $b = \pm 3$. For $p = 5$ and $b = \pm 2$, no such integer k exists.

Proof. Since $2b \neq 0$, the inequality $|2b|_p < 2$ is equivalent to $2b = \pm 1$. In this case

$$|3b|_p = (p-3)/2 \geq 3 \qquad (p \geq 11).$$

Both 2 and 3 are at most $(p-1)/2$ for $p \geq 11$. For the remaining primes,

$$|2 \cdot 2|_5 = 1, \qquad |2 \cdot 2|_7 = 3, \qquad |2 \cdot 3|_7 = 1, \qquad |3 \cdot 3|_7 = 2.$$

□

Theorem 30.7. *In the setting of Theorem 30.3, suppose there is an integer $2 \leq k \leq (p-1)/2$ with $|kb|_p \geq k$. Then evaluation*

$$\pi_1 = \pi|_{F_1} : F_1 \longrightarrow X$$

is a smooth embedding. Its image is exactly

$$N = \{x \in X : d(x, hx) = \ell\}.$$

The manifold N has codimension two and is a mod-p cohomology sphere. It carries a smooth unit field V whose orbits are ambient geodesics of prime period 2π. The form $\alpha = g(V, \cdot)|_{TN}$ is contact with Reeb field V. The time-ℓ map of V is $h|_N$.

The squared displacement $f(x) = d(x, hx)^2$ is smooth in a neighborhood of N. It is Morse–Bott there, with critical manifold N and a positive definite Hessian normal to N.

These conclusions hold for every $p \geq 11$. For these primes, evaluation also embeds the circle F_b as a closed geodesic in X.

Proof. For $v \in F_1$ based at x, its geodesic segment of length $k\ell$ joins x to $h^k x$. Since $|k|_p = k$ and $|kb|_p \geq k$, Theorem 30.4 gives

$$d(x, h^k x) = k\ell = \min_{y \in X} d(y, h^k y).$$

This segment, and each of its subsegments, minimizes distance.

Suppose $v, w \in F_1$ have the same base point x. Concatenate the first length-ℓ segment of the geodesic of v with the h-translate of the first length-$(k-1)\ell$ segment of the geodesic of w. This path joins x to $h^k x$ and has length $k\ell$, so it minimizes distance. Its tangent vectors at the junction hx are $dh(v)$ and $dh(w)$. A minimizing path has no corner; hence $v = w$.

For a vertical vector $\xi \in T_v F_1$, the associated Jacobi field satisfies $J(0) = J(\ell) = 0$. Apply Lemma 13.4 to the minimizing segment of length $T = k\ell$, with $t_0 = \ell < T$. It gives $J = 0$, hence $\xi = 0$. Thus π_1 is an injective immersion. Since F_1 is compact, it is an embedding.

Every point of $\pi(F_1)$ has displacement ℓ. Conversely, $\ell = \min_y d(y, hy)$ by Theorem 30.4. Lemma 13.1 therefore gives $G_\ell v = \widehat{hv}$ for the unit initial vector of any minimizing segment from $x \in N$ to hx. Hence $v \in F_1$ and $\pi(F_1) = N$. The dimension and cohomology of N are those of F_1.

Let R_λ be the geodesic vector field on SX. Since F_1 is flow-invariant, $V = (\pi_1)_*(R_\lambda|_{F_1})$ is a smooth unit field on N. Its flow is

$$\Phi_t = \pi_1 \circ G_t \circ \pi_1^{-1}, \qquad \Phi_\ell = h|_N.$$

Thus its orbits are ambient geodesics of prime period 2π. For the canonical contact form λ on SX,

$$\pi_1^*\big(g(V, \cdot)|_{TN}\big) = \lambda|_{F_1}.$$

The latter is contact by Lemma 30.1; its Reeb field is $R_\lambda|_{F_1}$.

For $x \in N$, the segment from x to hx extends to a minimizing segment of length $k\ell > \ell$. Hence hx lies outside the cut locus of x. On some open neighborhood $U \subset X$ of N, the function $L(y) = d(y, hy)$ and the vector field $W_y = \exp_y^{-1}(hy)$ are therefore smooth [63, proof of Lemma 1.1.2, p. 413]. The minimizing segments are

$$\gamma_y(t) = \exp_y(t\, W_y/\ell), \qquad 0 \leq t \leq \ell.$$

Fix $x \in N$, put $v = \pi_1^{-1}(x)$, and write $\gamma = \gamma_x$. Let D_t denote covariant differentiation along γ, and take the curvature convention $D_t^2 J + R(J, \dot\gamma)\dot\gamma = 0$ for Jacobi fields. For $w \in T_x X$, put $\beta(s) = \exp_x(sw)$ and choose $\varepsilon > 0$ such that $\beta(s) \in U$ for $|s| < \varepsilon$. Put

$$J_w(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_{\beta(s)}(t).$$

Both β and $h \circ \beta$ are geodesics. The Jacobi field J_w satisfies

$$J_w(0) = w, \qquad J_w(\ell) = dh(w).$$

The segment γ_y has constant speed $L(y)/\ell$, and γ has unit speed. Since $dL_x = 0$,

$$\langle D_t J_w(t), \dot\gamma(t)\rangle = \ell^{-1} dL_x(w) = 0 \qquad (0 \leq t \leq \ell).$$

With

$$I(Y, Z) = \int_0^\ell \big(\langle D_t Y, D_t Z\rangle - \langle R(Y, \dot\gamma)\dot\gamma, Z\rangle\big)\, dt,$$

the unit-speed specialization of [63, (1.3.1), p. 416] gives

$$\mathrm{Hess}_x\, L(w, w) = I(J_w, J_w).$$

Since $f = L^2$, $L(x) = \ell$, and $dL_x = 0$, polarization gives

$$\mathrm{Hess}_x\, f(w, z) = 2\ell I(J_w, J_z).$$

Integration by parts gives

$$I(J_w, J_z) = \langle dh^{-1} D_t J_w(\ell) - D_t J_w(0), z \rangle.$$

The Levi–Civita connection identifies $T_v TX$ with $T_x X \oplus T_x X$. The vector $\xi_w = (w, D_t J_w(0))$ belongs to $T_v SX$, since $\langle D_t J_w(0), v \rangle = 0$. For $K = \widehat{h}^{-1} G_\ell$, one has

$$dK_v(\xi_w) = \big(w, dh^{-1} D_t J_w(\ell)\big).$$

Consequently

$$w \in \ker \mathrm{Hess}_x f \quad \Longleftrightarrow \quad dK_v(\xi_w) = \xi_w.$$

The maps $\widehat{h}$ and G_ℓ commute, and $K^p = I$. Consequently

$$\ker(dK_v - I) = T_v F_1$$

by the fixed-set theorem for an invariant auxiliary metric [51, Chapter II, Theorem 5.1, p. 59]. It follows that $\ker \mathrm{Hess}_x f \subset T_x N$. Since $f \geq \ell^2$ and $f|_N = \ell^2$, the Hessian is nonnegative and vanishes on $T_x N$. Hence

$$\ker \mathrm{Hess}_x f = T_x N,$$

and its restriction to the normal space is positive definite. In coordinates (y, z) with $N = \{z = 0\}$, one has $\partial_z f(y, 0) = 0$ and $D_z \partial_z f(y, 0) > 0$. The implicit function theorem gives $\{\partial_z f = 0\} = \{z = 0\}$ near each point of N. Thus N is the critical set near N, and f is Morse–Bott there.

For $p \geq 11$, Lemma 30.6 supplies k. Put $a = b^{-1} \in \mathbb{F}_p^*$ and $h' = h^a$. On F_b one has $\widehat{h}' = G_\ell$, and Theorem 30.4 gives, for $1 \leq j < p$,

$$\min_{y \in X} d(y, (h')^j y) = \ell \min\{|ja|_p, |j|_p\}.$$

Since $a \neq \pm 1$, Lemma 30.6 gives an integer $2 \leq k' \leq (p-1)/2$ with $|k'a|_p \geq k'$. Every length-$k'\ell$ arc from F_b therefore minimizes distance. The preceding injectivity and immersion arguments, with (h', F_b, k') in place of (h, F_1, k), show that evaluation embeds F_b. By Theorem 30.3, F_b is one orbit of the geodesic circle, so its image is a closed geodesic. □

Corollary 30.8. *Under the embedding hypothesis of Theorem 30.7,*

$$\widetilde{H}_i(X \setminus N; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Proof. The manifold X is an oriented integral homology sphere, and N is a compact locally contractible subspace. Integral Alexander duality [37, Theorem 3.44 and the proof of Corollary 3.45], followed by the universal coefficient theorem, gives

$$\widetilde{H}_i(X \setminus N; \mathbb{F}_p) \cong \widetilde{H}^{n-i-1}(N; \mathbb{F}_p).$$

Since N is a mod-p cohomology $(n-2)$-sphere, the latter group is $\mathbb{F}_p$ for $i = 1$ and zero otherwise. □

Lemma 30.9. *Let $p \geq 11$ be prime and $b \in \mathbb{F}_p^* \setminus \{\pm 1\}$. Then the function*

$$\delta(j) = \min\{|j|_p, |bj|_p\}, \qquad j \in \mathbb{F}_p,$$

is not subadditive. More precisely, some r with $\delta(r) = 1$ and some $s \in \{1, -1\}$ satisfy $\delta(r + s) > 2$.

Proof. Changing b to $-b$ does not change δ. Put

$$a = |b|_p, \qquad r = |a^{-1}|_p, \qquad ar \equiv \varepsilon \pmod{p}, \quad \varepsilon \in \{1, -1\}.$$

Then $2 \leq a, r \leq (p-1)/2$ and $\delta(r) = \delta(1) = 1$. If $r \geq 4$, take $s = \varepsilon$. Since $3 \leq r + s \leq (p+1)/2$ and $3 \leq a + 1 \leq (p+1)/2$,

$$|r + s|_p \geq 3, \qquad |a(r+s)|_p = |a+1|_p \geq 3.$$

Thus $\delta(r+s) \geq 3$.

If $r = 3$ and $\varepsilon = 1$, take $s = 1$; then $|4|_p \geq 3$ and $|a(r+s)|_p = |a+1|_p \geq 3$. If $r = 3$ and $\varepsilon = -1$, then $a \geq 4$: the possibilities $a = 2, 3$ would make p divide 7 or 10. Again $s = 1$ gives

$$|r+s|_p = |4|_p \geq 3, \qquad |a(r+s)|_p = |a-1|_p = a-1 \geq 3.$$

Finally, if $r = 2$, then $4 \leq 2a \leq p-1$ excludes $2a \equiv 1 \pmod{p}$. Hence $2a = p-1$ and, for $s = 1$,

$$\delta(r+s) = \delta(3) = \min\{3, (p-3)/2\} = 3.$$

In each case $\delta(r+s) > \delta(r) + \delta(s) = 2$. □

Theorem 30.10. *In the setting of Theorem 30.3, if $p \geq 7$, the sets N_0 and C_0 are disjoint. For $p \geq 11$, put $N = N_0$ and $C = C_0$; these are embedded submanifolds. With any orientations on N and C, their integer linking number in the oriented homotopy sphere X satisfies*

$$\operatorname{lk}(C, N) \not\equiv 0 \pmod{p}.$$

Proof. Suppose $x \in N_0 \cap C_0$. Arcs on the two geodesics through x give $d(x, h^j x) \leq \ell\delta(j)$, while Theorem 30.4 gives the reverse inequality. Thus

$$d(h^u x, h^v x) = \ell\delta(v-u) \qquad (u, v \in \mathbb{F}_p).$$

For $p \geq 11$, the triangle inequality would make δ subadditive, contrary to Lemma 30.9.

For $p = 7$, replacing b by $-b$ if necessary gives $b \in \{2, 3\}$. If $b = 2$, put $c = 2$ and $j = 4$; if $b = 3$, put $c = -3 = 4$ and $j = 2$. In both cases

$$cj = 1 \text{ in } \mathbb{F}_7, \qquad \delta(1) = \delta(j) = 1, \qquad \delta(1+j) = 2.$$

Choose $v \in F_1$ and $w \in F_c$ based at x. Their length-ℓ segments join x to hx and $h^j x$, respectively. Concatenating the first segment with the h-translate of the second gives a path from x to $h^{1+j}x$ of length $2\ell = d(x, h^{1+j}x)$. It has no corner at hx, so $dh_x v = dh_x w$, and hence $v = w$. But $F_1 \cap F_c = \emptyset$: otherwise $G_{(c-1)\ell} v = v$, contrary to the prime period 2π. This proves disjointness.

Suppose now $p \geq 11$. By Theorem 30.7, N and C are embedded, and C is a circle. They are invariant under C_p. Set $Z = X \setminus N$. Choose an invariant closed tubular neighborhood U of N, disjoint from C, and put $K = X \setminus \operatorname{int} U$. Then K is a compact manifold with boundary, and Z retracts equivariantly onto K. Corollary 30.8 shows that K is a mod-p cohomology circle. The actions on C and K are free and induce the identity on their one-dimensional $H^1(-; \mathbb{F}_p)$, since $\operatorname{Hom}(C_p, \mathbb{F}_p^*) = 0$.

Choose generators $a_C \in H^1(C; \mathbb{F}_p)$ and $a_K \in H^1(K; \mathbb{F}_p)$. These determine $\mathbb{F}_p$-orientations of the Borel fibrations [47, Proposition 1.3, p. 281]; write $\eta_C, \eta_K \in H^2(BC_p; \mathbb{F}_p)$ for their Euler classes. By [47, Theorem 2.3, pp. 283–284], $d_2 a_K = \eta_K$. If $\eta_K = 0$, the Borel spectral sequence of K, whose only nonzero rows have degrees zero and one, would collapse. Its base row would then give nonzero cohomology in arbitrarily large degrees. This contradicts $K_{C_p} \simeq K/C_p$, since the latter is a compact manifold with boundary. Thus $\eta_K \neq 0$.

For the equivariant inclusion $i : C \hookrightarrow K$, define $D \in \mathbb{F}_p$ by $i^* a_K = D a_C$. The equivariant degree formula [47, Theorem 1.9, p. 282] gives

$$D\eta_C = \eta_K \neq 0.$$

Hence $D \neq 0$, and $i_* : H_1(C; \mathbb{F}_p) \to H_1(K; \mathbb{F}_p)$ is nonzero.

