SPHERES AND PROJECTIVE SPACES DETERMINED BY THE ONE-FORM SPECTRUM

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Abstract. The ordinary one-form spectrum determines the standard spheres, real, complex and quaternionic projective spaces, and the Cayley plane among closed connected Riemannian manifolds. The same conclusion holds in the complementary degree without an orientability assumption; for the complex projective plane it holds in each degree $0 \le p \le 4$. Above dimension four, partial spectral matchings suffice in degree one and in the complementary degree under quantitative bounds on omitted eigenvalue copies and matching cost. In sufficiently large dimension, a matching after independent finite omissions with error $o(\nu^{-1})$, uniformly as the model eigenvalue $\nu \to \infty$, determines every d-dimensional model in a common set of form degrees. As $d \to \infty$, the lower limit of their proportion among $0, \dots, d$ is at least $\sqrt{8/21}$. Heat identities and geodesic returns give averaged curvature bounds; for nonreal projective targets of dimension greater than four, equality in a periodic index estimate then determines the Jacobi operators.

1. Introduction

Let (M, g) be a closed connected Riemannian manifold. Let d denote exterior differentiation and d^* its L^2-adjoint, and put $\delta = d^*$. For an integer $p \ge 0$, the Hodge Laplacian

$$\Delta_p = d\delta + \delta d$$

acts on ordinary complexified p-forms. Its spectrum, denoted by $\mathrm{Spec}_p(M)$, is counted with multiplicity; zero is included when present. For $p > \dim M$, the space of p-forms is zero and $\mathrm{Spec}_p(M) = \emptyset$. Write Vol for Riemannian volume. For integers $d, n \ge 1$, the model spaces are the sphere and the real, complex, quaternionic and Cayley projective spaces, denoted respectively by

$$S^d(1), \qquad \mathbb{R}\mathrm{P}^d(1), \qquad \mathbb{C}\mathrm{P}^n, \qquad \mathbb{H}\mathrm{P}^n, \qquad \mathbb{O}\mathrm{P}^2. \tag{1.1}$$

The sphere has radius one and the real projective metric is its quotient by the antipodal map. The nonreal projective metrics have maximal sectional curvature four. In particular, $\mathrm{Vol}(\mathbb{C}\mathrm{P}^n) = \pi^n/n!$ [6, Table 1]. The projective lines $\mathbb{C}\mathrm{P}^1$ and $\mathbb{H}\mathrm{P}^1$ are round spheres of radius $1/2$.

Theorem A (One-forms). *Let X be one of the spaces* (1.1)*, and put $d = \dim X$. If a closed connected Riemannian manifold M satisfies*

$$\mathrm{Spec}_p(M) = \mathrm{Spec}_p(X)$$

for $p = 1$ or $p = d - 1$, then M is isometric to X.

No orientability or additional geometric structure is prescribed on M. Weyl's law gives $\dim M = d$. In degree $d - 1$, Hodge star intertwines the ordinary Laplacian with the one-form Laplacian with coefficients in the complexified orientation line. For $\mathbb{C}\mathrm{P}^2$, the conclusion holds in each ordinary degree $0 \le p \le 4$ (Corollary 5.5).

The comparison can be weakened in dimensions greater than four. A partial matching is a bijection between submultisets of the two spectra; thus individual eigenvalue copies are matched. Write (λ, ν) for a matched pair, with ν an eigenvalue of the model. For $R > 0$, let $D(R)$ count the unmatched copies on both sides with eigenvalue at most R^2. For $\eta > 0$, put

$$c_\eta(\lambda, \nu) = \begin{cases} |\lambda - \nu|, & |\lambda - \nu| < \eta, \\ 1 + \min(\lambda, \nu), & |\lambda - \nu| \ge \eta. \end{cases} \tag{1.2}$$

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Theorem B (Partial spectra). *Let X be one of the spaces* (1.1)*, with $d = \dim X > 4$, and let $p \in \{1, d-1\}$. Suppose that the ordinary p-spectra of a closed connected Riemannian manifold M and X admit a partial matching such that*

$$D(R) = o(R^{d-4}), \qquad \sum_{\substack{(\lambda,\nu) \text{ matched} \\ \min(\lambda,\nu)\le R^2}} c_{1/4}(\lambda,\nu) = o(R^{d-2}) \qquad (R \to \infty). \tag{1.3}$$

Then M is isometric to X.

Exact equality satisfies (1.3). Thus Theorem B implies Theorem A for $d > 4$; the lower-dimensional cases are given in Section 5.

Fix now an ordinary form degree p. Consider a bijection between the two spectra after independent finite omissions, with

$$\lim_{R\to\infty} \sup_{\substack{(\lambda,\nu) \text{ matched} \\ \nu\ge R}} (1+\nu)|\lambda-\nu| = 0. \tag{1.4}$$

Set

$$\rho_- = \frac{1}{2}\left(1 - \sqrt{\frac{11+6\sqrt{2}}{21}}\right), \qquad \rho_+ = \frac{1}{2}\left(1 - \sqrt{\frac{11-6\sqrt{2}}{21}}\right), \qquad \rho_0 = \frac{1-1/\sqrt{3}}{2}. \tag{1.5}$$

Theorem C (Proportional degrees). *For every compact set $K \subset (\rho_-, \rho_+) \setminus \{\rho_0\}$ there is an integer $d_K \ge 17$ with the following property. Let $d \ge d_K$, let X be one of the spaces* (1.1) *of dimension d, and let $p \in \{0, \dots, d\}$ satisfy*

$$\frac{\min(p, d-p)}{d} \in K.$$

If the ordinary p-spectra of a closed connected Riemannian manifold M and X admit a matching after independent finite omissions satisfying (1.4)*, then M is isometric to X.*

Let $D_d^* \subset \{0, \dots, d\}$ consist of the degrees for which (1.4) determines every model in (1.1) of dimension d, among closed connected Riemannian manifolds. Theorem C gives

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \ge 2(\rho_+ - \rho_-) = \sqrt{\frac{8}{21}} = 0.617213399848\ldots. \tag{1.6}$$

The Cayley plane, of dimension sixteen, occurs in Theorems A and B.

Earlier results. The first three heat coefficients of the Hodge Laplacians are integrals of curvature polynomials [39, pp. 270, 277, Proposition 2.1]; see also [16]. Patodi also proved that equality of the function and one-form spectra preserves constant sectional curvature and the Einstein condition [39, Proposition 4.1(II)(a),(b), p. 281]. From equality of the function and one-form spectra with those of an Einstein locally symmetric manifold, Ii deduced local symmetry [24, p. 558]. Tanno obtained one-form recognition of round spheres among orientable competitors in dimensions $2, 3$ and 16 through 93, and recognition of $\mathbb{CP}^n$ among Kähler competitors for $8 \le n \le 51$ [46, Theorem A(i) and its corollary, pp. 125–126; Theorem C and its corollary, p. 129]. Tsagas obtained recognition of the round sphere among orientable competitors in degree $p = \lfloor d/3 \rfloor$ for $d \ge 6$, and in degree zero for $2 \le d \le 5$ [48, Theorem 3.1, pp. 379–381, and Corollary 3.2, p. 381]. Bettiol–Lauret–Piccione proved that two manifolds underlying compact rank-one symmetric spaces, each endowed with a homogeneous metric, have the same scalar spectrum if and only if they are isometric [6, Theorem C, p. 4].

Within the Kähler class, Li proved recognition of $\mathbb{CP}^n$ for even degrees $2 \le p \le 2n-2$ satisfying $p^2 - 2np + n(2n-1)/3 \ne 0$ [32, arXiv version, Theorem 1.2]. In particular, the two-form spectrum determines $\mathbb{CP}^n$ in this class for every $n \ge 1$ [32, arXiv version, Corollary 1.4]. Huang–Liu–Xu–Zhi obtained further degree ranges, including proportional ranges as $n \to \infty$, without a parity restriction on p [23, Corollary 2.4 and p. 280]. Their limiting interval strictly contains (ρ_-, ρ_+), with ρ_0 excluded from both proportional ranges; see the final remark in Section 6.

Write $(\lambda_j)_{j\ge1}$ and $(\nu_j)_{j\ge1}$ for the spectra of the competitor and the model, respectively, listed in nondecreasing order with multiplicity. Liu–Xu proved recognition in every degree within the Kähler class [35, Theorem 1]. They also treated comparisons $|\lambda_j-\nu_j|=o(\nu_j^{-1})$ of the ordered spectra for $n\ge2$, and exact equality of the spectral multisets above a finite cutoff for $n\ge3$ [35, Theorem 12 and Proposition 13].

Wu introduced power bounds for differences of ordered scalar eigenvalues [50, p. 1374]. Li–Sun–Zhu obtained curvature characterizations in real dimension d from ordered p-spectra satisfying $|\lambda_j-\nu_j|=O(j^{-\alpha})$, $\alpha>2/d$, under specified degree and dimension restrictions [33, arXiv version, Definition 2.7 and Theorem 2.10]. For a real space form reference, the competing manifold is Einstein or a gradient shrinking Ricci soliton; for a complex space form reference, it is Kähler and its first Chern class is proportional, in real cohomology, to its Kähler class. The competitors in Theorems A–C are arbitrary closed connected Riemannian manifolds. In Theorem B, the unmatched copies need not be finite, and (1.3) bounds the total matching cost.

Curvature and geodesic flow. In Theorems B and C, the spectral hypotheses imply $\dim M=d>4$. The criterion for a common geodesic period in Proposition 2.4 follows from [37, Theorem 3.1]. Write dV_g for the Riemannian density, and let s, Ric and Rm denote the scalar, Ricci and Riemann curvatures of g. Let s_X be the scalar curvature of the model metric g_X. Norms are full tensor norms. Write $\mathrm{Ric}^\circ=\mathrm{Ric}-(s/d)g$, and put

$$J(g)=\int_M\left(|\mathrm{Ric}|^2+\frac{3}{2}|\mathrm{Rm}|^2\right)\,dV_g. \tag{1.7}$$

For round targets with $16\le d\le93$, Tanno's curvature argument and the volume give Theorem B from the first three one-form heat coefficients; see the proof of Proposition 4.9. In the remaining cases of that theorem, spectral multiplicities and half-wave returns give equality of the first three scalar heat coefficients (Section 4). Together with the one-form heat coefficients, these equalities imply

$$\mathrm{Vol}(M)=\mathrm{Vol}(X),\qquad \int_M s\,dV_g=s_X\,\mathrm{Vol}(X),$$
$$J(g)-J(g_X)=-4\int_M(s-s_X)^2\,dV_g\le0$$

by (3.7). For the degrees in Theorem C, the same volume and total scalar curvature are obtained. The coefficients u,v in (6.1) are nonpositive, and Lemma 6.2 gives

$$J(g)-J(g_X)=u\int_M|\mathrm{Ric}^\circ|^2\,dV_g+v\int_M(s-s_X)^2\,dV_g\le0.$$

For $x\in M$ and a unit vector $w\in T_xM$, let $K_w:w^\perp\to w^\perp$ be defined by $\langle K_we,f\rangle=\mathrm{Rm}(e,w,w,f)$ for $e,f\in w^\perp$, and let I denote the identity on $w^\perp$. The angular identities (3.1) bound the first two moments of K_w. For round targets they give $K_w=I$. For the nonreal projective targets, the unit-speed geodesic flow G_t on the unit tangent bundle of M satisfies $G_\pi=\mathrm{id}$. The normal parallel return $H_w:w^\perp\to w^\perp$ over $[0,\pi]$ satisfies

$$H_w^2=I,\qquad \dim\ker(H_w-I)=r,\qquad r=1,3,7$$

for $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$, respectively. Let $K_w(t)$ denote the Jacobi operator along the geodesic with initial velocity w, viewed on $w^\perp$ by parallel transport, and let P_w be the orthogonal projection onto $\ker(H_w-I)$. Lemma 3.3 gives $K_w(t)=I+3P_w$. Spherical-rank rigidity and the volume then give isometry with X (Proposition 3.4).

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2. Spectral data and geodesic returns

2.1. **Curvature and heat coefficients.** Let SM be the unit tangent bundle of M, and let G_t be its unit-speed geodesic flow. Let dV_g be the Riemannian density, and let $d\sigma_x$ be the normalized Riemannian measure on the unit sphere S_xM. The Liouville probability measure is $d\ell = \mathrm{Vol}(M)^{-1} dV_g\, d\sigma_x$. We write Tr for the operator trace and tr for the trace of a finite-dimensional endomorphism. For $w \in S_xM$, define $K_w : w^\perp \to w^\perp$ by $\langle K_w u, z\rangle = \mathrm{Rm}(u, w, w, z)$. The curvature convention is such that $K_w = I$ on the unit sphere; I denotes the identity on the indicated vector space.

Let X be a nonreal projective model of real dimension d. Set $r = 1, 3, 7$ for $X = \mathbb{CP}^n, \mathbb{HP}^n, \mathbb{OP}^2$, respectively. In the normalization of Section 1, K_w has eigenvalue 4 with multiplicity r and eigenvalue 1 with multiplicity $d - 1 - r$; the latter eigenspace may be zero [7, pp. 31, 33, 35 and 41]. Hence

$$\mathrm{Ric}(w, w) = \mathrm{tr}\, K_w = d - 1 + 3r, \qquad s = d(d - 1 + 3r).$$

The projective lines $\mathbb{CP}^1$, $\mathbb{HP}^1$ and $\mathbb{OP}^1$ are round spheres of radius $1/2$ and dimensions $2, 4, 8$, respectively. Every unit-speed geodesic of a nonreal projective model lies in a totally geodesic projective line [7, p. 31], and therefore has prime period π. On $S^d(1)$ and $\mathbb{RP}^d(1)$ the prime periods are 2π and π, respectively; for $d = 1$ these are the lengths of the two circles. For every $u > 0$,

$$\Delta_{p,u^2 g} = u^{-2} \Delta_{p,g}.$$

For comparison with a d-dimensional model, fix an integer $0 \le p \le d$ and put $m = \dim M$. For $p > m$, the space of p-forms is zero and its spectrum is empty. For integers $a \ge 0$ and $b \notin \{0, \dots, a\}$, set $\binom{a}{b} = 0$. Write

$$\mathrm{Tr}(e^{-t\Delta_p}) \sim (4\pi t)^{-m/2} \sum_{j \ge 0} a_{j,p} t^j \qquad (t \downarrow 0). \tag{2.1}$$

The leading coefficient is

$$a_{0,p} = \binom{m}{p} \mathrm{Vol}(M). \tag{2.2}$$

The coefficients are integrals of local invariants of Δ_p [16, Lemma 1.7.4]. Tensoring with a flat Hermitian line equipped with a compatible flat connection leaves these coefficients unchanged: in a local parallel unit frame, the twisted and ordinary operators have the same expression. Equality of nonempty spectra determines the dimension and then every $a_{j,p}$. All curvature norms are real tensor norms; in particular, $|\mathrm{Rm}|^2 = \sum_{i,j,k,l} R_{ijkl}^2$ in an orthonormal frame.

For $m \ge 4$, set $C_0 = \binom{m}{p}$, $C_1 = \binom{m-2}{p-1}$ and $C_2 = \binom{m-4}{p-2}$. Patodi's formulas [39, Proposition 2.1, p. 277], in the form [33, arXiv version, Lemma 3.3, (3.7)–(3.8)], are

$$a_{1,p} = (C_0/6 - C_1) \int_M s\, dV_g, \tag{2.3}$$

$$a_{2,p} = \int_M \left(\alpha |\mathrm{Rm}|^2 + \beta |\mathrm{Ric}|^2 + \gamma s^2\right) dV_g, \tag{2.4}$$

where

$$\begin{aligned} \alpha &= C_0/180 - C_1/12 + C_2/2, \\ \beta &= -C_0/180 + C_1/2 - 2C_2, \\ \gamma &= C_0/72 - C_1/6 + C_2/2. \end{aligned} \tag{2.5}$$

If M is nonorientable, apply these formulas to its orientation double cover. Each heat coefficient and each curvature integral on the cover is twice the corresponding quantity on M.

2.2. **Partial spectral matching.**

Lemma 2.1. *Let X be a model of dimension $d > 4$, and fix $0 \le p \le d$. Let the spectra of the p-form Laplacians on M and X, each possibly tensored with a Hermitian line carrying a compatible flat connection, admit a partial matching. Fix $\eta > 0$ and assume* (1.3) *with $c_{1/4}$ replaced by c_η from* (1.2). *Delete the matched pairs with $|\lambda - \nu| \ge \eta$. Denote the retained multiset of pairs by $\mathcal{R}$, and the unmatched submultisets on the two sides by $\mathcal{U}_M$ and $\mathcal{U}_X$. All counts and sums below include multiplicities. For $y \ge 0$, let $D_0(y)$ count the copies in $\mathcal{U}_M$ and $\mathcal{U}_X$ not exceeding y, and put*

$$S(y) = \sum_{\substack{(\lambda,\nu)\in\mathcal{R}\\ \min(\lambda,\nu)\le y}} |\lambda - \nu|.$$

Then, as $y \to \infty$,

$$D_0(y) = o(y^{d/2-2}), \qquad S(y) = o(y^{d/2-1}).$$

Moreover, as $t \downarrow 0$,

$$\sum_{\lambda\in\mathcal{U}_M} e^{-t\lambda} + \sum_{\nu\in\mathcal{U}_X} e^{-t\nu} = o(t^{2-d/2}),$$
$$\sum_{(\lambda,\nu)\in\mathcal{R}} |e^{-t\lambda} - e^{-t\nu}| = o(t^{2-d/2}),$$

and

$$\dim M = d, \qquad \mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad a_{j,p}(M) = a_{j,p}(X) \quad (0 \le j \le 2).$$

Proof. Set $a = d/2 - 2 > 0$. For a matched pair put $x = \min(\lambda, \nu)$, and let $C(y)$ be the sum of $1 + x$ over the deleted pairs with $x \le y$. The hypothesis gives $C(y) = o(y^{a+1})$. Stieltjes integration by parts gives

$$\#\{\text{deleted pairs} : x \le y\} = \int_{[0,y]} \frac{dC(s)}{1+s} = \frac{C(y)}{1+y} + \int_0^y \frac{C(s)}{(1+s)^2}\, ds = o(y^a).$$

Here $C(0-) = 0$. Indeed, if $C(s) \le \varepsilon(1+s)^{a+1}$ for $s \ge Y$, the sum of the last two terms is at most $O_Y(1) + (1 + a^{-1})\varepsilon(1+y)^a$ for $y \ge Y$. Divide by y^a, let $y \to \infty$, and then let $\varepsilon \downarrow 0$. Each deleted pair contributes at most two unmatched copies through y, and any such pair has $x \le y$. Thus

$$D_0(y) \le D(\sqrt{y}) + 2\#\{\text{deleted pairs} : x \le y\} = o(y^a).$$

On the retained pairs the cost is $|\lambda - \nu|$, so $S(y) = o(y^{a+1})$.

For a nondecreasing right-continuous function F with $F(0-) = 0$ and $F(y) = o(y^b)$, $b > 0$, integration by parts gives

$$\int_{[0,\infty)} e^{-ty}\, dF(y) = t \int_0^\infty e^{-ty} F(y)\, dy = o(t^{-b}).$$

In fact, if $F(y) \le \varepsilon y^b$ for $y \ge Y$, then

$$t^b \int_{[0,\infty)} e^{-ty}\, dF(y) \le t^b F(Y) + \varepsilon \int_0^\infty e^{-u} u^b\, du.$$

First let $t \downarrow 0$ and then $\varepsilon \downarrow 0$. Apply this estimate to D_0 with $b = a$ and to S with $b = a + 1$. The unmatched sum is $\int_{[0,\infty)} e^{-ty}\, dD_0(y)$, whereas

$$\sum_{(\lambda,\nu)\in\mathcal{R}} |e^{-t\lambda} - e^{-t\nu}| \le t \int_{[0,\infty)} e^{-ty}\, dS(y),$$

by $|e^{-t\lambda} - e^{-t\nu}| \le t|\lambda - \nu| e^{-t\min(\lambda,\nu)}$. Both sums are $o(t^{-a})$.