The contact form orients the connected manifold N. Integral Alexander duality [37, Theorem 3.44, pp. 254–255], together with the exact sequence of (X, Z), gives

$$H_1(Z; \mathbb{Z}) \cong H^{n-2}(N; \mathbb{Z}) \cong \mathbb{Z}.$$

Let $\mu \in H_1(Z; \mathbb{Z})$ be the meridian generator, represented by the boundary of a positively oriented normal two-disk to N. The inclusion $C \hookrightarrow Z$ sends its oriented fundamental class to $\operatorname{lk}(C, N)\mu$. Reduction identifies $H_1(Z; \mathbb{Z}) \otimes \mathbb{F}_p$ with $H_1(Z; \mathbb{F}_p)$. Since the retraction $Z \to K$ fixes C, the nonzero homomorphism above gives $\operatorname{lk}(C, N) \not\equiv 0 \pmod{p}$. Reversing either orientation changes only the sign. □

Proposition 30.11. *Under* (30.1)*, let* $c \in \mathbb{F}_p^*$ *with* $F_c \neq \emptyset$*, and write* $\dim F_c = 2d_c - 1$*. Then, for every* $v \in F_c$*,*

$$\dim \ker(d\pi|_{T_v F_c}) \leq \min\left\{ d_c - 1, \left\lfloor \frac{n-1}{p-1} \right\rfloor \right\}.$$

In particular evaluation is an immersion whenever $p > n$*.*

Proof. Replace h by $h^{c^{-1}}$, so that F_c becomes the phase-1 family. For $v \in F_1$, let γ be its unit-speed geodesic and put $r = \dim \ker(d\pi|_{T_v F_1})$. A vertical vector $\xi \in T_v F_1$ gives a normal Jacobi field with

$$J_\xi(0) = 0, \qquad J_\xi(t + \ell) = dh\, J_\xi(t).$$

The map $\xi \mapsto J_\xi$ is injective by uniqueness for the Jacobi equation. Thus each of the distinct interior instants $j\ell$, $1 \leq j \leq p-1$, has conjugate multiplicity at least r. The Morse index theorem [61, Theorem 15.1, p. 83] and Lemma 11.2 give

$$(p-1)r \leq \operatorname{ind}_D(\gamma|_{[0,2\pi]}) \leq n - 1.$$

The endpoint 2π is not included in the conjugate-point count.

The vertical subspace of $T_v E$ lies in $\ker \lambda_v$ and is isotropic for $d\lambda_v$. Its intersection with $T_v F_1$ is therefore isotropic in the symplectic space $T_v F_1 \cap \ker \lambda_v$, of dimension $2d_c - 2$. Hence $r \leq d_c - 1$ as well. If $p > n$, the integer bound is zero. □

30.4. **Intrinsic index forms.**

Lemma 30.12. *Let* $N \subset X$ *be the embedded minimum submanifold of Theorem 30.7, with its induced metric. Let* $\gamma : [0, T] \to N$*,* $T > 0$*, be an orbit arc of* V*, and let* I_N *and* I_X *denote its intrinsic and ambient index forms. They agree on* $H^1([0, T], \gamma^* TN)$*:*

$$I_N(W, Z) = I_X(W, Z).$$

Consequently the endpoint-fixed intrinsic index on a full period is at most the ambient endpoint-fixed index. If $\gamma|_{[0,T]}$ *minimizes ambient distance, an intrinsically normal Jacobi field vanishing at both ends of a proper subinterval* $[a, b] \subsetneq [0, T]$*, with* $a < b$*, vanishes identically.*

Proof. Write II for the second fundamental form of $N \subset X$, and D_t^N, D_t^X and R_N, R_X for the intrinsic and ambient covariant derivatives and curvature tensors. Since the orbit is an ambient geodesic, $D_t^N V = 0$ and $\mathrm{II}(V, V) = 0$. Take the convention that $\langle R(W, V)V, W\rangle$ is positive on the round sphere. The Gauss equation and the Gauss formula along a curve [58, Theorem 8.4, p. 136; Lemma 8.5, p. 138] give

$$\langle R_N(W, V)V, W\rangle = \langle R_X(W, V)V, W\rangle - |\mathrm{II}(V, W)|^2,$$

whereas

$$|D_t^X W|^2 = |D_t^N W|^2 + |\mathrm{II}(V, W)|^2.$$

Subtracting the curvature terms gives

$$|D_t^N W|^2 - \langle R_N(W, V)V, W\rangle = |D_t^X W|^2 - \langle R_X(W, V)V, W\rangle.$$

Integration and polarization give the equality for smooth fields; continuity extends it to H^1 fields. The intrinsic Dirichlet form domain is a subspace of the ambient one, which gives the index inequality.

Ambient minimality makes the common Dirichlet form nonnegative on every subinterval of $[0, T]$. If $b < T$, apply Lemma 13.4 to the intrinsic form on $[a, T]$. If $b = T$, then $a > 0$; apply that lemma to the reversed interval $[0, T]$. In either case the Jacobi field is zero. □

31. Spectral comparison at prime return times

Let $m \in \mathbb{Z}$, $m \geq 2$, and let $p \geq 5$ be prime. Put $n = 2m + 1$ and $\ell = 2\pi/p$. Fix an integer a with $2 \leq a \leq (p-1)/2$, and put

$$Q = L(p; 1^m, a), \qquad \zeta = e^{2\pi i/p}.$$

For a closed Riemannian manifold Y, let Δ_Y be its nonnegative scalar Laplacian and put $P_Y = \sqrt{\Delta_Y + m^2}$. Let M be a closed connected Riemannian manifold, and put $R_K = m + K + 1/2$ for $K \in \mathbb{Z}_{\geq 0}$.

Proposition 31.1. *Suppose that, for a fixed $0 \leq \eta < 1/4$ and a sequence of integers $K_\nu \geq 0$ tending to infinity,*

$$E_\eta(R_{K_\nu}; P_M, P_Q) = o(R_{K_\nu}).$$

Then $\dim M = n$*, and the universal Riemannian cover $X \to M$ is a closed smooth homotopy n-sphere satisfying* (30.1)*. Its deck group is cyclic of order p. There is a self-adjoint integer quantizer A strongly commuting with P_M, with $A - P_M \in \Psi^{-1}_{\mathrm{cl}}$, such that, for $R_\nu = R_{K_\nu}$,*

$$\mathrm{Tr}\big(e^{ij\ell A}\chi(A/R_\nu)\big) - \mathrm{Tr}\big(e^{ij\ell P_Q}\chi(P_Q/R_\nu)\big) = o(R_\nu) \tag{31.1}$$

for every $j \in \{0, \dots, p-1\}$ and $\chi \in C_c^\infty((0,1))$.

Proof. Theorem 15.6, with $q = p$, gives $\dim M = n$, a smooth homotopy-sphere cover X, covering degree p, prime period 2π, and volume V_n. A group of prime order is cyclic. The error is also $o(R_\nu^n)$, so Proposition 10.1 applies with lattice spacing one. Its quantizer A satisfies (10.3), which gives

$$\big|\mathrm{Tr}(e^{itA}\chi(A/R_\nu)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_\nu))\big| \leq \|\chi\|_\infty\big(E_\eta(R_\nu; P_M, P_Q) + O(1)\big) = o(R_\nu).$$

Take $t = j\ell$. □

For the rest of this section, let X, h, $M = X/C_p$, and the families F_c, B_c be as in Section 30. In particular, (30.1) holds. Suppose that a self-adjoint operator A strongly commuting with P_M satisfies

$$\mathrm{Spec}\, A \subset \mathbb{Z}, \qquad A - P_M \in \Psi^{-1}_{\mathrm{cl}},$$

and that (31.1) holds along $R_\nu \to \infty$.

Corollary 31.2. *The largest component of* $\mathrm{Fix}(G_\ell : SM \to SM)$ *has dimension $n-2$. Consequently, after a change of deck generator, the nonempty families are $F_{\pm 1}$ and $F_{\pm b}$, with dimensions $n-2, n-2, 1, 1$, where $b \not\equiv 0, \pm 1 \pmod p$.*

Proof. Lemma 15.1, with $L = 2\pi$ and $s = p$, gives $\mathrm{Fix}(G_\ell : SM \to SM) \neq \emptyset$. Let d be the maximum dimension of its components. For a fixed nonzero $0 \leq \chi \in C_c^\infty((0,1))$, Lemma 15.2 gives

$$\mathrm{Tr}(e^{i\ell A}\chi(A/R)) = c_M R^{(d+1)/2} + o(R^{(d+1)/2}), \qquad c_M \neq 0.$$

For every nonidentity deck power on the round reference, the two rotation blocks have distinct eigenvalue pairs, since $a \neq \pm 1$ modulo p. By [69, Theorem 1, p. 291; Lemma 1, p. 294], the length-ℓ return components are the two Hopf families on S^{2m-1} and the two orientations of S^1, modulo C_p. Applying the same reciprocal-trace lemma to P_Q gives

$$\mathrm{Tr}(e^{i\ell P_Q}\chi(P_Q/R)) = c_Q R^m + o(R^m), \qquad c_Q \neq 0.$$

If $(d+1)/2 \neq m$, divide (31.1) at $j = 1$ by $R_\nu^{\max\{(d+1)/2, m\}}$. The difference of the two expansions has a nonzero limit, whereas the right side tends to zero. Hence $d = 2m - 1 = n - 2$. Theorem 30.3 gives the four families. □

31.1. **The transverse determinant.** Choose the generator h as in Corollary 31.2. Let $F = F_1$, $B = B_1 = F/S^1$, and let $p_F : F \to B$ be the circle projection. Write $\alpha = \lambda|_F$ and $\omega_B = \omega|_B$, so that $p_F^*\omega_B = d\alpha$. Orient F by $\alpha \wedge (d\alpha)^{m-1}$. The Reeb flow on F has prime period 2π. The integral Euler class $e_B = e|_B$ has real image $[\omega_B/(2\pi)]$. With the symplectic orientation of B, put

$$D = \langle e_B^{m-1}, [B] \rangle \in \mathbb{Z}_{>0}. \tag{31.2}$$

Lemma 31.3. *Let Y be a closed Riemannian manifold, let α be the canonical contact form on SY, and suppose $G_T = I$ for some $T > 0$. Let $t \in T\mathbb{Q}$ and let $C \subset \mathrm{Fix}(G_t)$ be a connected component of dimension $2d + 1$. Then C is contact. For $\rho \in C$, put*

$$\mathcal{N}_\rho = (T_\rho C \cap \ker \alpha_\rho)^{\perp_{d\alpha}} \subset \ker \alpha_\rho, \qquad S_\rho = dG_t(\rho)|_{\mathcal{N}_\rho}.$$

The map S_ρ has finite order and no eigenvalue one. The Duistermaat–Guillemin density on C is

$$dv_C = \frac{|\alpha \wedge (d\alpha)^d|/d!}{\sqrt{\det_{\mathbb{R}}(I - S_\rho)}}. \tag{31.3}$$

The denominator is positive and constant on C.

Proof. Identify SY with S^*Y by the metric. Choose an integer $q_0 \geq 1$ with $q_0 t \in T\mathbb{Z}$. Then $G_t^{q_0} = I$. By Section 10.1, the fixed cone $\mathcal{C} = \{r\rho : r > 0,\ \rho \in C\}$ is a clean symplectic submanifold of $T^*Y \setminus 0$, and C is contact. Let ω be the canonical symplectic form on T^*Y. At $\rho \in C$, put $L = dG_t(\rho)$ and $W = \ker(I - L) = T_\rho\mathcal{C}$. Since $L^{q_0} = I$, the map L is semisimple and

$$T_\rho(T^*Y) = W \oplus W^\omega, \qquad (I - L)|_{W^\omega} \text{ is invertible.}$$

Both summands are L-invariant. The radial field and the Reeb field span a symplectic plane in W; its symplectic complement is $\ker \alpha_\rho$. Hence $W^\omega = \mathcal{N}_\rho$, and $I - L$ induces $I - S_\rho$ on $T_\rho(T^*Y)/W$. Put $\omega_C = \omega|_C = d(r\alpha)$. With symplectic density $|\omega_C^{d+1}|/(d+1)!$, the clean density formula [21, Lemma 4.4, p. 60], in the normalization of [76, Lemma 2.3, p. 8], gives

$$dv_C = \frac{|\omega_C^{d+1}|/(d+1)!}{\sqrt{\det_{\mathbb{R}}(I - S_\rho)}}.$$

Moreover,

$$\frac{\omega_C^{d+1}}{(d+1)!} = r^d\, dr \wedge \frac{\alpha \wedge (d\alpha)^d}{d!}.$$

Put $q(y, \xi) = |\xi|$. On the cone, $q = r$, so division by $|dq| = |dr|$ at $r = 1$ gives (31.3), by [21, p. 60]. The eigenvalues of S_ρ are q_0th roots of unity different from one. Their multiplicities are locally constant, hence constant on C. Nonreal eigenvalues occur in conjugate pairs, and the only possible real eigenvalue is -1. Thus $\det_{\mathbb{R}}(I - S_\rho) > 0$ and is constant on C. □

Let S be the restriction of dG_t to the symplectic normal bundle of the image of F in $S(X/C_p)$. This bundle has real rank $2n - 2 - (\dim F - 1) = 2(m + 1)$. Choose $w_1, \dots, w_{m+1} \in \mathbb{F}_p^*$ so that

$$\zeta^{w_1}, \zeta^{-w_1}, \dots, \zeta^{w_{m+1}}, \zeta^{-w_{m+1}}$$

are the eigenvalues of $S \otimes \mathbb{C}$. Write $[w]$ for the image of $w \in \mathbb{F}_p^*$ in $\mathbb{F}_p^*/\{\pm 1\}$. The multiset $\{[w_1], \dots, [w_{m+1}]\}$ is independent of the ordering and the choice of one eigenvalue from each conjugate pair.

Theorem 31.4. *Under* (31.1)*, one has*

$$D = 1, \qquad \{[w_1], \dots, [w_{m+1}]\} = \{[2]^{m-1}, [a-1], [a+1]\}. \tag{31.4}$$

Thus the complexified characteristic polynomial of the symplectic normal return agrees with that of the round reference along a $(2m-1)$-dimensional return component. The equality includes all collisions among the displayed residue classes and is valid also for $p = 5$ and $p = 7$.

Proof. Integration along the upstairs circle gives

$$\int_F \frac{\alpha \wedge (d\alpha)^{m-1}}{(m-1)!} = \frac{(2\pi)^m}{(m-1)!} D.$$

The deck group acts along these circles, so passing to F/C_p divides this integral by p. Lemma 31.3 therefore gives

$$\int_{F/C_p} dv = \frac{(2\pi)^m D}{p(m-1)! \prod_{i=1}^{m+1} |1-\zeta^{w_i}|}. \tag{31.5}$$

Reversal interchanges the images of F_1 and F_{-1}, preserving their positive densities and the absolute normal determinants. Both periodic Morse indices are zero, since their length is ℓ. Their leading clean-trace contributions consequently add with the same phase. The two one-dimensional components have leading order R, below R^m because $m \geq 2$.

The proof of Lemma 15.2, with $L = 2\pi$ and $s = p$, gives the same leading coefficient for A and P_M, since their principal and subprincipal symbols agree. Equation (31.1) therefore equates the leading R^m coefficients for P_M and P_Q.

For the round reference, the corresponding family is the Hopf bundle $S^{2m-1} \to \mathbb{CP}^{m-1}$. Its positive Euler class is the integral generator, so $D_Q = 1$. Apply [69, Lemma 6, p. 299] at length ℓ, with weight 1 of multiplicity m and weight a of multiplicity one. The normal eigenvalue pairs have weights

$$[2] \quad (m-1 \text{ times}), \qquad [a-1], \qquad [a+1].$$

The remaining eigenvalue one has multiplicity $2(m-1)$; it is tangent to the fixed family in the contact hyperplane. The pairs with weights ± 2 belong to the tangential summand of [69, Theorem 2, p. 292], but are normal to the fixed family and occur in (31.5).