It follows that $(4\pi t)^{d/2}$ times the difference of the two heat traces is $o(t^2)$. The normalized model trace tends to $\binom{d}{p}\operatorname{Vol}(X) > 0$. The normalized trace on M has the same limit, so its form bundle is nonzero and $p \le m$. Its leading term is

$$(4\pi t)^{(d-m)/2}\binom{m}{p}\operatorname{Vol}(M).$$

A finite positive limit forces $m = d$, and equality of the limits gives $\operatorname{Vol}(M) = \operatorname{Vol}(X)$. Comparing the coefficients of t and t^2 in (2.1) proves the remaining equalities. □

Corollary 2.2. *Let X be a model of dimension $d > 4$, and fix $0 \le p \le d$. A comparison* (1.4) *in degree p, with Hermitian lines carrying compatible flat connections permitted on either manifold, satisfies* (1.3) *with $c_{1/4}$ replaced by c_η, for every fixed $\eta > 0$. Consequently the dimension, volume and heat coefficients through order two agree with those of X.*

Proof. The unmatched count is bounded. For any $\varepsilon > 0$, all but finitely many matched pairs satisfy $|\lambda - \nu| \le \varepsilon(1+\nu)^{-1} < \eta$ and $|\lambda - \nu| < 1$. If $\min(\lambda, \nu) \le y$, such a pair has $\nu \le y+1$. Let Δ be the model Laplacian with the prescribed flat twist, and let $N_X(y)$ count its eigenvalue copies not exceeding y. The heat expansion (2.1) gives $N_X(y) \le e\ \operatorname{Tr}(e^{-\Delta/y}) = O(y^{d/2})$ as $y \to \infty$. Stieltjes integration therefore gives

$$\sum_{\nu \le y+1} \frac{1}{1+\nu} = \frac{N_X(y+1)}{y+2} + \int_0^{y+1} \frac{N_X(s)}{(1+s)^2}\, ds = O(y^{d/2-1}).$$

The sum of c_η through y is thus $O_\varepsilon(1) + O(\varepsilon y^{d/2-1})$. Divide by $y^{d/2-1}$, let $y \to \infty$, and then let $\varepsilon \downarrow 0$. Together with $D(R) = O(1) = o(R^{d-4})$, this proves the hypothesis of Lemma 2.1. □

2.3. **Flat lines and half-wave operators.** All flat Hermitian lines below carry compatible flat connections. For such a line $F \to M^m$, let $\Omega^q(M;F)$ denote the smooth F-valued q-forms, and write $\Omega^q(M) = \Omega^q(M;\mathbb{C})$ for ordinary complexified forms. Let d_F be the covariant exterior derivative. Flatness gives $d_F^2 = 0$. Put $\delta_F = d_F^*$ and $\Delta_{q,F} = d_F\delta_F + \delta_F d_F$ for $0 \le q \le m$. For $\lambda > 0$, Hodge decomposition [16, Theorem 1.5.2] gives the isometry

$$\lambda^{-1/2} d_F \,:\, \ker(\Delta_{0,F} - \lambda) \longrightarrow \ker(\Delta_{1,F} - \lambda) \cap \operatorname{im} d_F,$$

with inverse $\lambda^{-1/2}\delta_F$. The orthogonal complement of its image in $\ker(\Delta_{1,F} - \lambda)$ is $\ker(\Delta_{1,F} - \lambda) \cap \operatorname{im}\delta_F$. Let $\mathfrak{o}_M^{\mathbb{C}}$ be the complexified orientation line, with its canonical flat metric connection. Hodge star gives a unitary map

$$*\colon \Omega^{m-1}(M) \longrightarrow \Omega^1(M;\mathfrak{o}_M^{\mathbb{C}}), \qquad * \Delta_{m-1} = \Delta_{1,\mathfrak{o}_M^{\mathbb{C}}} *\,.$$

Let M^m be closed and connected, with $m \ge 2$, and let $E \to M$ be a Hermitian vector bundle of positive rank. For a nonnegative self-adjoint Laplace-type operator L on E, with principal symbol $|\xi|_g^2 I$, set

$$P_{L,c} = (L + c)^{1/2}, \qquad U_{L,c}(t) = e^{-itP_{L,c}}, \qquad c \ge 0.$$

For $L = \Delta_0$, write $P_{c,M}$ and $U_{c,M}(t)$. The map $u \mapsto u\,(dV_g)^{1/2}$ identifies sections with half-density-valued sections. The half-density bundle $|\Lambda^m T^*M|^{1/2}$ is defined without an orientation. Write $\Psi^q_{\mathrm{cl}}(M;E)$ for the classical pseudodifferential operators of order q in this realization, and $\sigma_q(A)$ for the homogeneous principal symbol of $A \in \Psi^q_{\mathrm{cl}}(M;E)$. For $c > 0$,

$$P_{L,c} \in \Psi^1_{\mathrm{cl}}(M;E), \qquad \sigma_1(P_{L,c})(x,\xi) = |\xi|_g I$$

by [27, arXiv version, §3.1, p. 10]. For $c = 0$, the orthogonal projection Π_0 onto $\ker L$ is smoothing by elliptic regularity, and

$$P_{L,0} = (L + \Pi_0)^{1/2} - \Pi_0.$$

Thus the same conclusion holds for $c = 0$. For $c > 0$,

$$P_{L,c} - P_{L,0} = c(P_{L,c} + P_{L,0})^{-1} \in \Psi^{-1}_{\mathrm{cl}}(M; E).$$

The scalar operator $P_{c,M}$ has zero subprincipal symbol. Let $S^*M = \{(x, \xi) \in T^*M : |\xi|_g = 1\}$. The metric identifies SM with S^*M; under this identification, G_t is the Hamiltonian flow of $|\xi|_g$. The same notation denotes its homogeneous extension to $T^*M \setminus 0$, and $H_{|\xi|_g}$ denotes its Hamiltonian vector field.

Since L has compact resolvent, the spectral theorem gives, for every bounded Borel function $f : [0, \infty) \to \mathbb{C}$,

$$f(L) \text{ is compact} \quad \Longleftrightarrow \quad \lim_{\substack{\lambda \in \mathrm{Spec}(L) \\ \lambda \to \infty}} f(\lambda) = 0. \tag{2.6}$$

Proposition 2.3. *Let ∇ be the unitary connection and V the Hermitian endomorphism determined by $L = \nabla^*\nabla + V$. Suppose that $G_T = \mathrm{id}$ for some $T > 0$. For $(x, \xi) \in S^*M$, let $\tau^E_T(x, \xi) : E_x \to E_x$ be parallel transport over $[0, T]$ along the geodesic with initial covector (x, ξ). Then $U_{L,c}(T) \in \Psi^0_{\mathrm{cl}}(M; E)$, and there is a constant $\mu_T \in \mathbb{C}$, independent of $c \geq 0$, such that*

$$\sigma_0\big(U_{L,c}(T)\big)(x, \xi) = \mu_T \, \tau^E_T(x, \xi), \qquad |\mu_T| = 1. \tag{2.7}$$

The constant μ_T is the principal symbol of $U_{c,M}(T)$.

Proof. Since $\sigma_1(P_{L,c}) = |\xi|_g I$ is scalar, the half-wave construction of [13, Theorem 1.1 and p. 44] has scalar phase and matrix amplitudes. It gives a classical Fourier integral operator of order zero with canonical relation the graph of G_t. For bundle-valued half-densities, the subprincipal partial connection of $P_{L,c}$ is induced by ∇ along $H_{|\xi|_g}$ [27, arXiv version, Proposition 3.2]. The endomorphism $V + c$ enters only at order -1. Write τ^E_t for parallel transport over $[0, t]$ along the same geodesic. The principal transport equation [14, Theorem 5.3.1, p. 190] therefore has bundle factor τ^E_t and the same scalar factor as the scalar half-wave. These local equations act on half-densities and hence require no orientation. At $t = T$ the canonical graph is the diagonal, so $U_{L,c}(T) \in \Psi^0_{\mathrm{cl}}(M; E)$. When $c = 0$, replacing L by $L + \Pi_0$ changes the propagator by $(e^{-it} - 1)\Pi_0$, a smoothing operator.

Let α_T be the Maslov index of the period-T Hamiltonian flow. The scalar operator $(T/2\pi)P_{c,M}$ has zero subprincipal symbol and time-2π flow equal to the identity. The scalar return formula [12, proof of Theorem 1.1, p. 510] gives

$$U_{c,M}(T) = e^{-i\pi\alpha_T/2} I \pmod{\Psi^{-1}_{\mathrm{cl}}}.$$

A common period suffices for this formula. The index α_T is locally constant on S^*M, which is connected because M is connected and $m \geq 2$. Thus $\mu_T = e^{-i\pi\alpha_T/2}$ is constant. Neither the Hamiltonian nor the principal transport changes with c, which proves (2.7). □

For the Hodge Laplacian on p-forms, the connection is induced by the Levi-Civita connection. At a positive common period T, let $H_T(v) : T_xM \to T_xM$ be tangent parallel transport along the geodesic with initial vector $v \in S_xM$, over $[0, T]$. Identifying tangent and cotangent spaces by the metric gives

$$\sigma_0\Big(e^{-iT\sqrt{\Delta_p + c}}\Big)(v) = \mu_T \Lambda^p H_T(v). \tag{2.8}$$

Proposition 2.4. *Let M^m be closed and connected, with $m \geq 2$, and let L be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle $E \to M$ of positive rank. Fix $c \geq 0$, $T \in \mathbb{R}$ and $z \in \mathbb{C}$ with $|z| = 1$, and put $U = U_{L,c}(T)$. If*

$$\lim_{t \downarrow 0} t^{m/2} \operatorname{Tr}\big((U - zI)^*(U - zI)e^{-tL}\big) = 0,$$

then $G_T = \mathrm{id}$ and $U - zI \in \Psi^{-1}_{\mathrm{cl}}(M; E)$.

Proof. For $T = 0$, the leading heat coefficient gives $z = 1$. For $T < 0$, taking adjoints replaces (T, z) by $(-T, \bar{z})$ and leaves the squared distances on each eigenspace unchanged. It suffices to consider $T > 0$. Enumerate the eigenvalues λ_j of L with multiplicity, and put $\mu_j = \sqrt{\lambda_j + c}$ and $\nu_j = \sqrt{\lambda_j + c + 1}$. For $\mu_j \le R$, one has $e^{-\lambda_j/R^2} \ge e^{-1}$. The hypothesis at $t = R^{-2}$ therefore gives

$$\sum_{\mu_j \le R} |e^{-iT\mu_j} - z|^2 = o(R^m).$$

For every $\delta > 0$,

$$\#\{j \,:\, \mu_j \le R,\ |e^{-iT\mu_j} - z| \ge \delta\} \le \delta^{-2} \sum_{\mu_j \le R} |e^{-iT\mu_j} - z|^2 = o(R^m).$$

Moreover,

$$|e^{-iT\nu_j} - e^{-iT\mu_j}| \le T(\nu_j - \mu_j) = \frac{T}{\nu_j + \mu_j} \longrightarrow 0.$$

Fix $0 < \kappa < 1$. Choose $\delta > 0$ with $2\delta < \sqrt{2(1-\kappa)}$. Apart from finitely many indices, the last difference is less than δ. Since $\nu_j \le R$ implies $\mu_j \le R$, all but $o(R^m)$ of the copies with $\nu_j \le R$ satisfy

$$|e^{iT\nu_j} - \bar{z}| < \sqrt{2(1-\kappa)}, \qquad \operatorname{Re}(z e^{iT\nu_j}) > \kappa.$$

Weyl's law gives $\#\{j \,:\, \nu_j \le R\} \sim CR^m$ for some $C > 0$. Choose $\beta \in \mathbb{R}$ with $e^{-i\beta} = z$ and apply [37, Theorem 3.1, author's version] to $P_{L,c+1} = (L + c + 1)^{1/2} > 0$. The theorem gives $G_T = \mathrm{id}$, and $e^{iTP_{L,c+1}}$ has a unitary principal symbol v. For an endomorphism A, write $\operatorname{Re} A = (A + A^*)/2$. The same theorem gives $\operatorname{Re}(zv) \ge \kappa I$. Since v is independent of κ, letting $\kappa \uparrow 1$ gives

$$0 \le (v - \bar{z}I)^*(v - \bar{z}I) = 2I - 2\operatorname{Re}(zv) \le 0.$$

Thus $v = \bar{z}I$. Proposition 2.3 shows that U has order zero and the same principal symbol as $U_{L,c+1}(T)$. Hence $\sigma_0(U) = v^* = zI$, so $U - zI \in \Psi^{-1}_{\mathrm{cl}}(M; E)$. □

Proposition 2.5. *Let M^m be closed and connected, with $m \ge 2$, and let L be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle $E \to M$ of positive rank. Fix $c \ge 0$. For $T > 0$ and $z \in \mathbb{C}$, the condition*

$$U_{L,c}(T) - zI \quad \text{is compact on } L^2(M; E) \tag{2.9}$$

implies $|z| = 1$ and $G_T = \mathrm{id}$. If $G_T = \mathrm{id}$, the universal Riemannian cover $p : X \to M$ has finite degree and is compact. Its geodesic flow $\widetilde{G}_t$ satisfies $\widetilde{G}_{2T} = \mathrm{id}$, and $H^*(X; \mathbb{Z})$ is isomorphic, as a graded ring, to the integral cohomology ring of a simply connected compact rank-one symmetric space.

Proof. By (2.6), $e^{-iT\sqrt{\lambda+c}} \to z$ as $\lambda \to \infty$ in $\operatorname{Spec}(L)$; hence $|z| = 1$. For every $\varepsilon > 0$, spectral calculus gives

$$\operatorname{Tr}\big((U_{L,c}(T) - zI)^*(U_{L,c}(T) - zI)e^{-tL}\big) \le O_\varepsilon(1) + \varepsilon^2 \operatorname{Tr}(e^{-tL}).$$

Multiply by $t^{m/2}$, let $t \downarrow 0$, and then let $\varepsilon \downarrow 0$. Since $\operatorname{Tr}(e^{-tL}) = O(t^{-m/2})$, Proposition 2.4 gives $G_T = \mathrm{id}$.

Suppose $G_T = \mathrm{id}$. Every geodesic is closed, so M is a Besse manifold. By [2, Corollary 4.7], its fundamental group is finite. Thus its universal cover X has finite degree and is compact. The subgroup

$$A = \{t \in \mathbb{R} \,:\, G_t = \mathrm{id}\}$$

contains T and has no nonzero element of absolute value less than the injectivity radius of M. Hence $A = T_0\mathbb{Z}$ for a least positive common period T_0, and $T = kT_0$ for an integer $k \ge 1$. For $x \in M$, the proof of [2, Corollary 4.7] shows that the class in $\pi_1(M, x)$ of a geodesic based at x and traversed for time T_0 has square one. Its $2k$-fold iterate, traversed for time $2T$, is therefore null-homotopic. The lift returns to its initial point; its terminal velocity equals its initial velocity because the covering is a local isometry. Thus $\widetilde{G}_{2T} = \mathrm{id}$.

The cover X is a simply connected compact Besse manifold. The Bott–Samelson–McCleary theorem [2, Theorem 1.1], [41, p. 912] gives the asserted integral cohomology ring. □

2.4. **Index and Maslov phase.** Let M^m be closed and connected, with $m \geq 2$, and suppose $G_T = \mathrm{id}$ for some $T > 0$. For $v \in SM$, let γ_v be the unit-speed geodesic with initial velocity v. The periodic Morse index is the index of the Hessian of

$$E_T(\gamma) = \frac{1}{2}\int_0^T |\dot\gamma(t)|^2\,dt, \qquad \gamma \in H^1(\mathbb{R}/T\mathbb{Z}, M).$$

A normal Jacobi field J along γ_v is determined by $(J(0), \nabla_t J(0)) \in v^\perp \oplus v^\perp$. These initial data define tangent vectors to SM. Since $dG_T = I$, every normal Jacobi field is periodic together with its first covariant derivative; their space has dimension $2(m-1)$. The splitting into normal and tangential fields is orthogonal for the index form. On tangential fields $f\dot\gamma_v$, the index form is $\int_0^T |f'|^2\,dt$, with f periodic. Its index is zero and its nullity is one. Thus the full periodic nullity is $2m-1$, and the normal and full indices agree. For nearby closed geodesics, parallel transport along the short minimizing segments joining corresponding points identifies their pullback tangent bundles over $\mathbb{R}/T\mathbb{Z}$ and hence their spaces of H^1 periodic fields. The full index forms vary continuously in these identifications. Constant nullity gives locally constant index by the min–max principle [41, p. 922]. Since SM is connected, the index is constant on SM. The splitting by $\dot\gamma_v$ is defined without a choice of orientation.

Lemma 2.6. *Let M^m be closed and connected, $m \geq 2$. Let $T > 0$, suppose $G_T = \mathrm{id}$, and let b be the common Morse index of these geodesics for the energy on $H^1(\mathbb{R}/T\mathbb{Z}, M)$. For the scalar half-wave with negative time exponent,*

$$\mu_T = \exp\bigl(-i\pi(b+m-1)/2\bigr). \tag{2.10}$$

Proof. Let $\varpi : T^*M \setminus 0 \to M$ be the projection and $V = \ker d\varpi$ the vertical Lagrangian distribution. For $\zeta = (x,\xi) \in S^*M$, the spaces

$$\mathcal{L}_t = (dG_t(\zeta))^{-1} V_{G_t\zeta}, \qquad 0 \leq t \leq T,$$

form a loop in the Lagrangian Grassmannian of $T_\zeta(T^*M \setminus 0)$. With the convention of [13, pp. 68–69, Eqs. (6.13)–(6.16)], its Maslov index is the phase index α_T of Proposition 2.3.

The fixed manifold S^*M is clean and has dimension $2m-1$. The reduced quadratic form Q of [13, Eq. (6.12)] is defined on $\mathbb{R}^{2m-2}$ and has nullity $\dim S^*M - 1 = 2m-2$; hence $Q = 0$. The strictly positive scalar operator $P_{1,M}$ has principal symbol $q(x,\xi) = |\xi|_g$. For $\eta \in T^*_xM$,

$$d^2_\xi q(\eta,\eta) = \frac{|\eta|^2_g}{|\xi|_g} > 0 \qquad (0 \neq \eta \perp \xi).$$

Since $T > 0$, the index formula [13, Eq. (6.16) and pp. 69–70] identifies the periodic Morse index as

$$b = \alpha_T + \operatorname{ind} Q - (m-1) = \alpha_T - (m-1).$$

Proposition 2.3 gives $\mu_T = e^{-i\pi\alpha_T/2} = e^{-i\pi(b+m-1)/2}$. □

Lemma 2.7. *Suppose $G_T = \mathrm{id}$, where $T > 0$. For $v \in SM$, let $H_T(v)$ be tangent parallel transport around the geodesic with initial vector v. Then*

$$\det H_T(v) = \mu_T^2. \tag{2.11}$$

Proof. Let $\zeta \in S^*M$ be the metric dual of v, and let $\mathcal{L}_t$ be the loop in the proof of Lemma 2.6. Identify its ambient symplectic vector space with $\mathbb{C}^m$ so that $\mathcal{L}_0 = \mathbb{R}^m$. Choose a lift $A : [0,T] \to U(m)$ with $\mathcal{L}_t = A(t)\mathbb{R}^m$ and $A(0) = I$; then $A(T) \in O(m)$. Choose a continuous real function θ such that $\det A(t) = e^{i\theta(t)}$ and $\theta(0) = 0$. The description of the Maslov class by the square of the determinant [4, §§1.2–1.4] gives

$$\alpha_T \equiv \theta(T)/\pi \pmod 2, \qquad \det A(T) = (-1)^{\alpha_T}.$$

Since $dG_T = I$, the maps dG_t and the canonical identifications $V_{(x,\xi)} \simeq T^*_xM$ identify the tautological bundle of $\mathcal{L}_t$ with $\gamma_v^* T^*M$ over $\mathbb{R}/T\mathbb{Z}$. Cotangent parallel transport is dual to $H_T(v)^{-1}$ and has the

same determinant as $H_T(v)$. The orientation monodromy of this bundle is therefore both $\det A(T)$ and $\det H_T(v)$. Therefore

$$\det H_T(v) = (-1)^{\alpha_T} = \mu_T^2.$$

□

2.5. **The complex-projective model return.** Let g_{FS} be the Fubini–Study metric on $\mathbb{CP}^n$ with maximal sectional curvature 4.

Lemma 2.8. *Let $n \geq 1$, let p be an integer with $0 \leq p \leq 2n$, and let $c > 0$. On $(\mathbb{CP}^n, g_{\mathrm{FS}})$,*

$$e^{-2\pi i(\Delta_p + c)^{1/2}} - I \in \Psi_{\mathrm{cl}}^{-1}.$$

Proof. Let J be the complex structure and $h : S^{2n+1}(1) \to \mathbb{CP}^n$ the Hopf projection. Every unit-speed geodesic has the form $\gamma = h \circ z$, where

$$z(t) = \cos(t) z_0 + \sin(t) z_1$$

and $z_0, z_1 \in \mathbb{C}^{n+1}$ are orthonormal for the standard Hermitian inner product. Put $v = \dot\gamma(0)$, and denote by ∇^S the Levi-Civita connection of $S^{2n+1}(1)$. A constant vector W complex-orthogonal to z_0, z_1 is horizontal along z and satisfies $\nabla^S_{\dot z} W = 0$. The connection formula for a Riemannian submersion [30, Problem 5-6(b), Eq. (5.27), p. 146] gives

$$\nabla_{\dot\gamma}\big(dh_{z(t)} W\big) = dh_{z(t)}\big(\nabla^S_{\dot z} W\big) = 0.$$

The identities $z(t + \pi) = -z(t)$ and $h(-z) = h(z)$ give $\gamma(t + \pi) = \gamma(t)$ and $dh_{-z_0}(W) = -dh_{z_0}(W)$. Moreover $\nabla J = 0$ and γ is a geodesic, so $\dot\gamma$ and $J\dot\gamma$ are parallel and return at time π. Thus $H_\pi(v)$ is I on $\mathrm{span}_{\mathbb{R}}\{v, Jv\}$ and $-I$ on its orthogonal complement. In particular, $H_{2\pi} = I$.

The metric in [25, §5, p. 529] is $4g_{\mathrm{FS}}$. Rescaling [25, Theorem 5.2(b)] therefore gives the scalar spectrum $\lambda_k = 4k(k + n)$, $k \geq 0$. Since $\sqrt{\lambda_k + n^2} = 2k + n$, one has $U_{n^2, \mathbb{CP}^n}(2\pi) = I$ and hence $\mu_{2\pi} = 1$. Formula (2.8) gives the identity principal symbol in every degree and for every $c > 0$. Therefore $e^{-2\pi i(\Delta_p + c)^{1/2}} - I \in \Psi_{\mathrm{cl}}^{-1}$. □

2.6. **A trace estimate.**

Proposition 2.9. *Let D be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle E over a closed Riemannian m-manifold, $m \geq 1$. Let F be a unitary classical Fourier integral operator on $L^2(M; E)$ of order zero whose canonical relation is the graph of a homogeneous canonical transformation $\chi : T^*M \setminus 0 \to T^*M \setminus 0$ preserving S^*M. If $\mathrm{Fix}(\chi) \cap S^*M$ has Liouville measure zero, then*

$$\lim_{t \downarrow 0} t^{m/2} \operatorname{Tr}(F e^{-tD}) = 0. \tag{2.12}$$

Proof. Write $\|\cdot\|_1$ for the trace norm. The construction of [20, Proposition 1.4], with smooth complex trivializations and a smooth partition of unity, gives a smooth complement E' and an isomorphism $E \oplus E' \simeq M \times \mathbb{C}^N$. Equip E' with a Hermitian metric and make this isomorphism unitary by polar decomposition. Choose a unitary connection ∇' on E', and put

$$\widetilde D = D \oplus \nabla'^* \nabla', \qquad D_0 = \Delta_{0,M} \otimes I_N, \qquad \widetilde F = F \oplus 0.$$

The operators $\widetilde D$ and D_0 have the common Sobolev domain $H^2(M; \mathbb{C}^N)$ and principal symbol $|\xi|_g^2 I_N$. Hence $B = \widetilde D - D_0$ has order at most one, and $(1 + \widetilde D)^{-1/2} B$ and $B(1 + D_0)^{-1/2}$ extend to bounded operators on $L^2(M; \mathbb{C}^N)$. Constants denoted by $C > 0$ depend only on the fixed operators. The heat expansion gives, for $A \in \{\widetilde D, D_0\}$ and $0 < v \leq 1$,

$$\|(1 + A)^{1/2} e^{-vA}\|_1 \leq C v^{-1/2} \operatorname{Tr}(e^{-vA/2}) = O(v^{-(m+1)/2}).$$

Differentiating $e^{-(t-u)\widetilde D} e^{-uD_0}$ for $0 < u < t$ gives Duhamel's identity

$$e^{-t\widetilde D} - e^{-tD_0} = -\int_0^t e^{-(t-u)\widetilde D} B e^{-uD_0}\, du.$$

The two bounded products give, for $0 < t \le 1$,

$$\begin{aligned}\|e^{-t\widetilde{D}} - e^{-tD_0}\|_1 &\le C\int_0^{t/2}(t-u)^{-(m+1)/2}\,du + C\int_{t/2}^{t}u^{-(m+1)/2}\,du\\ &= O(t^{(1-m)/2}).\end{aligned}$$

In particular, the integral converges in trace norm.