For the signed deck weights, identify the round geodesic space with the quadric of isotropic lines by $(x, v) \mapsto [x - iv]$. The geodesic flow acts on $x - iv$ by e^{it}, and for $z = (x - iv)/\sqrt{2}$ one has $\mathrm{Im}(\bar{z} \cdot dz) = v \cdot dx = \alpha$. Thus the complex structure on the quadric is compatible with the positive reduced symplectic form. At a line of deck weight 1, the projective tangent weights are

$$0 \quad (m-1 \text{ times}), \qquad -2 \quad (m \text{ times}), \qquad a-1, \qquad -a-1.$$

The differential of the quadric equation removes one direction of weight -2. Hence the signed normal weights of the deck action on the geodesic space are

$$-2 \quad (m-1 \text{ times}), \qquad a-1, \qquad -a-1.$$

The quotient return at time ℓ lifts to $\widehat{h}^{-1} G_\ell$ and induces the inverse deck map on the geodesic space. Its signed normal weights are therefore the negatives of these deck weights; their classes modulo sign agree.

Comparing (31.5) on the two sides, all universal factors and the common index phase cancel. Put $u = (2, \dots, 2, a-1, a+1)$, a list of length $m+1$. Then

$$\frac{D}{\prod_i |1-\zeta^{w_i}|} = \frac{1}{\prod_i |1-\zeta^{u_i}|}.$$

Squaring gives an equality in $K = \mathbb{Q}(\zeta)$:

$$D^2 \prod_i (1-\zeta^{u_i})(1-\zeta^{-u_i}) = \prod_i (1-\zeta^{w_i})(1-\zeta^{-w_i}). \tag{31.6}$$

Write $N_{K/\mathbb{Q}}$ for the field norm and Φ_p for the pth cyclotomic polynomial. For every nonzero $s \pmod p$, $N_{K/\mathbb{Q}}(1-\zeta^s) = \Phi_p(1) = p$. Taking norms in (31.6) and cancelling the $2(m+1)$ identical factors gives $D^{2(p-1)} = 1$. Since $D > 0$, $D = 1$.

For $s \in \mathbb{F}_p^*$, let c_s be the multiplicity of s in the signed list $(w_1, -w_1, \dots, w_{m+1}, -w_{m+1})$ minus its multiplicity in the analogous list for u. Then

$$c_s = c_{-s}, \qquad \sum_s c_s = 0, \qquad \prod_s (1-\zeta^s)^{c_s} = 1.$$

Every nontrivial pth root is a Galois conjugate of ζ, so the last equality holds at every such root. Franz's independence lemma [26, §§9, 11, pp. 251–252] (see also [20, Chapter I, §5, pp. 11–12]) gives $c_s = 0$ for all s. This is exactly (31.4). The multiplicities in the signed lists include all coincidences among the u_i. □

Corollary 31.5. *For $F = F_1$ or F_{-1}, put $\alpha_F = \lambda|_F$ and give F its contact orientation. Then*

$$\int_F \frac{\alpha_F \wedge (d\alpha_F)^{m-1}}{2^{m-1}(m-1)!} = V_{2m-1}.$$

Proof. The fiber integral in the proof of Theorem 31.4, with $D = 1$, gives the value $(2\pi)^m/(2^{m-1}(m-1)!) = V_{2m-1}$ for F_1. Reversal preserves the positive contact density, so the integral has the same value on F_{-1}. □

31.2. **Annular coefficient polynomials.** Write $h_a(k)$ for the multiplicity of the frequency $k+m$ on Q and $H_a(z) = \sum_{k\ge 0} h_a(k) z^k$. Lemma 8.1 gives

$$H_a(z) = \frac{1-z^2}{p} \sum_{r=0}^{p-1} \frac{1}{(1-z\zeta^r)^m (1-z\zeta^{-r})^m (1-z\zeta^{ar})(1-z\zeta^{-ar})}. \tag{31.7}$$

For $j = 1, \dots, p-1$, let $C_{j,s}(a)$, $1 \le s \le m$, be the partial-fraction coefficients at $z = \zeta^j$:

$$H_a(z) = \sum_{s=1}^{m} \frac{C_{j,s}(a)}{(1-z\zeta^{-j})^s} + \text{a function regular at } \zeta^j.$$

Define the polynomial

$$T_{j,a}(x) = \zeta^{jm} \sum_{s=1}^{m} C_{j,s}(a) \binom{x-m+s-1}{s-1} = \sum_{r=0}^{m-1} t_{j,a,r} x^r. \tag{31.8}$$

Proposition 31.6. *For every $\chi \in C_c^\infty((0,1))$, every $j = 1, \dots, p-1$, and every $N > 0$, as $R \to +\infty$,*

$$\operatorname{Tr}\big(e^{ij\ell P_Q} \chi(P_Q/R)\big) = \sum_{r=0}^{m-1} t_{j,a,r} R^{r+1} \int_0^1 \chi(u) u^r \, du + O_{\chi,j,N}(R^{-N}).$$

The leading coefficient of $T_{j,a}$ is

$$t_{j,a,m-1} = \frac{2\zeta^{jm}}{p(m-1)!(1-\zeta^{2j})^{m-1}(1-\zeta^{(a+1)j})(1-\zeta^{(1-a)j})}. \tag{31.9}$$

Proof. At $z = \zeta^j$, the two terms $r = j, -j$ have a pole of order m. The two terms $r = a^{-1}j, -a^{-1}j$ have simple poles. They are distinct from $j, -j$ because $a \not\equiv \pm 1$. All other terms are regular. Multiplication by $(1-z\zeta^{-j})^m$ and evaluation at $z = \zeta^j$ therefore gives

$$C_{j,m}(a) = \frac{2}{p(1-\zeta^{2j})^{m-1}(1-\zeta^{(a+1)j})(1-\zeta^{(1-a)j})}.$$

This remains valid for $m = 2$, since the other poles have order one. The rational function (31.7) is proper, and every pole is a pth root of unity. For $\xi^p = 1$, let s_ξ be its pole order at ξ, with $s_\xi = 0$ if it is regular there, and let $C_{\xi,s}$ be the coefficients of $(1-z/\xi)^{-s}$ in its principal part. Partial fractions give

$$h_a(k) = \sum_{\xi^p=1} \sum_{s=1}^{s_\xi} C_{\xi,s} \binom{k+s-1}{s-1} \xi^{-k}.$$

At time $j\ell$, the phase $\zeta^{j(k+m)}$ is nonoscillatory precisely when $\xi = \zeta^j$. The proof of Lemma 8.4 applies to each of these finitely many terms: the terms with $\xi \ne \zeta^j$ are $O(R^{-N})$ for every N, and the remaining sum equals

$$\int_{\mathbb{R}} T_{j,a}(x) \chi(x/R) \, dx + O(R^{-N}).$$

Indeed $x = k + m$; since χ is supported in a compact subset of $(0, 1)$, the integers $x < m$ make no contribution for large R. Expanding $T_{j,a}$ proves the trace formula. The leading coefficient of $T_{j,a}$ is $\frac{\zeta^{jm} C_{j,m}(a)}{(m-1)!}$. □

Theorem 31.7. *Let A be a self-adjoint integer quantizer strongly commuting with P_M, with $A - P_M \in \Psi_{\mathrm{cl}}^{-1}$. Under* (31.1)*, for every $j = 1, \dots, p-1$ the coefficients of $R^m, R^{m-1}, \dots, R$ in the annular clean trace of A equal those in Proposition 31.6.*

Proof. Since $G_\ell^p = I$ and j is a unit modulo p, $\mathrm{Fix}(G_{j\ell}) = \mathrm{Fix}(G_\ell)$. Its components are clean, of dimensions $2m - 1$ and one. The operator A is scalar and classical, with principal symbol $|\xi|_g$. A finite-rank smoothing change makes A positive and leaves all annular traces unchanged for large R. The full clean trace expansion [21, Theorem 4.5, equations (4.6)–(4.7)], paired with the annular cutoff as in Section 10.1, gives constants $\tau_{j,r}$ independent of χ such that

$$\mathrm{Tr}(e^{ij\ell A}\chi(A/R)) = \sum_{r=0}^{m-1} \tau_{j,r} R^{r+1} \int_0^1 \chi(u) u^r \, du + O_{\chi,j}(1).$$

The integer exponents follow from the odd dimensions of the fixed components. The first omitted frequency-symbol term has degree -1; its annular integral is bounded because the support of χ is bounded away from zero.

Choose $0 \le \chi \not\equiv 0$. If some $\tau_{j,r} \neq t_{j,a,r}$, let r_0 be the largest such index. Subtract the expansion in Proposition 31.6 and divide by $R_\nu^{r_0+1}$. The limit is

$$(\tau_{j,r_0} - t_{j,a,r_0}) \int_0^1 \chi(u) u^{r_0} \, du \neq 0,$$

contrary to (31.1). Thus $\tau_{j,r} = t_{j,a,r}$ for every r. Since these constants are independent of χ, the equality holds for every cutoff. □

The coefficient of R is the sum of the principal contributions of the one-dimensional families and the degree-zero frequency-symbol terms of the $(2m-1)$-dimensional families.

Corollary 31.8. *Under* (31.1)*, if X is round, then X/C_p is isometric to $L(p; 1^m, a)$.*

Proof. The period normalization makes X isometric to the unit round sphere. By [45, §1, Example (ii), p. 77], a free order-p orthogonal action is a direct sum of rotation planes with nonzero weights modulo p. Group equal weights modulo sign. A block of multiplicity s gives two fixed components $\mathbb{CP}^{s-1}$ in the geodesic space. Corollary 31.2 therefore gives precisely one block of multiplicity m and one of multiplicity one. After changing the generator and conjugating individual coordinates, the quotient is $L(p; 1^m, b)$ with $2 \le b \le (p-1)/2$.

By (31.9), the absolute leading coefficient at $j = 1$ is

$$|t_{1,b,m-1}| = \frac{2}{p(m-1)![2\sin(2\pi/p)]^{m-1} 4[\sin^2(\pi b/p) - \sin^2(\pi/p)]}.$$

The leading coefficient for A equals that for P_M by the proof of Lemma 15.2. Equation (31.1) therefore gives $|t_{1,b,m-1}| = |t_{1,a,m-1}|$. The expression in brackets is strictly increasing for $2 \le b \le (p-1)/2$, so $b = a$. The normalized orthogonal representations are then conjugate. Their quotients are isometric by [45, Lemma 1.2, p. 78]. □

32. Signed normal weights and marked phases

For a symplectic vector space V, write $\mathrm{Mp}(V)$ for its metaplectic group. In a Schrödinger realization, its fourfold extension is the group $\mu_4 \mathrm{Mp}(V)$, where $\mu_4 = \{1, -1, i, -i\}$ acts by scalar multiplication [16, §11.2, p. 2455].

Lemma 32.1. *Let P be a scalar, self-adjoint, classical pseudodifferential operator of order one on half-densities over a smooth manifold Y. Assume that its principal symbol q is real and positively homogeneous of degree one, and that its invariant subprincipal symbol vanishes. Put $U(t) = \exp(-itP)$, and let Φ_t be the flow of the Hamiltonian vector field H_q of q. Suppose that, for an odd positive integer p and $\ell > 0$,*

$$\Phi_{p\ell} = \mathrm{id}, \qquad U(p\ell) = I \pmod{\Psi^{-1}}.$$

For $\rho \in \mathrm{Fix}(\Phi_\ell) \subset T^*Y \setminus 0$, the principal action of $U(\ell)$ on coherent-state profiles at ρ is the unique metaplectic lift of $L = d\Phi_\ell(\rho)$ whose pth power is the identity. It is also the unique such lift in the fourfold extension.

Proof. Since $\Phi_\ell(\rho) = \rho$, one has $L^p = I$. Let $d = \dim Y$ and let $\hbar \to 0^+$. Choose cotangent coordinates $\rho = (x_0, \xi_0)$ and $\chi_0 \in C_c^\infty(Y)$ supported in this coordinate chart, with $\chi_0 = 1$ near x_0. For $f \in \mathcal{S}(\mathbb{R}^d)$, put

$$T_\rho^\hbar f(x) = \hbar^{-d/4} \chi_0(x) e^{i\xi_0 \cdot (x - x_0)/\hbar} f\big((x - x_0)/\sqrt{\hbar}\big) |dx|^{1/2}.$$

Then

$$\lim_{\hbar \to 0^+} \|T_\rho^\hbar f\|_{L^2(Y)} = \|f\|_{L^2(\mathbb{R}^d)}. \tag{32.1}$$

The unnormalized packet $\hbar^{d/4} T_\rho^\hbar f$ belongs to the class $I^0(\{\rho\})$ of [34, Definition 4.1, p. 21, arXiv version]. We apply the coherent-state results to this packet and then multiply by $\hbar^{-d/4}$.

The trajectory $\{\Phi_t(\rho) : 0 \le t \le p\ell\}$ is compact and disjoint from the zero section. In cotangent charts covering it, the semiclassical Weyl symbol of $\hbar P$ is $q + O(\hbar^2)$. After inserting microlocal cutoffs, extend the real symbol in each chart to a proper real symbol, quadratic outside a compact set, with self-adjoint Weyl quantization. The coherent-state propagation theorem [34, Theorem 4.2, pp. 21–22, arXiv version] applies to these local extensions. The principal transport depends only on the symbol near the trajectory, so a finite subdivision of $[0, p\ell]$ gives the transport for $U(t)$.

Let J be an open time interval containing $[0, p\ell]$ on which the trajectory is defined, and let τ denote the cotangent variable dual to t. With the canonical symplectic form $d\tau \wedge dt + d\xi \wedge dx$, one has $H_q = \partial_\xi q \cdot \partial_x - \partial_x q \cdot \partial_\xi$. Put

$$\Sigma_\rho = \{(t, -q(\rho); \Phi_t \rho) : t \in J\} \subset T^*\mathbb{R} \times T^*Y.$$

Let F denote the microlocal propagator from half-densities on Y to half-densities on $J \times Y$, given by $Fu(t) = U(t)u\,|dt|^{1/2}$. Its canonical relation is parameterized by

$$(t, \rho') \longmapsto \big((t, -q(\rho'); \Phi_t \rho'), \rho'\big).$$

Projection to the input covector is $(t, \rho') \mapsto \rho'$, a submersion. Theorem 4.3 of [34, p. 22, arXiv version] therefore gives $F(\hbar^{d/4} T_\rho^\hbar f) \in I(\Sigma_\rho)$. Apply the second transport equation [34, Theorem 3.12, p. 17, arXiv version] to $-i\hbar\partial_t + \hbar P$ on half-densities on $\mathbb{R} \times Y$ along this curve. Its principal symbol $\tau + q$ vanishes on Σ_ρ, and its Hamilton field $\partial_t + H_q$ is tangent to this curve. The symplectic identification of $T_{\Phi_t \rho}(T^*Y)$ with the symplectic normal of Σ_ρ is

$$v \longmapsto [(0, -dq_{\Phi_t \rho}(v); v)].$$

It intertwines the normal transport with $d\Phi_t$. The normal bundle over the interval J admits a metaplectic structure; choose the lift equal to the identity at $t = 0$. Write $(x(s), \xi(s)) = \Phi_s \rho$ in each cotangent chart. The scalar transport factors are

$$\exp\left(\frac{i}{\hbar} \int_0^t (\xi \cdot \dot{x} - q)\, ds\right), \qquad \exp\left(-i \int_0^t \mathrm{sub}(P)(\Phi_s \rho)\, ds\right).$$

They equal one: the second by hypothesis, and the first because $\xi \cdot \partial_\xi q = q$. The remaining local profile map is the metaplectic transport determined by $d\Phi_t$.

Under a cotangent coordinate change $x = \psi(y)$, let $x_0 = \psi(y_0)$, $A = D\psi(y_0)$, and $B_{ij} = \xi_0 \cdot \partial_{ij}\psi(y_0)$. Taylor expansion of the phase and the half-density transformation give the principal change of profile

$$f(v) \longmapsto |\det A|^{1/2} e^{iv^{\mathsf{T}}Bv/2} f(Av).$$

The quadratic multiplier is a metaplectic shear. The dilation belongs to the metaplectic group when $\det A > 0$. For a reflection in one coordinate, the corresponding operator is $i\exp[-i\pi(N + \frac{1}{2})]$, where $N = \frac{1}{2}(-\partial_x^2 + x^2 - 1)$ is the one-dimensional oscillator number operator. Thus the dilation for arbitrary A belongs to the fourfold extension of [16, §11.2, p. 2455]. Expressing the initial and final profiles in the same chart therefore gives a lift $\widetilde{L}$ of L in this extension.

The symbol expansion and the norm estimate [34, Theorem 3.15, p. 18, arXiv version] give, with a compact microlocal cutoff equal to one near ρ understood on the left,

$$U(\ell) T_\rho^\hbar f = T_\rho^\hbar(\widetilde{L} f) + O_{L^2}(\hbar^{1/2}).$$

Composition of the principal profile maps at the fixed point gives the same formula for $U(p\ell)$ with $\widetilde{L}^p$. An operator of order -1 is $O_{L^2}(\hbar)$ on these microlocalized packets. The hypothesis on $U(p\ell)$ consequently gives

$$T_\rho^\hbar\big((\widetilde{L}^p - I) f\big) = O_{L^2}(\hbar^{1/2}).$$

Equation (32.1) implies $(\widetilde{L}^p - I) f = 0$ for every Schwartz profile, hence $\widetilde{L}^p = I$.

Every other lift in the fourfold extension is $c\widetilde{L}$ with $c \in \mu_4$. Since p is odd, $c \mapsto c^p$ is a bijection of μ_4, so $\widetilde{L}$ is the unique lift whose pth power is one. For either metaplectic lift T of L, one has $T^p \in \{I, -I\}$; exactly one of $T, -T$ has pth power one. It must equal $\widetilde{L}$. A change of the initial chart conjugates the profile map and preserves its pth power, so this pointwise conclusion requires no global metaplectic structure on T^*Y. □

For a metaplectic operator T covering a symplectic map with no fixed vectors, let $\Theta(T) \in \mathbb{C}^\times$ denote its oscillatory trace in the quadratic-kernel normalization of [16, §11.3 and Example 11.1, p. 2456]. Its phase is $\Theta(T)/|\Theta(T)|$.

Lemma 32.2. *Let S and S_0 be symplectic maps of real dimension $2d$, with $S^p = S_0^p = I$ for an odd prime p, and with no fixed vectors. Choose invariant compatible complex structures, and let their complex eigenvalues be*

$$e^{2\pi i r_i/p}, \qquad e^{2\pi i r_i^0/p}, \qquad r_i, r_i^0 \in \{1, \dots, p-1\}.$$

Suppose that their unsigned weight multisets agree. Let T and T_0 be their unique metaplectic lifts with $T^p = T_0^p = I$. If

$$\frac{\Theta(T)}{|\Theta(T)|} = \frac{\Theta(T_0)}{|\Theta(T_0)|},$$

then

$$\prod_{i=1}^{d} r_i = \prod_{i=1}^{d} r_i^0 \quad \text{in } \mathbb{F}_p.$$

Proof. Choose a compatible unitary basis, with symplectic form $d\xi \wedge dx$ and complex coordinate $x - i\xi$ on each plane. A rotation through $\theta = 2\pi r/p$ has continuous metaplectic lift $\exp[-i\theta(N + \frac{1}{2})]$, where $N = \frac{1}{2}(-\partial_x^2 + x^2 - 1)$ has the simple eigenvalues $0, 1, 2, \dots$. Its pth power is $(-1)^r$. The lift of S whose pth power is one is therefore the tensor product of these lifts multiplied by $(-1)^{\sum_i r_i}$.

In the oscillator representation with compact weights $-k - 1/2$, $k \in \mathbb{Z}_{\geq 0}$, the rank-one compact character formula [3, §2, p. 4; Proposition 3.18, pp. 11–12, author's version] gives

$$\Theta\big(\exp[-i\theta(N + \tfrac{1}{2})]\big) = \frac{1}{2i\sin(\theta/2)}.$$

This is Adams's convention $\psi_-(x) = e^{-2\pi i x}$, whose compact generator is $\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$. Here $0 < \theta < 2\pi$ and $\theta \neq \pi$, since p is odd, so each rank-one element lies in the regular compact set of the cited formula. The quadratic kernels and their oscillatory traces factor under tensor products. Consequently

$$\Theta(T) = \frac{(-1)^{\sum_i r_i} i^{-d}}{\prod_i 2\sin(\pi r_i/p)}. \tag{32.2}$$

The denominator is positive. Reversing the Hamiltonian or Fourier convention conjugates both characters and preserves equality of their phases.

Match the occurrences of each unsigned class in the two multisets. After permutation, each r_i is r_i^0 or $p - r_i^0$, including when a class occurs more than once. Each replacement $r_i^0 \mapsto p - r_i^0$ changes the parity of $\sum_i r_i$ and multiplies the product in $\mathbb{F}_p$ by -1. Equality of phases in (32.2) implies an even number of replacements and hence equality of the products. For $d = 0$, both products and both characters equal one. □

Corollary 32.3. *Under the hypotheses of Theorem 31.4, the product of the signed symplectic normal weights at $B_1 = F_1/S^1$ agrees with the corresponding product for the round reference. This holds for both the quotient return and the deck action on the normal bundle in the space of oriented geodesics.*

Proof. On scalar half-densities, $P_M = \sqrt{\Delta_M + m^2}$ has principal symbol $|\xi|_g$ and zero subprincipal symbol [21, p. 58]. Since A has integer spectrum and strongly commutes with P_M, (10.8) gives

$$e^{-2\pi i P_M} = e^{2\pi i (A - P_M)} = I \pmod{\Psi^{-1}}.$$

The frequencies of P_Q are integers, so $e^{-2\pi i P_Q} = I$. Both homogeneous geodesic flows have period 2π. Lemma 32.1 therefore applies to P_M and P_Q with $\ell = 2\pi/p$.

Identify SM with S^*M by the metric, and let C be the cone over the image of F_1 in SM. For $\rho \in C$, put $W = T_\rho C$. This is a symplectic space of dimension $2m$, and the derivative of the return has the invariant splitting

$$T_\rho(T^*M) = W \oplus W^\omega, \qquad dG_\ell(\rho) = I_W \oplus S.$$

By uniqueness in Lemma 32.1, its lift whose pth power is the identity is $I \otimes \widetilde{S}$, where $\widetilde{S}$ is the corresponding lift of S on W^ω.

In the clean stationary-phase formula, the transverse quadratic factor is $\Theta(\widetilde{S})$ [16, Theorems 11–12, pp. 2454–2456, and §11.4]. The fixed factor is positive. Indeed, identify W symplectically with $T^*\mathbb{R}^m$, write ω_W for its symplectic form, and let $\mathrm{Op}_\hbar^w$ denote Weyl quantization for $\hbar > 0$. For $0 \leq a_W \in C_c^\infty(W)$, $a_W \not\equiv 0$, one has

$$\mathrm{Tr}(\mathrm{Op}_\hbar^w(a_W)) = (2\pi\hbar)^{-m} \int_W a_W \, \frac{\omega_W^m}{m!} > 0.$$

Thus the phase of the local principal clean-trace coefficient is the phase of $\Theta(\widetilde{S})$.

Every length-ℓ geodesic has periodic Morse index zero, by Lemma 15.1. Conjugating the component formula in Section 10.1 gives, for the negative half-wave, the phase

$$\exp\left(-\frac{i\pi(m-1)}{2}\right)$$

on each $(2m-1)$-dimensional return component of M and Q. Their normal metaplectic traces therefore have the same phase. Their unsigned normal weight multisets agree by Theorem 31.4. Lemma 32.2, with $d = m + 1$, gives equality of the signed products for the quotient returns.

On F_1, the quotient return induces $d\bar{h}^{-1}$ on the normal space of B_1 in $\mathcal{G}$. Replacing the quotient return by the deck derivative negates all $m + 1$ weights. Both products acquire the factor $(-1)^{m+1}$, so their equality persists. □

Lemma 32.4. *Under the hypotheses of Theorem 31.4, normalize the generator so that a $(2m-1)$-dimensional family has phase 1, and let $b \in \mathbb{F}_p^* \setminus \{\pm 1\}$ be the phase of a one-dimensional family. Let*

$v \in H^2(BC_p;\mathbb{F}_p)$ be the reduction modulo p of the first Chern class of the character $h \mapsto \zeta$. Give the normal bundle of B_1 in $\mathcal{G}$ an invariant compatible complex structure, and let Π_b be the product of its signed deck weights relative to v. Then

$$\Pi_b = (-2)^{m-1}(1-b^2) \quad \text{in } \mathbb{F}_p. \tag{32.3}$$

In particular,

$$(1-b^2)^2 = (1-a^2)^2 \quad \text{in } \mathbb{F}_p, \tag{32.4}$$

and hence

$$b^2 = a^2 \quad \text{or} \quad b^2 = 2-a^2.$$

If $2-a^2$ is zero or a quadratic nonresidue modulo p, then $b = \pm a$. This holds for all allowed values of a when $p = 5$ or $p = 7$.

Proof. Let $\mathcal{L} \to \mathcal{G}$ be the line bundle associated to the geodesic circle bundle by

$$(G_t u, z) \sim (u, e^{-it} z).$$

For a local section s of $E \to \mathcal{G}$, the frame $[s,1]$ has connection form $-i\,s^*\lambda$ and curvature $-i\omega$. The Chern-class normalization of [7, Appendix D, p. 126, author's version] gives

$$c_1(\mathcal{L})_{\mathbb{R}} = \left[\frac{i(-i\omega)}{2\pi}\right] = \left[\frac{\omega}{2\pi}\right] = e_{\mathbb{R}}.$$

The group $H^2(\mathcal{G};\mathbb{Z})$ is torsion-free by Lemma 12.1, so $c_1(\mathcal{L}) = e$. The deck action on $\mathcal{L}$ is $h[u,z] = [\widehat{h}u, z]$. Above B_c, it acts on the fibers by ζ^{-c}.

All cohomology groups below have coefficients in $\mathbb{F}_p$. Put $\xi = c_1^{C_p}(\mathcal{L}) \bmod p$ and $e_c = e|_{B_c} \bmod p$. Additivity of the first Chern class [38, Proposition 3.10, p. 86] gives

$$\xi|_{B_c} = e_c - cv.$$

Write

$$H^*(BC_p) = \mathbb{F}_p[v] \otimes \Lambda(\epsilon), \qquad |v| = 2, \quad |\epsilon| = 1, \qquad K = \mathbb{F}_p(v).$$

By Lemma 29.2 and Corollary 31.2,

$$\mathcal{G}^{C_p} = B_1 \sqcup B_{-1} \sqcup B_b \sqcup B_{-b}.$$

Give $\mathcal{G}$ and these components their symplectic orientations, and give their normal bundles invariant compatible complex structures. Write $e_{C_p}(\nu_B)$ for the equivariant Euler class of the normal bundle of a fixed component B. For its inclusion i_B, the Thom identity gives

$$i_B^* i_{B!}(x) = e_{C_p}(\nu_B)x$$

[37, Corollary 4D.9, Theorem 4D.10, and p. 444]. The part of $e_{C_p}(\nu_B)$ of degree zero on B is $v^{\operatorname{rank}_{\mathbb{C}} \nu_B}$ times the product of the nonzero normal weights. The remaining part has positive degree on B and is nilpotent. Thus this Euler class is invertible after extending scalars to K. Lemma 13.2 and equivariant pushforward therefore give

$$\int_{\mathcal{G}} x = \sum_{B \subset \mathcal{G}^{C_p}} \int_B \frac{i_B^* x}{e_{C_p}(\nu_B)}.$$

The class ξ and the normal Euler classes have even degree. Since each fixed component has no odd cohomology by Lemma 30.1, the restrictions and reciprocal Euler classes in this formula have no ϵ term.

Let z be an indeterminate. Define the formal Laurent series at infinity

$$\mathcal{R}(z) = v^{2m+1} \int_{\mathcal{G}} \frac{1}{zv - \xi} = \sum_{k \geq 0} z^{-k-1} v^{2m-k} \int_{\mathcal{G}} \xi^k. \tag{32.5}$$

The pushforward $\int_{\mathcal{G}} \xi^k$ belongs to $H^{2k-4m}(BC_p)$. It is zero for $k < 2m$ and is a scalar multiple of v^{k-2m} for $k \geq 2m$. Thus every coefficient of $\mathcal{R}$ belongs to $\mathbb{F}_p$. For $c = \pm 1$, Lemma 30.1 and Theorem 31.4, with reversal for $c = -1$, give

$$e_c^m = 0, \qquad \int_{B_c} e_c^{m-1} = 1.$$

Hence

$$\frac{1}{zv - \xi|_{B_c}} = \sum_{j=0}^{m-1} \frac{e_c^j}{v^{j+1}(z+c)^{j+1}}.$$

The corresponding contribution to $\mathcal{R}$ is rational, with its only possible pole at $-c$ and order at most m. The point components contribute simple poles at b and $-b$. These four points are distinct, so the denominator divides

$$Q_b(z) = (z^2 - 1)^m (z^2 - b^2).$$

The constant equivariant pushforward of ξ^{2m} equals the ordinary number $\int_{\mathcal{G}} e^{2m}$. Lemma 12.2, the prime period 2π and $\mathrm{Vol}(X) = V_n$ give this number as 2. Since Q_b is monic of degree $2m + 2$, the expansion at infinity therefore gives

$$\mathcal{R}(z) = \frac{2z + C}{Q_b(z)}, \qquad C \in \mathbb{F}_p.$$

Let $\bar{J}$ be the involution of $\mathcal{G}$ induced by geodesic reversal. It commutes with C_p and inverts the geodesic circle action, so $\bar{J}^*\mathcal{L} \simeq \mathcal{L}^*$ equivariantly and $\bar{J}^*\xi = -\xi$. Moreover, $\bar{J}^*\omega = -\omega$. Since $\dim \mathcal{G} = 4m$, the map $\bar{J}$ preserves orientation. Thus

$$\int_{\mathcal{G}} \xi^k = (-1)^k \int_{\mathcal{G}} \xi^k.$$

For odd k this pushforward is zero because p is odd. Consequently $\mathcal{R}(-z) = -\mathcal{R}(z)$, and $C = 0$:

$$\mathcal{R}(z) = \frac{2z}{(z^2 - 1)^m (z^2 - b^2)}. \tag{32.6}$$

Write $r_1, \dots, r_{m+1}$ for the signed normal deck weights at B_1. Its contribution to the coefficient of $(z+1)^{-m}$ is

$$v^{2m+1} \int_{B_1} \frac{e_1^{m-1}}{v^m v^{m+1} \prod_i r_i} = \frac{1}{\Pi_b}.$$

The other components have no pole at -1. Since e_1^{m-1} has top degree on B_1, every positive-degree term of the reciprocal normal Euler class vanishes in this integral. The same coefficient in (32.6) is

$$\left. \frac{2z}{(z-1)^m (z^2 - b^2)} \right|_{z=-1} = \frac{(-1)^{m-1}}{2^{m-1}(1 - b^2)}.$$

Taking reciprocals proves (32.3).