For $1 \le a \le N$, the diagonal entry $\widetilde{F}_{aa}$ is a scalar classical Fourier integral operator of order zero associated with χ. Since M has no boundary, it belongs to the class $\mathcal{A}_0$ of [26, §4.5]. Let $\Delta_{0,M}\varphi_j = \lambda_j\varphi_j$ be an orthonormal scalar eigenbasis. The zero-measure case of [26, Theorem 6.2] and the scalar counting bound give

$$A_a(y) := \sum_{\lambda_j \le y}\langle \widetilde{F}_{aa}\varphi_j, \varphi_j\rangle = o(y^{m/2}) \qquad (y \to \infty).$$

Stieltjes integration by parts, with $A_a(0-) = 0$, yields

$$\bigl|\mathrm{Tr}(\widetilde{F}_{aa}e^{-t\Delta_{0,M}})\bigr| \le t\int_0^\infty e^{-ty}|A_a(y)|\,dy = o(t^{-m/2}).$$

Summing over a and applying the trace-norm estimate gives

$$\mathrm{Tr}(Fe^{-tD}) = \mathrm{Tr}(\widetilde{F}e^{-t\widetilde{D}}) = \sum_{a=1}^{N}\mathrm{Tr}(\widetilde{F}_{aa}e^{-t\Delta_{0,M}}) + O(t^{(1-m)/2}).$$

Multiplication by $t^{m/2}$ gives (2.12). □

3. Curvature moments and the periodic index

3.1. **Curvature moments.** For a closed Riemannian manifold (M, g), put

$$J(g) = \int_M \bigl(|\,\mathrm{Ric}\,|^2 + \tfrac{3}{2}|\,\mathrm{Rm}\,|^2\bigr)\,dV_g.$$

Angular brackets denote integration over SM against the Liouville probability measure $d\ell$. For an endomorphism A of a Euclidean space, write $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}(A^*A)$.

Lemma 3.1. *Let (M^d, g) be closed, with $d \ge 2$. Then*

$$\bigl\langle \mathrm{tr}\,K_w\bigr\rangle = \frac{1}{d\,\mathrm{Vol}(M)}\int_M s\,dV_g, \qquad \bigl\langle \mathrm{tr}\,K_w^2\bigr\rangle = \frac{J(g)}{d(d+2)\,\mathrm{Vol}(M)}. \tag{3.1}$$

Let X be a normalized model of dimension d, and put $r = 0$ for $S^d(1)$ and $\mathbb{RP}^d(1)$, and $r = 1, 3, 7$ for $\mathbb{CP}^{d/2}$, $\mathbb{HP}^{d/4}$ and $\mathbb{OP}^2$, respectively. If

$$\mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad \int_M s\,dV_g = s_X\,\mathrm{Vol}(X), \qquad J(g) \le J(g_X), \tag{3.2}$$

then

$$\bigl\langle \mathrm{tr}(K_w - I)\bigr\rangle = 3r, \qquad \bigl\langle \|K_w - I\|_{\mathrm{HS}}^2\bigr\rangle \le 9r. \tag{3.3}$$

Proof. At $x \in M$, choose an orthonormal basis $e_1, \dots, e_d$, write $w = \sum_a w_a e_a$, and set $R_{iabj} = \mathrm{Rm}(e_i, e_a, e_b, e_j)$. Extend K_w to T_xM by $K_w w = 0$. Its matrix entries are $\sum_{a,b} R_{iabj}w_a w_b$. For the probability measure on S_xM, the sphere integrals in [18, proof of Theorem 3.3, pp. 167–168] give

$$\int_{S_xM} w_a w_b\,d\sigma_x = \frac{\delta_{ab}}{d}, \qquad \int_{S_xM} w_a w_b w_c w_e\,d\sigma_x = \frac{\delta_{ab}\delta_{ce} + \delta_{ac}\delta_{be} + \delta_{ae}\delta_{bc}}{d(d+2)}.$$

Together with [18, Lemma 2.2, (2.7), p. 161], these formulas give

$$\begin{aligned}\int_{S_xM} \operatorname{tr} K_w \, d\sigma_x &= \frac{1}{d}\sum_{i,a} R_{iaai} = \frac{s}{d},\\ d(d+2)\int_{S_xM} \operatorname{tr} K_w^2 \, d\sigma_x &= \sum_{i,j}\left(\sum_a R_{iaaj}\right)^2 + \sum_{i,a,b,j} R_{iabj}^2 + \sum_{i,a,b,j} R_{iabj}R_{ibaj}\\ &= |\operatorname{Ric}|^2 + \tfrac{3}{2}|\operatorname{Rm}|^2.\end{aligned}$$

Integration against $dV_g/\operatorname{Vol}(M)$ proves (3.1).

For $w \in SX$, the model Jacobi spectrum in Section 2 gives $\operatorname{tr}(K_w - I) = 3r$ and $\|K_w - I\|_{\mathrm{HS}}^2 = 9r$ on $w^\perp$. For every $w \in SM$, also on $w^\perp$,

$$\|K_w - I\|_{\mathrm{HS}}^2 = \operatorname{tr} K_w^2 - 2\operatorname{tr} K_w + d - 1.$$

Thus (3.2) and (3.1) give (3.3). □

For a closed Riemannian manifold (M^d, g) with $d \geq 2$, integration of Patodi's scalar and one-form formulas [39, Proposition 2.1, (21)–(23), p. 277] gives

$$\begin{aligned}360\,a_{2,0} &= \int_M (5s^2 - 2|\operatorname{Ric}|^2 + 2|\operatorname{Rm}|^2)\, dV_g,\\ 360\,a_{2,1} &= \int_M \big((5d-60)s^2 + (180-2d)|\operatorname{Ric}|^2 + (2d-30)|\operatorname{Rm}|^2\big)\, dV_g.\end{aligned} \tag{3.4}$$

The divergence terms integrate to zero since M is closed. Combining the two formulas gives [39, proof of Proposition 4.1, p. 281]

$$J(g) + 4\int_M s^2\, dV_g = 6\big((60-d)a_{2,0}(M) + a_{2,1}(M)\big). \tag{3.5}$$

Suppose that (X^d, g_X) is closed, has constant scalar curvature s_X, and satisfies

$$a_{j,0}(M) = a_{j,0}(X) \quad (0 \leq j \leq 2), \qquad a_{2,1}(M) = a_{2,1}(X). \tag{3.6}$$

Since $a_{0,0}(M) = \operatorname{Vol}(M)$ and $a_{1,0}(M) = \frac{1}{6}\int_M s\, dV_g$, with the same formulas for X, the equalities for $j = 0, 1$ give the two equalities in (3.2). Subtracting (3.5) for the two metrics gives

$$\begin{aligned}J(g) - J(g_X) &= -4\left(\int_M s^2\, dV_g - s_X^2 \operatorname{Vol}(X)\right)\\ &= -4\int_M (s - s_X)^2\, dV_g \leq 0.\end{aligned} \tag{3.7}$$

3.2. **The index estimate.** Along the unit-speed geodesic γ_v, identify the normal spaces with $v^\perp$ by parallel transport. Write $K_v(t)$ for the matrix of $K_{\dot\gamma_v(t)}$ in this frame; the normal Jacobi equation is $u'' + K_v u = 0$. Put

$$\overline{f} = \frac{1}{\pi}\int_0^\pi f(t)\, dt,$$

and take all L^2 inner products on $[0, \pi]$ with respect to dt/π. Set

$$V_v(t) = K_v(t) - I, \qquad e(v) = \overline{\operatorname{tr}(V_v^2)}.$$

Lemma 3.2. *Let M^d be closed and connected, with $d \geq 2$ and $G_\pi = \mathrm{id}$. Let H_v be normal parallel return through π. Suppose that $H_v^2 = I$ for every $v \in SM$ and that $q = \dim\ker(H_v - I)$ is independent of v. Put $m = d - 1$. The normal Morse index i is constant, and*

$$i + q \leq \frac{32}{135} e(v) \qquad (v \in SM). \tag{3.8}$$

Proof. Fix $v \in SM$, and let H be the normal parallel return. On $\mathcal{H} = L^2([0,\pi];\mathbb{R}^m, dt/\pi)$, the normal index operator has, since $H^{-1} = H$, the form

$$L = -\partial_t^2 - K_v(t),$$
$$\mathcal{D}(L) = \{u \in H^2([0,\pi];\mathbb{R}^m) \,:\, u(\pi) = Hu(0),\ u'(\pi) = Hu'(0)\}.$$

On the two eigenspaces of H, $-\partial_t^2$ is the periodic or antiperiodic Laplacian. Bounded symmetric perturbation by $-K_v$ gives a self-adjoint operator L, bounded below and with compact resolvent. Its kernel consists of the periodic normal Jacobi fields. The discussion preceding Lemma 2.6 gives $\dim\ker L = 2m$ and constancy of the normal index on SM.

Let $D = -\partial_t^2 - I$ have the same domain. Its nonpositive eigenspace W consists of the constant fields in $\ker(H-I)$ and the frequency-one sine and cosine fields in $\ker(H+I)$. Thus $\dim W = 2m - q$. Let P be the orthogonal projection onto $W^\perp$ and put $D_+ = D|_{W^\perp}$. Then $D_+ \geq 3I$, and

$$E = \operatorname{ran} \mathbf{1}_{(-\infty,0]}(L) \cap W^\perp \quad \text{satisfies} \quad \dim E \geq (i+2m) - (2m-q) = i + q.$$

Regard V_v as a multiplication operator on $\mathcal{H}$. The operator

$$C = D_+^{-1/2} P V_v P D_+^{-1/2} \,:\, W^\perp \longrightarrow W^\perp$$

is compact and self-adjoint. For $u \in E \subset \mathcal{D}(D)$, put $z = D_+^{1/2} u$. Then

$$\langle Cz, z\rangle - \|z\|^2 = -\langle Lu, u\rangle \geq 0.$$

The map $u \mapsto z$ is injective. The min–max principle [47, Theorem 4.10, p. 119], applied to $-C$, therefore gives at least $i+q$ eigenvalues of C in $[1,\infty)$, counted with multiplicity.

Let (e_j) be an orthonormal eigenbasis of D_+, with $D_+ e_j = \lambda_j e_j$. Symmetry of $|\langle V_v e_j, e_k\rangle|^2$ in j,k and $2/(xy) \leq x^{-2} + y^{-2}$ give

$$\begin{aligned}\sum_{j,k} \frac{|\langle V_v e_j, e_k\rangle|^2}{\lambda_j \lambda_k} &\leq \sum_j \lambda_j^{-2} \|P V_v e_j\|^2 \\ &\leq \sum_j \lambda_j^{-2} \|V_v e_j\|^2.\end{aligned}$$

The sums are increasing limits of nonnegative finite partial sums. Set $P_+ = (I+H)/2$ and $P_- = I - P_+$. The sum of the squares of the normalized sine and cosine functions of any positive integral frequency is 2. Hence the last sum equals

$$\kappa_+ \overline{\operatorname{tr}(P_+ V_v^2)} + \kappa_- \overline{\operatorname{tr}(P_- V_v^2)},$$

where

$$\kappa_+ = 2\sum_{k\geq 1} (4k^2-1)^{-2}, \qquad \kappa_- = 2\sum_{k\geq 1} \big((2k+1)^2 - 1\big)^{-2}.$$

Termwise comparison gives $\kappa_- < \kappa_+$. Since $4k^2 - 1 \geq 15k^2/4$ for $k \geq 2$,

$$\begin{aligned}\kappa_+ &\leq \frac{2}{9} + \frac{32}{225} \sum_{k\geq 2} k^{-4} \\ &\leq \frac{2}{9} + \frac{32}{225}\left(2^{-4} + \int_2^\infty x^{-4}\, dx\right) = \frac{32}{135}.\end{aligned}$$

Since $\operatorname{tr}(P_\pm V_v^2) \geq 0$, the double sum is finite and at most $(32/135)e(v)$. Thus C is Hilbert–Schmidt and $i + q \leq \|C\|_{\mathrm{HS}}^2 \leq (32/135)e(v)$. □

Lemma 3.3. *Let M^d be closed and connected, with $d \geq 2$ and $G_\pi = \mathrm{id}$. Suppose that its normal parallel return H_v satisfies $H_v^2 = I$ and $\dim\ker(H_v - I) = r \in \{1,3,7\}$ for every $v \in SM$. Put $m = d-1$, and suppose that $m - r$ is even. If*

$$\int_{SM} \overline{\operatorname{tr} V_v}\, d\ell = 3r, \qquad \int_{SM} e(v)\, d\ell(v) \leq 9r,$$

then, on every geodesic,

$$K_v(t) = I + 3P_+,$$

where $P_+ = (I + H_v)/2$ *in the parallel frame.*

Proof. Let i be the normal Morse index. Integrating (3.8) gives $i + r \le (32/135)9r$, hence

$$i \le 17r/15 < r + 1.$$

Since $r \in \{1, 3, 7\}$ and i is an integer, $i \le r$.

Fix $v \in SM$, and write $H = H_v$. Choose an orthogonal complex structure A on $\ker(H + I)$ and set $A = 0$ on $\ker(H - I)$. Then $A^*A = I - P_+$, and $F(t) = \exp(tA)$ is orthogonal with $F(\pi) = H$. Put $B(t) = F(t)^* V_v(t) F(t)$. For $x \in \mathbb{R}^m$, set $u_x(t) = F(t)x$. Since $AH = HA$, $u_x(\pi) = Hu_x(0)$ and $u_x'(\pi) = Hu_x'(0)$. The index form on these fields is

$$\overline{|u_x'|^2 - \langle K_v u_x, u_x\rangle} = -\langle (P_+ + \overline{B})x, x\rangle.$$

The map $x \mapsto u_x$ is injective, so $P_+ + \overline{B}$ has at most i strictly positive eigenvalues. Since $\overline{B} \le P_+ + \overline{B}$, the same is true of $\overline{B}$ by the min–max principle.

Let Q be the orthogonal projection onto r orthonormal eigenvectors for the largest eigenvalues of $\overline{B}$. All remaining eigenvalues are nonpositive, whence

$$\operatorname{tr}(Q\overline{B}) \ge \operatorname{tr}\overline{B} = \overline{\operatorname{tr} V_v} =: \tau(v).$$

It follows that

$$\begin{aligned} 0 \le \overline{\|B - 3Q\|_{\mathrm{HS}}^2} &= e(v) - 6\operatorname{tr}(Q\overline{B}) + 9r \\ &\le e(v) - 6\tau(v) + 9r. \end{aligned}$$

Thus $h(v) = e(v) - 6\tau(v) + 9r$ is nonnegative. The functions e and τ are time averages of the smooth functions $\|K_w - I\|_{\mathrm{HS}}^2$ and $\operatorname{tr}(K_w - I)$ on SM; hence h is continuous. The hypotheses imply $\int_{SM} h\, d\ell \le 0$. Liouville measure has full support, so $h = 0$ on SM. On the fixed orbit, $\overline{\|B - 3Q\|_{\mathrm{HS}}^2} = 0$; continuity gives $B(t) = 3Q$ for every t.

The positive semidefinite matrix $P_+ + 3Q$ has rank at most $i \le r$ and kernel $\ker P_+ \cap \ker Q$. Both projections have rank r, whereas $\dim(\operatorname{ran} P_+ + \operatorname{ran} Q) = \operatorname{rank}(P_+ + 3Q) \le r$. Hence $Q = P_+$. Since $F(t)$ fixes $\operatorname{ran} P_+$ pointwise,

$$V_v(t) = F(t)(3P_+)F(t)^* = 3P_+.$$

Since $K_v(t + \pi) = HK_v(t)H$, the identity holds for all $t \in \mathbb{R}$. □

3.3. **The metric.** Let M^d be closed and connected, with $d \ge 2$ and $\sec_g \equiv 1$. Then, up to isometry,

$$\begin{aligned} \operatorname{Vol}(M) = \operatorname{Vol}(S^d(1)) \quad &\Longrightarrow \quad M \cong S^d(1), \\ \operatorname{Vol}(M) = \tfrac{1}{2}\operatorname{Vol}(S^d(1)) \quad &\Longrightarrow \quad M \cong \mathbb{R}\mathrm{P}^d(1). \end{aligned} \tag{3.9}$$

By the Killing–Hopf theorem [30, Theorem 12.4 and Corollary 12.5, pp. 348–349], M is isometric to $S^d(1)/\Gamma$, where $\Gamma \subset O(d+1)$ is finite and acts freely on the sphere. Moreover, $\operatorname{Vol}(M) = \operatorname{Vol}(S^d(1))/|\Gamma|$. Thus $|\Gamma| = 1$ or 2, respectively. In the second case, the nonidentity element $A \in \Gamma$ satisfies $A^2 = I$. Since its action on $S^d(1)$ is free, $\ker(A - I) = 0$ and $A = -I$.

Proposition 3.4. *Let* X^d *be a normalized model, with* $d > 4$*, and let* (M^d, g) *be closed and connected. Suppose that* (3.2) *holds. If* X *is nonreal projective, suppose also that* $G_\pi = \mathrm{id}$ *and that normal parallel return* H_v *through* π *satisfies* $H_v^2 = I$ *and* $\dim\ker(H_v - I) = r$ *for every* $v \in SM$*, where* $r = 1, 3, 7$ *for the complex, quaternionic and octonionic models, respectively. Then* (M, g) *is isometric to* X*.*

Proof. Lemma 3.1 gives (3.3). For a round model, $r = 0$, so the continuous nonnegative function $\|K_w - I\|_{\mathrm{HS}}^2$ has zero Liouville integral. Hence $K_w = I$ for all $w \in SM$. Thus M has constant sectional curvature one, and (3.9) gives the isometry.

Now let X be nonreal projective. Its value of $d - 1 - r$ is even. Invariance of Liouville measure under G_t gives

$$\int_{SM} \overline{\operatorname{tr} V_v}\, d\ell = \big\langle \operatorname{tr}(K_w - I) \big\rangle, \qquad \int_{SM} e(v)\, d\ell(v) = \big\langle \|K_w - I\|_{\mathrm{HS}}^2 \big\rangle.$$

Lemma 3.3 therefore gives $K_v(t) = I + 3P_+$ in a parallel frame along every geodesic. Let $(Y, \widetilde{g})$ be the Riemannian universal cover of (M, g). It is complete and simply connected. Let γ be a $\widetilde{g}$-unit-speed geodesic, and write $K(t)$ for its Jacobi matrix in a parallel frame. For the metric $4\widetilde{g}$, $\widetilde{\gamma}(s) = \gamma(s/2)$ has unit speed, and its Jacobi matrix is $\frac{1}{4}K(s/2)$. Thus its normal eigenvalues are 1 and $1/4$, with parallel eigenspaces of multiplicities r and $d - 1 - r$. In particular, $\sec \le 1$, and every geodesic admits a nonzero parallel normal field spanning with its velocity a plane of sectional curvature one. By [42, Corollary 2, p. 67], $(Y, 4\widetilde{g})$ is a compact rank-one symmetric space. Since $0 < r < d-1$, the dimension and the two Jacobi multiplicities identify $(Y, \widetilde{g})$ with X in the original normalization [7, p. 41]. The cover $Y \longrightarrow M$ has a finite number N of sheets because Y is compact, and

$$N = \frac{\operatorname{Vol}(Y, \widetilde{g})}{\operatorname{Vol}(M, g)} = 1.$$

The covering map is therefore an isometry. □

4. One-form recovery

4.1. **Commuting operators.** For a Hermitian bundle $E \to M$ over a closed Riemannian manifold, write $H^a(E) = H^a(M; E)$ for the Sobolev space of order $a \in \mathbb{R}$. Operator norms are taken between the indicated Hilbert spaces.

Lemma 4.1. *Let $P = (L + c)^{1/2}$, where L is a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle E over a closed Riemannian manifold (M^d, g) and $c \ge 0$. Suppose that $Z \in \Psi^0_{\mathrm{cl}}$ and P have a common orthonormal eigenbasis, with eigenvalues (ω_j, z_j). For an integer $q \ge 0$, suppose that, for every $u > 0$,*

$$\#\{j : \omega_j \le R,\ \omega_j^q |z_j| > u\} = o(R^d) \qquad (R \to \infty). \tag{4.1}$$

Then $Z \in \Psi^{-q-1}_{\mathrm{cl}}$.

Proof. For $d = 0$ every operator is smoothing. Assume $d \ge 1$. Suppose $0 \le k \le q$ and $Z \in \Psi^{-k}_{\mathrm{cl}}$. The operator $B = P^k Z \in \Psi^0_{\mathrm{cl}}$ has eigenvalues $\omega_j^k z_j$. For $R \ge 0$ and $u > 0$, put

$$N_P(R) = \#\{j : \omega_j \le R\}, \qquad F_B(R) = \operatorname{Tr}(B^* B \mathbf{1}_{[0,R]}(P)),$$
$$E_u(R) = \#\{j : \omega_j \le R,\ \omega_j^q |z_j| > u\}.$$

For $\omega_j \ge 1$, the inequality $\omega_j^k |z_j| > u$ implies $\omega_j^q |z_j| > u$. The frequencies below one form a finite set. Thus, for every $u > 0$,

$$F_B(R) \le u^2 N_P(R) + \|B\|^2 E_u(R) + O(1).$$

The heat formula [31, Eqs. (1.22)–(1.23)] gives

$$N_P(R) \le e \operatorname{Tr}(e^{-P^2/R^2}) = O(R^d) \qquad (R \ge 1).$$

Since $E_u(R) = o(R^d)$, dividing the bound for F_B by R^d and letting first $R \to \infty$ and then $u \downarrow 0$ gives $F_B(R) = o(R^d)$. Extend F_B by zero for $R < 0$. Stieltjes integration and the substitution $x = \sqrt{t} R$ give

$$t^{d/2} \operatorname{Tr}(B^* B e^{-tP^2}) = 2 \int_0^\infty x e^{-x^2} t^{d/2} F_B(x/\sqrt{t})\, dx \longrightarrow 0.$$

Indeed, $F_B(R) \le C(1+R)^d$, so dominated convergence applies. With weight B^*B, the same heat formula identifies this limit with

$$(4\pi)^{-d/2}\operatorname{Vol}(M)\int_{S^*M}\operatorname{tr}\big(\sigma_0(B)^*\sigma_0(B)\big)\,d\ell.$$

The integrand is continuous and nonnegative, and $d\ell$ has full support, so $\sigma_0(B) = |\xi|^k\sigma_{-k}(Z) = 0$. Consequently $Z \in \Psi_{\mathrm{cl}}^{-k-1}$. Iteration for $k = 0, \dots, q$ gives $Z \in \Psi_{\mathrm{cl}}^{-q-1}$. □

Let $E \to M$ be a Hermitian bundle over a closed manifold. An elliptic $A \in \Psi_{\mathrm{cl}}^0(M;E)$ which is invertible on $L^2(E)$ has inverse in Ψ_{cl}^0, and inversion is continuous in the classical topology [3, Example 2, Theorem 5.4, and proof of Theorem 7.6, arXiv version]. The bundle case follows by adding a complementary bundle. For $A \in \Psi_{\mathrm{cl}}^0(M;E)$, let Γ be a positively oriented simple closed contour in its resolvent set. Suppose that $zI - \sigma_0(A)$ is invertible on S^*M for every $z \in \Gamma$. Then the Riesz projection

$$R_\Gamma(A) = \frac{1}{2\pi i}\int_\Gamma (zI - A)^{-1}\,dz \in \Psi_{\mathrm{cl}}^0(M;E)$$

is orthogonal when A is normal.

Lemma 4.2. *Let $E \to M$ be a Hermitian bundle over a closed Riemannian manifold. Let $P \in \Psi_{\mathrm{cl}}^1(M;E)$ be positive and self-adjoint on $H^1(E)$, with principal symbol $|\xi|I$. Let $\Pi \in \Psi_{\mathrm{cl}}^0$ be an orthogonal projection commuting with P. Fix $T > 0$, $\beta \in \mathbb{R}$ and an integer $s \ge 1$. Suppose*

$$W := e^{iT\beta}\Pi e^{-iTP}\Pi + (I - \Pi), \qquad W - I \in \Psi_{\mathrm{cl}}^{-s}.$$

On $\operatorname{ran}\Pi$*, round every sufficiently high eigenvalue of P to the nearest member of $\beta + (2\pi/T)\mathbb{Z}$, and assign positive weights in this lattice to the remaining eigenspaces. The resulting self-adjoint operator N has domain $\Pi H^1(E)$ and satisfies*

$$N = P|_{\operatorname{ran}\Pi} + K|_{\operatorname{ran}\Pi}, \qquad K \in \Psi_{\mathrm{cl}}^{-s}, \qquad K = \Pi K\Pi, \qquad [K,P] = 0.$$

Changing the assigned positive lattice weights on finitely many eigenspaces changes K by a smoothing operator.

Proof. The operator W is unitary and commutes with P and Π. Since $W - I$ is compact, for a sufficiently large R the smoothing projection $F = \mathbf{1}_{[0,R]}(P)$ satisfies

$$\widehat{W} = W(I - F) + F, \qquad \|\widehat{W} - I\| < 1.$$

Increase R so that every nearest lattice point to $\omega > R$ is positive. On $\operatorname{ran}\Pi$, the displayed norm bound excludes the phase -1, so the nearest lattice point is unique. The operator $\widehat{W}$ is unitary, commutes with P and Π, and equals the identity on $\ker\Pi$. Put $A = \widehat{W} - I$. As in the logarithmic construction of [29, Lemma 3.4, p. 248, and proof of Theorem 5.1, p. 255], put

$$K_0 = -\frac{i}{T}\log\widehat{W}, \qquad \log(I + A) = Af(A), \qquad f(z) = \frac{\log(1+z)}{z}, \quad f(0) = 1.$$

Here $A \in \Psi_{\mathrm{cl}}^{-s}$. On the circle $|z| = r$, with $\|A\| < r < 1$, the operator $zI - A$ is elliptic and invertible. Holomorphic functional calculus therefore gives $f(A) \in \Psi_{\mathrm{cl}}^0$ and $K_0 \in \Psi_{\mathrm{cl}}^{-s}$. The principal logarithm of $\widehat{W}$ is skew-adjoint, so $K_0 = K_0^*$, $K_0 = \Pi K_0\Pi$, and $[K_0, P] = 0$. On a high eigenvalue ω of $P|_{\operatorname{ran}\Pi}$, the logarithm is $-iT(\omega - h)$ for the nearest lattice point h; hence $K_0 = h - \omega$ there. For the prescribed low weights h_ω, put

$$S = \sum_{\omega \le R}(h_\omega - \omega)\Pi\mathbf{1}_{\{\omega\}}(P), \qquad K = K_0 + S.$$

Since $K_0F = 0$, the operator $P + K$ has the prescribed eigenvalues on $\operatorname{ran}\Pi$. The operator S is self-adjoint and smoothing, satisfies $S = \Pi S\Pi$, and commutes with P. Bounded self-adjoint perturbation gives $\mathcal{D}(N) = \Pi H^1(E)$. □

Lemma 4.3. *Let N_0, N_1 be self-adjoint operators with compact resolvent on Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1$, with spectra in the same lattice $\beta + \delta\mathbb{Z}$, where $\beta \in \mathbb{R}$ and $\delta > 0$. Let $B : \mathcal{H}_0 \to \mathcal{H}_1$ be bounded and Fredholm, suppose that $B\mathcal{D}(N_0) \subset \mathcal{D}(N_1)$, and suppose that $N_1 B - BN_0$ extends to a compact operator. Then*

$$\dim\ker(N_0 - h) = \dim\ker(N_1 - h)$$

for every lattice weight h outside a finite set.

Proof. Let K be the compact extension of $N_1B - BN_0$. The identity $N_1Bu = BN_0u + Ku$ shows that B is bounded for the graph norms. Differentiation on $\mathcal{D}(N_0)$ and integration give

$$e^{itN_1}Be^{-itN_0} - B = i\int_0^t e^{isN_1}Ke^{-isN_0}\,ds.$$

The integrand is norm-continuous because K is compact. Thus the displayed difference is compact and norm-continuous in t. Set $T = 2\pi/\delta$. The operators $e^{it(N_j-\beta I)}$ define unitary actions of $\mathbb{R}/T\mathbb{Z}$. Compact-group Fredholm averaging [21, §8, pp. 643–644] shows that the norm integral

$$\overline{B} = \frac{1}{T}\int_0^T e^{itN_1}Be^{-itN_0}\,dt$$

is an intertwining Fredholm compact perturbation of B. Its kernel and cokernel are finite-dimensional circle representations, so contain only finitely many weights. For every other weight h, $\overline{B}$ is injective on $\ker(N_0 - h)$ and $\ker(N_1 - h) \subset \operatorname{ran}\overline{B}$. Writing $E_{j,h} = \mathbf{1}_{\{h\}}(N_j)$ for the weight projections, the identity $\overline{B}E_{0,h} = E_{1,h}\overline{B}$ gives surjectivity on the weight spaces as well. □

Let $X_g = H_{|\xi|_g}|_{S^*M}$ be the vector field generating G_t.

Lemma 4.4. *Let (M, g) be a connected Riemannian manifold of dimension $d \geq 3$, let $E \to M$ be a real vector bundle with a metric connection, and let $\pi : S^*M \to M$. Suppose that a real metric line subbundle $\mathcal{L} \subset \pi^*E$ is preserved by geodesic parallel transport. There are a real metric line ℓ on M and an isometric inclusion $j : \pi^*\ell \to \pi^*E$ with image $\mathcal{L}$. For the unique metric connection on ℓ, which is flat, $\nabla_{X_g} j = 0$, where ∇ is the induced connection on* $\operatorname{Hom}(\pi^*\ell, \pi^*E)$.

Proof. The fiber S^{d-1} is connected and simply connected. The homotopy sequence of the sphere bundle gives $\pi_1(S^*M) \simeq \pi_1(M)$ [19, Theorem 4.41, p. 376]. Hence pullback is an isomorphism on $H^1(\,\cdot\,;\mathbb{F}_2)$. Classification by the first Stiefel–Whitney class [20, Proposition 3.10, p. 86] gives a real line ℓ with $\pi^*\ell \simeq \mathcal{L}$. Real line bundles are associated to double coverings. Over a smooth base, these coverings have unique compatible smooth structures, and their covering isomorphisms are smooth. Choose a metric on ℓ and normalize a smooth isomorphism to obtain j. Fix $v \in S^*M$, and write τ_t^E and τ_t^ℓ for parallel transport along the geodesic with initial covector v. The maps $\tau_t^E j_v$ and $j_{G_t v}\tau_t^\ell$ are isometries between the same real lines. Their quotient is a continuous sign equal to one at $t = 0$, so they agree whenever defined. Differentiation at $t = 0$ gives $\nabla_{X_g} j = 0$. □

Lemma 4.5. *Let $Y \to M$ be a finite connected normal Riemannian cover of a closed manifold. Suppose that the geodesic flow on Y has period $T > 0$. Fix $c \geq 0$. If the pullbacks of flat Hermitian lines F_1, F_2 admit parallel unit sections on Y, the operators $e^{-iT\sqrt{\Delta_{0,F_j}+c}}$ have the same complete local symbols modulo $S^{-\infty}$ in common coordinates and parallel unit frames.*

Proof. Let Γ be the deck group, let Δ_Y be the scalar Laplacian, and let K_Y be the kernel of $e^{-iT\sqrt{\Delta_Y+c}}$ relative to dV_Y. For $c = 0$, the finite-rank modification in the proof of Proposition 2.3 changes the return by a smoothing operator. The fixed-time half-wave construction [13, Theorem 1.1 and p. 44] gives a Fourier integral operator whose canonical relation at time T is the diagonal. Thus $e^{-iT\sqrt{\Delta_Y+c}} \in \Psi^0_{\mathrm{cl}}$. Choose an evenly covered ball U and a lift $s : U \to Y$ such that $\overline{s(U)} \cap \overline{\gamma s(U)} = \emptyset$ for $\gamma \neq 1$. In parallel unit frames on the cover, sections of F_j correspond to functions satisfying $f(\gamma y) = \chi_j(\gamma)f(y)$, where $\chi_j : \Gamma \to U(1)$ is a unitary character. Let K_{F_j} be the corresponding twisted

return kernel on U, relative to dV_g in the induced parallel unit frame. Decomposing the integral over the sheets gives

$$K_{F_j}(x,z) = \sum_{\gamma\in\Gamma} \chi_j(\gamma) K_Y(s(x), \gamma\, s(z)).$$

The identity term has coefficient $\chi_j(1) = 1$. Every other term is smooth on $U \times U$, since K_Y is smooth off the diagonal. Thus the two localized kernels differ by a smooth kernel, and their complete symbols agree modulo $S^{-\infty}$. □

Lemma 4.6. *Let L be a nonnegative self-adjoint Laplace-type operator on a Hermitian bundle E over a closed d-manifold, where $d > 4$. Let $H \in \Psi^{-3}_{\mathrm{cl}}$ be self-adjoint. Then*

$$\operatorname{Tr}(e^{-t(L+H)}) - \operatorname{Tr}(e^{-tL}) = O(t^{(5-d)/2}) = o(t^{2-d/2}) \qquad (t \downarrow 0).$$

Proof. Choose $a > \|H\| + 1$ and put $A_0 = L + aI$, $A_1 = L + H + aI$. These are positive self-adjoint classical operators of order two on $H^2(E)$, with principal symbol $|\xi|^2 I$. Their homogeneous symbols of degrees $2, 1, 0, -1, -2$ agree. With common local coordinates, frames, and symbol extensions for small ξ, take the principal extension bounded below by $\max\{\inf \operatorname{Spec} A_0, \inf \operatorname{Spec} A_1\}$. The resolvent construction [17, (2.2) and (3.35), arXiv v4] therefore gives the same first five local heat-kernel terms. The remainder estimate [17, Theorem 2.1(2), (2.6), arXiv v4], with operator order 2 and $M = 5$, is $O(t^{(5-d)/2})$ for both kernels, since $2 - 5 > -d$. Integrating their fiber traces on the diagonal and multiplying by e^{at} gives the trace estimate. Finally, $t^{(5-d)/2} = t^{1/2} t^{2-d/2}$. □

4.2. **Complex projective space.**

Lemma 4.7. *Let M^{2n} be a closed connected Riemannian manifold homotopy equivalent to $\mathbb{CP}^n$, where $n \geq 2$ is even. If its geodesic flow has a positive common period, the scalar return at every positive common period has principal symbol $+1$.*

Proof. Put $d = 2n$. Let T_0 be the least positive common period, b_0 the common free-loop index at T_0, and $B = b_0 + d - 1 > 0$. Degrees in loop homology are shifted by d. Every closed geodesic has index $> 3 - d$, since $d \geq 4$. Corollary 2.14(a),(c) of [8, §2.5, pp. 18–19, arXiv v3] and its proof, with coefficients in $\mathbb{F}_2$, give a homogeneous element θ of shifted degree B in the reduced loop algebra such that $\theta^k \neq 0$ for every $k \geq 1$. The sign local systems are trivial over $\mathbb{F}_2$.

Orient M so that a fixed homotopy equivalence to $\mathbb{CP}^n$ has degree one. Homotopy invariance of the loop product [11, Theorem 1] and its value on $\mathbb{CP}^n$ [10, Theorem 3], with field coefficients as in [9, Corollary 4.6, arXiv v1], give, since $n + 1$ is odd, the shifted algebra over $\mathbb{F}_2$

$$\mathbb{F}_2[c, w, z]/(c^{n+1}, w^2, wc^n, zc^n), \qquad |c| = -2, \quad |w| = -1, \quad |z| = 2n.$$

The reduced algebra is the quotient by $\chi(M)[\mathrm{pt}] = c^n$ over $\mathbb{F}_2$ [8, Example 2.10(a), arXiv v3]; it is

$$\mathbb{F}_2[c, w, z]/(c^n, w^2).$$

The ideal (c, w) is nilpotent and the quotient is $\mathbb{F}_2[z]$. The nonnilpotent homogeneous element θ has nonzero image there, so $2n \mid B$. Since $4 \mid 2n$, Lemma 2.6 gives $\mu_{T_0} = i^{-B} = 1$. Every positive common period is an integer multiple of T_0, and its phase is the corresponding power of μ_{T_0}. □

Proposition 4.8. *Let $X = \mathbb{CP}^n$ have holomorphic sectional curvature four, with $n \geq 3$ and $d = 2n$. Let M be closed and connected, and let E be a flat Hermitian line on M. If the spectra of $\Delta_{1,E}$ on M and Δ_1 on X admit a partial matching satisfying* (1.3) *with $\eta = 1/4$, then M is isometric to X and E has a global parallel unit section.*

Proof. Lemma 2.1 gives $\dim M = d$, $\operatorname{Vol}(M) = V = \pi^n/n!$ and equality of the first three one-form heat coefficients. Delete the pairs with $|\lambda - \nu| \geq 1/4$. Let $\mathcal{R}$ be the retained multiset, and let $\mathcal{U}_M, \mathcal{U}_X$ be the unmatched submultisets. Write $D_0(y)$ for their total count through eigenvalue y, and put

$$S(y) = \sum_{\substack{(\lambda,\nu)\in\mathcal{R} \\ \min(\lambda,\nu)\leq y}} |\lambda - \nu|.$$

Then

$$D_0(y) = o(y^{d/2-2}), \qquad S(y) = o(y^{d/2-1}). \tag{4.2}$$

Put $\epsilon = (-1)^n$.

4.2.1. *The two spectral families.* For type $(1,0)$, the table preceding [25, Theorem 5.2] has parameters $(r,s) = (1,0)$ and $(2,0)$, each with representation multiplicity one. Replacing its index by $k-1$ and multiplying eigenvalues by four gives the normalization fixed by the scalar spectrum $4k(k+n)$ of Section 2. Complex conjugation gives type $(0,1)$. The one-form families are therefore

$$\begin{aligned} A_k &= 4k(k+n), & \sqrt{A_k + n^2} &= 2k+n, \\ B_k &= 4(k+1)(k+n), & \sqrt{B_k + (n-1)^2} &= 2k+n+1 \qquad (k \geq 1). \end{aligned}$$

The scalar representation at A_k has dimension

$$m_k = \binom{k+n}{n}^2 - \binom{k+n-1}{n}^2,$$

the dimension of harmonic polynomials of bidegree (k,k) on $\mathbb{C}^{n+1}$ [25, §7]. It occurs once in each complex one-form type, so its one-form multiplicity is $2m_k$. The identities $B_k - A_k = 4(k+n)$ and $A_{k+1} - B_k = 4(k+1)$ show that the families are disjoint.

Let H_A, H_B be their model heat traces. Then

$$H_A(t) = 2\big(\operatorname{Tr}(e^{-t\Delta_{0,X}}) - 1\big), \qquad H_B(t) = \operatorname{Tr}(e^{-t\Delta_{1,X}}) - H_A(t),$$

and

$$(4\pi t)^{d/2} H_A(t) \longrightarrow 2V, \qquad (4\pi t)^{d/2} H_B(t) \longrightarrow (d-2)V.$$

For $C \in \{A, B\}$, let $\mathcal{R}_C$ consist of the retained pairs whose model eigenvalue belongs to family C, and put

$$\widetilde{H}_C(t) = \sum_{(\lambda,\nu)\in\mathcal{R}_C} e^{-t\lambda}.$$

Lemma 2.1 gives

$$\begin{aligned} |\widetilde{H}_C(t) - H_C(t)| &\leq \sum_{\nu\in\mathcal{U}_X} e^{-t\nu} + \sum_{(\lambda,\nu)\in\mathcal{R}_C} |e^{-t\lambda} - e^{-t\nu}| \\ &= o(t^{2-d/2}). \end{aligned}$$

Consequently the same leading limits hold for $\widetilde{H}_A$ and $\widetilde{H}_B$.

Put

$$P_1 = (\Delta_{1,E} + n^2)^{1/2}, \qquad U(t) = e^{-itP_1}, \qquad \omega = \sqrt{\lambda + n^2}, \quad \omega_* = \sqrt{\nu + n^2}.$$

For every retained pair, $|\omega - \omega_*| < 1/[4(\omega + \omega_*)]$. On the A-family $\omega_* = 2k+n$; on the B-family,

$$\omega_* - (2k+n+1) = \frac{2n-1}{\omega_* + 2k + n + 1}.$$

Hence $e^{-i\pi\omega}$ tends uniformly at high retained frequencies to ϵ on the A-family and to $-\epsilon$ on the B-family. Put $\epsilon_A = \epsilon$, $\epsilon_B = -\epsilon$,

$$H_M(t) = \operatorname{Tr}(e^{-t\Delta_{1,E}}), \qquad H_{\mathrm{miss}}(t) = \sum_{\lambda\in\mathcal{U}_M} e^{-t\lambda},$$

$$r_K = \max_{C\in\{A,B\}} \sup_{\substack{(\lambda,\nu)\in\mathcal{R}_C \\ \nu\geq K}} |e^{-i\pi\omega} - \epsilon_C|.$$

Then $r_K \to 0$ and $|e^{-2\pi i\omega} - 1| \le 2|e^{-i\pi\omega} - \epsilon_C|$ on $\mathcal{R}_C$. For a constant C_K independent of $t > 0$,

$$\begin{aligned}\operatorname{Tr}\big((U(2\pi) - I)^*(U(2\pi) - I)e^{-t\Delta_{1,E}}\big) \\ \le C_K + 4r_K^2 H_M(t) + 4H_{\mathrm{miss}}(t),\\ \Big|\operatorname{Tr}\big(U(\pi)e^{-t\Delta_{1,E}}\big) - \epsilon(\widetilde{H}_A(t) - \widetilde{H}_B(t))\Big| \\ \le C_K + r_K H_M(t) + H_{\mathrm{miss}}(t).\end{aligned}$$

First let $t \downarrow 0$ after multiplication by $t^{d/2}$, then let $K \to \infty$. The first estimate and Proposition 2.4 give $G_{2\pi} = \mathrm{id}$ and $\sigma_0(U(2\pi)) = I$. For a common period T, write $\mathrm{Hol}_T(E)(v)$ for the scalar of modulus one defined by parallel transport in E along the geodesic with initial vector v, over $[0, T]$. By (2.7),

$$\mu_{2\pi}\,\mathrm{Hol}_{2\pi}(E)(v)H_{2\pi}(v) = I.$$

Since $H_{2\pi}(v)v = v$, the scalar factor is one and $H_{2\pi}(v) = I$.

The second estimate gives

$$\lim_{t\downarrow 0}(4\pi t)^{d/2}\operatorname{Tr}\big(U(\pi)e^{-t\Delta_{1,E}}\big) = \epsilon(4-d)V \neq 0.$$

The map G_π is an involution. For any Riemannian metric h on SM, the metric $h + G_\pi^* h$ is G_π-invariant. The fixed set of G_π is a finite union of submanifolds [16, Lemma 4.5.1]. A fixed component of full dimension makes the isometry the identity on connected SM. Otherwise the fixed set has Liouville measure zero, contradicting Proposition 2.9 and the displayed limit. Thus $G_\pi = \mathrm{id}$; cf. [35, §§5.1–5.2, arXiv v1].

Write $H_v = H_\pi(v)$ and $\alpha(v) = \mu_\pi\,\mathrm{Hol}_\pi(E)(v)$. The group law gives $H_v^2 = I$ and $\alpha(v)^2 = 1$. Put $a(v) = \dim_{\mathbb{R}} \ker(H_v - I)$. The functions $a(v)$ and $\alpha(v)$ are continuous and integer-valued on connected SM, so are constant. Denote their values by a and α. If τ_t denotes tangent parallel transport along γ_v, periodicity gives

$$H_{G_t v} = \tau_t H_v \tau_t^{-1}.$$

Thus both eigenspace bundles are preserved by geodesic parallel transport. The weighted heat formula [31, Eqs. (1.22)–(1.23)] gives

$$\epsilon(4-d) = \int_{SM} \operatorname{tr}(\alpha H_v)\, d\ell(v) = \alpha(2a - d).$$

Hence

$$(\alpha, a) = (\epsilon, 2) \quad \text{or} \quad (\alpha, a) = (-\epsilon, d-2). \tag{4.3}$$

Both alternatives have $\det H_v = 1$.