The quotient return on F_1 induces the inverse deck derivative on the normal space of B_1. Inversion preserves unsigned weights, so Theorem 31.4 gives

$$\Pi_b^2 = 2^{2m-2}(a-1)^2(a+1)^2.$$

Squaring (32.3) and cancelling the nonzero factor 2^{2m-2} proves (32.4). Factoring the difference of squares in $\mathbb{F}_p$ gives the two alternatives. Since $b \neq 0$, the second is impossible when $2 - a^2$ is zero or a quadratic nonresidue. For $p = 5$, $a = 2$ gives the nonresidue 3. For $p = 7$, $a = 2, 3$ give respectively the nonresidue 5 and zero. □

Theorem 32.5. *Under the hypotheses of Theorem 31.4, the phase of the one-dimensional family satisfies*

$$b \equiv \pm a \pmod{p}.$$

Consequently, after normalizing the generator by a $(2m-1)$-dimensional component, the nonempty phase families are $F_{\pm 1}, F_{\pm a}$, and

$$\min_{x \in X} d(x, h^j x) = \frac{2\pi}{p} \min\{|j|_p, |ja|_p\} \qquad (1 \le j < p).$$

Proof. Choose the same associated-line and character conventions on both sides, with a phase-one $(2m-1)$-dimensional component on each. Corollary 32.3 gives equality of the signed normal deck-weight products. Applying (32.3) to M and Q gives

$$(-2)^{m-1}(1-b^2) = (-2)^{m-1}(1-a^2).$$

The common factor is nonzero in $\mathbb{F}_p$, so $b^2 = a^2$ and $b = \pm a$. Theorem 30.4 gives the displacement formula. □

33. Integral topology and signed prime return maps

Let $m \ge 2$ be an integer, let X^{2m+1} be a normalized Zoll homotopy sphere, and let h generate a free isometric action of prime order $p \ge 5$. Let d be the Riemannian distance on X. Put $C_p = \langle h \rangle$, $\ell = 2\pi/p$ and $\zeta = e^{2\pi i/p}$. Let $\pi : SX \to X$ be the unit tangent bundle projection, G_t the geodesic flow on SX, and $\widehat{h} = dh|_{SX}$. The flow defines a free action of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Write $\mathcal{G} = SX/S^1$, and let $\bar{h}$ be the induced map on $\mathcal{G}$. For $c \in \mathbb{F}_p$, put

$$F_c = \{v \in SX : \widehat{h}v = G_{c\ell}v\}.$$

For $r \in \mathbb{R}$, write $|r|_p = \min_{k \in \mathbb{Z}} |r - kp|$; this also defines a function on $\mathbb{R}/p\mathbb{Z}$.

Assume that the nonempty families are F_1, F_{-1}, F_a, F_{-a}, where a is an integer satisfying $2 \le a \le (p-1)/2$, with dimensions $2m-1, 2m-1, 1, 1$. Assume that they are connected mod-p homology spheres and that

$$\min_{x \in X} d(x, h^j x) = \ell \min\{|j|_p, |aj|_p\} \qquad (1 \le j < p). \tag{33.1}$$

At F_1, assume that the unsigned transverse weights are

$$[2]^{m-1}, \quad [a-1], \quad [a+1],$$

where $[r] = \{r, -r\} \subset \mathbb{F}_p^\times$ and exponents denote multiplicities. These hypotheses hold in the spectral setting of Proposition 31.1, by Corollary 31.2 and Theorems 30.3, 31.4, and 32.5.

Let λ be the canonical contact form on SX. For $v \in F_c$, the relative return is the symplectic map $d_v(\widehat{h}^{-1} G_{c\ell})|_{\ker \lambda_v}$, with symplectic form $d\lambda$. The Euler class of a geodesic circle bundle has the positive convention of Section 12.

Theorem 33.1. *The families $F_{\pm 1}$ are homotopy $(2m-1)$-spheres, and*

$$F_1/S^1 \simeq \mathbb{CP}^{m-1}, \qquad F_1/C_p \simeq L(p; 1, \dots, 1).$$

Let $e \in H^2(F_1/S^1; \mathbb{Z})$ be its positive Euler class. Then $H^*(F_1/S^1; \mathbb{Z}) = \mathbb{Z}[e]/(e^m)$ and $\langle e^{m-1}, [F_1/S^1] \rangle = 1$ in the symplectic orientation. If $m = 2$, then

$$F_1/S^1 \cong S^2, \qquad F_1 \cong S^3, \qquad F_1/C_p \cong L(p; 1, 1)$$

diffeomorphically.

All four relative returns are symplectically conjugate to the corresponding returns for the round action $(z, w) \mapsto (\zeta z, \zeta^a w)$ on $S^{2m+1} \subset \mathbb{C}^m \oplus \mathbb{C}$. With invariant compatible complex structures J satisfying $d\lambda(u, Ju) > 0$ for $u \ne 0$, their nontrivial signed weights are

family	*relative return*	*weights in $\mathbb{F}_p$*
F_1	$d(\widehat{h}^{-1} G_\ell)$	$2^{m-1},\ 1-a,\ 1+a$
F_a	$d(\widehat{h}^{-1} G_{a\ell})$	$(a-1)^m,\ (a+1)^m.$

(33.2)

Reversal gives the other two maps. On $\mathcal{G}$, the relative return induces $d\bar{h}^{-1}$.

33.1. **Boundary indices and orientations.** For $j \in \mathbb{F}_p^\times$, consider the energy on the twisted path space $\Lambda_{h^j}X$ defined in Section 29. A critical path extends by h^j to a closed geodesic. Its length is $r\ell$ for an integer $r \geq 1$: its p-fold extension has length in $2\pi\mathbb{Z}_{>0}$. Moreover, $p \nmid r$, since h^j has no fixed point. Its initial unit state lies in $F_{rj^{-1}}$, and conversely a state v in that family gives the critical path $t \mapsto \pi G_{r\ell t}v$, $0 \leq t \leq 1$. Thus the critical lengths have the four distinct residues $j, -j, aj, -aj$ modulo p.

The initial-state parametrization is a smooth embedding, by uniqueness for the geodesic and Jacobi equations and compactness of the family, as in the proof of Theorem 29.6. The relative return $\widehat{h^{-j}}G_{r\ell}$ has pth power equal to the identity. Its fixed tangent space is therefore the tangent space of its fixed set. The normal Hessian kernel is this fixed space in $\ker\lambda$; the tangential kernel consists of the constant multiples of $\dot\gamma$. Thus the Hessian kernel is exactly the tangent space of the critical manifold. Lemma 29.7 applies. Since $\deg(h^j) = 1$, the evaluation fibration in its proof gives connectedness and

$$\pi_k(\Lambda_{h^j}X) = 0 \qquad (1 \leq k \leq 2m-1).$$

Fix $F = F_c$, where $c \in \{1, -1, a, -a\}$, and write $\dim F = 2d-1$. Put $g = h^{c^{-1}}$. Along a geodesic with initial state in F, one has $\gamma(t+\ell) = g\gamma(t)$. Take the Hermitian metric, covariant derivative D_t and curvature convention of Lemma 29.8. For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, let $j(\theta)$ be the complex negative index of the normal form

$$I_\theta(J) = \int_0^\ell \left(|D_tJ|^2 - \langle R(J,\dot\gamma)\dot\gamma, J\rangle\right) dt$$

on complex normal H^1 fields satisfying $J(\ell) = e^{i\theta}dg\,J(0)$. Let $i(q)$ be the real Morse index of the q-fold arc with endpoint twist g^q, for an integer $q \geq 1$. Equation (33.1) gives minimum displacement at least ℓ for every nonidentity power of h. The short arc in F attains this bound for g, so $j(0) = 0$.

Put

$$T = d(\widehat{g}^{-1}G_\ell)|_{\ker\lambda}, \qquad E_s = \ker(T - \zeta^s I) \quad (0 \leq s < p),$$

where E_s is taken in the complexified contact space. Then $T^p = I$ and its real fixed space has dimension $2d-2$. The crossing calculation in the proof of Lemma 29.8 depends only on the twisted boundary condition. With h replaced by g, it gives on E_s the Hermitian form

$$Q_s(u,v) = -i\omega(u,\bar v), \qquad \omega = d\lambda,$$

where ω is extended complex bilinearly. Semisimplicity of T makes the pairing of E_s with $\overline{E_s}$ nondegenerate, so Q_s is nondegenerate. On E_0 its positive and negative indices are both $d-1$. Consequently $j(0+) = j(2\pi-) = d-1$.

For $1 \leq s < p$, let A_s and B_s be the negative and positive indices of Q_s, respectively, and put $\Delta_s = A_s - B_s$. The index at $2\pi s/p$ is its left limit minus B_s, or its right limit minus A_s. The character decomposition in the proof of Lemma 29.8, with h replaced by g, gives

$$i(q) = \sum_{z^q=1} j(\arg z). \tag{33.3}$$

The hypotheses of [31, Lemma 2.3 and (2.4), pp. 37–38] hold, since $g^p = I$ and $\gamma(t+\ell) = g\gamma(t)$. The tangential index form is nonnegative.

For $p \nmid q$, the number of sample points strictly beyond $2\pi s/p$ is $q - 1 - \lfloor qs/p\rfloor$. Hence

$$i(q) = (q-1)(d-1) + \sum_{s=1}^{p-1}\Delta_s\left(q - 1 - \left\lfloor\frac{qs}{p}\right\rfloor\right). \tag{33.4}$$

For $p \mid q$, every nonzero eigenphase is sampled, and the right side must be decreased by $\sum_{s=1}^{p-1}B_s$.

The normal Jacobi space has real dimension $4m$. Conjugation gives $B_{p-s} = A_s$, and therefore

$$\sum_{s=1}^{p-1}\Delta_s = 0, \qquad \sum_{s=1}^{p-1}B_s = 2m - d + 1.$$

The p-fold arc is a prime closed geodesic, so Lemma 11.2 gives $i(p) = 2m$. Substitution in the corrected formula at $q = p$ yields

$$\sum_{s=1}^{p-1}(p-1-s)\Delta_s = 4m - p(d-1).$$

The identities

$$\left\lfloor \frac{(q+p)s}{p} \right\rfloor = \left\lfloor \frac{qs}{p} \right\rfloor + s, \qquad \left\lfloor \frac{qs}{p} \right\rfloor + \left\lfloor \frac{(p-q)s}{p} \right\rfloor = s-1 \quad (1 \le q < p)$$

now give

$$\begin{aligned} i(q+p) &= i(q) + 4m \qquad (q \ge 1), \\ i(q) + i(p-q) &= 4m - 2d + 2 \quad (1 \le q < p). \end{aligned} \tag{33.5}$$

The change of variable $t \mapsto q\ell - t$ identifies the index form of an arc with that of its reversal, with g replaced by g^{-1}. Thus the same indices apply to the opposite family with the inverse endpoint twist.

Put $M = X/C_p$, and let $\rho : X \to M$ be the quotient map. The manifold M is closed and Besse. Every deck transformation preserves orientation, since its degree has odd order. The orientation of X therefore descends to M. For $f \in C_p$, including $f = I$, let $\Lambda_{[f]}M$ be the component of the free loop space corresponding to f under $\pi_1(M) = C_p$. The map

$$\Lambda_f X \longrightarrow \Lambda_{[f]}M, \qquad \gamma \longmapsto \rho \circ \gamma,$$

is a p-sheeted covering: a loop has one lift for each initial point in a fiber of ρ, and its endpoint transformation is unchanged by a deck transformation because C_p is abelian. Its differential identifies fields with $J(1) = df\,J(0)$ with periodic fields along the projected loop. Since ρ is a local isometry, this identification intertwines covariant derivatives and the energy Hessians. The quotient has odd dimension, so [84, Theorem 3, p. 283] gives

$$i(q) \equiv 0 \pmod 2 \qquad (q \ge 1), \tag{33.6}$$

by taking $f = g^q$.

By the transfer theorem [37, Proposition 3G.1, p. 321], the pullback

$$\rho^* : H^2(M;\mathbb{F}_2) \longrightarrow H^2(X;\mathbb{F}_2) = 0$$

is injective, since p is odd. Thus the second Stiefel–Whitney class $w_2(M)$ vanishes, and the oriented manifold M is spin. By [70, Corollary C, p. 914], the negative bundle over every critical manifold in the free loop space of M is orientable. The Hessian identification above makes the negative bundle over each lifted critical manifold its pullback. In particular, the negative bundles over the circle critical manifolds in $\Lambda_{h^j}X$ are orientable.

33.2. **The circle indices and the large family.**

Lemma 33.2. *The homotopy types, integral cohomology, Euler number and dimension-three diffeomorphisms are those in Theorem 33.1. In a twisted path space whose minimum is F_1 or F_{-1}, each of the two circle critical manifolds below 2π has index $2m$. If $b = a^{-1} \in \mathbb{F}_p^*$, $g = h^b$, and $d_q = |b\,q|_p$, the q-fold arc in F_a with endpoint twist g^q has index*

$$i_C(q) = \begin{cases} 0, & q < d_q, \\ 2m, & d_q < q < p - d_q, \\ 4m, & p - d_q < q, \end{cases} \qquad 1 \le q < p. \tag{33.7}$$

Proof. Fix a sector $\Lambda_{h^j}X$ whose full minimum is F_ε, with $\varepsilon \in \{1,-1\}$. Its minimum index is zero. Reversal and (33.5), with $d = m$, give index $2m+2$ for the opposite family. With $d = 1$, the same identity gives circle indices r and $4m - r$. Subsequent occurrences of each family add $4m$ to its index. The four critical residues are distinct, so each critical value corresponds to a single family.