The finite universal cover Y has the integral cohomology of a simply connected compact rank-one space, by Proposition 2.5. Write

$$H^*(Y;\mathbb{Q}) = \mathbb{Q}[u]/(u^{h+1}), \qquad |u| \text{ even}, \quad h|u| = d.$$

A deck transformation D has $D^*u = \zeta u$, $\zeta = \pm 1$, since D^* preserves the integral cohomology lattice. Let $L(D)$ denote its Lefschetz number. If D preserves orientation, then $\zeta^h = 1$ and

$$L(D) = \sum_{j=0}^{h} \zeta^j = \begin{cases} h+1, & \zeta = 1, \\ 1, & \zeta = -1,\ h \text{ even}. \end{cases}$$

The Lefschetz theorem [19, Theorem 2C.3] gives a fixed point. A deck transformation with a fixed point is the identity. Thus the orientation character of $\pi_1(M)$ is injective. Every period-π geodesic has orientation character $\det H_v = 1$, so is nullhomotopic. Its lift to Y is closed with its initial velocity. In particular, $G_\pi^Y = \mathrm{id}$, $\mathrm{Hol}_\pi(E) = 1$ and $\alpha = \mu_\pi$.

4.2.2. *The scalar return sign.* Suppose the second alternative of (4.3) holds. Lift the family (H_v) to SY and put $\mathcal{F}_v = \ker(H_v + I)$. This is a real plane in $v^\perp = T_v S_x Y$. Its restriction to $S_x Y = S^{2n-1}$ is orientable, since $H^1(S^{2n-1}; \mathbb{F}_2) = 0$. Regarded as a complex line, it is trivial, since $H^2(S^{2n-1}; \mathbb{Z}) = 0$ [20, Propositions 3.10 and 3.11]. It would give two independent tangent vector fields. For odd n, the Radon–Hurwitz number of $2n$ is two; Adams's theorem [1, Theorem 1.1] excludes this.

Let n be even. The sphere fibration gives $\pi_1(SY) = 0$, so choose an orientation of $\mathcal{F}$. Reversal $\rho(v) = -v$ satisfies $H_{-v} = H_v^{-1} = H_v$. The resulting identification $\mathcal{F}_v = \mathcal{F}_{-v}$ has constant orientation sign on SY. If it reversed orientation, its restriction to a tangent fibre would descend to a plane bundle over $\mathbb{RP}^{d-1}$ with orientation system $\mathbb{Z}_{\mathrm{sign}}$, the integer local system with monodromy -1. In the cellular cochain complex, the differential from degree two to degree three is multiplication by two, so

$$H^2(\mathbb{RP}^{d-1}; \mathbb{Z}_{\mathrm{sign}}) = 0.$$

This group contains the only obstruction to a section of its unit circle bundle, since $\pi_j(S^1) = 0$ for $j > 1$ [20, §3.3, pp. 101–102]. A unit section would lift to an even map $f : S^{d-1} \to S^{d-1}$ with $f(v) \perp v$. The homotopy $\cos(t)v + \sin(t)f(v)$, $0 \le t \le \pi/2$, gives $\deg f = 1$. But f factors through the oriented double cover $S^{d-1} \to \mathbb{RP}^{d-1}$, so its degree is even. Therefore reversal preserves the orientation of $\mathcal{F}$.

Suppose $H^2(Y; \mathbb{Z}) = 0$. Since Y is simply connected, TY is orientable. As $2 < d - 1$, the Gysin sequence gives $H^2(SY; \mathbb{Z}) = 0$ [19, p. 438]. Regarded as a Hermitian line, $\mathcal{F}$ has a global unit section σ. Let $\varpi : SY \to Y$ be the projection, let ∇^Y be the Levi-Civita connection, and let $P_{\mathcal{F}}$ be the orthogonal projection of $\varpi^* TY$ onto $\mathcal{F}$. Give $\mathcal{F}$ the connection $\nabla^{\mathcal{F}} = P_{\mathcal{F}} \varpi^* \nabla^Y$. Define the real one-form $A \in \Omega^1(SY; \mathbb{R})$ by $\nabla^{\mathcal{F}} \sigma = iA \otimes \sigma$. Geodesic parallel transport preserves $\mathcal{F}$ and has holonomy -1 on it through π. Hence

$$I(v) := \int_{c_v} A \in \pi + 2\pi\mathbb{Z}, \qquad c_v(t) = G_t v, \quad 0 \le t \le \pi.$$

The function I is continuous on connected SY, so is constant.

Reversal intertwines the projected connections because $\varpi\rho = \varpi$ and $P_{\mathcal{F}_{-v}} = P_{\mathcal{F}_v}$. It is complex linear by the orientation conclusion. Write $\sigma(\rho v) = e^{i\theta(v)}\sigma(v)$. A real-valued lift θ exists since SY is simply connected, and $\rho^* A = A + d\theta$. Since $\rho G_t = G_{-t}\rho$,

$$I(v) = \int_{c_v} (A + d\theta) = \int_{c_v} \rho^* A = -I(\rho v).$$

This contradicts constancy and $I(v) \in \pi + 2\pi\mathbb{Z}$.

Thus Y has the integral cohomology ring $\mathbb{Z}[u]/(u^{n+1})$, $|u| = 2$. Since n is even, every deck transformation preserves orientation; the preceding Lefschetz argument gives $M = Y$. Let $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$ be the hyperplane class. Represent u by a map $M \to \mathbb{CP}^\infty$. Since $\dim M = 2n$, cellular approximation gives $f : M \to \mathbb{CP}^n$ with $f^*(x^j) = u^j$ for $0 \le j \le n$ [19, Example 4.50 and Theorems 4.57 and 4.8]. Both spaces have free integral homology, so f is an integral homology equivalence. They are simply connected, so it is a homotopy equivalence [19, Corollary 4.33]. Lemma 4.7 gives $\mu_\pi = 1$, whereas the second alternative gives $\mu_\pi = -\epsilon = -1$. Consequently the first alternative holds. For $v \in SM$, define $\mathcal{L}_v = \ker(H_v - I) \cap v^\perp$. The constant rank and the parallel-transport identity above make $\mathcal{L}$ a smooth line bundle preserved by geodesic parallel transport. Thus

$$\mu_\pi = \epsilon, \qquad \ker(H_v - I) = \mathbb{R}v \oplus \mathcal{L}_v, \qquad \dim_{\mathbb{R}} \mathcal{L}_v = 1. \tag{4.4}$$

4.2.3. *The scalar coefficients.* Let $\pi : S^*M \to M$ be the projection, and identify $\mathcal{L}_v$ with a cotangent line by the metric. Lemma 4.4 gives a real flat metric line ℓ and an isometry

$$j : \pi^*\ell \longrightarrow \mathcal{L} \subset \pi^* T^* M, \qquad \nabla_{X_g} j = 0.$$

For a flat Hermitian line F, put

$$P_F = (\Delta_{0,F} + n^2)^{1/2}, \qquad U_F = e^{-\pi i P_F}.$$

The period loops are nullhomotopic; hence $\sigma_0(U_F) = \epsilon$ by (2.7), and $U_F - \epsilon I \in \Psi_{\mathrm{cl}}^{-1}$. At each positive scalar E-eigenvalue λ, choose a one-form eigenbasis containing the image of a scalar eigenbasis under $d_E/\sqrt{\lambda}$, and assign the matched copies to this basis. At most $D_0(y) + O(1)$ scalar copies through eigenvalue y are unmatched. Since $U_E - \epsilon I$ is compact, no sufficiently high scalar copy is paired with a B_k: its phase would tend to $-\epsilon$. For a retained A_k pair, put $h_k = 2k + n$ and $\omega = \sqrt{\lambda + n^2}$. Then

$$\omega^3 |e^{-\pi i \omega} - \epsilon| \le \pi \frac{\omega^3}{\omega + h_k} |\lambda - A_k| \le \pi R^2 |\lambda - A_k| \qquad (\omega \le R).$$

For every $u > 0$, therefore,

$$\#\{\omega \le R \,:\, \omega^3 |e^{-\pi i \omega} - \epsilon| > u\} \le O(1) + D_0(R^2) + \pi u^{-1} R^2 S(R^2) = o(R^d). \tag{4.5}$$

Lemma 4.1, with $q = 3$, gives $U_E - \epsilon I \in \Psi_{\mathrm{cl}}^{-4}$.

The finite universal cover Y has period π, and every flat Hermitian line becomes parallel trivial on Y. Lemma 4.5 consequently gives

$$U_F - \epsilon I \in \Psi_{\mathrm{cl}}^{-4} \qquad \text{for every flat Hermitian line } F. \tag{4.6}$$

The essential spectrum of $U(\pi)$ is contained in $\{\epsilon, -\epsilon\}$. Choose a circle about ϵ, of radius less than one, in its resolvent set, and let Π be the Riesz projection. Then $\Pi \in \Psi_{\mathrm{cl}}^{0}$ is orthogonal, commutes with P_1, and has principal symbol

$$(v \otimes v + jj^*) \otimes I_E.$$

The orthogonal projection onto exact one-forms is

$$\Pi_e = d_E \Delta_{0,E}^{\dagger} \delta_E,$$

where the dagger denotes the inverse off the kernel. It commutes with P_1 and Π. Hence $\Pi_\ell = \Pi(I - \Pi_e)$ is an orthogonal projection with symbol $jj^* \otimes I_E$.

The operator $Z_A = \Pi(U(\pi) - \epsilon I)\Pi$ has order -1 and vanishes on $\ker \Pi$. At sufficiently high frequency, Π is the identity on every retained A_k copy and zero on every retained B_k copy; cf. [35, Lemmas 9–10, arXiv v1]. Thus the eigenvalues of Z_A satisfy (4.5), with $|e^{-\pi i\omega} - \epsilon|$ replaced by $|z|$. Lemma 4.1 gives $Z_A \in \Psi_{\mathrm{cl}}^{-4}$.

Round the high eigenvalues of P_F and $P_1|_{\operatorname{ran}\Pi}$ to $h_k = 2k + n$, $k \ge 0$. Assign the finitely many remaining eigenspaces weights in the same set. By Lemma 4.2, with $T = \pi$ and $s = 4$, the resulting operators satisfy

$$\begin{aligned} &N_F = P_F + K_F, \qquad K_F \in \Psi_{\mathrm{cl}}^{-4}, \\ &N_A = P_1|_{\operatorname{ran}\Pi} + K_A|_{\operatorname{ran}\Pi}, \qquad K_A \in \Psi_{\mathrm{cl}}^{-4}, \qquad K_A = \Pi K_A \Pi. \end{aligned}$$

The operators N_A, K_A commute with Π_ℓ. Their domains are respectively $\Pi H^1(T^*M \otimes E)$ and $L^2(T^*M \otimes E)$; the domain of N_F is $H^1(F)$. The operators N_F^2 and $\Delta_{0,F}$ have domain $H^2(F)$, and

$$H_F := N_F^2 - n^2 - \Delta_{0,F} = 2P_F K_F + K_F^2 \in \Psi_{\mathrm{cl}}^{-3}. \tag{4.7}$$

Since K_F commutes with P_F, the operator H_F is bounded, self-adjoint, and commutes with $\Delta_{0,F}$.

Put $\ell_{\mathbb{C}} = \ell \otimes_{\mathbb{R}} \mathbb{C}$ and $E' = \ell_{\mathbb{C}} \otimes E$. Extend j complex linearly and quantize $j \otimes I_E$ as $C \in \Psi_{\mathrm{cl}}^{0}(E', T^*M \otimes E)$. Then

$$B = \Pi_\ell C \,:\, L^2(E') \longrightarrow \operatorname{ran}\Pi_\ell$$

is Fredholm, since $B^*B - I$ and $BB^* - \Pi_\ell$ have order -1. The subprincipal connection formula [27, arXiv version, Proposition 3.2 and (12), (14)], applied on half-densities to $(T^*M \otimes E) \oplus E'$, gives

$$\sigma_0(P_1 B - B P_{E'}) = \frac{1}{i} \nabla_{X_g}(j \otimes I_E) = 0.$$

Consequently $P_1B - BP_{E'} \in \Psi_{\mathrm{cl}}^{-1}$ and $N_AB - BN_{E'}$ is compact. Moreover, B maps $H^1(E')$ into $\Pi_\ell H^1(T^*M \otimes E)$. Since Π_ℓ reduces N_A, the restriction $N_\ell := N_A|_{\operatorname{ran}\Pi_\ell}$ is self-adjoint with compact resolvent and domain $\Pi_\ell H^1(T^*M \otimes E)$. Lemma 4.3, applied to $N_{E'}, N_\ell, B$ and the lattice $n + 2\mathbb{Z}$, gives

$$\dim\ker(N_{E'} - h_k) = \dim\bigl(\operatorname{ran}\Pi_\ell \cap \ker(N_A - h_k)\bigr) \qquad (k \gg 1).$$

Put $m_0 = 1$ and $s_k = \dim\ker(N_A - h_k)$. Since $U_E - \epsilon I$ is compact, all sufficiently high exact forms belong to $\operatorname{ran}\Pi$. The isometry $d_E/\sqrt{\lambda}$ intertwines the square roots and their nearest rounded weights. Thus

$$\dim\ker(N_E - h_k) + \dim\ker(N_{E'} - h_k) = s_k \qquad (k \gg 1). \tag{4.8}$$

Every retained high A_k pair has the same rounded weight h_k on both sides, since $|\omega - h_k| \le 1/[4(\omega + h_k)]$ and consecutive weights differ by two. Retained high B_k copies lie outside $\operatorname{ran}\Pi$. Let u_k count the unmatched copies on M in $\operatorname{ran}\Pi$ rounded to h_k, and let v_k count the unmatched model copies at A_k. Then

$$s_k - 2m_k = u_k - v_k \qquad (k \gg 1).$$

For $h_k \le R$, every copy counted by u_k has frequency at most $R + \|K_A\|$, and every copy counted by v_k has eigenvalue at most R^2. Thus, for a fixed $c_0 > \|K_A\|$,

$$\sum_{h_k \le R} |s_k - 2m_k| \le O(1) + D_0((R + c_0)^2) = o(R^{d-4}).$$

For $y \ge 0$, put $R = \sqrt{y + n^2}$. Equation (4.8) and the preceding bound give a cumulative absolute multiplicity discrepancy $o(y^{d/2-2})$ between the two rounded scalar spectra and two copies of the scalar model spectrum. The finitely many remaining weights contribute $O(1)$, and $d/2 - 2 > 0$. Laplace–Stieltjes integration, as in Lemma 2.1, gives

$$\operatorname{Tr} e^{-t(N_E^2 - n^2)} + \operatorname{Tr} e^{-t(N_{E'}^2 - n^2)} - 2\operatorname{Tr} e^{-t\Delta_{0,X}} = o(t^{2-d/2}).$$

Apply Lemma 4.6 to $(\Delta_{0,F}, H_F)$ for $F = E, E'$. The heat coefficients of a flat line twist are the ordinary scalar coefficients. Hence

$$a_{j,0}(M) = a_{j,0}(X) \qquad (0 \le j \le 2).$$

The one-form equalities were obtained at the start of the proof. Equation (3.7) gives $J(M) \le J(X)$. Together with (4.4), these equalities satisfy the hypotheses of Proposition 3.4, with positive normal rank one. Thus M is isometric to X. Since X is simply connected, E has a parallel unit section. □

4.3. **The remaining models.**

4.3.1. *Eigenvalue families.* For a compact symmetric pair G/K with G connected and semisimple, the form Laplacian for the metric induced by minus the Killing form is minus the Casimir [25, Propositions 2.2–2.3]. Rescaling the metric multiplies its eigenvalues by the same factor in every degree. At maximal sectional curvature four, the first positive scalar eigenvalues of $\mathbb{HP}^n$ and $\mathbb{OP}^2$ are $8(n + 1)$ and 48 [6, Table 1].

For the models other than complex projective spaces, the one-form eigenvalues have the form

$$h^2 - \beta_X^2 + \delta, \qquad h \in \beta_X + \mathbb{Z}, \tag{4.9}$$

where

X	β_X	δ
S^d, $\mathbb{RP}^d$ $(d \ge 3)$	$(d-1)/2$	$0,\ d-2$
$\mathbb{HP}^n$ $(n \ge 2)$	$2n+1$	$0,\ 8,\ 4n+1$
$\mathbb{OP}^2$	11	$0,\ 21,\ 24.$

The ranges and parity restrictions are as follows.

On the unit sphere the exact and coexact families are

$$j(j + d - 1), \qquad (j + 1)(j + d - 2), \qquad j \ge 1,$$

respectively [25, Theorem 4.2]. Both have $h = j + \beta_X$. The antipodal map acts by $(-1)^j$ on the exact family, since differentiation commutes with pullback. On the coexact family it acts by $(-1)^{j+1}$: the coefficients of a tangential polynomial one-form have homogeneous degree j [25, Theorem 6.8(3),(4), p. 537], and pullback changes the sign of each coordinate differential. On $\mathbb{RP}^d$, the trivial line retains character 1, and the orientation line retains character $(-1)^{d+1}$. The exact family has the same parity restriction as the corresponding twisted scalar spectrum.

For $\mathbb{HP}^n$, the three one-form series are given in [43, §2, p. 97]. In the minus-Killing metric the scalar eigenvalues are $j(j+2n+1)/(2(n+2))$. Comparing the first positive eigenvalue with $8(n+1)$ gives the scale factor $8(n+2)$. Put $A_j = 4j(j+2n+1)$. Let $e_1, \dots, e_{n+1}$ be a basis of the weight space of $\mathrm{Sp}(n+1)$ with simple roots $e_i - e_{i+1}$ $(1 \le i \le n)$ and $2e_{n+1}$. Write $\sum_i a_i e_i$ as $(a_1, \dots, a_{n+1})$. The series are

eigenvalue	$\mathrm{Sp}(n+1)$ highest weight	range
A_j	$(j, j, 0, \dots, 0)$	$j \ge 1$
$A_{j+1} + 8$	$(j+2, j, 0, \dots, 0)$	$j \ge 0$
$4(j+1)(j+2n+1)$	$(j+1, j, 1, 0, \dots, 0)$	$j \ge 1$.

The branching formula [49, Theorem 1.3, pp. 410–411; §3, p. 418], applied to the complexified isotropy representation, gives multiplicity one for each listed representation in one-forms; see also [28, p. 144, case CII(i), with $p = 1$ and $q = n$]. The scalar type $(j, j, 0, \dots, 0)$ occurs once on functions [49, Theorem 1.3]. Equation (4.9) follows from

$$\begin{aligned} A_j + \beta_X^2 &= (2j + 2n + 1)^2, \\ A_{j+1} + 8 + \beta_X^2 &= (2j + 2n + 3)^2 + 8, \\ 4(j+1)(j+2n+1) + \beta_X^2 &= (2j + 2n + 2)^2 + 4n + 1. \end{aligned}$$

For $\mathbb{OP}^2 = F_4/\mathrm{Spin}(9)$, the one-form decomposition is given in [28, p. 161, case FII]; the scalar and one-form spectra are tabulated in [36, §4.2, Table 2, p. 393]. The first positive scalar eigenvalue there is 12; hence multiplication by four gives the normalization fixed above. With $\beta_X = 11$, the three one-form families are

eigenvalue	eigenvalue $+ \beta_X^2$	range
$4k(k+11)$	$(2k+11)^2$	$k \ge 1$
$4(k^2+13k+18)$	$(2k+13)^2 + 24$	$k \ge 0$
$4(k^2+14k+24)$	$(2k+14)^2 + 21$	$k \ge 0$.

The same table gives multiplicity one for each listed irreducible F_4-representation in one-forms and for each scalar representation in functions. The first family consists of the nonconstant scalar representations.

On $\mathbb{HP}^n$ and $\mathbb{OP}^2$, the spectral Hodge isometry of Section 2.3 is equivariant. It identifies each positive scalar representation with its unique copy in one-forms. Thus, for each model the zero-offset family is the image of the positive scalar spectrum, with the prescribed flat twist on $\mathbb{RP}^d$. At sufficiently large weights it has no intersection with the other families. Indeed, distinct positive weights $h, h' \in \beta_X + \mathbb{Z}$ satisfy $|h^2 - (h')^2| \ge 2\min(h, h') + 1$, whereas the offsets belong to a fixed finite set; equal weights with distinct offsets give distinct eigenvalues. Thus every sufficiently high zero-offset multiplicity is exactly the scalar multiplicity.

4.3.2. *The projective return.* On $\mathbb{CP}^n, \mathbb{HP}^n, \mathbb{OP}^2$, the shifted scalar frequencies are $2j + \beta_X$, $j \ge 0$, with $\beta_X = n, 2n+1, 11$, respectively. Thus the scalar return at π is $(-1)^{\beta_X} I$. The model has $G_\pi = \mathrm{id}$ by Section 2; (2.7) gives scalar phase $(-1)^{\beta_X}$. For $\mathbb{CP}^n$, the tangent return is positive on the velocity and one normal direction by Lemma 2.8. For $\mathbb{HP}^n$ and $\mathbb{OP}^2$, symmetry gives $\nabla \mathrm{Rm} = 0$. Their normal Jacobi operators have eigenvalues four and one of multiplicities r and $d - 1 - r$, where $r = 3$ and 7. Fix $v \in SX$, and let τ_t denote tangent parallel transport along the geodesic with initial velocity v. For $u \in v^\perp$ with $K_v u = q^2 u$, $q = 1, 2$, the Jacobi field with initial data $(u, 0)$ is $J(t) = \cos(qt)\tau_t u$. Since $dG_\pi = I$, $J(\pi) = u$, and hence $\tau_\pi u = (-1)^q u$. Thus tangent parallel transport at π is positive on the

curvature-four space and the velocity, and negative on the curvature-one space. In all three cases the tangent return has trace

$$2 + 2r - d, \qquad r = 1, 3, 7. \tag{4.10}$$

Proposition 4.9. *Let X be S^d, $\mathbb{RP}^d$, $\mathbb{HP}^n$ ($n \geq 2$) or $\mathbb{OP}^2$, with $d = \dim X > 4$. Let M be a closed connected Riemannian manifold. Let F, F_* be the trivial real metric lines on M, X, or their respective orientation lines. The same symbols denote their complexifications. If the spectra of $\Delta_{1,F}$ and Δ^X_{1,F_*} admit a partial matching satisfying* (1.3)*, then M is isometric to X.*

Proof. Lemma 2.1 gives $\dim M = d$, $\mathrm{Vol}(M) = \mathrm{Vol}(X) = V$ and equality of the first three one-form coefficients; flat line twists have the same local coefficients.

If X is round and $16 \leq d \leq 93$, the proof of [46, Theorem A(i), (2.3) and (2.7), pp. 126–128] gives constant sectional curvature on M. The local curvature formulas and their integrals are valid without orientability. Equation (2.3) fixes the curvature value to one, and (3.9) gives $M \cong X$.

For the remaining cases, discard the pairs of discrepancy at least $1/4$. Let $\mathcal{R}$ be the retained multiset of pairs, and let $\mathcal{U}_M, \mathcal{U}_X$ be the unmatched multisets. Write $D_0(y)$ and $S(y)$ as in Lemma 2.1, so that $D_0(y) = o(y^{d/2-2})$ and $S(y) = o(y^{d/2-1})$. Put

$$\beta = \beta_X, \qquad P_1 = (\Delta_{1,F} + \beta^2)^{1/2}, \qquad \zeta = e^{-2\pi i\beta}.$$

For a retained pair $\nu = h^2 - \beta^2 + \delta$, put $\omega = \sqrt{\lambda + \beta^2}$. Then

$$|\omega - h| = \frac{|\delta + \lambda - \nu|}{\omega + h} = O(h^{-1}).$$

Put $U = e^{-2\pi i P_1}$. Since $e^{-2\pi i h} = \zeta$, the retained phases tend uniformly to ζ as $h \to \infty$. The squared-return estimate in Section 4.2.1, with reference phase ζ, and the unmatched heat bound in Lemma 2.1 give

$$t^{d/2} \operatorname{Tr}\big((U - \zeta I)^*(U - \zeta I)e^{-t\Delta_{1,F}}\big) \longrightarrow 0.$$

Proposition 2.4 therefore gives

$$G_{2\pi} = \mathrm{id}, \qquad \sigma_0(e^{-2\pi i P_1}) = \zeta I.$$

For $v \in SM$, write $\mathrm{Hol}_{2\pi}(F)(v)$ for the scalar parallel transport in F along the corresponding periodic geodesic. Equation (2.7) gives

$$\mu_{2\pi} \mathrm{Hol}_{2\pi}(F)(v) H_{2\pi}(v) = \zeta I.$$

Since $H_{2\pi}(v)v = v$, the scalar factor is ζ and $H_{2\pi}(v) = I$. The holonomy of F along this loop is one: for the orientation line it is $\det H_{2\pi}(v) = 1$. Hence $\mu_{2\pi} = \zeta$.

4.3.3. *The zero-offset projection.* Fix a Borel function $f_\beta : [0, \infty) \to (\beta + \mathbb{Z}) \cap (0, \infty)$ which assigns to λ a nearest positive lattice point to $\sqrt{\lambda + \beta^2}$, with a fixed choice at ties. Put $N_1 = f_\beta(\Delta_{1,F})$. The scalar rounded operators below are defined by the same function, so each acts as a scalar on every eigenspace of its Laplacian. Lemma 4.2 gives

$$N_1 - P_1 \in \Psi^{-1}_{\mathrm{cl}}, \qquad Q_1 = P_1^2 - N_1^2 \in \Psi^0_{\mathrm{cl}}.$$

The domains of N_1 and N_1^2 are $H^1(T^*M \otimes F)$ and $H^2(T^*M \otimes F)$, respectively. Since N_1 and P_1 are commuting spectral functions, Q_1 extends to a bounded self-adjoint operator.

The principal symbol $q_1 = \sigma_0(Q_1)$ is real symmetric. Take local orthonormal frames of the real bundle whose complexification is $T^*M \otimes F$, and work on half-densities. For $0 < \hbar \leq 1$, the coefficients of $1, \hbar, \hbar^2$ in the left semiclassical symbol of $\hbar^2 P_1^2$ are real, imaginary and real. The square-root recursion preserves this alternation. Its leading coefficient is $|\xi| I$; the equations for the next two coefficients have scalar coefficient $2|\xi|$, and a term with j symbol derivatives has the factor i^{-j}. For a real phase ϕ with $d_x\phi \neq 0$ and a smooth matrix amplitude b, this gives

$$e^{-i\phi/\hbar}\hbar P_1(e^{i\phi/\hbar}b) = |d_x\phi|b + i\hbar D_1 b + \hbar^2 D_2 b + O(\hbar^3),$$

where D_1, D_2 are differential operators with real coefficients. If $\partial_t\phi + |d_x\phi| = 0$ and $b \sim b_0 + \hbar b_1 + \cdots$, the first two transport equations are

$$(\partial_t - D_1)b_0 = 0, \qquad i(\partial_t - D_1)b_1 = D_2 b_0.$$

Their initial amplitudes $I, 0$ give real b_0 and imaginary b_1; see [29, §2, (2.1)–(2.3), p. 243]. Along a fixed ray, compose finitely many short-time parametrices with real nondegenerate homogeneous phases, using real cutoffs equal to one near its successive images. The composition and left-symbol formulas [22, (2.1.4), p. 104, and (2.1.9), p. 106] preserve this alternation. Under changes of real homogeneous phase, the jth stationary-phase correction is i^j times a real differential expression, after removing the common signature factor [22, Theorem 3.1.6, §3.2, (3.2.5)–(3.2.7), Lemma 3.2.3]. Thus the leading and next amplitudes retain this alternation up to their common Maslov factor. The homogeneous action is zero, since $\xi \cdot \partial_\xi|\xi| - |\xi| = 0$. At the return the first two coefficients in the diagonal left symbol have the form $m(B_0 + i\hbar B_1)$, with B_0, B_1 real and m the common Maslov factor. Since $mB_0 = \zeta I$, m/ζ is real. Consequently

$$\sigma_{-1}\big(\zeta^{-1}e^{-2\pi i P_1} - I\big) \quad \text{is purely imaginary.}$$

The logarithmic formula in Lemma 4.2 gives

$$q_1 = \frac{i|\xi|}{\pi}\sigma_{-1}\big(\zeta^{-1}e^{-2\pi i P_1} - I\big). \tag{4.11}$$

Thus q_1 is real, and self-adjointness makes it symmetric.

Let $\boldsymbol{D}$ be the finite model offset set and put $p(s) = \prod_{\delta\in\boldsymbol{D}}(s-\delta)$. On each sufficiently high retained copy, $N_1 = h$ and the eigenvalue q of Q_1 is $\delta + \lambda - \nu$. Hence $|p(q)| \le C_p|\lambda - \nu|$ for a constant C_p independent of the copy. For every $u > 0$,

$$\#\{\omega \le R \,:\, |p(q)| > u\} \le O(1) + D_0(R^2) + C_p u^{-1} S(R^2) = o(R^d).$$

Lemma 4.1, with $q = 0$, gives

$$\prod_{\delta\in\boldsymbol{D}}(Q_1 - \delta I) \in \Psi^{-1}_{\mathrm{cl}}, \qquad \prod_{\delta\in\boldsymbol{D}}(q_1 - \delta I) = 0.$$

Hence the essential spectrum of Q_1 is contained in $\boldsymbol{D}$. Choose $1/4 < \rho < 1/2$ so that $|z| = \rho$ lies in the resolvent set, and let Π be the corresponding Riesz projection. Its contour avoids the eigenvalues of q_1; thus $\Pi \in \Psi^0_{\mathrm{cl}}$ is orthogonal and commutes with P_1. The offsets are separated by at least one. Every high retained zero-offset copy lies in $\operatorname{ran}\Pi$, and every other high retained copy lies outside it. The zero-offset model multiplicities are the scalar multiplicities. Restriction to these pairs leaves at most $D_0(y) + O(1)$ unmatched copies through y. The sum of $|\lambda - \nu|$ over the retained pairs with $\min(\lambda, \nu) \le y$ is at most $S(y)$. The heat-trace estimates in Lemma 2.1 give

$$\operatorname{Tr}(\Pi e^{-t\Delta_{1,F}}) - \operatorname{Tr} e^{-t\Delta^X_{0,F_*}} = o(t^{2-d/2}).$$

The weighted heat formula therefore gives

$$V = \lim_{t\downarrow 0}(4\pi t)^{d/2}\operatorname{Tr}(\Pi e^{-t\Delta_{1,F}}) = V\int_{S^*M}\operatorname{rank}\sigma_0(\Pi)\,d\ell.$$

Since S^*M is connected, $\sigma_0(\Pi)$ has rank one. By (4.11), its range is the complexification of a real metric line $\mathcal{L}$. Let $\pi : S^*M \to M$ be the projection.

The relation $[\Pi, P_1] = 0$ and the subprincipal connection formula imply that $\mathcal{L}$ is preserved by geodesic parallel transport. Lemma 4.4 gives a real flat metric line L on M and an isometry

$$j : \pi^*L \longrightarrow \mathcal{L} \subset \pi^*(T^*M \otimes F), \qquad \nabla_{X_g} j = 0.$$

At 2π the ambient parallel transport is the identity. The identity $\tau_t^{T^*M\otimes F} j_v = j_{G_t v}\tau_t^L$ shows that each period-2π loop has trivial L-holonomy. Write also L for its complexification, and extend j complex linearly.

4.3.4. *Scalar multiplicities and the constant offset.* For $J = F, L$, put

$$P_J = (\Delta_{0,J} + \beta^2)^{1/2}, \qquad N_J = f_\beta(\Delta_{0,J}), \qquad Q_J = P_J^2 - N_J^2.$$

The scalar return symbol at 2π is ζ for both lines. Lemma 4.2 gives $N_J - P_J \in \Psi_{\mathrm{cl}}^{-1}$, with domains $H^1(J)$ for N_J and $H^2(J)$ for N_J^2. The operator Q_J is bounded and self-adjoint. Let C be an order-zero quantization of j, and put $B = \Pi C : L^2(L) \to \operatorname{ran}\Pi$. The differences $B^*B - I$ and $BB^* - \Pi$ have order -1, so B is Fredholm. On half-densities, the subprincipal connection formula [27, Proposition 3.2, (12) and (14), arXiv version 1], applied to $P_1 \oplus P_L$, gives

$$\sigma_0(P_1 B - B P_L) = \frac{1}{i} \nabla_{X_g} j = 0.$$

Hence $P_1 B - B P_L \in \Psi_{\mathrm{cl}}^{-1}$ and $N_1 B - B N_L \in \Psi_{\mathrm{cl}}^{-1}$. The map B takes $H^1(L)$ to $\Pi H^1(T^*M \otimes F)$. Since Π reduces N_1, the restriction $N_\Pi = N_1|_{\operatorname{ran}\Pi}$ is self-adjoint with compact resolvent and domain $\Pi H^1(T^*M \otimes F)$. Lemma 4.3, applied to N_L, N_Π, B and the lattice $\beta + \mathbb{Z}$, yields

$$\dim\ker(N_L - h) = \dim\big(\Pi \ker(N_1 - h)\big) \qquad (h \gg 1). \tag{4.12}$$

For $h \in \beta + \mathbb{Z}$, put

$$m_*(h) = \dim\ker\big((\Delta_{0,F_*}^X + \beta^2)^{1/2} - h\big), \qquad s(h) = \dim\big(\Pi \ker(N_1 - h)\big).$$

Every high retained zero-offset pair has the same rounded weight on both sides. After restriction to those pairs, let u_h count the unmatched copies in $\Pi\ker(N_1 - h)$ and v_h those in the model scalar space at weight h. For all sufficiently large h,

$$s(h) - m_*(h) = u_h - v_h.$$

For $h \le R$, an unmatched copy on M has eigenvalue at most $R^2 + \|Q_1\|$, and a model copy has eigenvalue at most R^2. Choose $c_0 > 0$ with $c_0^2 \ge \|Q_1\|$. Then

$$\sum_{h \le R} |\dim\ker(N_L - h) - m_*(h)| \le O(1) + D_0((R + c_0)^2) = o(R^{d-4}). \tag{4.13}$$

On $(\ker\Delta_{0,F})^\perp$, the operator $T = d_F \Delta_{0,F}^{-1/2}$ is an isometry onto the exact one-form subspace. Extend T by zero on $\ker\Delta_{0,F}$. Then $T \in \Psi_{\mathrm{cl}}^0$ with principal symbol $i|\xi|^{-1}\xi \otimes I_F$. The common rounding function gives $N_1 T = T N_F$ on the H^1 domains. Hence $Q_1 T = T Q_F$ as bounded maps from $L^2(F)$ to $L^2(T^*M \otimes F)$. On $(\ker\Delta_{0,F})^\perp$,

$$T^* \prod_{\delta \in \boldsymbol{D}} (Q_1 - \delta I) T = \prod_{\delta \in \boldsymbol{D}} (Q_F - \delta I).$$

The left side is compact. The scalar kernel is finite dimensional, so the polynomial on the right is compact on the full scalar space. Its principal symbol vanishes. Thus the smooth scalar symbol $\sigma_0(Q_F)$ takes values in $\boldsymbol{D}$. Connectedness gives a constant $c \in \boldsymbol{D}$ such that $Q_F - cI \in \Psi_{\mathrm{cl}}^{-1}$.

At each positive scalar eigenvalue, choose a one-form eigenbasis containing the image under T of a scalar eigenbasis, and assign the matched copies to these vectors. On a high retained scalar copy, the model offset is c: the eigenvalues of $Q_F - cI$ tend uniformly to zero, whereas the other offsets are at least one from c and $|\lambda - \nu| < 1/4$. Write z for the eigenvalue of $Q_F - cI$ on a scalar copy. On every sufficiently high retained copy, $z = \lambda - \nu$. At most $D_0(y) + O(1)$ scalar copies through eigenvalue y are unmatched. With $\omega = \sqrt{\lambda + \beta^2}$, for every $u > 0$ we obtain

$$\#\{\omega \le R : \omega^2 |z| > u\} \le O(1) + D_0(R^2) + u^{-1} R^2 S(R^2) = o(R^d).$$

Lemma 4.1, with $q = 2$, gives

$$Q_F - cI \in \Psi_{\mathrm{cl}}^{-3}. \tag{4.14}$$

Let $\chi_J : \pi_1(M) \to \{\pm 1\}$, $J = F, L$, be the holonomy characters. Let $Y \to M$ be the connected normal cover corresponding to $\ker\chi_F \cap \ker\chi_L$. Its degree is at most four. Both lines admit parallel unit

sections there, and every period-2π geodesic lifts closed with its velocity, since its two holonomies are trivial. Lemma 4.5 gives equality of the complete local return symbols for F and L. Their square-root symbols also agree in parallel unit frames. The logarithmic formula in Lemma 4.2, whose finite spectral modifications are smoothing, gives the same complete local symbols for $N_F - P_F$ and $N_L - P_L$. Consequently

$$Q_L - cI \in \Psi^{-3}_{\mathrm{cl}}.$$

Put $B_L = Q_L - cI \in \Psi^{-3}_{\mathrm{cl}}$. This operator is bounded, self-adjoint, and commutes with P_L. On $H^2(L)$,

$$\Delta_{0,L} - B_L = N_L^2 - \beta^2 + cI.$$

Lemma 4.6, applied to $\Delta_{0,L}$ and $-B_L$, gives

$$\operatorname{Tr} e^{-t\Delta_{0,L}} - e^{-ct} \operatorname{Tr} e^{-t(N_L^2-\beta^2)} = o(t^{2-d/2}).$$

For $y \geq 0$, set $R = \sqrt{y+\beta^2}$ in (4.13). The cumulative absolute multiplicity discrepancy through y is $o(y^{d/2-2})$. The finitely many negative values of $h^2 - \beta^2$ contribute $O(1) = o(t^{2-d/2})$ to the heat trace. Laplace–Stieltjes integration as in Lemma 2.1 therefore compares the rounded trace with the model scalar trace to the same order. The local scalar coefficients for F and L agree. Consequently

$$\operatorname{Tr} e^{-t\Delta_{0,F}} - e^{-ct} \operatorname{Tr} e^{-t\Delta^X_{0,F_*}} = o(t^{2-d/2}). \tag{4.15}$$

Comparison of the coefficients gives

$$a_{j,0}(M) = \sum_{\ell=0}^{j} \frac{(-c)^\ell}{\ell!}\, a_{j-\ell,0}(X), \qquad 0 \leq j \leq 2.$$

In particular, $a_{1,0}(M) = a_{1,0}(X) - cV$. By (2.3), $a_{1,1} = (d-6)a_{1,0}$. Equality of the one-form coefficients therefore gives $(d-6)cV = 0$, and hence $c = 0$ when $d \neq 6$.

In dimension six the present models are round and $\boldsymbol{D} = \{0, 4\}$. Their scalar coefficients per unit volume at orders one and two are 5 and 12. If $c = 4$, (4.15) would give $a_{1,0}(M) = V$ and $a_{2,0}(M) = 0$. Let W denote the Weyl curvature tensor of M. Equation (3.4) and the curvature decomposition give

$$360\, a_{2,0}(M) = \int_M \left(2|W|^2 + \frac{24}{5}s^2\right) dV_g \geq \frac{24}{5V}\left(\int_M s\, dV_g\right)^2 = \frac{864}{5}V > 0,$$

since $\int_M s\, dV_g = 6V$. This contradiction gives $c = 0$ also in dimension six. Thus the first three ordinary scalar and one-form coefficients agree with those of X, and (3.7) gives $J(M) \leq J(X)$.

4.3.5. *The quaternionic and Cayley returns.* For a round target, these heat equalities suffice for Proposition 3.4. Suppose now that $X = \mathbb{HP}^n$, $n \geq 2$, or $X = \mathbb{OP}^2$. The relation $Q_1 T = T Q_F$ and $c = 0$ give

$$q_1(\xi \otimes F) = 0.$$

The zero eigenspace of q_1 has rank one. The orthogonal projection onto exact one-forms $\Pi_e = d_F \Delta^{\dagger}_{0,F} \delta_F$ has principal symbol the projection onto $\mathbb{R}\xi \otimes F$. Therefore $\Pi - \Pi_e \in \Psi^{-1}_{\mathrm{cl}}$. Since Π is a spectral function of $\Delta_{1,F}$, it commutes with Π_e. The two operators $\Pi(I - \Pi_e)$ and $(I - \Pi)\Pi_e$ are compact orthogonal projections, hence have finite rank. They commute with $\Delta_{1,F}$; thus $\Pi = \Pi_e$ on all sufficiently high eigenspaces.

Restricting the retained pairs to the zero-offset families therefore gives a scalar matching with unmatched count at most $D_0(y) + O(1)$ and cost at most $S(y) + O(1)$. The shifted model scalar frequencies are $2j + \beta$, with $\beta = 2n + 1$ or 11. Their phases at π are $(-1)^\beta$. The retained scalar phase errors tend uniformly to zero at high frequency. The squared-return estimate in Section 4.2.1, with time π and reference phase $(-1)^\beta$, and Lemma 2.1 give the hypothesis of Proposition 2.4. Hence

$$G_\pi = \mathrm{id}, \qquad \sigma_0(e^{-i\pi P_F}) = (-1)^\beta.$$

For $v \in SM$, put $H_v = H_\pi(v)$. The one-form return symbol divided by the scalar F-return symbol is H_v. The equality $H_{2\pi}(v) = I$ gives $H_v^2 = I$. For a retained pair, set

$$f_t(\lambda) = e^{-i\pi\sqrt{\lambda+\beta^2}}e^{-t\lambda}, \qquad x = \min(\lambda, \nu).$$

Then

$$|f_t(\lambda) - f_t(\nu)| \le \left(\pi\left|\sqrt{\lambda+\beta^2} - \sqrt{\nu+\beta^2}\right| + t|\lambda-\nu|\right)e^{-tx}.$$

For $x \ge 1$, the factor in parentheses is at most $C_1(x^{-1/2}+t)$, with C_1 independent of the retained pair. The number of retained pairs with $x \le y$ is at most the model counting function through $y + 1/4$, hence is $O(1 + y^{d/2})$. Stieltjes integration gives

$$t^{d/2}\sum_{(\lambda,\nu)\in\mathcal{R}} |f_t(\lambda) - f_t(\nu)| = O(t^{1/2}) + O(t).$$

The omitted heat mass is $o(t^{-d/2})$. Thus the one-form return traces, multiplied by $t^{d/2}$, have the same limit on M and X. Put $r = 3$ for $X = \mathbb{HP}^n$ and $r = 7$ for $X = \mathbb{OP}^2$. The weighted local Weyl formula, the common scalar return symbol and (4.10) give

$$\begin{aligned}\lim_{t\downarrow 0}(4\pi t)^{d/2}\operatorname{Tr}(e^{-i\pi P_1}e^{-t\Delta_{1,F}}) &= (-1)^\beta V\int_{S^*M}\operatorname{tr} H_v\, d\ell(v)\\ &= (-1)^\beta V(2+2r-d).\end{aligned}$$

Since H_v is a smooth involution on connected S^*M, its trace is constant. Thus $\dim\ker(H_v - I) = (d + \operatorname{tr} H_v)/2 = r + 1$. It fixes the velocity, so its positive normal rank is r. Its negative rank $d - 1 - r$ is even. Thus $\det H_v = 1$, the orientation-line holonomy along each period-π geodesic is trivial, and the ordinary scalar return phase is $(-1)^\beta$. All hypotheses of Proposition 3.4 now hold. Thus M is isometric to X. □

Proof of Theorem B. Apply Lemma 2.1 in the measured ordinary degree. It gives $\dim M = d$ before Hodge star is applied. For $p = 1$ take the trivial lines F, F_*; for $p = d - 1$ take the complexified orientation lines. Hodge star gives the corresponding comparison of one-form spectra. If $X = \mathbb{CP}^n$, then $d > 4$ gives $n \ge 3$, and Proposition 4.8 applies. Every other target of dimension greater than four occurs in Proposition 4.9. □

5. Dimensions at most four

Write $\chi(M)$ for the Euler characteristic of M. For a finite covering $Y \to M$, denote its degree by $[Y : M]$.