Put $q_0 = \min\{r, 4m - r\}$ and suppose $q_0 < 2m$. Every other attachment above the minimum then has rank at least $2m + 2$. If $q_0 = 0$, attachment of its negative disk bundle adds a disjoint circle. Attachments of rank at least $2m + 2$ preserve the set of connected components. A path between the resulting components would have bounded energy and hence lie in a bounded sublevel. This contradicts the connectedness of $\Lambda_{h^j} X$.

If $q_0 > 0$, (33.6) gives $2 \le q_0 \le 2m - 2$. Choose regular sublevels $A \subset D$ immediately before and after the index-q_0 circle. All earlier positive-rank attachments have rank at least $2m + 2$. Since F_ε is a mod-p homology $(2m - 1)$-sphere,

$$H_{q_0-1}(A; \mathbb{F}_p) = 0.$$

The negative bundle of the circle is orientable, so the Thom isomorphism gives

$$H_{q_0}(D, A; \mathbb{F}_p) = H_0(S^1; \mathbb{F}_p) = \mathbb{F}_p.$$

The exact sequence of the pair makes the map $H_{q_0}(D; \mathbb{F}_p) \to H_{q_0}(D, A; \mathbb{F}_p)$ surjective. Every later attachment has rank at least $2m + 2$, so it preserves H_{q_0}. A singular bounding chain has bounded energy; hence $H_{q_0}(\Lambda_{h^j} X; \mathbb{F}_p) \neq 0$. This contradicts the homotopy vanishing from Lemma 29.7 and the Hurewicz theorem. Thus both circle indices are $2m$ in every such sector.

In $\Lambda_h X$, equation (33.1) gives the full minimum F_1, since $1 < a$. All attachments above it now have rank at least $2m$. Lemma 29.7 gives

$$\pi_j(\Lambda_h X, F_1) = 0 \quad (j < 2m).$$

Its homotopy sequence and the vanishing of $\pi_j(\Lambda_h X)$ for $1 \le j \le 2m - 1$ give

$$\pi_j(F_1) = 0 \qquad (1 \le j \le 2m - 2).$$

By Lemma 29.2, F_1 is a closed contact $(2m - 1)$-manifold, oriented by $\lambda|_{F_1} \wedge (d\lambda|_{F_1})^{m-1}$. The Hurewicz isomorphism supplies a map $S^{2m-1} \to F_1$ representing its integral orientation class. It is an integral homology equivalence between simply connected spaces of CW type, hence a homotopy equivalence [37, Corollary 4.33, p. 367]. Reversal gives $F_{-1} \simeq S^{2m-1}$.

Put $B = F_1/S^1$, and let $e \in H^2(B; \mathbb{Z})$ be the positive Euler class of this circle bundle. The circle action is free, since its prime period is 2π. The quotient is a closed smooth $(2m - 2)$-manifold, and the homotopy sequence of the circle bundle gives $\pi_1(B) = 0$. The integral Gysin sequence [37, Section 4.D, p. 438] gives

$$H^*(B; \mathbb{Z}) = \mathbb{Z}[e]/(e^m).$$

Indeed, cup product with e is an isomorphism between the successive even groups through degree $2m - 2$, and the odd groups vanish, since $H^*(F_1; \mathbb{Z})$ is supported in degrees 0 and $2m - 1$. Thus e^{m-1} generates top cohomology. Its symplectic representative is positive, so $\langle e^{m-1}, [B] \rangle = 1$.

Let $\mathcal{L}_B$ be the line bundle associated by

$$(G_t w, z) \sim (w, e^{-it} z), \qquad w \in F_1.$$

With the associated-line convention in Section 32, one has $c_1(\mathcal{L}_B) = e$. Let $u = c_1(\mathcal{O}(1))$ be the positive generator on $\mathbb{CP}^\infty$. Complex line-bundle classification [38, Proposition 3.10, p. 86] gives a map $f : B \to \mathbb{CP}^\infty$ with $\mathcal{L}_B \cong f^*\mathcal{O}(1)$ and $f^* u = e$. The smooth compact manifold B has a finite CW structure of dimension $2m - 2$. Cellular approximation [37, Theorem 4.8, p. 349] therefore makes f take values in $\mathbb{CP}^{m-1}$. It induces an integral cohomology isomorphism. The universal coefficient theorem and the homological Whitehead theorem [37, Corollary 4.33, p. 367] give $B \simeq \mathbb{CP}^{m-1}$.

Taking dual unit-circle bundles gives an S^1-equivariant bundle map

$$\widetilde{f} : F_1 \longrightarrow S^{2m-1}$$

over f, with the standard scalar action on the target. Its fiber map has degree one. The homotopy sequences of the two bundles and Whitehead's theorem [37, Theorem 4.5, p. 346] show that $\widetilde{f}$ is a homotopy equivalence. On F_1, the deck generator is G_ℓ. Consequently $\widetilde{f}$ descends to

$$F_1/C_p \longrightarrow L(p; 1, \dots, 1).$$

The quotient map induces the identity between the deck groups on fundamental groups. Its lift to the universal covers is $\widetilde{f}$, so it induces isomorphisms on all higher homotopy groups as well. The same theorem gives the stated quotient homotopy equivalence.

If $m = 2$, the surface B is closed, oriented and simply connected. Choose an orientation-preserving diffeomorphism $f : B \to \mathbb{CP}^1 = S^2$. Since $\langle e, [B]\rangle = 1$, it satisfies $f^*u = e$. The associated line bundles are topologically isomorphic. By [41, Chapter 4, Section 3, Theorem 3.5, p. 101], their underlying smooth real plane bundles admit a smooth isomorphism T. It preserves fiber orientation: otherwise $e = -e$ in $H^2(B;\mathbb{Z}) = \mathbb{Z}e$. Choose Hermitian metrics and write T^* for the fiberwise adjoint. The polar part $T(T^*T)^{-1/2}$ is an oriented fiberwise isometry, hence is complex linear. Taking dual unit circles gives an S^1-equivariant diffeomorphism $F_1 \cong S^3$, and therefore $F_1/C_p \cong L(p;1,1)$.

Finally, let $b = a^{-1}$, $g = h^b$ and $d_q = |bq|_p$. Equation (33.1) gives

$$\min_x d(x, g^q x) = \ell \min\{|q|_p, d_q\}.$$

For $1 \le q \le (p-1)/2$, the inequality $q < d_q$ means that the q-fold arc in F_a attains the global minimum, so its index is zero. If $d_q < q \le (p-1)/2$, the minimum is F_1 or F_{-1}, and the circle index is $2m$ by the first part of the proof. For $(p+1)/2 \le q < p$, use $d_{p-q} = d_q$ and (33.5) with $d = 1$. These give all three cases of (33.7). An equality $q = d_q$ or $q = p - d_q$ would give $bq \equiv \pm q \pmod p$, hence $b = \pm 1$, contrary to $a \neq \pm 1$. □

33.3. **The return along F_1.** Here $g = h$ and $d = m$. Put $x = p\theta/(2\pi) \in \mathbb{R}/p\mathbb{Z}$ and $H = (p-1)/2$. For the round reference, write $j_R(x)$ for the normal boundary index at phase $2\pi x/p$. Let $\mathbf{1}_A$ denote the indicator of a set A. One has

$$j_R(x) = (m-1) \sum_{\epsilon=\pm1} \mathbf{1}_{\{|x-\epsilon|_p<1\}} + \sum_{\epsilon=\pm1} \mathbf{1}_{\{|x-\epsilon a|_p<1\}}. \tag{33.8}$$

Indeed, on a round real normal two-plane of deck weight w, the Fourier modes of $-D_t^2 - I$ have eigenvalues $(pk + x \pm w)^2 - 1$, $k \in \mathbb{Z}$. There are $m-1$ planes with $w = 1$ and one with $w = a$. The strict inequalities exclude the zero eigenvalues. In particular, $\sum_{j=0}^{p-1} j_R(j) = 2m$.

The actual and round returns have the same unsigned weights by hypothesis. Thus their possible crossing forms differ only in the positive and negative indices at these eigenvalues. For $1 \le r \le H$, replacing one positive direction of Q_r by a negative one changes the boundary index by 2 on $(r, p-r)$ and by 1 at each endpoint. The change in its integer-grid sum is $2(p-2r)$.

Suppose first that $2 \le a \le (p-3)/2$. In the round model, Q_2 has $m-1$ positive directions from the repeated weights, and the exceptional weights give one negative direction at $a-1$ and one positive direction at $a+1$. Let $k \in \{0, \dots, m-1\}$ count the repeated directions whose signs change, and let $\epsilon_b, \epsilon_c \in \{0,1\}$ be the indicators of a sign change at $a-1$ and $a+1$, respectively. These descriptions allow coincident eigenvalues; the resulting inertia is the sum of their contributions. Both full-period indices equal $2m$, so

$$k(p-4) - \epsilon_b(p-2a+2) + \epsilon_c(p-2a-2) = 0. \tag{33.9}$$

If $\epsilon_b = 0$, both remaining coefficients are positive, and $k = \epsilon_c = 0$. If $\epsilon_b = \epsilon_c = 1$, then $k(p-4) = 4$, impossible for a prime $p \ge 7$. If $(\epsilon_b, \epsilon_c) = (1,0)$, then

$$0 < p - 2a + 2 < 2(p-4), \qquad k(p-4) = p - 2a + 2.$$

Hence $k = 1$ and $a = 3$. The two changes then occur at the same eigenvalue ζ^2, with opposite signs, and leave its inertia unchanged.

Suppose now that $a = (p-1)/2$. Both exceptional directions are negative, at $r = (p-3)/2$ and $r = (p-1)/2$, and the equation is

$$k(p-4) = 3\epsilon_b + \epsilon_c. \tag{33.10}$$

For $p \ge 11$, only the zero solution occurs. For $p = 7$, the only other solution is $(k, \epsilon_b, \epsilon_c) = (1,1,0)$, whose two changes cancel at ζ^2. For $p = 5$, one has $a = 2$ and $k = 3\epsilon_b + \epsilon_c$. The solution $(1,0,1)$ again

preserves the inertia at ζ^2. The remaining solutions are $(3,1,0)$ and $(4,1,1)$, subject to $k \le m-1$, and give the same distinct inertia. They require $m \ge 4$.

For this distinct inertia, let $j_*(x)$ denote the actual boundary index at phase $2\pi x/5$. Relative to j_R, its change is -2 on $(1,4)$ and $+6$ on $(2,3)$, with half these changes at the respective endpoints. Thus

x	$(0,1)\cup(4,5)$	$(1,2)\cup(3,4)$	$(2,3)$
$j_*(x)$	$m-1$	$m-2$	$6.$

At the integer points, $j_*(1) = j_*(4) = m-2$ and $j_*(2) = j_*(3) = 2$. Formula (33.3) gives the indices 6 and $2m-4$ for the large-family arcs of lengths 2ℓ and 3ℓ.

In $\Lambda_{h^2}X$, the four critical families below 2π are F_{-a}, F_1, F_{-1}, F_a, at lengths $\ell, 2\ell, 3\ell, 4\ell$. The first is the full minimum set by (33.1). Complementarity with $d=1$ gives index $4m$ for the last one. Their indices would therefore be

$$0, \quad 6, \quad 2m-4, \quad 4m,$$

and every later occurrence adds $4m$ by (33.5). Since $m \ge 4$, all attachments after the minimum circle have rank at least four. Lemma 29.7 would give

$$\pi_1(\Lambda_{h^2}X) \cong \pi_1(F_{-a}) \cong \mathbb{Z},$$

contrary to the vanishing already proved. Thus the crossing inertias at F_1 agree with the round ones for every $p \ge 5$.

33.4. **An invertible finite convolution.**

Lemma 33.3. *Put $H = (p-1)/2$. If real numbers $c_1, \dots, c_H$ satisfy*

$$\sum_{r=1}^{H} c_r \left(q - 1 - 2\left\lfloor \frac{qr}{p} \right\rfloor \right) = 0 \qquad (1 \le q < p),$$

then every c_r is zero.

Proof. Taking $q = p-1$, for which $\lfloor (p-1)r/p \rfloor = r-1$, gives

$$\sum_{r=1}^{H} (p-2r)c_r = 0.$$

Put $G = (\mathbb{Z}/p\mathbb{Z})^\times$. For $u \in G$, let $\langle u \rangle \in \{1, \dots, p-1\}$ be its representative and put $B(u) = \langle u \rangle / p - 1/2$. Extend c to G by $c(r) = c_r$ and $c(-r) = -c_r$ for $1 \le r \le H$. Since

$$q - 1 - 2\left\lfloor \frac{qr}{p} \right\rfloor = \frac{q(p-2r)}{p} + 2B(qr),$$

and $B(-u) = -B(u)$, the hypotheses give $\sum_{u \in G} c(u)B(qu) = 0$ for every $q \in G$. For a character $\chi : G \to \mathbb{C}^\times$, multiplication by $\chi(q)$ and summation over q, followed by $v = qu$, give

$$0 = \beta_\chi \sum_{u \in G} c(u)\overline{\chi(u)}, \qquad \beta_\chi = \sum_{v \in G} \chi(v)B(v). \tag{33.11}$$

If $\chi(-1) = -1$, its extension by zero on $p\mathbb{Z}$ is a nonprincipal Dirichlet character of conductor p. Therefore

$$\beta_\chi = \frac{1}{p} \sum_{v=1}^{p-1} v\chi(v) \neq 0$$

by [82, Proposition 4.1, p. 31; Theorem 4.9, p. 37]. Thus (33.11) makes every odd Fourier coefficient of c zero. The even coefficients vanish because c is odd. Character orthogonality [20, Exposé I, Proposition (1.2), p. 2] now gives $c = 0$. □

33.5. **The return along F_a.** Let $b \in \{1, \dots, p-1\}$ satisfy $ab \equiv 1 \pmod p$, and put $g = h^b$. Then $\widehat{g}v = G_\ell v$ for $v \in F_a$. Put $H = (p-1)/2$. For the return $d(\widehat{g}^{-1}G_\ell)|_{\ker\lambda}$, write $\Delta_r = A_r - B_r$ and $d_r = A_r + B_r$ for $1 \le r \le H$, with the crossing conventions above. Its fixed contact space is zero, since $\dim F_a = 1$. Hence

$$\sum_{r=1}^{H} d_r = 2m, \qquad d_r \ge |\Delta_r|.$$

Pairing r with $p-r$ in (33.4) gives

$$i_C(q) = \sum_{r=1}^{H} \Delta_r \left(q - 1 - 2\left\lfloor \frac{qr}{p} \right\rfloor \right) \qquad (1 \le q < p). \tag{33.12}$$

For $x = p\theta/(2\pi)$, the corresponding round boundary index is

$$j_R(x) = m\left(\mathbf{1}_{\{|x-b|_p<1\}} + \mathbf{1}_{\{|x+b|_p<1\}}\right).$$

Let Δ_r^R be its jumps. Formula (33.3) gives its q-fold index as

$$i_R(q) = m \sum_{\epsilon=\pm1} \#\{k \in \mathbb{Z} : |pk + \epsilon bq| < q\}.$$

For $1 \le q < p$, put $d_q = |bq|_p$. For either sign ϵ, the distances below p from ϵbq to $p\mathbb{Z}$ are d_q and $p - d_q$. The two endpoints do not equal q, as in Lemma 33.2. Each count is therefore 0, 1, or 2 according as $q < d_q$, $d_q < q < p - d_q$, or $p - d_q < q$. Consequently $i_R(q) = i_C(q)$ by (33.7). The round jumps satisfy (33.12) with i_C replaced by i_R. Lemma 33.3, applied to $c_r = \Delta_r - \Delta_r^R$, gives

$$\Delta_r = \Delta_r^R \qquad (1 \le r \le H).$$

The four endpoints of the round intervals have the two unsigned classes $[b-1]$ and $[b+1]$. These classes are nonzero and distinct: $b \ne \pm1$, and their equality would give $2 = 0$ or $2b = 0$ in $\mathbb{F}_p$. Each of the two positive-half jumps has absolute value m. Thus

$$\sum_{r=1}^{H} |\Delta_r| = 2m = \sum_{r=1}^{H} d_r.$$

The inequalities $d_r \ge |\Delta_r|$ imply

$$d_r = |\Delta_r|, \qquad A_r = \frac{|\Delta_r| + \Delta_r}{2}, \qquad B_r = \frac{|\Delta_r| - \Delta_r}{2}.$$

These are the round inertias. Conjugation gives the same conclusion at the phases $p - r$.

Completion of the proof of Theorem 33.1. The topology is Lemma 33.2. Let T be the relative return at F_1 for h and time ℓ, or the relative return at F_a for $g = h^b$ and time ℓ. It is a real symplectic map with $T^p = I$, hence is semisimple and has no eigenvalue -1. Let $T_{\mathbb{C}}$ be its complexification, and extend $\omega = d\lambda$ complex bilinearly. For a nonreal eigenvalue z, put

$$E_z = \ker(T_{\mathbb{C}} - zI), \qquad H_z(u,v) = i\omega(u,\bar{v}), \quad u, v \in E_z.$$

For $z = \zeta^s$, this form is $-Q_s$, so its positive and negative indices are A_s and B_s. In [36, Theorem 1.2, p. 115], the form $\widehat{Q}_1^z$ is exactly H_z. Semisimplicity gives $\ker((T_{\mathbb{C}} - zI)^k) = E_z$ for every $k \ge 1$ and makes all the higher forms zero. The fixed summand is the identity, of the same dimension as in the round model. Thus the equal crossing inertias proved above give the required real symplectic conjugacy for each of these two returns. At F_a, taking ath powers gives the return in the theorem, since

$$(\widehat{g}^{-1}G_\ell)^a = \widehat{h}^{-1}G_{a\ell}.$$

For the round metric, identify normal Jacobi fields with their initial data $(x, y) = (J(0), D_tJ(0))$ in a parallel normal frame. Then

$$d\lambda((x, y), (x', y')) = \langle y, x' \rangle - \langle x, y' \rangle, \qquad \mathcal{J}(x, y) = (y, -x).$$

One has $d\lambda(v, \mathcal{J}v) = |x|^2 + |y|^2 > 0$ for $v = (x, y) \neq 0$, and Jacobi evolution is $e^{t\mathcal{J}}$. With the complex coordinate $x - iy$, the inverse deck rotation on a real normal two-plane of weight w has weights $w, -w$. It commutes with $\mathcal{J}$, so the time-ℓ relative return has weights $1 - w, 1 + w$. At F_a, the m normal two-planes have weight b for g. The short return has weights $1 - b, 1 + b$, each repeated m times; its ath power has weights $a - 1, a + 1$ modulo p. At F_1, the $m - 1$ normal two-planes of weight one give weights $0, 2$, and the plane of weight a gives $1 - a, 1 + a$. Removing the fixed summand gives (33.2).

Let $\mathcal{R}(x, v) = (x, -v)$ be reversal. It satisfies

$$\mathcal{R}^* d\lambda = -d\lambda, \qquad \mathcal{R}(\widehat{h}^{-1} G_{c\ell})\mathcal{R}^{-1} = \widehat{h}^{-1} G_{-c\ell}.$$

Applying reversal on both the comparison and round sides gives symplectic conjugacies for the other two returns, with the negatives of the displayed signed weights. Since G_t acts trivially on $\mathcal{G}$, the relative return induces $d\bar{h}^{-1}$. □

34. Integral linking and displacement extrema

Retain the hypotheses of Theorem 33.1, and suppose $p \geq 11$. Put $n = 2m + 1$, $\ell = 2\pi/p$, and $H = (p - 1)/2$. Let $\pi : SX \to X$ be the unit tangent bundle projection. Proposition 12.3 gives $\mathrm{Vol}(X) = \mathrm{Vol}(S^n)$, where S^n has its unit round metric. Thus (30.1) holds, and Theorems 30.7 and 30.10 give disjoint embedded submanifolds

$$N = \pi(F_1) = \pi(F_{-1}), \qquad C = \pi(F_a) = \pi(F_{-a}).$$

By Lemma 33.2, N is a homotopy $(n-2)$-sphere; C is a circle. Fix orientations on X, N, C. For disjoint closed oriented submanifolds $A, B \subset X$ of positive dimensions with $\dim A + \dim B = n - 1$, write $\mathrm{lk}(A, B)$ for their integral linking number, defined by intersecting B with an integral chain bounding A. For an isometry f of X, put $\delta_f(x) = d(x, fx)$, where d is the Riemannian distance. Choose $b \in \{1, \dots, p - 1\}$ with $ab \equiv 1 \pmod{p}$.

Theorem 34.1. *One has* $|\mathrm{lk}(N, C)| = 1$. *The inclusions*

$$N \longrightarrow X \setminus C, \qquad C \longrightarrow X \setminus N$$

are respectively a homotopy equivalence and an integral homology equivalence.

At least one of the following conditions holds for some integers q, r. *If*

$$1 \leq q < r \leq H, \qquad |aq|_p < q < r < |ar|_p, \tag{34.1}$$

then, for $f = h^q$,

$$\min_{x \in X} \delta_f(x) = |aq|_p \ell, \qquad \operatorname*{argmin}_{x \in X} \delta_f(x) = C,$$

$$\max_{x \in X} \delta_f(x) = q\ell, \qquad \operatorname*{argmax}_{x \in X} \delta_f(x) = N.$$

If

$$1 \leq q < r \leq H, \qquad |bq|_p < q < r < |br|_p, \tag{34.2}$$

then, for $f = h^{bq}$,

$$\min_{x \in X} \delta_f(x) = |bq|_p \ell, \qquad \operatorname*{argmin}_{x \in X} \delta_f(x) = N,$$

$$\max_{x \in X} \delta_f(x) = q\ell, \qquad \operatorname*{argmax}_{x \in X} \delta_f(x) = C.$$

In either case, $u_f = \frac{1}{2}\delta_f^2$ *is smooth near its maximum set. Its Hessian has that set's tangent space as kernel and is negative definite on the normal space.*

Lemma 34.2. *Let $p \geq 11$ be prime, put $H = (p-1)/2$, and let $c \in \mathbb{F}_p^{\times} \setminus \{1, -1\}$. Define*

$$\pi_c(i) = |ci|_p, \qquad \pi_{c^{-1}}(i) = |c^{-1}i|_p, \qquad 1 \leq i \leq H.$$

Then some $\pi \in \{\pi_c, \pi_{c^{-1}}\}$ and integers $1 \leq q < r \leq H$ satisfy $\pi(q) < q < r < \pi(r)$.

Proof. Multiplication by c permutes the pairs $\{i, -i\}$, so $\pi_{c^{-1}} = \pi_c^{-1}$. Neither permutation has a fixed point, since $ci = \pm i$ would give $c = \pm 1$. Suppose neither has the required pair. For either permutation π, an index q with $\pi(q) < q$ cannot precede an index r with $\pi(r) > r$. Since there are no fixed points, one has

$$\{i \,:\, \pi_c(i) > i\} = \{1, \dots, R\}, \qquad \{j \,:\, \pi_c^{-1}(j) > j\} = \{1, \dots, S\}.$$

Both sets are nonempty and proper: a fixed-point-free permutation of $\{1, \dots, H\}$ sends 1 above itself and H below itself. For $j = \pi_c(i)$, one has

$$i \leq R \iff j > i \iff \pi_c^{-1}(j) < j \iff j > S.$$

Hence $R + S = H$ and $\pi_c(\{1, \dots, R\}) = \{H - R + 1, \dots, H\}$. Replacing c by c^{-1} if necessary, assume $1 \leq R \leq H/2$.

Regard the following integer intervals as subsets of $\mathbb{F}_p$:

$$A = \{-R, \dots, R\}, \qquad J = \{H - R + 1, \dots, H + R\}, \qquad B = cA = \{0\} \cup J.$$

For subsets $U, V \subset \mathbb{F}_p$, write $U + V = \{u + v \,:\, u \in U,\ v \in V\}$. Then

$$A + A = \{-2R, \dots, 2R\}, \qquad J + J = \{-(2R-1), \dots, 2R-1\}, \qquad B + B = c(A + A) = J \cup (J + J).$$

Since $4R + 1 \leq p$, one has $|A + A| = 4R + 1$. The intervals J and $J + J$ are disjoint if $3R \leq H$; their union is $\mathbb{F}_p$ if $3R \geq H + 1$. Thus

$$4R + 1 = |B + B| = \min(p, 6R - 1).$$

It follows that $R = 1$ or $p = 4R + 1$. If $R = 1$, then $c = \pm H$ in $\mathbb{F}_p$. Since $H \geq 5$,

$$\pi_c(2) = 1 < 2 < 3 < H - 1 = \pi_c(3),$$

contrary to the assumption.

Suppose $p = 4R+1$. Then multiplication by c sends $A \setminus \{0\}$ to its complement in $\mathbb{F}_p^{\times}$. For $j = 2, 4$, put $S_j = \sum_{x \in A} x^j$. The finite-field power-sum identity [72, Chapter I, §2.1, Lemma, p. 5] gives $\sum_{x \in \mathbb{F}_p^{\times}} x^j = 0$, since $0 < j < p - 1$. Hence $c^j S_j = -S_j$. In $\mathbb{F}_p$ one has $R = -1/4$ and

$$S_2 = \frac{R(R+1)(2R+1)}{3} = -\frac{1}{32}, \qquad S_4 = \frac{R(R+1)(2R+1)(3R^2+3R-1)}{15} = \frac{5}{512}.$$

The denominators and both sums are nonzero because $p \geq 11$. Thus $c^2 = c^4 = -1$, whereas $(c^2)^2 = 1$, a contradiction. □

34.1. **Negative bundles and evaluation.** Let f be an isometry of a closed oriented homotopy n-sphere X. On the twisted path space

$$\Lambda_f X = \{\gamma \in H^1([0,1], X) \,:\, \gamma(1) = f\gamma(0)\},$$

put

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}|^2 \, dt, \qquad \mathrm{ev}(\gamma) = \gamma(0).$$

For $s \in \mathbb{R}$, write $E^s = \{\gamma \,:\, E(\gamma) \leq s\}$. At a critical path γ, let I_γ be the energy Hessian, represented with the H^1 inner product in the proof of Lemma 29.7. Its negative spectral spaces form the negative bundle over any compact Morse–Bott critical manifold.

Lemma 34.3. *Let $F \subset \Lambda_f X$ be a Morse–Bott critical manifold, and suppose that* $\mathrm{ev}|_F$ *is a diffeomorphism onto an embedded submanifold $B \subset X$. Suppose every $\gamma \in F$ minimizes between its fixed endpoints. Let E_γ^- be the negative spectral space of I_γ. Then evaluation induces an injection*

$$E_\gamma^- \longrightarrow T_{\gamma(0)}X / T_{\gamma(0)}B, \qquad Y \longmapsto Y(0) \bmod T_{\gamma(0)}B.$$

It is an isomorphism if $\dim E_\gamma^- = \operatorname{codim} B$.

Proof. Suppose $Y \in E_\gamma^-$ and $Y(0) \in T_{\gamma(0)}B$. Since evaluation restricts to a diffeomorphism on F, there is $J \in T_\gamma F = \ker I_\gamma$ with $J(0) = Y(0)$. The twisted endpoint condition gives $(Y - J)(0) = (Y - J)(1) = 0$. Fixed-endpoint minimality therefore gives

$$0 \le I_\gamma(Y - J, Y - J) = I_\gamma(Y, Y).$$

Negative definiteness on E_γ^- implies $Y = 0$. Equality of dimensions gives the last assertion. □

For a finite singular chain ξ, let $|\xi|$ denote the union of the images of its singular simplices with nonzero coefficients.

Lemma 34.4. *Let $L > 0$ and $c = L^2/2$. Suppose that the critical set of E at and below c is the disjoint union of the full minimum set F_0 and a compact Morse–Bott manifold F, and that F_0 is Morse–Bott. Suppose their evaluations are diffeomorphisms onto disjoint closed connected oriented submanifolds A, B, with*

$$\operatorname{ind}(F) = \operatorname{codim} B = r, \qquad \dim A = r - 1, \qquad 2 \le r \le n - 1.$$

Assume that $d(y, fy) = L$ on B and that every path in F minimizes between its fixed endpoints. Then

$$|\operatorname{lk}(A, B)| = 1, \qquad d(x, fx) \le L \quad (x \in X), \qquad d(x, fx) < L \quad (x \notin B).$$

Proof. Every path in F is a constant-speed minimizing geodesic of length L, so $E(F) = c > \min E$. Let $\nu^- F$ be the negative bundle, with a smooth fiber metric and norm $\|\cdot\|$. For $\epsilon > 0$, let $D_\epsilon(\nu^- F)$ denote its closed disk bundle of radius ϵ. A smooth tubular chart along F restricts, for sufficiently small $\epsilon > 0$, to an embedding

$$\iota : D_\epsilon(\nu^- F) \longrightarrow \Lambda_f X$$

which is the identity on the zero section and whose differential there is the inclusion on $TF \oplus \nu^- F$. Compactness of F and negative definiteness of the Hessian give $\epsilon, \kappa > 0$ such that

$$E(\iota(\gamma, v)) \le c - \kappa\|v\|^2 \qquad (\|v\| \le \epsilon).$$

Put $D = D_\epsilon(\nu^- F)$ and $S = \partial D$. Choose a regular value $c_- < c$ greater than both $\min E$ and $\max_S(E \circ \iota)$. Give $\Lambda_f X$ the complete Riemannian metric of [30, Example 1.2, p. 208]. The energy is smooth, bounded below and satisfies the Palais–Smale condition by Lemma 29.7. Choose a regular $c_0 \in (\min E, c_-)$ sufficiently close to $\min E$. The Morse–Bott attachment at the minimum makes $F_0 \hookrightarrow E^{c_0}$ a homotopy equivalence. The interval $[c_0, c_-]$ has no critical value, so the closed-sublevel deformation theorem [64, Theorem 9.3.2, pp. 193–194] deforms E^{c_-} onto E^{c_0}. Thus $F_0 \hookrightarrow E^{c_-}$ is a homotopy equivalence. Choose a homotopy $R_t : E^{c_-} \to E^{c_-}$, $0 \le t \le 1$, with $R_0 = \mathrm{id}$ and $R_1(E^{c_-}) \subset F_0$.