5.1. **Round manifolds.** For a closed four-manifold M, the Gauss–Bonnet formula in the full tensor norm is

$$32\pi^2\chi(M) = \int_M \left(|\operatorname{Rm}|^2 - 4|\operatorname{Ric}|^2 + s^2\right) dV_g \tag{5.1}$$

[16, p. 106; Theorem 2.4.8, p. 122]. For a nonorientable M, apply the oriented formula to its orientation cover and divide by two.

Corollary 5.1. *Let $X = S^d(1)$ or $\mathbb{RP}^d(1)$, where $1 \le d \le 4$, and let $p \in \{1, d-1\}$. If a closed connected Riemannian manifold M satisfies $\operatorname{Spec}_p(M) = \operatorname{Spec}_p(X)$, then M is isometric to X.*

Proof. The case $X = \mathbb{RP}^4(1)$ is Lemma 5.2 below. For the other models, Weyl's law gives $\dim M = d$ and $\operatorname{Vol}(M) = \operatorname{Vol}(X)$. If $d = 1$, a closed connected one-manifold is a circle, and its length determines its metric up to isometry.

Suppose $d \ge 2$. Let F be the trivial complex line if $p = 1$ and the complexified orientation line otherwise. For $p \ne 1$, the model X is orientable; Hodge star intertwines Δ_p with $\Delta_{1,F}$ on M and with Δ_1 on X. Thus $\operatorname{Spec}_{1,F}(M) = \operatorname{Spec}_1(X)$ in either case. The first three heat coefficients of $\Delta_{1,F}$ have the

ordinary one-form curvature formulas, since F is flat of rank one. For $d = 2, 3$, the proof of Tanno's Theorem A(i) [46, p. 127] applies to these coefficients and gives constant sectional curvature one. Equation (3.9) then identifies M with X.

Let $X = S^4(1)$. The same heat coefficients and (5.1) give $\chi(M) \geq 2$ by the proof of Tanno's Theorem B [46, p. 128]. The positive one-form eigenvalues of X are $k(k+3)$ and $(k+1)(k+2)$, $k \geq 1$ [25, Theorem 4.2]. Their square roots after addition of $9/4$ are $k + 3/2 + O(k^{-1})$. Thus

$$U = \exp\bigl(-2\pi i\sqrt{\Delta_{1,F} + 9/4}\bigr) \quad \text{satisfies} \quad U + I \text{ compact.}$$

Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$ and a compact simply connected universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Since

$$[Y : M]\chi(M) = \chi(Y) \in \{2, 3\},$$

we have $Y = M$. In particular, M is simply connected and F is trivial.

Suppose $\chi(M) = 3$. Then $H^*(M; \mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, $|u| = 2$. The classifying-map construction in Section 4.2.2, with $n = 2$, gives a homotopy equivalence $M \to \mathbb{CP}^2$. Lemma 4.7 gives $\mu_{2\pi} = 1$. On the other hand, tangent parallel transport fixes the velocity. Since F is trivial, the principal symbol of U on the metric dual of this velocity is $\mu_{2\pi}$ by Proposition 2.3. Compactness of $U + I$ therefore gives $\mu_{2\pi} = -1$. This contradiction proves $\chi(M) = 2$. Now M is orientable, $\mathrm{Spec}_1(M) = \mathrm{Spec}_1(S^4(1))$, and $\chi(M) = \chi(S^4)$. Tanno's Theorem A(ii) [46, pp. 126 and 128] gives constant sectional curvature one. Equation (3.9) gives $M \simeq S^4(1)$ isometrically. □

Let $Y \to M$ be a finite universal covering of a closed connected four-manifold. If $H^*(Y; \mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, $|u| = 2$, every deck transformation fixes u^2 and preserves orientation. The Lefschetz calculation in Section 4.2.1, with $h = 2$, gives

$$H^*(Y; \mathbb{Z}) = \mathbb{Z}[u]/(u^3), \quad |u| = 2 \quad \Longrightarrow \quad [Y : M] = 1. \tag{5.2}$$

Lemma 5.2. *Let $p = 1$ or 3. A closed connected Riemannian manifold M with the ordinary p-spectrum of $\mathbb{RP}^4(1)$ is isometric to $\mathbb{RP}^4(1)$.*

Proof. Weyl's law gives $\dim M = 4$ and $\mathrm{Vol}(M) = \mathrm{Vol}(S^4(1))/2$. Put $X = \mathbb{RP}^4(1)$ and $z = e^{-3\pi i/2} = i$. For $Z = M, X$ and $t \in \mathbb{R}$, set $U_Z(t) = e^{-it\sqrt{\Delta_p^Z}}$. On X, the tangent parallel return at time π is $1 \oplus (-I_3)$: a geodesic lifts to a semicircle, and the antipodal differential identifies its endpoints. The even spherical harmonics give the scalar eigenvalues $2k(2k+3)$, $k \geq 0$ [25, Theorem 4.2], and hence the scalar phase z. Proposition 2.3 and the leading local heat coefficient consequently give

$$U_X(2\pi) + I \text{ compact}, \qquad \lim_{t \downarrow 0} \frac{\mathrm{Tr}(U_X(\pi)e^{-t\Delta_p^X})}{\mathrm{Tr}(e^{-t\Delta_p^X})} = \frac{z\ \mathrm{tr}\,\Lambda^p(1 \oplus (-I_3))}{\binom{4}{p}} = z(-1)^p\frac{4-2p}{4} \neq 0.$$

Equality of spectra gives the same two statements for U_M. Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$ and a compact simply connected universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$.

The involution G_π on the connected manifold SM is an isometry for the metric $h + G_\pi^* h$, where h is any metric. If $G_\pi \neq \mathrm{id}$, its fixed locus has measure zero [16, Lemma 4.5.1], and Proposition 2.9 makes the displayed limit zero. Hence $G_\pi = \mathrm{id}$.

Let H be tangent parallel transport around a geodesic of length π, with initial velocity v, and put $\mu = \mu_\pi$. Since $U_M(2\pi) = U_M(\pi)^2$, the principal symbols give

$$\mu^2 \Lambda^p(H^2) = -I.$$

The map H^2 preserves orientation and fixes v. For $p = 1$, $\Lambda^p(H^2)$ fixes $v^\flat$; for $p = 3$, it fixes $* v^\flat$, where $*$ is defined by any orientation of the initial tangent space. Hence $\mu^2 = -1$, and Lemma 2.7 gives $\det H = -1$.

The orientation line has holonomy -1 along this geodesic, so M is nonorientable. Its universal cover factors through its orientation double cover; hence $[Y : M]$ is even. Equation (5.2) therefore

excludes the cohomology ring of $\mathbb{CP}^2$. Thus Y is an integral homology four-sphere. By Hurewicz and Whitehead [19, Theorem 4.32 and Corollary 4.33], it is a smooth homotopy four-sphere. The manifold M is Besse, and Lin–Schmidt's theorem [34, Theorem 2] gives constant sectional curvature K. For $p = 1, 3$, the coefficient $a_{1,p}$ equals $-\frac{1}{3}\int_M s\, dV_g$ by (2.3). Its equality with the model coefficient, together with equality of volumes and $s = 12K$, gives $K = 1$. Equation (3.9) gives the assertion. □

Since $\mathbb{CP}^1$ and $\mathbb{HP}^1$ are round spheres of radius $1/2$, simultaneous dilation of the two metrics in a spectral comparison also gives Theorem A for these projective lines.

5.2. **The complex projective plane.** Write $X_2 = (\mathbb{CP}^2, g_{\mathrm{FS}})$, so that $\mathrm{Vol}(X_2) = \pi^2/2$ and $s_{X_2} = 24$. On a four-manifold, let W be the Weyl tensor and put $E = \mathrm{Ric} - sg/4$. The curvature decomposition and (5.1) give

$$32\pi^2\chi(M) = \int_M \left(|W|^2 - 2|E|^2 + s^2/6\right) dV_g. \tag{5.3}$$

For an oriented M, let W^+ and W^- be the self-dual and anti-self-dual Weyl tensors, and let $\tau(M)$ be its signature. In the same tensor norm, the signature formula [16, §2.2, pp. 98–99, and Theorem 3.1.1] is

$$48\pi^2\tau(M) = \int_M \left(|W^+|^2 - |W^-|^2\right) dV_g. \tag{5.4}$$

Lemma 5.3. *Let M be a closed connected Riemannian four-manifold whose universal cover Y is compact and has the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Fix $p \in \{0, 1, 2, 3, 4\}$ and suppose*

$$a_{j,p}(M) = a_{j,p}(X_2) \qquad (0 \le j \le 2).$$

If $p \in \{1, 3\}$ or $H^2(Y; \mathbb{Z}) \neq 0$, then M is isometric to X_2.

Proof. The coefficients of $\int s$ in (2.3), in degrees $0, 1, 2, 3, 4$, are $1/6, -1/3, -1, -1/3, 1/6$. Hence

$$V = \mathrm{Vol}(M) = \pi^2/2, \qquad \int_M s\, dV_g = 24V.$$

For $p = 1, 3$, the local heat coefficients coincide with the one-form coefficients. The proof of Tanno's Theorem B [46, p. 128], together with (5.3), gives $\chi(M) \ge 3$. On the other hand, $\chi(M) = \chi(Y)/[Y : M] \le 3$. Thus $\chi(M) = 3$, and M is Einstein by the equality case. Moreover, $\chi(M) = 3$ forces $[Y : M] = 1$ and the cohomology ring of $\mathbb{CP}^2$.

In any degree, if $H^2(Y; \mathbb{Z}) \neq 0$, the cohomology ring is $\mathbb{Z}[u]/(u^3)$, and (5.2) gives $[Y : M] = 1$. Thus in every case under consideration, M is simply connected with the integral cohomology ring of $\mathbb{CP}^2$. Orient it so that $\tau(M) = 1$, and set

$$A = \int_M |E|^2\, dV_g, \qquad B = \int_M (s - 24)^2\, dV_g.$$

Subtracting (5.4) from (5.3) gives

$$B = 12A - 12\int_M |W^-|^2\, dV_g. \tag{5.5}$$

For $p = 0$, the first three scalar heat coefficients agree with those of the Einstein manifold X_2. Both manifolds are orientable and have Euler characteristic three. By Tanno's Proposition 8 [45, p. 398], M is Einstein. The case $p = 4$ reduces to this case by Hodge star. Thus M is Einstein for $p = 0, 1, 3, 4$. Schur's theorem and $\int_M s\, dV_g = 24V$ give $s = 24$, whence $A = B = 0$. Equation (5.5) gives $W^- = 0$.

For $p = 2$, substitution of $|\mathrm{Ric}|^2 = |E|^2 + s^2/4$ and $|\mathrm{Rm}|^2 = |W|^2 + 2|E|^2 + s^2/6$ in (2.4)–(2.5), followed by (5.3) and $\chi(M) = 3$, gives

$$0 = 120\big(a_{2,2}(M) - a_{2,2}(X_2)\big) = 52A - B. \tag{5.6}$$

Together with (5.5), this yields

$$40A + 12\int_M |W^-|^2\, dV_g = 0.$$

Hence $A = B = 0$ and $W^- = 0$ in this case as well.

Consequently M is compact, simply connected, self-dual and Einstein, with $\mathrm{Ric} = 6g$. By Hitchin's positive-scalar-curvature classification [5, Theorem 13.30(1), p. 376], M is homothetic to the round S^4 or to Fubini–Study $\mathbb{CP}^2$. The cohomology excludes S^4, and $s = 24$ fixes the stated Fubini–Study normalization. □

Lemma 5.4. *Let Y be a compact simply connected Besse four-manifold with the integral cohomology of S^4. All prime geodesics have a common length L. For every $c > 0$,*

$$e^{-iL\sqrt{\Delta_{0,Y}+c}} + I \in \Psi_{\mathrm{cl}}^{-1}(Y).$$

Proof. By Hurewicz and Whitehead [19, Theorem 4.32 and Corollary 4.33], Y is a homotopy four-sphere. By Freedman's theorem [15, Theorem 1.6, p. 371], it is homeomorphic to S^4. Radeschi–Wilking's theorem [41, Theorem A, p. 913] gives a common prime length L; in particular, $G_L = \mathrm{id}$. By the argument preceding Lemma 2.6, the full periodic nullity at this period is seven and the prime Morse index has a common value b.

Let $i(Y)$ denote the least Morse index among nonconstant closed geodesics. Radeschi–Wilking give $i(Y) = 3$ [41, pp. 941–942]. Every prime geodesic is regular in the sense of [41, Definition 4.1], since its length is L. Thus [41, Corollary 4.5, with $q = 1$] gives, for every prime geodesic γ,

$$\mathrm{ind}(\gamma^m) \geq b + (4-1) + i(Y) = b + 6 \qquad (m \geq 2).$$

Every nonconstant closed geodesic is a prime geodesic or an iterate of one. Hence $b = i(Y) = 3$. Lemma 2.6 gives

$$\mu_L = \exp\bigl(-i\pi(3+4-1)/2\bigr) = -1.$$

Proposition 2.3, applied to the trivial line bundle, gives the assertion for every $c > 0$. □

Corollary 5.5. *Let M be a closed connected Riemannian manifold. If, for some $p \in \{0,1,2,3,4\}$,*

$$\mathrm{Spec}_p(M) = \mathrm{Spec}_p(X_2),$$

then M is isometric to X_2.

Proof. Weyl's law gives $\dim M = 4$. If $p = 4$, the equality $b_4(M) = 1$ gives orientability [19, Theorem 3.26]; Hodge star then gives $\mathrm{Spec}_0(M) = \mathrm{Spec}_0(X_2)$. It therefore suffices to consider $p \in \{0,1,2,3\}$.

Equality of the heat traces gives $a_{j,p}(M) = a_{j,p}(X_2)$ for $0 \leq j \leq 2$. Lemma 2.8 and (2.6) give compactness of $e^{-2\pi i\sqrt{\Delta_p+4}} - I$ on M. Proposition 2.5 supplies a compact simply connected Besse universal cover Y with the integral cohomology ring of S^4 or $\mathbb{CP}^2$. Lemma 5.3 applies directly for $p = 1, 3$.

For $p = 2$, the zero eigenspace gives $b_2(M) = 1$. Pullback $H^2(M;\mathbb{Q}) \to H^2(Y;\mathbb{Q})$ is injective by transfer for the finite cover [19, Proposition 3G.1]. Hence $H^2(Y;\mathbb{Z}) \neq 0$, and Lemma 5.3 applies.

It remains to treat $p = 0$. Suppose that Y has the integral cohomology of S^4. Let L be its common prime geodesic length, given by Lemma 5.4. By that lemma, $e^{-iL\sqrt{\Delta_{0,Y}+4}} + I$ is compact. Pullback by the Riemannian covering $Y \to M$ is injective and intertwines the scalar Laplacians. Thus every scalar eigenvalue of M occurs on Y. These eigenvalues are $4k(k+2)$, $k \geq 0$, and their shifted square roots are $2(k+1)$. Equation (2.6) therefore gives

$$e^{-2ijL} \longrightarrow -1 \qquad (j \to \infty).$$

Consecutive quotients yield $e^{-2iL} = 1$, which would give $e^{-2ijL} = 1$ for every j. This contradiction excludes the spherical cohomology ring. Lemma 5.3 completes the proof. □

For Kähler competitors, the scalar case of Corollary 5.5 follows from Tanno [45, Theorem D, p. 392], and the two-form case from Li [32, Theorem 1.2, p. 4, arXiv v2]. Perrone [40, Theorem 1, p. 871] proved the conclusion of Corollary 5.5 for $p = 1, 3$ when (M, g, J) is Kähler and $c_1(M,J) \in \mathbb{R}[\omega_g] \subset H^2(M;\mathbb{R})$, where $c_1(M,J)$ is the first Chern class in real cohomology and ω_g is the Kähler form.

Proof of Theorem A. For $d > 4$, the identity matching satisfies (1.3) with no omissions and zero cost, so Theorem B applies. For $d \le 4$, the round spheres and real projective spaces are covered by Corollary 5.1. The projective lines follow by dilation from that corollary. The only remaining model is $\mathbb{CP}^2$, to which Corollary 5.5 applies. □

6. Proportional degrees

6.1. **A finite criterion.** Fix integers $d > 4$ and $1 \le p \le d-1$. Put

$$x = \frac{p(d-p)}{d(d-1)}, \qquad y = \frac{p(p-1)(d-p)(d-p-1)}{d(d-1)(d-2)(d-3)},$$
$$\mathsf{A} = 2 - 30x + 180y, \qquad \mathsf{B} = -2 + 180x - 720y, \qquad \mathsf{C} = 5 - 60x + 180y.$$

When $\mathsf{A} \ne 0$, define

$$u = 1 - \frac{3\mathsf{B}}{2\mathsf{A}}, \qquad v = \frac{1}{d} - \frac{3}{2\mathsf{A}}\left(\frac{\mathsf{B}}{d} + \mathsf{C}\right). \tag{6.1}$$

For a polynomial f, write $[t^p]f(t)$ for its coefficient of t^p. For an integer $0 \le j \le d$, set

$$U_p(j) = \binom{d}{p}^{-1} [t^p](1-t)^j(1+t)^{d-j}. \tag{6.2}$$

Thus $\binom{d}{p}U_p(j)$ is the trace of the pth exterior power of an orthogonal involution whose negative eigenspace has dimension j.

Theorem 6.1. *Suppose that*

$$x \ne \frac{1}{6}, \qquad \mathsf{A} \ne 0, \qquad u \le 0, \qquad v \le 0. \tag{6.3}$$

Let M be a closed connected Riemannian manifold. If its ordinary p-form spectrum admits a matching with that of $S^d(1)$ or $\mathbb{RP}^d(1)$ after finitely many omissions, satisfying (1.4)*, then M is isometric to that model.*

Let (X, r) be $(\mathbb{CP}^n, 1)$ with $d = 2n$, or $(\mathbb{HP}^n, 3)$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose also that

$$\left.\begin{aligned} j \in 2\mathbb{Z}, \qquad 1 \le j \le 1 + \frac{32r}{15}, \\ |U_p(j)| = |U_p(r+1)| \end{aligned}\right\} \quad \Longrightarrow \quad j = r+1. \tag{6.4}$$

If the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying (1.4)*, then M is isometric to X. For $r = 1$, condition* (6.4) *is automatic; for $r = 3$, the possible values of j are* $2, 4, 6$.

For a d-dimensional metric g, write $\mathrm{Ric}^\circ = \mathrm{Ric} - (s/d)g$, and let J be as in (1.7).

Lemma 6.2. *Let M be a closed connected Riemannian manifold, and let X be one of the models in Theorem 6.1. Suppose that the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying* (1.4)*. If $x \ne 1/6$ and $\mathsf{A} \ne 0$, then*

$$\dim M = d, \qquad \mathrm{Vol}(M) = \mathrm{Vol}(X), \qquad \int_M s\, dV_g = s_X \,\mathrm{Vol}(X),$$

and

$$J(g) - J(g_X) = u \int_M |\,\mathrm{Ric}^\circ\,|^2\, dV_g + v \int_M (s - s_X)^2\, dV_g. \tag{6.5}$$

Proof. Corollary 2.2 gives the dimension, volume and equalities $a_{j,p}(M) = a_{j,p}(X)$ for $0 \le j \le 2$. The ratios C_1/C_0 and C_2/C_0 in (2.3)–(2.5) are x and y. Since $x \neq 1/6$, equation (2.3) gives

$$\int_M s\, dV_g = s_X \operatorname{Vol}(X).$$

Equations (2.4), (2.5) and (6.1) give

$$\frac{540\, a_{2,p}(M)}{\mathsf{A}\binom{d}{p}} = J(g) - u \int_M |\operatorname{Ric}^\circ|^2\, dV_g - v \int_M s^2\, dV_g.$$

The same identity holds on X, which is Einstein. Moreover, the equal volume and total scalar curvature give

$$\int_M s^2\, dV_g - s_X^2 \operatorname{Vol}(X) = \int_M (s - s_X)^2\, dV_g.$$

Subtraction gives (6.5). □

6.2. **The projective return.**

Lemma 6.3. *Let $X = \mathbb{CP}^n$ with $d = 2n$, or $X = \mathbb{HP}^n$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose $d > 4$ and $1 \le p \le d-1$. Let M be a closed connected Riemannian manifold whose ordinary p-form spectrum admits a matching with that of X after finitely many omissions, with $|\lambda - \nu| \to 0$ uniformly as the matched model eigenvalue ν tends to infinity. Then*

$$\dim M = d, \qquad \operatorname{Vol}(M) = \operatorname{Vol}(X), \qquad G_\pi = \mathrm{id}\,.$$

For every $v \in SM$, tangent parallel return $H = H_\pi(v)$ satisfies $H^2 = I$ and $\det H = 1$.

Proof. The matching forces $p \le \dim M$, since the model spectrum is infinite. The matched errors are bounded. If N_M and N_X are the counting functions, including multiplicities, there are constants C, C_0 such that, for all sufficiently large y,

$$N_X(y - C) - C_0 \le N_M(y) \le N_X(y + C) + C_0.$$

Weyl's law gives $\dim M = d$. The form bundles then have the same rank $\binom{d}{p}$, so $\operatorname{Vol}(M) = \operatorname{Vol}(X)$.

For $\mathbb{CP}^n$, Hodge duality and the Lefschetz decomposition reduce all form degrees to the integral eigenvalue families in [25, Eq. (1.5), §3 and Theorem 5.2(b)]. The scalar family $k(k+n)$ in that normalization becomes $4k(k+n)$ here; the same factor four multiplies every form eigenvalue.

For $\mathbb{HP}^n$, the form Laplacian of the metric induced by minus the Killing form equals minus the Casimir [25, Propositions 2.2–2.3, p. 519]. A metric dilation rescales all form Laplacians by the same factor. In orthonormal coordinates of type C_{n+1}, an occurring highest weight $b = (b_1, \dots, b_{n+1})$ satisfies $b_1 \ge \cdots \ge b_{n+1} \ge 0$ and $b_j \in \mathbb{Z}$ [49, §1, p. 410]. Its Laplace eigenvalue is proportional to

$$S(b) = \sum_{j=1}^{n+1} b_j\big(b_j + 2(n+2-j)\big),$$

with the same proportionality constant in every form degree. For the scalar weight $(1, 1, 0, \dots, 0)$, one has $S(b) = 4(n+1)$, whereas the eigenvalue is $8(n+1)$ by Section 4.3.1. Hence every form eigenvalue in the present normalization is $2S(b) \in 2\mathbb{Z}$. Thus, in both cases, $\operatorname{Spec}_p(X) \subset 2\mathbb{Z}$.