For every $\eta \in \Lambda_f X$,

$$E(\eta) \ge \tfrac{1}{2} \operatorname{length}(\eta)^2 \ge \tfrac{1}{2} d(\eta(0), f\eta(0))^2. \tag{34.3}$$

Thus a path of energy below c cannot evaluate to B. Fix $\gamma \in F$ and orient its fiber disk D_γ^r. The evaluation of this disk meets B only at the centre. Lemma 34.3 makes this intersection transverse, with sign ± 1. The homotopy R_t moves the boundary of this disk into F_0 through paths in E^{c_-}. Attach the resulting cylinder to the disk and evaluate. This gives a map of pairs

$$\Psi : (D^r, S^{r-1}) \longrightarrow (X, A)$$

whose algebraic intersection with B is ± 1. Let $[\Psi] \in H_r(X, A; \mathbb{Z})$ be the image of the relative fundamental class of the oriented disk. The homology sequence of (X, A) gives an isomorphism

$$\partial : H_r(X, A; \mathbb{Z}) \longrightarrow H_{r-1}(A; \mathbb{Z}) = \mathbb{Z}[A],$$

since $H_r(X; \mathbb{Z}) = H_{r-1}(X; \mathbb{Z}) = 0$. Write $\partial[\Psi] = k[A]$, with $k \in \mathbb{Z}$. By the intersection definition of linking,

$$k \operatorname{lk}(A, B) = [\Psi] \cdot [B] = \pm 1.$$

Consequently $|\operatorname{lk}(A, B)| = |k| = 1$.

Put $\nu_X B = TX|_B / TB$. The maps in Lemma 34.3 vary smoothly with γ and give a vector-bundle isomorphism

$$\nu^- F \cong (\operatorname{ev}|_F)^* \nu_X B.$$

The orientations of X and B orient $\nu_X B$; pull this orientation back to $\nu^- F$, and orient F through $\operatorname{ev}|_F$. Then D is a compact oriented n-manifold with boundary S. Choose a finite relative fundamental chain z of (D, S), regarded through ι as a chain in $\Lambda_f X$. Its boundary lies in E^{c-}. Since

$$H_{n-1}(E^{c-}; \mathbb{Z}) = H_{n-1}(A; \mathbb{Z}) = 0,$$

there is a finite singular n-chain w in E^{c-} with $\partial w = \partial z$. Thus $\Gamma = z - w$ is an integral n-cycle, and

$$|\Gamma| \subset E^c, \qquad |\Gamma| \cap E^{-1}(c) \subset F.$$

Fix $y \in B$ and put $\gamma_y = (\operatorname{ev}|_F)^{-1}(y)$. Then

$$(\operatorname{ev} \circ \iota)^{-1}(y) = \{(\gamma_y, 0)\}, \qquad y \notin \operatorname{ev}(|w|).$$

At $(\gamma_y, 0)$ the differential of $\operatorname{ev} \circ \iota$ is invertible: the tangent directions to F map isomorphically onto $T_y B$, and the negative directions map isomorphically onto $T_y X / T_y B$. Hence the image of $\operatorname{ev}_*[\Gamma]$ in $H_n(X, X \setminus \{y\}; \mathbb{Z})$ is a local orientation generator. The local-to-global fundamental-class isomorphism [37, Theorem 3.26(a), p. 236] gives

$$\operatorname{ev}_*[\Gamma] = \pm[X].$$

For every $x \in X$, the punctured manifold has $H_n(X \setminus \{x\}; \mathbb{Z}) = 0$ [37, Proposition 3.29, p. 239]. If $x \notin \operatorname{ev}(|\Gamma|)$, the evaluated cycle would be a cycle in $X \setminus \{x\}$, contrary to $\operatorname{ev}_*[\Gamma] = \pm[X]$. Thus $\operatorname{ev}(|\Gamma|) = X$. For $x \notin B$ and $\eta \in |\Gamma|$ with $\operatorname{ev}(\eta) = x$, one has $\eta \notin F$ and therefore $E(\eta) < c$. Equation (34.3) proves the displacement bounds. □

34.2. **Proof of the linking and extremum theorem.**

Proof of Theorem 34.1. Apply Lemma 34.2 with $c = a$. It gives (34.1) or (34.2).

Suppose first that (34.1) holds, and put $s = |aq|_p$. In $\Lambda_{h^q} X$, the critical lengths below 2π and their evaluated images are

length	$s\ell$	$q\ell$	$(p-q)\ell$	$(p-s)\ell$
image	C	N	N	C.

Equation (33.1) and (34.3) make the first critical manifold the full energy minimum: its arcs attain the lower bound, and any path attaining it is a critical path of that length. Corollary 30.5 gives $\operatorname{argmin}_{x \in X} \delta_{h^q}(x) = C$.

Since $r < |ar|_p$, every length-$r\ell$ arc in F_1 attains the marked minimum for h^r. Its length-$q\ell$ prefix therefore minimizes between its endpoints. Consequently $\delta_{h^q} = q\ell$ on N. Lemma 34.3 bounds the index of this second critical manifold by $\operatorname{codim} N = 2$. The index is even by (33.6). If it were zero, its negative disk bundle would add a disjoint component above the minimum. Every later attachment has even rank; rank zero adds a component, and rank at least two cannot join distinct components. A path joining two components has bounded energy and hence lies below finitely many critical levels,

by Lemma 29.7. This would contradict the connectedness of $\Lambda_{h^q} X$. Thus the index is two. Lemma 34.4, with $A = C$, $B = N$, and $L = q\ell$, gives

$$|\operatorname{lk}(C, N)| = 1, \qquad \max_{x \in X} \delta_{h^q}(x) = q\ell, \qquad \operatorname*{argmax}_{x \in X} \delta_{h^q}(x) = N.$$

Suppose instead that (34.2) holds. Put $g = h^b$ and $s = |b\,q|_p$. The first two critical lengths in $\Lambda_{g^q} X$ are $s\ell, q\ell$, with evaluated images N, C. The marked minimum and (34.3) again identify the first critical manifold with the full minimum. Corollary 30.5, applied with $j = b\,q$, gives its displacement minimum set N, since $|b\,q|_p < q = |a\,b\,q|_p$. For the second critical manifold, one has $s < q < p - s$, so (33.7) gives index $2m = n - 1 = \operatorname{codim} C$. The length-$r\ell$ arcs in F_a attain the marked minimum for g^r, since $r < |b\,r|_p$. Their length-$q\ell$ prefixes therefore minimize between their endpoints, and $\delta_{g^q} = q\ell$ on C. Lemma 34.4, with $A = N$, $B = C$, and $L = q\ell$, proves the second alternative and $|\operatorname{lk}(N, C)| = 1$. The two orders of linking have the same absolute value, so the linking assertion follows in either case.

Let B be the maximum set in the chosen alternative, put $L = q\ell$, and write f for the corresponding isometry. For $x \in B$, the selected arc from x to fx is a proper prefix of a minimizing arc. It is the unique minimizing arc to fx: replacing it by another and retaining the tail would give a minimizing curve; its velocities must agree at the junction, and uniqueness for the geodesic equation identifies the two prefixes. Lemma 13.4 excludes conjugacy at fx. Thus (x, fx) is outside the cut locus, and

$$u_f(y) = \tfrac{1}{2} d(y, fy)^2$$

is smooth near B [63, proof of Lemma 1.1.2, p. 413].

Parametrize the selected arc by unit speed, $\gamma : [0, L] \to X$, and let I be its index form. For $w \in T_x X$, nonconjugacy gives a unique Jacobi field J_w with $J_w(0) = w$ and $J_w(L) = df_x w$. Since $\dot\gamma(L) = df_x \dot\gamma(0)$, its tangential coefficient has equal endpoint values. The Jacobi equation makes this coefficient affine; hence

$$\langle D_t J_w, \dot\gamma \rangle = 0.$$

The first variation of δ_f vanishes at x. The second variation formula for geodesic endpoint variations, followed by polarization, gives

$$\operatorname{Hess}_x u_f(w, z) = LI(J_w, J_z)$$

[63, (1.3.1), p. 416]. Every H^1 field Y satisfying $Y(L) = df_x Y(0)$ decomposes uniquely as $Y = J_{Y(0)} + Z$, where $Z(0) = Z(L) = 0$. Integration by parts gives $I(J_w, Z) = 0$. The Dirichlet form is nonnegative by minimality and has zero kernel by nonconjugacy; it is therefore positive definite. Thus the endpoint form $(w, z) \mapsto I(J_w, J_z)$ has the index and nullity of the full twisted form, namely $\operatorname{codim} B$ and $\dim B$. The critical manifold evaluates diffeomorphically onto B, so its kernel is identified with $T_x B$. The Hessian of u_f is consequently negative definite on the normal space to B.

Integral duality for a compact locally contractible subset of a closed oriented manifold [37, Theorem 3.44, pp. 254–255], together with the relative homology sequences, gives

$$\widetilde{H}_i(X \setminus C; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = n - 2, \\ 0, & i \neq n - 2, \end{cases} \qquad \widetilde{H}_i(X \setminus N; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Here X is an integral homology sphere, C is a circle, and N is a homotopy $(n-2)$-sphere. In each complement, orient the meridian generator as the boundary of an oriented normal disk to the omitted submanifold. The two inclusion homomorphisms send $[N]$ and $[C]$ to $\operatorname{lk}(N, C)$ and $\operatorname{lk}(C, N)$ times these generators, respectively. Indeed, a bounding chain in X has that coefficient under the relative Thom class, and the boundary homomorphism sends a normal disk to its meridian. Both coefficients have absolute value one by linking symmetry. Thus both inclusions are integral homology equivalences.

To prove that $X \setminus C$ is simply connected, take a smooth loop in this complement and a continuous filling disk in X. Extend the filling to a neighborhood of the disk. Smooth approximation [41, Chapter 2, Section 3, Theorem 3.3, p. 57] and transverse approximation [41, Chapter 3, Section 2,

Theorem 2.1, p. 74] give a smooth disk transverse to C, with boundary sufficiently close to the original loop to be homotopic to it in $X \setminus C$. Since $2 + 1 - n < 0$, this disk is disjoint from C. Thus $X \setminus C$ is simply connected. The manifolds N and $X \setminus C$ have CW homotopy type, and N is simply connected because $n - 2 \geq 3$. The homological Whitehead theorem [37, Corollary 4.33, p. 367] therefore makes $N \hookrightarrow X \setminus C$ a homotopy equivalence.

□

34.3. **An upper bound for displacement.**

Theorem 34.5. *Let q be an integer with $1 \leq q \leq H$ and $|aq|_p < q$. Then*

$$\delta_{h^q}(x) \leq q\ell \quad (x \in X), \qquad \delta_{h^q}(x) < q\ell \quad (x \notin N).$$

Proof. Put $s = |aq|_p$. The critical lengths below 2π, their evaluated images, and their indices are

length	$s\ell$	$q\ell$	$(p-q)\ell$	$(p-s)\ell$
image	C	N	N	C
index	0	2	$n-1$	$2n-2.$

The first critical manifold is the full minimum by (33.1) and (34.3). For each $\epsilon \in \{1, -1\}$, one has

$$\#\{k \in \mathbb{Z} : |pk + \epsilon q| < q\} = 0, \qquad \#\{k \in \mathbb{Z} : |pk + \epsilon aq| < q\} = 1.$$

Indeed, the two least lattice distances are $q, p - q$ in the first count and $s, p - s$ in the second, with $s < q < p - q < p - s$. The large-family boundary index is (33.8), as proved in Section 33. The Bott formula (33.3) therefore gives index two at length $q\ell$. Reversal and (33.5) give the two opposite indices. The same formula increases every index by $4m = 2n - 2$ when a full period is added.

Put $c = (q\ell)^2/2$, and let F be the second critical manifold. Evaluation identifies F with N. Choose a sufficiently small negative disk bundle D in the Morse–Bott attachment at c. Its zero section has energy c, and its other points have energy below c. Choose regular values $c_- < c < c_+$ with no other critical value between them and with $\partial D \subset E^{c_-}$. Put $Z = E^{c_-}$ and $Y = E^{c_+}$. The first critical manifold is the full Morse–Bott minimum, and there is no other critical value below c. The minimum-sublevel argument in the proof of Lemma 34.4, with the same complete Riemannian metric and Palais–Smale condition, makes its inclusion in Z a homotopy equivalence.

The rank-two negative bundle over N is orientable, since N is simply connected. The integral Thom isomorphism [10, Chapter VI, Theorem 11.3, pp. 368–369] gives

$$H_n(Y, Z; \mathbb{Z}) \cong H_{n-2}(N; \mathbb{Z}) \cong \mathbb{Z}.$$

Since $H_n(Z; \mathbb{Z}) = H_{n-1}(Z; \mathbb{Z}) = 0$, the pair sequence gives $H_n(Y; \mathbb{Z}) \cong H_n(Y, Z; \mathbb{Z})$. The next attachment has an orientable negative bundle over N of rank $n - 1$. Its relative groups in degrees n and $n + 1$ are $H_1(N; \mathbb{Z})$ and $H_2(N; \mathbb{Z})$, respectively; both vanish because $N \simeq S^{n-2}$ and $n \geq 5$. Every later negative rank is at least $2n - 2 > n + 1$. The pair sequences therefore make all subsequent maps on H_n isomorphisms. Every finite singular chain has bounded energy, so

$$H_n(Y; \mathbb{Z}) \xrightarrow{\cong} H_n(\Lambda_{h^q} X; \mathbb{Z}).$$

Moreover, $\deg(h^q) = 1$ because h^q has odd order. Lemma 29.7 gives

$$\mathrm{ev}_* : H_n(\Lambda_{h^q} X; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z}).$$

Orient the negative bundle and N. Since $\partial D \subset Z$ and $H_{n-1}(Z; \mathbb{Z}) = 0$, the finite-chain construction in the proof of Lemma 34.4 caps D by a chain in Z. Denote the resulting cycle by Γ. Its image in $H_n(Y, Z; \mathbb{Z})$ is a Thom generator, so $[\Gamma]$ generates $H_n(Y; \mathbb{Z})$. Consequently $\mathrm{ev}_*[\Gamma] = \pm[X]$. The support satisfies

$$|\Gamma| \subset E^c, \qquad |\Gamma| \cap E^{-1}(c) \subset F, \qquad \mathrm{ev}(F) = N.$$

Proposition 3.29 of [37, p. 239] gives $\mathrm{ev}(|\Gamma|) = X$, since $H_n(X \setminus \{x\}; \mathbb{Z}) = 0$ for every $x \in X$. For $x \notin N$ and $\eta \in |\Gamma|$ with $\mathrm{ev}(\eta) = x$, one has $\eta \notin F$ and $E(\eta) < c$. Equation (34.3) proves the result.

□

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