Let $a = n$ for $\mathbb{CP}^n$ and $a = 2n+1$ for $\mathbb{HP}^n$, and put $P_Y = (\Delta_{p,Y} + a^2)^{1/2}$ for $Y = M, X$. On X, Section 4.3.2 gives $G_\pi = \mathrm{id}$, scalar phase $(-1)^a$ and involutive tangent parallel return. Hence $e^{-2\pi i P_X} - I \in \Psi^{-1}_{\mathrm{cl}}$ by Proposition 2.3. For a matched pair,

$$\sqrt{\lambda + a^2} - \sqrt{\nu + a^2} = \frac{\lambda - \nu}{\sqrt{\lambda + a^2} + \sqrt{\nu + a^2}} \longrightarrow 0.$$

By (2.6), $e^{-2\pi i P_M} - I$ is compact. Proposition 2.5 gives $G_{2\pi} = \mathrm{id}$. By Proposition 2.3, $e^{-2\pi i P_M} \in \Psi^0_{\mathrm{cl}}$. Its compact difference from I has zero principal symbol, so $e^{-2\pi i P_M} - I \in \Psi^{-1}_{\mathrm{cl}}$.

Write $P = P_M$. Apply Lemma 4.2 with $\Pi = I$, $T = 2\pi$, $\beta = a$ and $s = 1$. Since $a \in \mathbb{Z}$, it gives a positive self-adjoint operator N commuting with P, with $\operatorname{Spec} N \subset \mathbb{Z}$, such that

$$P - N \in \Psi_{\mathrm{cl}}^{-1}, \qquad Q = (P-N)(P+N) = P^2 - N^2 \in \Psi_{\mathrm{cl}}^{0}.$$

Initially defined on smooth forms, Q extends to a bounded self-adjoint operator on L^2. On a common eigenvector, write $\omega = \sqrt{\lambda + a^2}$ for the eigenvalue of P and $h \in \mathbb{Z}$ for that of N. Since integer squares have the same parity as their bases,

$$\begin{aligned} e^{-i\pi\omega} - (-1)^a e^{i\pi(\lambda + a^2 - h^2)} \\ = (-1)^h \left(e^{-i\pi(\omega - h)} - e^{i\pi\lambda} \right). \end{aligned}$$

Now $\omega - h \to 0$ and $\operatorname{dist}(\lambda, 2\mathbb{Z}) \to 0$. Equation (2.6) gives

$$e^{-i\pi P} - (-1)^a e^{i\pi Q} \quad \text{compact.}$$

For $C \in \Psi_{\mathrm{cl}}^{0}$ with scalar principal symbol, $[Q, C] \in \Psi_{\mathrm{cl}}^{-1}$. The identity

$$[e^{i\pi Q}, C] = i \int_0^\pi e^{isQ} [Q, C] e^{i(\pi - s)Q} \, ds$$

is a norm-convergent integral of compact operators. Consequently $e^{i\pi P} C e^{-i\pi P} - C$ is compact. Egorov's theorem [27, arXiv version, Theorem 1.4 and Proposition 3.3] gives $c \circ G_\pi = c$ for $c = \sigma_0(C)$. Smooth scalar symbols separate the points of S^*M, so $G_\pi = \mathrm{id}$.

Let μ be the scalar phase at π and fix $v \in SM$. For $H = H_\pi(v)$, the principal symbol of the squared return is

$$\mu^2 \Lambda^p(H^2) = I.$$

Since H^2 is orthogonal and $1 \le p \le d-1$, scalarity of $\Lambda^p(H^2)$ implies scalarity of H^2 [35, §5.2, p. 11]. As H fixes the velocity, $H^2 = I$ and $\mu^2 = 1$. Lemma 2.7 gives $\det H = 1$. □

Lemma 6.4. *Let M be a closed connected Riemannian manifold. Let $(X, r) = (\mathbb{CP}^n, 1)$ with $d = 2n$, or $(X, r) = (\mathbb{HP}^n, 3)$ with $d = 4n$, and give X its metric of maximal sectional curvature four. Suppose $d > 4$ and $1 \le p \le d-1$. Suppose that* (6.4) *holds and that the ordinary p-spectra of M and X admit a matching after finitely many omissions satisfying* (1.4). *If* (3.2) *holds, then M is isometric to X.*

Proof. Put $h = r + 1$ and $b = (d + h - 2)/2$. On X, tangent parallel return at π has positive rank h and negative rank $d - h$, and the scalar phase at π is $(-1)^b$ by Section 4.3.2. Lemma 6.3 gives $\dim M = d$, $G_\pi = \mathrm{id}$, $H_\pi(w)^2 = I$ and $\det H_\pi(w) = 1$ for every $w \in SM$. Put $a = \dim \ker(H_\pi(w) - I)$. This integer is constant on the connected bundle SM, and $a \ge 1$ because $H_\pi(w)$ fixes the velocity. Since d is even and $\det H_\pi(w) = (-1)^{d-a} = 1$, the integer a is even.

For $Y = M, X$, set

$$\begin{aligned} P_Y = (\Delta_{p,Y} + b^2)^{1/2}, \qquad & Z_Y(t) = \operatorname{Tr}(e^{-t\Delta_{p,Y}}), \\ T_Y(t) = \operatorname{Tr}(e^{-i\pi P_Y} e^{-t\Delta_{p,Y}}), \quad & L_Y(t) = T_Y(t)/Z_Y(t). \end{aligned}$$

Write $\zeta_\lambda = e^{-i\pi\sqrt{\lambda + b^2}}$. For each matched pair,

$$|\zeta_\lambda e^{-t\lambda} - \zeta_\nu e^{-t\nu}| \le |e^{-t\lambda} - e^{-t\nu}| + |\zeta_\lambda - \zeta_\nu| e^{-t\nu}.$$

Corollary 2.2 and Lemma 2.1 bound the omitted heat mass and the sum of the first terms by $o(t^{2-d/2})$. The square-root identity in the proof of Lemma 6.3 gives $|\zeta_\lambda - \zeta_\nu| \to 0$ uniformly on the matched tail. Thus, for each $\varepsilon > 0$,

$$\begin{aligned} |T_M(t) - T_X(t)| &\le o(t^{2-d/2}) + O_\varepsilon(1) + \varepsilon Z_X(t), \\ Z_M(t) \sim Z_X(t) &\sim (4\pi t)^{-d/2} \binom{d}{p} \operatorname{Vol}(X). \end{aligned}$$

Since $|T_X| \le Z_X$,

$$|L_M - L_X| \le \frac{|T_M - T_X| + |Z_M - Z_X|}{Z_M} \longrightarrow 0,$$

by first letting $t \downarrow 0$ and then $\varepsilon \downarrow 0$.

Proposition 2.3 gives $e^{-i\pi P_Y} \in \Psi^0_{\mathrm{cl}}$ for $Y = M, X$. The scalar phase μ_π on M is constant. The return symbol (2.7) and the local heat Weyl formula [31, Eqs. (1.22)–(1.23)] give

$$\lim_{t\downarrow 0} L_X(t) = (-1)^b U_p(d-h), \qquad \lim_{t\downarrow 0} L_M(t) = \mu_\pi U_p(d-a).$$

Since $|\mu_\pi| = 1$ and $U_p(d-j) = (-1)^p U_p(j)$, it follows that

$$|U_p(a)| = |U_p(h)|. \tag{6.6}$$

The positive normal rank is $a-1$. Lemma 3.1 gives (3.3). Lemma 3.2 gives $a - 1 \le i + a - 1 \le (32/135)e(w)$ for every $w \in SM$, where $i \ge 0$ is the normal Morse index. Invariance of Liouville measure under the geodesic flow gives

$$\begin{aligned} a - 1 &\le \frac{32}{135}\int_{SM} e(w)\, d\ell(w) \\ &= \frac{32}{135}\left\langle \|K_w - I\|^2_{\mathrm{HS}} \right\rangle \le \frac{32r}{15}. \end{aligned}$$

Thus (6.4) and (6.6) force $a = h$. The normal return has positive rank r, so Proposition 3.4 applies. □

Proof of Theorem 6.1. By Lemma 6.2 and (6.3), (3.2) holds. For a round target, Proposition 3.4 gives the conclusion. For a complex or quaternionic target, apply Lemma 6.4. □

6.3. **The interval of degrees.**

Proof of Theorem C. For $K = \emptyset$, take $d_K = 17$. Fix a nonempty compact set $K \subset (\rho_-, \rho_+) \setminus \{\rho_0\}$. First suppose $p \le d/2$, and put $\rho = p/d$, $z = \rho(1-\rho)$ and $\varepsilon = d^{-1}$. Then

$$x = \frac{z}{1-\varepsilon}, \qquad y = \frac{z^2 - \varepsilon z + \varepsilon^2 z}{(1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon)}.$$

Thus $x = z + O(d^{-1})$ and $y = z^2 + O(d^{-1})$, uniformly for $0 \le p \le d$. Put

$$\begin{aligned} A_\infty &= 180\left(z - \frac{1}{12}\right)^2 + \frac{3}{4}, \\ u_\infty &= \frac{10 - 600z + 2520z^2}{2A_\infty}, \quad v_\infty = -\frac{15(1-6z)^2}{2A_\infty}. \end{aligned} \tag{6.7}$$

Since $A_\infty \ge 3/4$, the definitions give $\mathsf{A} = A_\infty + O(d^{-1})$, $u = u_\infty + O(d^{-1})$ and $v = v_\infty + O(d^{-1})$, uniformly for $0 \le p \le d$. Moreover,

$$u_\infty < 0 \iff \frac{5 - 3\sqrt{2}}{42} < z < \frac{5 + 3\sqrt{2}}{42} \iff \rho_- < \rho < \rho_+.$$

In the range $0 \le \rho \le 1/2$, one has $v_\infty < 0$ precisely when $\rho \ne \rho_0$. Hence

$$c_K = \min_{\rho \in K}\left\{\left|z - \frac{1}{6}\right|, -u_\infty, -v_\infty\right\} > 0.$$

For all sufficiently large d, uniformly for $\rho \in K$,

$$\mathsf{A} \ge \frac{3}{8}, \qquad \left|x - \frac{1}{6}\right| \ge \frac{c_K}{2}, \qquad u, v \le -\frac{c_K}{2}.$$

These are the conditions (6.3).

Fix an integer $j \geq 0$ and suppose $d \geq 2j$. Write $(q)_\ell = q(q-1)\cdots(q-\ell+1)$ and $(q)_0 = 1$. At binomial parameter $1/2$, the generating and hypergeometric formulae for the Krawtchouk polynomials [38, Eqs. (18.20.6) and (18.23.3)], with the normalization (6.2), give

$$U_p(j) = \sum_{\ell=0}^{j} (-2)^\ell \binom{j}{\ell} \frac{(p)_\ell}{(d)_\ell}.$$

For $0 \leq p \leq d$ and $0 \leq \ell \leq j$, all factors $(p-k)/(d-k)$ have absolute value at most one, and

$$\left| \frac{(p)_\ell}{(d)_\ell} - \rho^\ell \right| \leq \sum_{k=0}^{\ell-1} \left| \frac{p-k}{d-k} - \rho \right| \leq \frac{\ell(\ell-1)}{d}.$$

Consequently,

$$U_p(j) = (1-2\rho)^j + O_j(d^{-1}), \tag{6.8}$$

uniformly for $0 \leq p \leq d$. Put $s = 1 - 2\rho$. Since $0 < s < 1$ on K,

$$\delta_K = \min_{\rho \in K} s^4(1-s^2) > 0.$$

The differences between the distinct powers s^2, s^4, s^6 are at least δ_K. Equation (6.8) therefore gives, for all sufficiently large d and $\rho \in K$,

$$\big| |U_p(j)| - |U_p(k)| \big| \geq \frac{\delta_K}{2} \qquad (j,k \in \{2,4,6\},\ j \neq k).$$

This gives (6.4) for $r = 3$; for $r = 1$ the condition is automatic.

The identities

$$x(d, d-p) = x(d,p), \qquad y(d,d-p) = y(d,p), \qquad U_{d-p}(j) = (-1)^j U_p(j)$$

show that (6.3) and (6.4) are invariant under $p \mapsto d-p$. They therefore hold for the degree p whenever $\min(p, d-p)/d \in K$. Choose a common threshold $d_K \geq 17$ for the preceding estimates. Theorem 6.1 applies to every model of dimension $d \geq d_K$ in (1.1), since the Cayley plane has dimension sixteen.

Choose closed intervals $I_- \subset (\rho_-, \rho_0)$ and $I_+ \subset (\rho_0, \rho_+)$. For all sufficiently large d, every integral degree p with p/d in $I_- \cup I_+$ or its reflection about $1/2$ lies in D_d^*. These four intervals are disjoint. For $I = [\alpha, \beta]$,

$$\#\{p \in \mathbb{Z} : p/d \in I\} = \lfloor d\beta \rfloor - \lceil d\alpha \rceil + 1 = d|I| + O(1).$$

It follows that

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \geq 2\big(|I_-| + |I_+|\big).$$

Letting the endpoints of I_- and I_+ tend to the endpoints of their respective open intervals gives

$$\liminf_{d\to\infty} \frac{|D_d^*|}{d+1} \geq 2(\rho_+ - \rho_-) = \sqrt{\frac{8}{21}} = 0.617213399848\ldots. \qquad \square$$

Remark 6.5. Put

$$\sigma_- = \frac{1}{2}\left(1 - \sqrt{\frac{15+\sqrt{185}}{30}}\right), \qquad \sigma_+ = \frac{1}{2}\left(1 - \sqrt{\frac{15-\sqrt{185}}{30}}\right).$$

Let $L \subset (\sigma_-, \sigma_+) \setminus \{\rho_0\}$ be compact. For all sufficiently large integers n and every integer p with $p/(2n) \in L$, a closed connected Kähler manifold of complex dimension n with the same ordinary p-spectrum as $\mathbb{CP}^n$ is holomorphically isometric to $\mathbb{CP}^n$ [23, Corollary 2.4 and p. 280]. One has

$$0 < \sigma_- < \rho_- < \rho_0 < \rho_+ < \sigma_+ < \frac{1}{2}.$$

References

[1] J. F. Adams, Vector fields on spheres, *Ann. of Math.* (2) **75** (1962), 603–632.

[2] M. Amann, C. Lange, and M. Radeschi, Odd-dimensional orbifolds with all geodesics closed are covered by manifolds, *Math. Ann.* **380** (2021), 1355–1386.

[3] B. Ammann, R. Lauter, V. Nistor and A. Vasy, Complex powers and non-compact manifolds, *Comm. Partial Differential Equations* **29** (2004), no. 5–6, 671–705. doi:10.1081/PDE-120037329. Preprint: https://arxiv.org/abs/math/0211305v1.

[4] V. I. Arnol'd, Characteristic class entering in quantization conditions, *Funct. Anal. Appl.* **1** (1967), no. 1, 1–13. doi:10.1007/BF01075861.

[5] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987. doi:10.1007/978-3-540-74311-8.

[6] R. G. Bettiol, E. A. Lauret and P. Piccione, The first eigenvalue of a homogeneous CROSS, *J. Geom. Anal.* **32** (2022), article 76, 63 pp.

[7] Q.-S. Chi, Curvature characterization and classification of rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), no. 1, 31–42.

[8] K. Cieliebak, N. Hingston and A. Oancea, Poincaré duality for loop spaces, *Compos. Math.* **161** (2025), 3140–3212. https://arxiv.org/abs/2008.13161v3.

[9] K. Cieliebak, A. Oancea and E. Shelukhin, Resonances and string point invertibility for compact rank one symmetric spaces, arXiv:2608.04691v1 (2026).

[10] R. L. Cohen, J. D. S. Jones and J. Yan, The loop homology algebra of spheres and projective spaces, in *Categorical Decomposition Techniques in Algebraic Topology*, Progr. Math. **215**, Birkhäuser, 2003, 77–92.

[11] R. L. Cohen, J. R. Klein and D. Sullivan, The homotopy invariance of the string topology loop product and string bracket, *J. Topol.* **1** (2008), 391–408.

[12] Y. Colin de Verdière, Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques, *Comment. Math. Helv.* **54** (1979), 508–522.

[13] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975), 39–79.

[14] J. J. Duistermaat and L. Hörmander, Fourier integral operators. II, *Acta Math.* **128** (1972), 183–269.

[15] M. H. Freedman, The topology of four-dimensional manifolds, *J. Differential Geom.* **17** (1982), 357–453.

[16] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, 1984.

[17] H. Gimperlein and G. Grubb, Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators, *J. Evol. Equ.* **14** (2014), 49–83. doi:10.1007/s00028-013-0206-2. Preprint: arXiv:1302.6529v4.

[18] A. Gray and L. Vanhecke, Riemannian geometry as determined by the volumes of small geodesic balls, *Acta Math.* **142** (1979), 157–198. doi:10.1007/BF02395060.

[19] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[20] A. Hatcher, *Vector Bundles & K-Theory*, version 2.2, November 2017. https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[21] N. Higson, The Baum–Connes conjecture, *Doc. Math.*, Extra Volume ICM 1998, II, 637–646. doi:10.4171/DMS/1-2/61.

[22] L. Hörmander, Fourier integral operators. I, *Acta Math.* **127** (1971), 79–183. doi:10.1007/BF02392052.

[23] X. Huang, K. Liu, H. Xu, and Y. Zhi, Asymptotic characterization of complex projective space, *Algebraic Geometry and Physics* **2** (2025), no. 2, 271–308.

[24] K. Ii, Curvature and spectrum of Riemannian manifold, *Tohoku Math. J.* (2) **25** (1973), 557–567.

[25] A. Ikeda and Y. Taniguchi, Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbb{C})$, *Osaka J. Math.* **15** (1978), 515–546.

[26] D. Jakobson, Y. Safarov and A. Strohmaier, The semiclassical theory of discontinuous systems and ray-splitting billiards, with an appendix by Y. Colin de Verdière, *Amer. J. Math.* **137** (2015), 859–906.

[27] D. Jakobson and A. Strohmaier, High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows, *Comm. Math. Phys.* **270** (2007), 813–833. Preprint: arXiv:math/0607616v1.

[28] E. Kaneda, The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces II, *J. Math. Kyoto Univ.* **24** (1984), no. 1, 141–162. doi:10.1215/kjm/1250521390.

[29] R. Kuwabara, Spectrum of the Laplacian on vector bundles over $C_{2\pi}$-manifolds, *J. Differential Geom.* **27** (1988), no. 2, 241–258. doi:10.4310/jdg/1214441782.

[30] J. M. Lee, *Introduction to Riemannian Manifolds*, second edition, Graduate Texts in Mathematics 176, Springer, 2018. doi:10.1007/978-3-319-91755-9.

[31] L. Li and A. Strohmaier, The local counting function of operators of Dirac and Laplace type, *J. Geom. Phys.* **104** (2016), 204–228.

[32] P. Li, The spectral rigidity of complex projective spaces, revisited, *Math. Z.* **290** (2018), 1115–1143. doi:10.1007/s00209-018-2055-8. Preprint: arXiv:1608.02737v2.

[33] P. Li, X. Sun, and A. Zhu, The spectral rigidity of Ricci soliton and Einstein-type manifolds, *Comm. Anal. Geom.*, published online 22 October 2026, 32 pp. doi:10.4310/CAG.261018010944. Preprint: arXiv:2312.07259v1 (2023).

[34] S. Lin and B. Schmidt, Real projective spaces with all geodesics closed, *Geom. Funct. Anal.* **27** (2017), 631–636, doi:10.1007/s00039-017-0407-x.

[35] K. Liu and H. Xu, A proof of the spectral rigidity for complex projective space, arXiv:2611.03536v1, 2 November 2026.
[36] K. Mashimo, Spectra of the Laplacian on the Cayley projective plane, *Tsukuba J. Math.* **21** (1997), no. 2, 367–396. doi:10.21099/tkbjm/1496163248.
[37] A. Nifa, *Spectral resolution and rigidity of spherical space forms*, preprint, version 2, 23 October 2026. doi:10.5281/zenodo.23586002. Author's version: CVGMT 8220.
[38] *NIST Digital Library of Mathematical Functions*, Release 1.2.8 of 2026-09-15, https://dlmf.nist.gov/.
[39] V. K. Patodi, Curvature and the fundamental solution of the heat operator, *J. Indian Math. Soc.* (N.S.) **34** (1970), 269–285.
[40] D. Perrone, On the spectral rigidity of $\mathbb{CP}^n$, *Proc. Amer. Math. Soc.* **104** (1988), no. 3, 871–875. doi:10.1090/S0002-9939-1988-0964867-5.
[41] M. Radeschi and B. Wilking, On the Berger conjecture for manifolds all of whose geodesics are closed, *Invent. Math.* **210** (2017), 911–962.
[42] K. Shankar, R. Spatzier and B. Wilking, Spherical rank rigidity and Blaschke manifolds, *Duke Math. J.* **128** (2005), 65–81.
[43] N. Sthanumoorthy, Spectral invariant of the zeta function of the Laplacian on $\mathrm{Sp}(r+1)/\mathrm{Sp}(1)\times\mathrm{Sp}(r)$, *Arch. Math. (Brno)* **27** (1991), 95–104. DML-CZ 107408.
[44] C. J. Sutton, On the Poisson relation for compact Lie groups, *Ann. Global Anal. Geom.* **57** (2020), no. 4, 537–589. Preprint: arXiv:1606.07426.
[45] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, *Tohoku Math. J.* (2) **25** (1973), 391–403.
[46] S. Tanno, The spectrum of the Laplacian for 1-forms, *Proc. Amer. Math. Soc.* **45** (1974), 125–129.
[47] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, vol. 99, Amer. Math. Soc., Providence, RI, 2009. https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf.
[48] G. Tsagas, The spectrum of the Laplace operator for a special Riemannian manifold, *Kodai Math. J.* **4** (1981), 377–382.
[49] C. Tsukamoto, Spectra of Laplace–Beltrami operators on $\mathrm{SO}(n+2)/\mathrm{SO}(2)\times\mathrm{SO}(n)$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(1)\times\mathrm{Sp}(n)$, *Osaka J. Math.* **18** (1981), 407–426.
[50] J.-Y. Wu, On almost isospectral manifolds, *Indiana Univ. Math. J.* **39** (1990), no. 4, 1373–1381.

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